Lionel London

An introduction to black hole perturbation theory

UK Royal Society

Kavli-Villum Summer School September 2023 Corfu

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Lecture Resources on GitHub

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Preface Organization and scope of lectures

- Some knowledge of the following will be assumed:
  - Differential calculus, General Relativity, Ordinary Differential Equations
  - There are no stupid questions (almost)
- Philosophy of the lecturer
  - Black hole perturbation theory is a story with a plot, characters, twists …
  - Let the story drive understanding of the details, not the other way around (!)
  - Concepts before calculations, forest before trees, horses before carts, the etc …
  - The tortoise beats the hare …
- Goal of the lectures
  - Help you formulate questions that amount to “What happens next?”
  - Provide a few tools that may help answer some of your future questions

Preface Organization and scope of lectures

Topics that will be covered

- Lecture #1
  - **1.1** “Belief in the possible presence of BHs” (Linear Stability)
  - **1.2** Einstein’s equations linearized around Kerr (Teukolsky’s Equations)
  - **1.3** Black hole specific special functions (Quasinormal Modes, QNMs)
- Lecture #2
  - **2.1** So many quasi-normal modes (Structure of the QNM solution space)
  - **2.2** Introduction to QNM Orthogonality
  - **2.3** What’s next? (Some known open questions)

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- Lecture #1
  - **1.1** "Belief in the possible presence of BHs" (Linear Stability)
  - **1.2** Einstein's equations linearized around Kerr (Teukolsky's Equations)
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  - **2.3** What's next? (Some known open questions)

Preface Organization and scope of lectures

Some overlapping topics mostly left to references and upcoming lectures

- Second order perturbations (nonlinear QNMs)
  - Campanelli+ 1999, London+ 2014, Lagos+Hui 2023, Cheung+ 2023, others
- Extreme mass ratio inspirals and self-force (Adam)
- Beyond GR (Lavinia)
- QNMs and GW signal modeling
  - Husa 2015, Ossokine+ 2020, Hamilton+ 2023, and many others …
- Tests of GR with QNMs (Gregorio)

Preface Organization and scope of lectures

My favorite topical reviews

- The characteristic `sound' of black holes and neutron stars (Nollert)
- Quasi-Normal Modes of Stars and Black Holes (Kokkotas+)
- Quasinormal modes of black holes and branes (Berti+)

"The discovery of pulsars and the conviction that they are neutron stars resulting from gravitational collapse have strengthened *the belief in the possible presence of Schwarzschild black holes*…"

Vishveshwara,19
70

1.1 On the possible presence of black holes

"We have equilibrium, but ***is it stable?***"

Regge and Wheeler, 1957

- By the 1950s, there were a number of known "black hole" solutions to Einstein's equations. Schwarzschild's (1915) was the first and simplest.
- But for black holes to be physical objects they must also survive perturbations.
- Example of an unstable object: sphere of water surrounded by a shell of mercury held together by self-gravity — *pluck it*, and mercury falls to the center.
- Regge and Wheeler studied metric perturbations of the Schwarzschild solution.

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$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \end{aligned}$$

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$$\underbrace{g'_{\alpha\beta}}_{\text{perturbed metric}} = \underbrace{g_{\alpha\beta}}_{\text{background metric}} + \underbrace{h_{\alpha\beta}}_{\text{perturbation}}$$

1.1 On the possible presence of black holes

1950s-1970s

"We have equilibrium, but *is it stable?*"

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.
- The expected key result is a differential equation for the metric perturbation.
- Guiding concept: If perturbation solutions grow in time, then the BH is unstable. If they shrink, then the BH is stable.

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1.1 On the possible presence of black holes

"We have equilibrium, but ***is it stable?***"

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

Notation: Perturbed quantities are primed, and $G = c = 1$.

$$\underbrace{g'_{\alpha\beta}}_{\text{perturbed metric}} = \underbrace{g_{\alpha\beta}}_{\text{background metric}} + \underbrace{h_{\alpha\beta}}_{\text{perturbation}}$$

1.1 On the possible presence of black holes

"We have equilibrium, but ***is it stable?***"

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric (compute the Ricci tensor)

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

First think about the unperturbed case.

1.1 On the possible presence of black holes

"We have equilibrium, but ***is it stable?***"

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric (compute the Ricci tensor)

$$G_{\mu\nu} = 0$$

The right-hand-side is zero because the BH is isolated.

1.1 On the possible presence of black holes

"We have equilibrium, but ***is it stable?***"

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric (compute the Ricci tensor)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

The left-hand-side is determined by the Ricci tensor and scalar.

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1. Re-derive the field equations using the perturbed metric (compute the Ricci tensor)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \underbrace{\cancel{R}}_{=0} = 0$$

R=0 for Schwarzschild.

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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric (compute the Ricci tensor)

$$R_{\mu\nu}(g_{\mu\nu}) = 0$$

Schematically, the Ricci scalar is a function of the metric.

1.1 On the possible presence of black holes

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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric (compute the Ricci tensor)

$$R_{\mu\nu}(g'_{\mu\nu}) = 0$$

The same is true for the perturbed case.

1.1 On the possible presence of black holes

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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric

$$R_{\mu\nu}(g_{\mu\nu} + h_{\mu\nu}) \;=\; R_{\mu\nu}(g_{\mu\nu}) + \delta R_{\mu\nu}(g_{\mu\nu}, h_{\mu\nu})$$

Expanding this gives …

1.1 On the possible presence of black holes

"We have equilibrium, but ***is it stable?***"

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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric

$$R_{\mu\nu}(g_{\mu\nu}+h_{\mu\nu}) = \cancelto{=0}{R_{\mu\nu}(g_{\mu\nu})} + \delta R_{\mu\nu}(g_{\mu\nu}, h_{\mu\nu})$$

We're only interested in the deviation.

1.1 On the possible presence of black holes

*"We have equilibrium, but **is it stable?**"*

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric

covariant derivative

$$\delta R_{\mu\nu} = -\delta\Gamma^{\beta}_{\mu\nu;\beta} + \delta\Gamma^{\beta}_{\mu\beta;\nu}$$

This can be written only in terms of the deviation of the Christoffel symbols

1.1 On the possible presence of black holes

"We have equilibrium, but ***is it stable?***"

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric

$$\delta\Gamma^{\alpha}_{\beta\gamma} \;=\; \frac{1}{2}g^{\alpha\nu}\left(h_{\beta\nu;\gamma}+h_{\gamma\nu;\beta}-h_{\beta\gamma;\nu}\right)$$

After turning the tensor crank, it can be shown that the deviation of the Christoffel symbols is …

1.1 On the possible presence of black holes

"We have equilibrium, but ***is it stable?***"

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric

$$\begin{aligned} \delta\Gamma^{\alpha}_{\beta\gamma} &= \frac{1}{2}g^{\alpha\nu}(h_{\beta\nu;\gamma} + h_{\gamma\nu;\beta} - h_{\beta\gamma;\nu}) \\ &= 0 \end{aligned}$$

We have arrived at the field equations, but more strategy is needed to proceed …

1.1 On the possible presence of black holes

"We have equilibrium, but ***is it stable?***"

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form (a summary)

- Make use of symmetry: Decompose the perturbation into tensor spherical harmonics so that the related moments are rationally invariant.
- Use a simplifying coordinate gauge, i.e. "Regge-Wheeler" gauge.
- Consider "odd" (even/polar) and "even" (odd/axial) parity terms separately

1.1 On the possible presence of black holes

"We have equilibrium, but ***is it stable?***"

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form

- Make use of symmetry

$$h_{\mu\nu} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{p}^{10} C_{\ell m}^{(p)}(t,r)\,(Y_{\ell m}^{(p)})_{\mu\nu}(\theta,\phi)$$

1.1 On the possible presence of black holes

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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form

- Make use of symmetry

e.g. $\frac{\partial}{\partial_\theta}\frac{\partial}{\partial_\phi}Y_{\ell m}(\theta,\phi)$

$$h_{\mu\nu} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{p}^{10} C_{\ell m}^{(p)}(t,r)\,\boxed{(Y_{\ell m}^{(p)})_{\mu\nu}(\theta,\phi)}$$

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"We have equilibrium, but ***is it stable?***"

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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form

Regge-Wheeler's "odd"

= Chandrasekhar's "polar"

- Handle different parity components separately

$$
\begin{aligned}
(Y^{(\text{odd})}_{\ell m})_{\mu\nu}(\pi-\theta,\phi+\pi) &= (Y^{(\text{polar})}_{\ell m})_{\mu\nu}(\pi-\theta,\phi+\pi) \\
&= (-1)^{\ell}(Y^{(\text{polar})}_{\ell m})_{\mu\nu}(\theta,\pi)
\end{aligned}
$$

1.1 On the possible presence of black holes

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Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form

Regge-Wheeler's "even"

= Chandrasekhar's "axial"

- Handle different parity components separately

$$\begin{aligned}(Y^{(\text{even})}_{\ell m})_{\mu\nu}(\pi-\theta,\phi+\pi) &= (Y^{(\text{axial})}_{\ell m})_{\mu\nu}(\pi-\theta,\phi+\pi)\\ &= (-1)^{\ell+1}(Y^{(\text{axial})}_{\ell m})_{\mu\nu}(\theta,\pi)\end{aligned}$$

1.1 On the possible presence of black holes

"We have equilibrium, but ***is it stable?***"

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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form

- The result is that, it became evident that perturbations can take the form

$$h_{\mu\nu} \propto \frac{1}{r} e^{-iwt} \; {}_{-2}Y_{\ell}(\cos(\theta))$$

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more on this in a moment

1.1 On the possible presence of black holes

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Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form (the final form is a *single* equation)

- Further, the remaining *r* dependence is determined by a single quantity which obeys a second order linear differential equation

$$\left[\boxed{p_0(r,w)} + p_1(r)\partial_r + p_2(r)\partial_r^2\right]\psi(r) = 0$$

1.1 On the possible presence of black holes

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3. Stability analysis …

- Modern perspective: Show that only damped and oscillatory time-domain are physical. (i.e. damped harmonic oscillation)
- Prior to 1970: Only show that purely growing but non-oscillatory time-domain modes are unphysical. (i.e. exclude pure exponential growth)

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3. Stability analysis' key idea: w must be consistent between time and radial sectors

$$h_{\mu\nu} \propto \frac{1}{r}\boxed{e^{-iwt}}\,{}_{-2}Y_\ell(\cos(\theta))$$

$$\left[\,\boxed{p_0(r,w)} + p_1(r)\partial_r + p_2(r)\partial_r^2\,\right]\psi(r) = 0$$

Equations describing the steps: Ricci scalar, canonical coordinate gauge, even vs odd, odd is easiest (Zerilli did even)

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Regge and Wheeler, 1957

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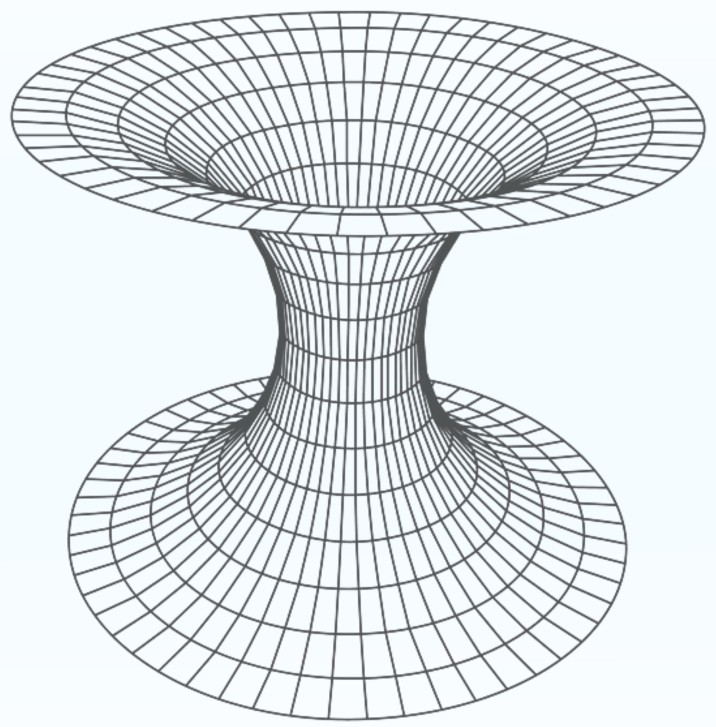
3. Stability analysis …

  - The Regge+Wheeler argument: The physical boundary conditions are that the radiation is regular at spatial infinity, and should be thought of as entering a *wormhole* at the BH event horizon. Their wormhole picture required that one should be able to "stitch" the radiation to outgoing radiation elsewhere …



1.1 On the possible presence of black holes

1950s-1970s

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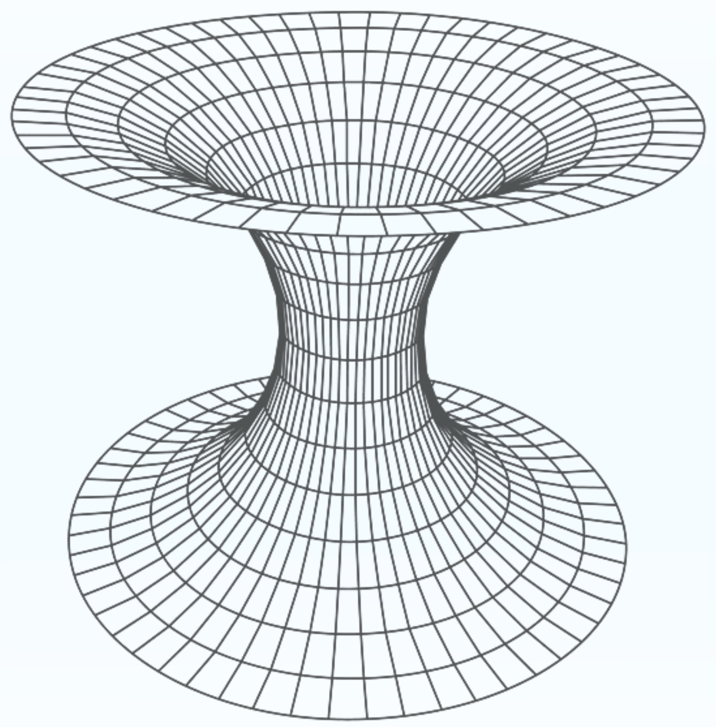
- Guiding concept: If perturbation solutions grow in time, then the BH is unstable. If they shrink, then the BH is stable.

3. Stability analysis …

- Regge+Wheeler claimed without explicit proof that ***if*** one assumed a purely imaginary frequency, ***then*** radial solutions vanish at the event horizon, meaning that the solution could not be smoothly stitched to non-zero radiation at the other end of the wormhole. ***Since such solutions are unphysical, Schwarzschild BHs are stable.***



"We have equilibrium, but *is it stable?*"
Regge and Wheeler, 19

- Guiding co cept: If perturbation solutions grow in time, then the BH is unstable. If they shrink, the the BH is sta

3. Stability analysis …

Does this argument make sense to us?

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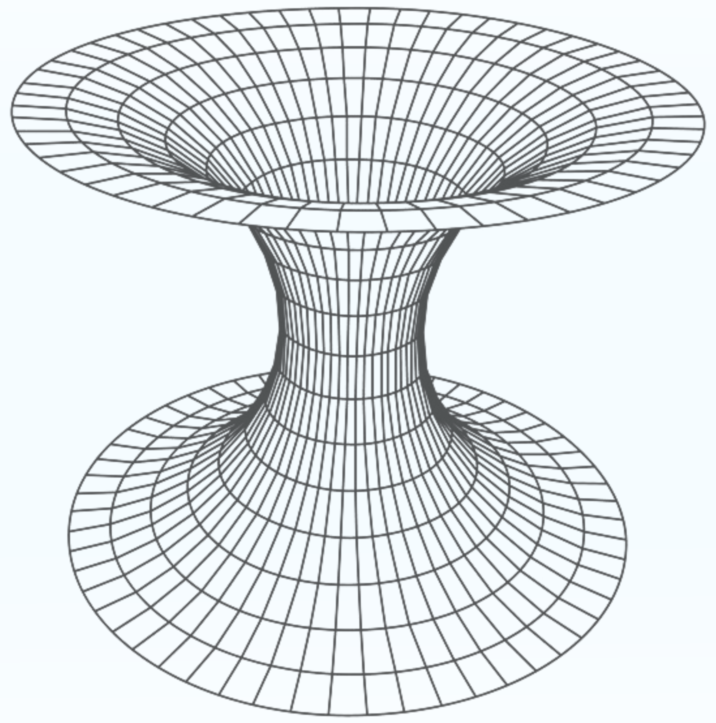
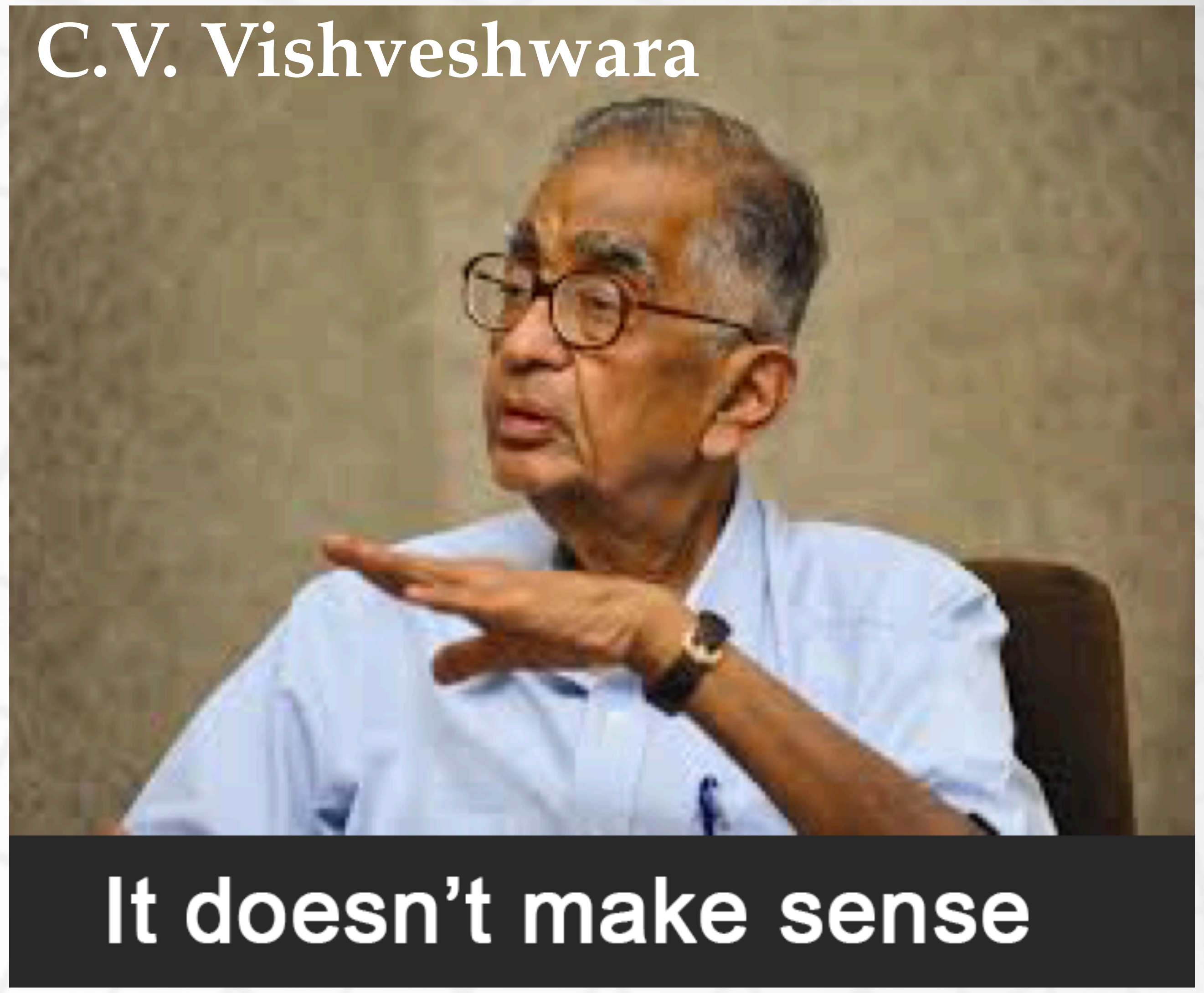
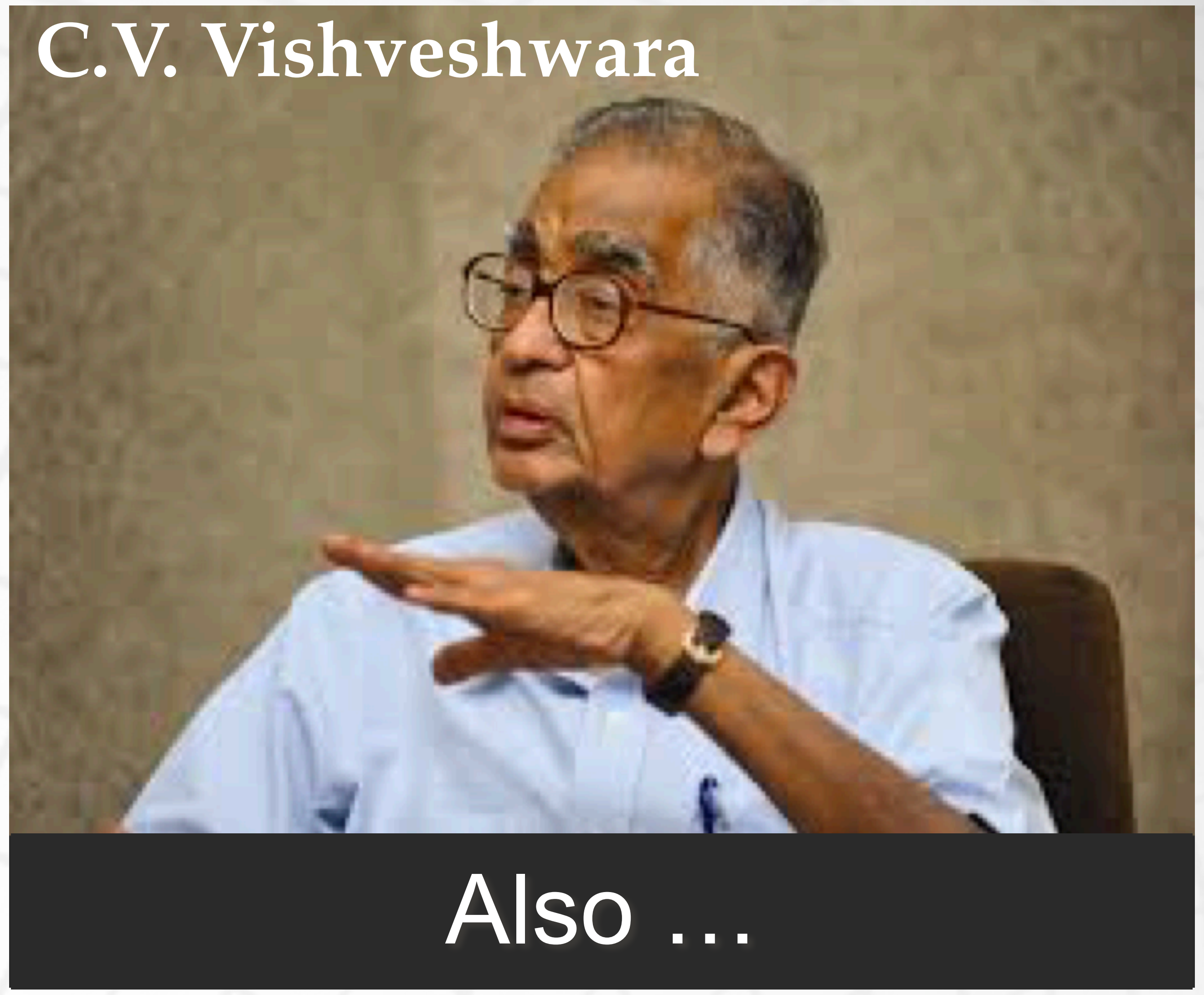


Vishveshwara, 1968 (PhD thesis)



- Using Schwarzschild coordinates leads to misleading and unphysical behavior at the event horizon.
- Instead, use Kruskal coordinates so that there is no coordinate singularity at the event horizon.
- In those coordinates, imposing regularity at spatial infinity is sufficient to exclude purely imaginary modes without relying on a wormhole argument.

Vishveshwara, 1970



- Even that picture is incomplete because it is still based in a *normal mode* perspective.
- Vishveshwara later showed that a "quasi-normal" mode picture is more appropriate since BH systems are dissipative.
- Also, along with Edelstein, he showed that there were various minor errors in the Regge-Wheeler derivation …

Theory is messy ✓

Skepticism is essential

A rough timeline of what followed

1. **1950s — 1960s**
   Linear perturbations of Schwarzschild
   Regge, Wheeler, Zerilli, Vishveshwara, others …

2. **1970s**
   Linear perturbations of Kerr
   Teukolsky, Press, Price, others

3. **1980s**
   Development of analytic methods
   Leaver, Whiting, Kokkotas, others

4. **1990s**
   Advanced numerical methods
   somewhat limited analytical methods
   Sasaki, Nakamura, Mano, Suzuki, Hughes, others

5. **2000-2020s**
   Interface with Numerical Relativity
   Ongoing advances in analytic understanding of QNMs
   Berti, Cardoso, Kamaretsos, London, Green, many others!

A rough timeline of what followed

1950s — 1960s Linear perturbations of Schwarzschild	1	Regge, Wheeler, Zerilli, Vishveshwara, others …
1970s Linear perturbations of Kerr	2	Teukolsky, Press, Price, others
1980s Development of analytic methods	3	Leaver, Whiting, Kokkotas, others
1990s Advanced numerical methods somewhat limited analytical methods	4	Sasaki, Nakamura, Mano, Suzuki, Hughes, others
2000-2020s Interface with Numerical Relativity Ongoing advances in analytic understanding of QNMs	5	Berti, Cardoso, Kamaretsos, London, Green, many others!

1.2 Einstein's equations linearized around Kerr

1970s

- While Regge, Wheeler and other were navigating the Schwarzschild problem …
  - Penrose, Newman, Pirani and others were developing gauge invariant tetrad formalisms (mostly during the early to mid 1960s)
  - Teukolsky was first to apply the Newman-Penrose formalism to the linear stability of Kerr (after the confusion with Schwarzschild, that perhaps seemed a worthwhile effort)
- The resulting linearization of Einstein's equations around Kerr is (in Boyer-Lindquist coords)

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}$$
$$- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}$$
$$- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 4\pi\Sigma T\,.$$

1.2 Einstein's equations linearized around Kerr

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Weyl scalar

$$- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}$$

$$- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = \overset{=0}{\cancel{4\pi\Sigma T}}\,.$$

1.2 Einstein's equations linearized around Kerr

1970s

$$
\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\,\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}
$$

Weyl scalar

$$
- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}
$$

$$
- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 4\pi\overset{=0}{\cancel{\Sigma T}}\,.
$$

Teukolsky's "master" equation

Separate with ansatz

1.2 Einstein's equations linearized around Kerr

1970s

The full form of a **single** QNM

$$\psi \;\propto\; \underline{R(r)\,S(\theta)\,e^{-i\omega t}\,e^{-im\phi}}$$

In more general scenarios, one should perform an integral transform.

ukolsky's "master" equation | Separate with ansatz | Teukolsky's Equa

1.3 Black hole specific special functions

1970s

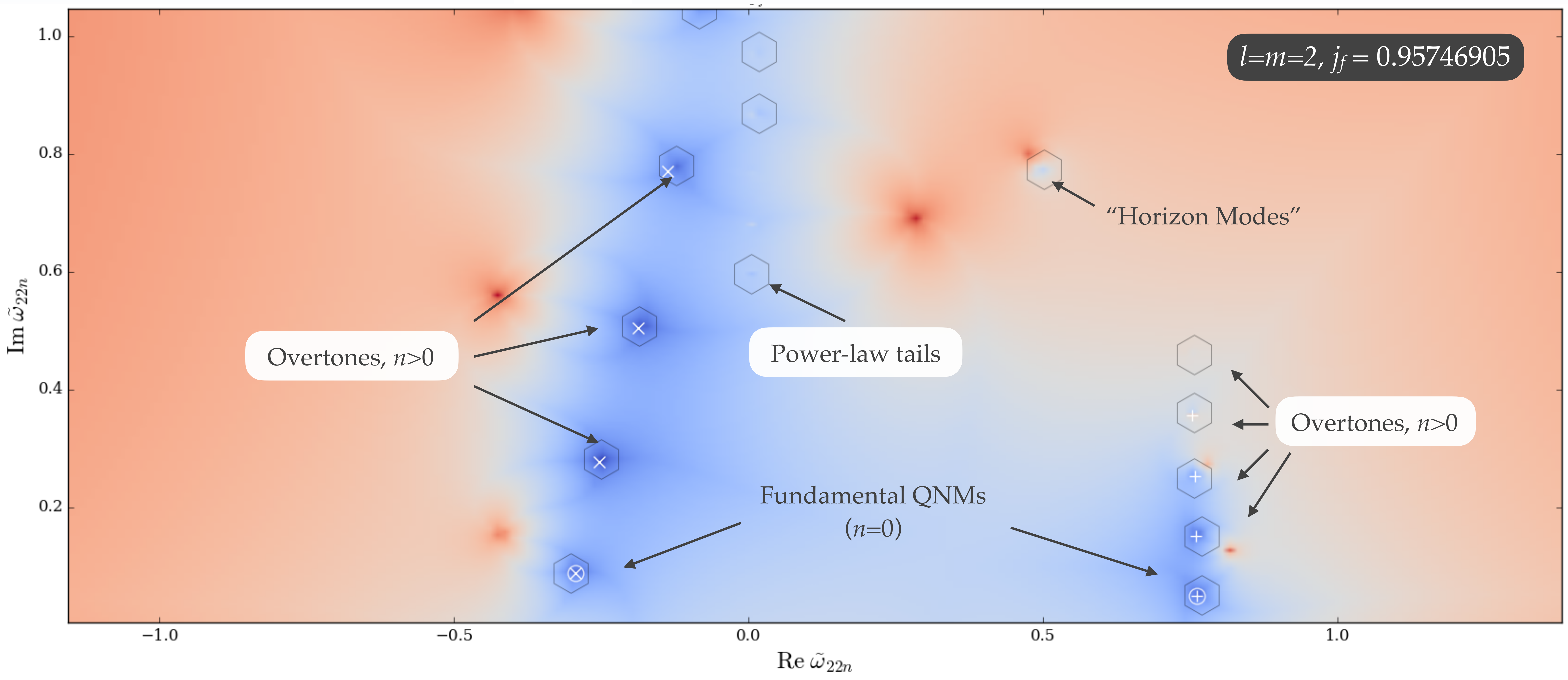
$$\mathcal{L}_u \, S(u) \;=\; -A \, S(u)$$
$$\mathcal{L}_r \, R(r) \;=\; +A \, R(r)$$

The QNMs' radial and angular functions are special functions that, when solved simultaneously, fully constrain a Kerr QNM

Separate with ansatz | **Teukolsky's Equations** | Next lecture

1.3 Black hole specific special functions

1980s-2000s

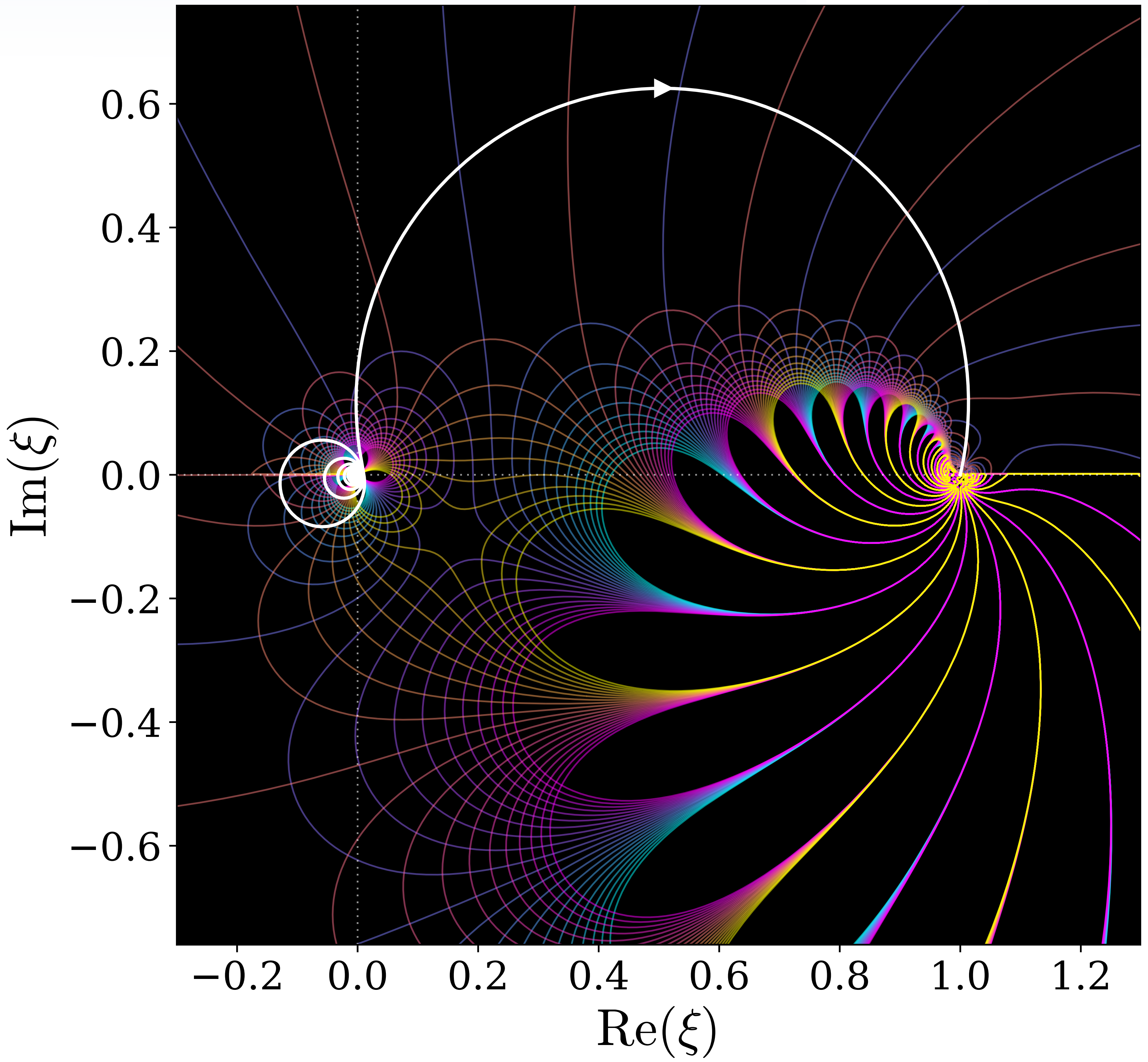


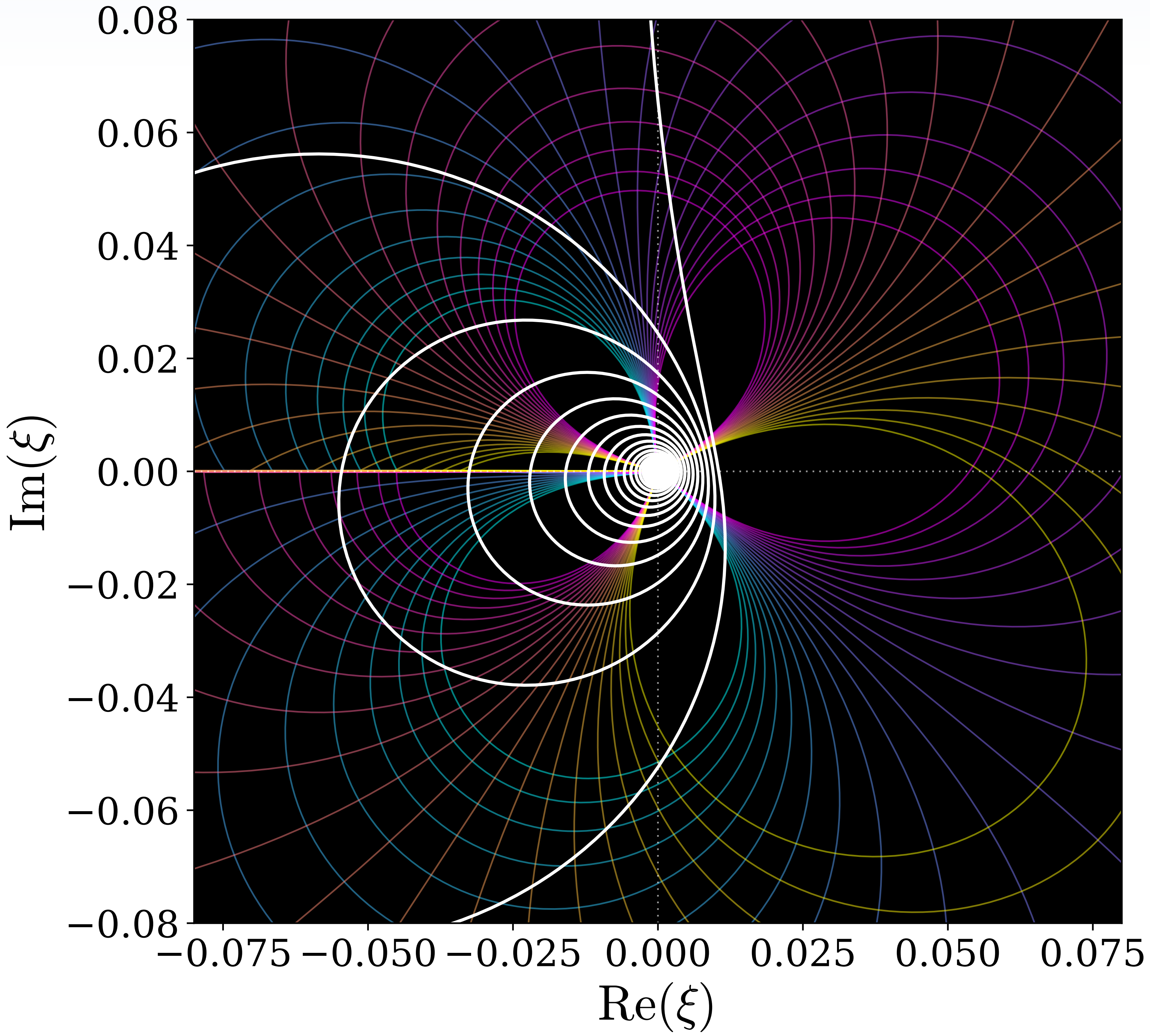
Teukolsky's Equations

Next lecture

1.3 Black hole specific special functions

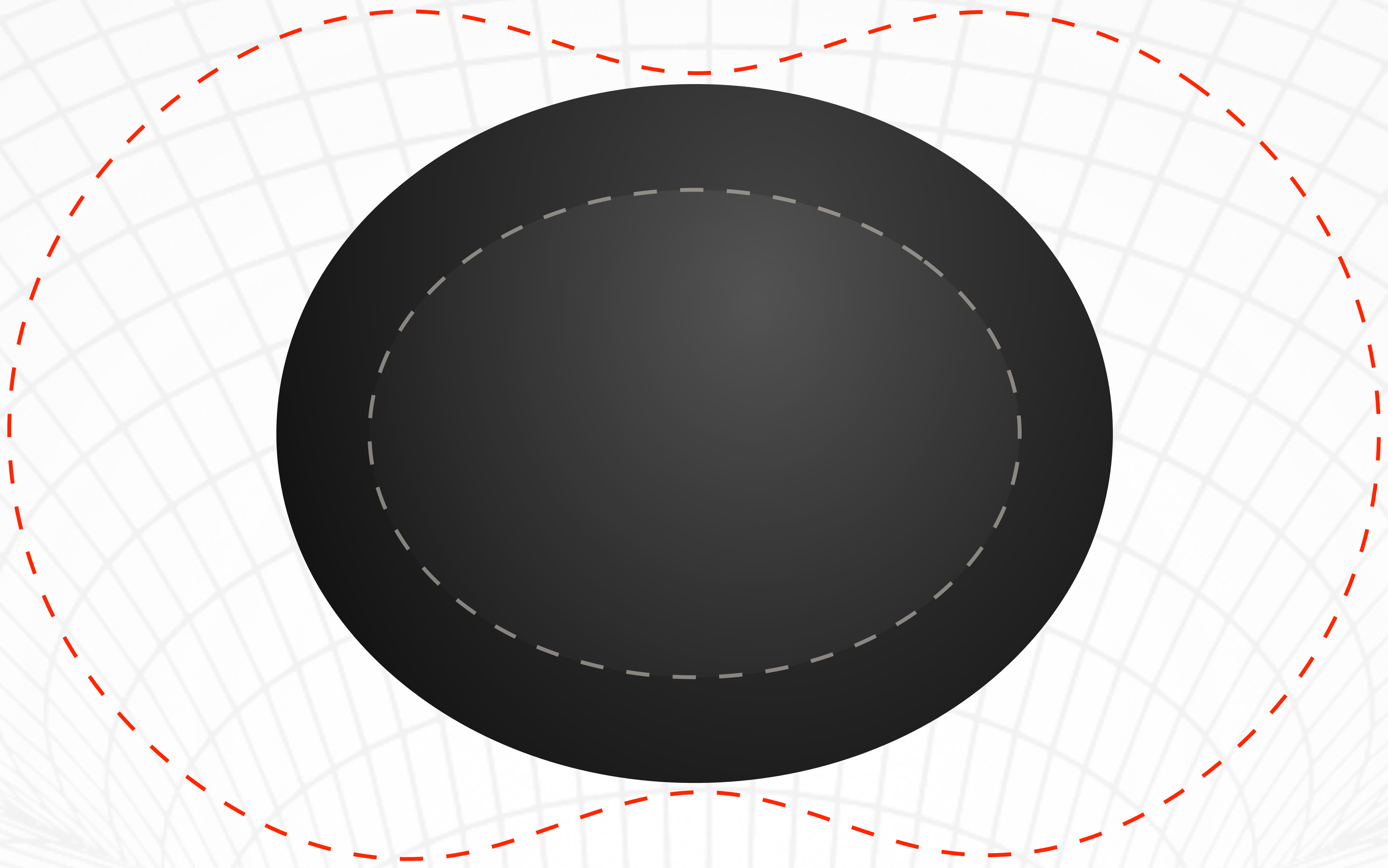
1980s-2000s





Teukolsky’s Equations

Next lecture



Thanks!