Stability of a Schwarzschild Singularity

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It is shown that a Schwarzschild singularity, spherically symmetrical and endowed with mass, will undergo small vibrations about the spherical form and will therefore remain stable if subjected to a small nonspherical perturbation.

I. INTRODUCTION AND SUMMARY

SCHWARZSCHILD found long ago the solution of the Einstein equations for the metric around a fixed spherically symmetrical center-of-mass:

$$ds^2=-(1-2m^*/r)dT^2+(1-2m^*/r)^{-1}dr^2 +r^2(d\theta^2+\sin^2\theta d\varphi^2)=g_{\mu\nu}dx^\mu dx^\nu, \quad (1)$$

with $x^0=T$, $x^1=r$, $x^2=\theta$, $x^3=\varphi$. Here the quantity

$$m^*(\mathrm{cm})=Gm/c^2=(0.74\times10^{-28}\ \mathrm{cm/g})m(\mathrm{g}) \quad (2)$$

denotes the value of the mass as expressed in the geometric units of length. A similar but more complicated expression was found by Reissner and Nordström[1] for the case where the system appears at a distance not only as a spherically symmetrical center of gravitational attraction but also as a spherically symmetrical center of electric lines of force. Such a charge-like solution permits interpretation in terms of a multiply connected space.[2] The lines of force can be considered to emerge from one mouth of a wormhole, the other end of which is located somewhere else in space. This interpretation has the following features: (1) There is no real charge present anywhere in space. Lines of force never end. The Maxwell field is free of singularity. However, the lines of force are trapped in the topology of space so that their number cannot change and the charge remains constant. (2) The other wormhole mouth can be supposed as far away as one pleases. We shall assume it to be infinitely far away, so that it does not disturb the spherical symmetry and dynamics of the mouth under consideration. Then we can use the Reissner-Nordström solution as an idealized representation of the metric down to the throat of the wormhole. (3) The electric field, being divergence-free, has a strength q/r^2 and an energy density $q^2/8\pi r^4$. Translated to the units of length, the charge has the value

$$q^*(\mathrm{cm})=(G^{\frac{1}{2}}/c^2)q(\text{electrostatic units}). \quad (3)$$

(4) The stress energy density of this electric field acts as a source of gravitational field in Einstein's field equations. There is no other source of the gravitational field in the usual way of writing these equations. However, the equations can be rearranged[3] in such a way as to bring into evidence an additional production of gravitational field by the stress energy tensor *of* the gravitational field. On this account the geometrized mass, m^*, is not uniquely determined by the geometrized charge, q^*; it only follows that m^* is no less than q^*. (5) One can therefore think of the field energy—or the mass and stress that goes with it—as in equilibrium under its own gravitational attraction.

We have equilibrium, but is it stable? A sphere of water held together by gravitational forces is stable against small departures from sphericity. A sphere of water surrounded by a spherical shell of liquid mercury is also an equilibrium configuration for gravitational forces, but a situation of unstable equilibrium. Initial small departures from sphericity at the water-mercury interface will grow exponentially, and the mercury will concentrate with a rush at the center of the sphere. Which situation will more closely correspond to the behavior of a Schwarzschild singularity subjected to a small initial perturbation?

We have investigated this question here up to terms of the first order in the departures from sphericity. In this approximation, as in every other kind of stability problem in physics, the equations are linear and it is possible to analyze the disturbance into proper modes and find for each its frequency, real (stability) or imaginary (instability). We have determined these proper frequencies and find that the Schwarzschild singularity is essentially stable because imaginary frequencies would require a clearly unrealistic spatial behavior of the initial perturbation.

We therefore conclude that the object in question, built out of the mass-free Einstein field, is stable against small departures from sphericity. A typical disturbance from the equilibrium configuration will not grow with time but will oscillate around equilibrium.

The analysis proceeds in the following way: In Sec. II we write down, following Komar and Eisenhart, the *linear* differential equations for small first-order departures from the Schwarzschild metric. These equations ensure that the perturbed field will satisfy the Einstein equations and represent a mass-free space

[1] H. Reissner, Ann. Physik **50**, 106 (1916); G. Nordström, Proc. Acad. Sci. Amsterdam **20**, 1238 (1918).

[2] J. A. Wheeler, Phys. Rev. **97**, 511 (1955); C. W. Misner and J. A. Wheeler, Ann. Phys. (to be published); also A. Einstein and N. Rosen, Phys. Rev. **48**, 73 (1935).

[3] S. N. Gupta, Phys. Rev. **96**, 1683 (1954).

of nearly spherical symmetry. We look for those solutions of these differential equations which can be expressed in the form of a product of four factors, each depending upon a single one of the quantities T, r, θ, φ.

We find mathematical expressions for a complete set of functions of this kind, by superposition of which one can represent any arbitrary small first-order perturbation that satisfies the appropriate boundary conditions.

The typical mode of disturbance of the Einstein field has the circular frequency ω. Here the quantity ω is an eigenvalue parameter to be chosen so that the disturbance satisfies the radial wave equation with the appropriate boundary conditions for small and large r.

In Sec. III we formulate the boundary conditions for this eigenvalue problem and analyze the radial dependence of the functions involved in the problem by way of the JWKB procedure. In this manner we can avoid the problem of securing an exact solution of the radial wave equation and still conclude that the proper frequencies are real.

II. DIFFERENTIAL EQUATIONS IN POLAR COORDINATES FOR SMALL FIRST-ORDER CHANGES AWAY FROM THE SCHWARZSCHILD METRIC

General Equations

We shall indicate here the background metric with $g_{\mu\nu}$ and the perturbation in it with $h_{\mu\nu}$. The quantity $g_{\mu\nu}$ will be later specialized to be the ordinary Schwarzschild metric. The perturbations $h_{\mu\nu}$ are supposed to be very small as compared with $g_{\mu\nu}$. The contracted Ricci tensor will be, as usual, called $R_{\mu\nu}$ if calculated from $g_{\mu\nu}$ and $R_{\mu\nu}+\delta R_{\mu\nu}$ if calculated from $g_{\mu\nu}+h_{\mu\nu}$. It is not difficult to derive an expression for $\delta R_{\mu\nu}$. The calculation has been made by Eisenhart and independently by Komar.[4] We shall follow here Eisenhart's point of view as being more suitable for our kind of calculation. His result can be expressed in the form

$$\delta R_{\mu\nu}=-\delta\Gamma^{\beta}{}_{\mu\nu;\beta}+\delta\Gamma^{\beta}{}_{\mu\beta;\nu}, \tag{4}$$

where the semicolons indicate covariant differentiation and where we use the symbol

$$\delta\Gamma^{\alpha}{}_{\beta\gamma}=\tfrac{1}{2}g^{\alpha\nu}(h_{\beta\nu;\gamma}+h_{\gamma\nu;\beta}-h_{\beta\gamma;\nu}). \tag{5}$$

Although $\Gamma^{\alpha}{}_{\beta\gamma}$ is not a tensor, its variation is a tensor. Putting (5) into (4), we get a second-order differential equation on the $h_{\mu\nu}$ from the condition $\delta R_{\mu\nu}=0$. This equation is a generalization in a curved space of the known Schrödinger equation for a massless particle of spin 2 in a flat space.[5] When the background metric is given by (1), then of course $R_{\mu\nu}=0$. Then the equation $\delta R_{\mu\nu}=0$ means that the perturbed space is also empty of matter or distributed energy. These equations we shall analyze and separate in polar coordinates.

Analysis into Spherical Harmonics

We first undertake to separate the solution into a product of four factors, each a function of a single coordinate. This separation is best achieved by generalizing to tensors the well-known development in spherical harmonics, already firmly established for vectors, scalars, and spinors. For all four cases the symmetry of the metric allows angular momentum to be defined. The angular momentum is investigated by studying rotations on the 2-dimensional manifold $x^0=T=$constant, $x^1=r=$constant. Under a rotation of the frame around the origin, the ten components of the perturbing tensor transform like 3 scalars (h_{00}, h_{01}, h_{11}), 2 vectors (h_{02}, h_{03}; h_{12}, h_{13}), and a second-order tensor, when considered as covariant quantities on the sphere. For the scalar and vector parts we know already how to develop into spherical harmonics. A typical scalar function has the form

$$\phi_L{}^M=\text{const}\,Y_L{}^M(x_2,x_3)=\text{const}\,Y_L{}^M(\theta,\varphi). \tag{6}$$

This term belongs to a wave of parity $(-)^L$ and of angular momentum L, whose projection on the z axis is M. For vectors we have two distinct types with opposite parity:

$$\psi_L{}^M{}_{,\mu}=\text{const}\frac{\partial}{\partial x^\mu}Y_L{}^M(x_2x_3), \qquad \text{parity } (-)^L; \tag{7}$$

$$\phi_L{}^M{}_{,\mu}=\text{const}\,\epsilon_\mu{}^\nu\frac{\partial}{\partial x_\nu}Y_L{}^M(x_2x_3), \quad \text{parity } (-)^{L+1}. \tag{8}$$

Here the labels μ and ν run over the values 2 and 3, when $x^2=\theta$, $x^3=\varphi$; and the $\epsilon_\mu{}^\nu$ represent the quantities $\epsilon_2{}^2=\epsilon_3{}^3=0$; $\epsilon_2{}^3=-1/\sin x^2$, $\epsilon_3{}^2=\sin x^2$. Similarly we shall later introduce the quantities $\gamma_{22}=1$, $\gamma_{23}=0=\gamma_{32}$; $\gamma_{33}=\sin^2 x^2$. Both types of vector carry an angular momentum L. For tensors similar properties hold. We outline here only the results. There are three fundamental types of tensor of angular momentum L:

$$\psi_L{}^M{}_{\mu\nu}=\text{const}\,Y_L{}^M{}_{;\mu\nu}(\text{covariant derivatives}), \qquad \text{parity } (-)^L; \tag{9}$$

$$\phi_L{}^M{}_{\mu\nu}=\text{const}\,\gamma_{\mu\nu}Y_L{}^M, \quad \text{parity } (-)^L; \tag{10}$$

$$\chi_L{}^M{}_{\mu\nu}=\tfrac{1}{2}\,\text{const}[\epsilon_\mu{}^\lambda\psi_L{}^M{}_{\lambda\nu}+\epsilon_\nu{}^\lambda\psi_L{}^M{}_{\lambda\mu}], \qquad \text{parity } (-)^{L+1}. \tag{11}$$

Any one of these terms can obviously be multiplied by an arbitrary function of r and T, without changing its transformation properties under a rotation. It is clear that term (10) is a simple combination of a scalar with the metric tensor on the sphere, $\gamma_{\mu\nu}=g_{\mu\nu}/r^2$. Term (8) can be considered as a "pseudogradient" of $Y_L{}^M(x_2x_3)$. Similarly (9) and (11) can be obtained by operating twice with gradients and pseudogradients. The trace

[4] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1926), Chap. VI; A. Komar, Ph.D. thesis, Princeton, 1956 (unpublished).

[5] See for example T. Regge, Nuovo cimento **5**, 325 (1957).

of (11) is identically 0. The trace of (9) has the value

$$g^{\mu\nu}\psi_L{}^M{}_{\mu\nu}=-L(L+1)Y_L{}^M(\theta,\varphi)\times\text{const.}$$

From the results that we have just outlined, it follows that the most general perturbation belonging to a given L, M, and parity $(-)^{L+1}$ is of the form:

$$h_{\mu\nu}=\left|\begin{array}{llll} 0 & 0 & -h_0(T,r)(\partial/\sin\theta\partial\varphi)Y_L{}^M & h_0(T,r)(\sin\theta\partial/\partial\theta)Y_L{}^M \\ 0 & 0 & -h_1(T,r)(\partial/\sin\theta\partial\varphi)Y_L{}^M & h_1(T,r)(\sin\theta\partial/\partial\theta)Y_L{}^M \\ \text{Sym} & \text{Sym} & h_2(T,r)(\partial^2/\sin\theta\partial\theta\partial\varphi-\cos\theta\partial/\sin^2\theta\partial\varphi)Y_L{}^M & \text{Sym} \\ \text{Sym} & \text{Sym} & \frac{1}{2}h_2(T,r)(\partial^2/\sin\theta\partial\varphi\partial\varphi+\cos\theta\partial/\partial\theta-\sin\theta\partial^2/\partial\theta\partial\theta)Y_L{}^M & -h_2(T,r)(\sin\theta\partial^2/\partial\theta\partial\varphi-\cos\theta\partial/\partial\varphi)Y_L{}^M \end{array}\right|. \tag{12}$$

Here the rows and columns are numbered in the order 0, 1, 2, 3 (T, r, θ, φ). The symbol "Sym" indicates that the missing components of $h_{\mu\nu}$ are to be found from the symmetry $h_{\mu\nu}=h_{\nu\mu}$. We shall refer to (12) as the "odd" type of perturbation.

Even Waves

The grouping of the terms of even parity yields the "even" perturbation:

$$h_{\mu\nu}=\left|\begin{array}{llll} (1-2m^*/r)H_0(T,r)Y_L{}^M & H_1(T,r)Y_L{}^M & h_0(T,r)(\partial/\partial\theta)Y_L{}^M & h_0(T,r)(\partial/\partial\varphi)Y_L{}^M \\ H_1(T,r)Y_L{}^M & (1-2m^*/r)^{-1}H_2(T,r)Y_L{}^M & h_1(T,r)(\partial/\partial\theta)Y_L{}^M & h_1(T,r)(\partial/\partial\varphi)Y_L{}^M \\ \text{Sym} & \text{Sym} & r^2[K(T,r)+G(T,r)(\partial^2/\partial\theta^2)]Y_L{}^M & \text{Sym} \\ \text{Sym} & \text{Sym} & r^2G(T,r)(\partial^2/\partial\theta\partial\varphi-\cos\theta\partial/\sin\theta\partial\varphi)Y_L{}^M & r^2[K(T,r)\sin^2\theta+G(T,r)(\partial^2/\partial\varphi\partial\varphi+\sin\theta\cos\theta\partial/\partial\theta)]Y_L{}^M \end{array}\right|. \tag{13}$$

Frequency Analysis; Specialization to $M=0$

Owing to the spherical symmetry of the background metric, Eqs. (4) and (5) cannot mix terms belonging to different L and parity. To apply quantum language to a classical problem, we can say that L, M, and the parity are constants of the motion. The existence of still another constant follows from the circumstance that the background metric (1) is independent of the cotime, $T=ct$. On this account we can consider a perturbation of a definite frequency, $\omega=kc$, so that every component of the perturbation $h_{\mu\nu}$ will have a time dependence of the form $\exp(-i\omega t)=\exp(-ikT)$. We therefore proceed to determine completely the form of the individual solution of specified parity, L and M values, and frequency. The general solution will be a superposition of these individual solutions with coefficients adjusted to fit the appropriate boundary conditions and initial values.

In working out the typical individual solution, we need not occupy ourselves with the angular dependence, which is completely specified by (12) and (13). Moreover, there is no need to work with an arbitrary M. For any specified choice of L, k, and partly all values of M ($M=-L, -L+1, \cdots L$) will lead to the same radial equation. We prefer to take $M=0$ with the advantage that φ will completely disappear from the calculations. We are still left with a considerable amount of labor. An odd wave contains three unknown functions of r. Worse, an even wave contains seven unknown functions. Here comes a fact which greatly simplifies and illuminates the calculation.

Gauge or Coordinate Transformations and the Simplifications They Can Bring

Different waves can represent the same physical phenomena viewed in different systems of coordinates. Consider an infinitesimal coordinate transformation:

$$x'^\alpha=x^\alpha+\xi^\alpha\quad(\xi^\alpha\ll x^\alpha). \tag{14}$$

The infinitesimal displacements ξ^α transform like a vector. In the new frame we shall have:

$$g_{\mu\nu}'+h_{\mu\nu}'=g_{\mu\nu}+\xi_{\mu;\nu}+\xi_{\nu;\mu}+h_{\mu\nu}. \tag{15}$$

Now $h_{\mu\nu}$ is defined as the difference between the perturbed metric and the Schwarzschild metric *written in spherical coordinates*. According to this definition, the difference in the new frame will have the value

$$h_{\mu\nu}{}^{\text{new}}=h_{\mu\nu}{}^{\text{old}}+\xi_{\mu;\nu}+\xi_{\nu;\mu}. \tag{16}$$

This result can be interpreted by saying that for infinitesimal changes in the coordinates the $h_{\mu\nu}$ undergo a "gauge transformation" quite similar to the well-known gauge transformation for the electromagnetic field. We can now use this important circumstance in order to simplify the description of the perturbation and to make it unique.

The gauge transformation (16) can be performed on any individual partial wave. Obviously no real simplification will result unless the resulting wave still belongs to the original eigenvalues. This requirement limits the possible choices for ξ^α. This vector turns out to be a spherical harmonic of the same L and parity as the partial wave in consideration. Such a gauge transformation allows us to impose additional simplifying conditions on the perturbation, $h_{\mu\nu}$. We have therefore chosen to eliminate those terms which contain the derivatives of the highest order with respect to the angles. Then the final radial equations simplify. Moreover, the desired gauge transformation ξ^α can then be found by the use of finite operations only, without arbitrary constants and boundary conditions.

The gauge vector ξ^α that simplifies the general odd wave (12) must have the form

$$\xi^0=0,\quad \xi^1=0,\quad \xi^\mu=\Lambda(T,r)\epsilon^{\mu\nu}(\partial/\partial x^\nu)Y_L{}^M(\theta,\varphi),\quad (\mu,\,\nu=2,3),\tag{17}$$

according to the foregoing arguments. Moreover, the radial function Λ can be adjusted to annul the radial factor $h_2(T,r)$ or that component of the perturbation $h_{\mu\nu}$ which has the form (11).

Canonical Form of Odd and Even Waves

The final canonical form for an odd wave of total angular momentum L and projection $M=0$ is then

$$h_{\mu\nu}=\begin{vmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ \text{Sym} & \text{Sym} & 0 & 0 \end{vmatrix} \times\exp(-ikT)(\sin\theta\partial/\partial\theta)P_L(\cos\theta).\tag{18}$$

In a similar way we write the gauge transformation that simplifies even waves in the form

$$\begin{aligned}&\xi^0=M_0(T,r)Y_L{}^M(\theta,\varphi);\quad \xi^1=M_1(T,r)Y_L{}^M(\theta,\varphi);\\ &\xi^2=M(T,r)(\partial/\partial\theta)Y_L{}^M(\theta,\varphi);\\ &\xi^3=M(T,r)(\partial/\sin^2\theta\partial\varphi)Y_L{}^M(\theta,\varphi).\end{aligned}\tag{19}$$

We adjust the factors M_0, M_1, and M to annul the factors G, h_0, and h_1 in expression (13), thereby obtaining the even wave in the canonical form

$$h_{\mu\nu}=\exp(-ikT)P_L(\cos\theta)\times\begin{vmatrix} H_0(1-2m^*/r) & H_1 & 0 & 0 \\ H_1 & H_2(1-2m^*/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2K & 0 \\ 0 & 0 & 0 & r^2K\sin^2\theta \end{vmatrix}\tag{20}$$

There are therefore two unknown functions of r in the odd case (h_0 and h_1) and four unknowns for the even case (H_0, H_1, H_2, and K).

Radial Wave Equations

We now substitute the expressions (18) and (20) for the first-order perturbation into the first variation of the Einstein field equations,

$$\delta\Gamma_{\mu\nu}{}^{\beta}{}_{;\beta}-\delta\Gamma_{\mu\beta}{}^{\beta}{}_{;\nu}=0.\tag{21}$$

The resulting equations for the radial factors may be derived and discussed separately for odd and even waves.

For odd waves the variation $\delta\Gamma_{\mu\beta}{}^\beta$ vanishes identically. Out of the 10 Einstein equations only 3 nontrivial ones can be obtained:

$$\begin{aligned}&(1-2m^*/r)^{-1}kh_0+(d/dr)(1-2m^*/r)h_1=0, && \text{from}\quad \delta R_{23}=0;\\ &(1-2m^*/r)^{-1}k(dh_0/dr-kh_1-2h_0/r)+(L-1)(L+2)h_1/r^2=0, && \text{from}\quad \delta R_{13}=0;\\ &(d/dr)(kh_1-dh_0/dr)+2kh_1/r=r^{-2}(1-2m^*/r)^{-1}\times(4m^*h_0/r-L(L+1)h_0), && \text{from}\quad \delta R_{03}=0.\end{aligned}\tag{22}$$

The last equation is a consequence of the other two. Define

$$Q=(1-2m^*/r)h_1/r,\tag{23}$$

eliminate h_0, and find for Q the second-order wave equation,

$$d^2Q/dr^{*2}+k_{\text{eff}}{}^2(r)Q=0.\tag{24}$$

Here we have made the abbreviations

$$dr^*=\exp(\tfrac{1}{2}\lambda-\tfrac{1}{2}\nu)dr,$$

and

$$k_{\text{eff}}{}^2=k^2-L(L+1)e^\nu/r^2+6m^*e^\nu/r^3,\tag{25}$$

where λ and ν are defined by the expression for the metric,

$$ds^2=-e^\nu dT^2+e^\lambda dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2);\tag{26a}$$

thus,

$$e^\nu=e^{-\lambda}=1-2m^*/r.\tag{26b}$$

For even waves the 10 Einstein field equations give 7 nontrivial conditions: one algebraic relation; 3 first-order differential equations; and 3 of the second order. We use one relation,

$$H_2=H_0\equiv H, \tag{27a}$$

to eliminate H_0 from the other 6 equations:

$$dK/dr+r^{-1}(K+H)-L(L+1)H_1/2kr^2 \\ -(1-2m^*/r)^{-1}m^*K/r^2=0, \tag{27b}$$

$$(d/dr)(1-2m^*/r)H_1=k(K-H), \tag{27c}$$

$$kH_1-(1-2m^*/r)(d/dr)(K+H)-(2m^*/r^2)H=0; \tag{27d}$$

$$(d/dr)(1-2m^*/r)(2rH+(d/dr)r^2K)-L(L+1)K \\ -2krH_1+k^2r^2(1-2m^*/r)^{-1}K=0; \tag{27e}$$

$$2k(1-2m^*/r)(dH_1/dr)+2km^*H_1/r^2+k^2H \\ -(1-2m^*/r)^2[(d/r^2dr)(r^2dH/dr) \\ +2(d/r^2dr)(r^2dK/dr)]-(1-2m^*/r)[L(L+1)H/r^2 \\ +4m^*(d/r^2dr)H+2m^*(d/r^2dr)K]=0; \tag{27f}$$

$$(1-2m^*/r)^2d^2H/dr^2+(2/r)(1-2m^*/r)dH/dr-k^2H \\ -L(L+1)(1-2m^*/r)(H/r^2)-2km^*H_1/r^2 \\ -2k(1-2m^*/r)(d/r^2dr)r^2H_1+2k^2K \\ +2(1-2m^*/r)(m^*/r^2)dK/dr=0. \tag{27g}$$

We are facing a system of 6 equations in 3 unknowns. One will expect that the 3 first-order Eqs. (27b,c,d) will suffice to determine the solutions, apart from the boundary conditions. Actually the second-order Eqs. (27e,f,g) contain an additional nontrivial piece of information about the solution that is consistent with (27b,c,d). Specifically, a rather elaborate investigation shows that (27e,f,g) can all be deduced from (27b,c,d) plus the algebraic relation

$$[6m^*/r+(L-1)(L+2)]H+[2kr-L(L+1)m^*/r^2k]H_1 \\ +[(2m^*/r)+(L-1)(L+2) \\ -2(1-2m^*/r)^{-1}(m^{*2}/r^2+k^2r^2)]K\equiv F=0; \tag{28}$$

and conversely, (27b,c,d) and (28) follow from (27e,f,g), provided that the frequency k is nonzero. The static case will be discussed separately.

It is remarkable that (28) is an *algebraic* relation consistent with (27b,c,d). It can properly be called a "first integral." Thus, let $K^\dagger$, $H^\dagger$, $H_1{}^\dagger$ be any solution of (27b,c,d) and let $F^\dagger$ be the corresponding function constructed from (28). We have the following identity:

$$dF^\dagger/dr+(m^*/r^2)(1-2m^*/r)^{-1}F^\dagger=0. \tag{29}$$

If now $F^\dagger$ vanishes at one point—as is possible by an appropriate choice of the arbitrary constants—then $F^\dagger=0$ everywhere.

Static Case and Its Physical Interpretation

In the static case where $k=0$ it follows from (27b,c) that H_1 vanishes. The other two radial factors can be shown to satisfy the relations

$$(d/dr)(K+H)+(2m^*/r^2)(1-2m^*/r)^{-1}H=0, \\ (dH/dr)+(2m^*/r^2)(1-2m^*/r)^{-1}H+2H/r \\ +(L-1)(L+2)(K+H)/2m^*=0. \tag{30}$$

For $L=0$ the solution is trivial. We get the difference between the Schwarzschild metrics for the two reduced masses m^* and $m^*+\delta m^*$. For $L=1$ we expect to find a solution that corresponds to a displacement of the center of attraction by the amount δz:

$$x^\mu\to x^\mu+h^\mu; \quad \theta\to\theta-\sin\theta\delta z/r; \\ r\to r+\cos\theta\delta z; \quad \varphi\to\varphi; \quad T\to T. \tag{31}$$

This transformation is not acceptable (1) because it assumes Euclidean rather than Schwarzschild geometry for the displacement and (2) because it leads to a change $h_{\mu\nu}$ in the metric which does not have the diagonal form of (20). The correct infinitesimal coordinate transformation has the form

$$r\to r+\cos\theta\delta z, \\ \theta\to\theta+(\sin\theta\delta z/2m^*)\ln(1-2m^*/r), \tag{32}$$

and the two radial factors that satisfy (30) have the values

$$H_2=H_0=H=-(2m^*/r^2)\delta z/(1-2m^*/r); \\ K=(\delta z/m^*)[(2m^*/r)+\ln(1-2m^*/r)]. \tag{33}$$

For $L>1$ we eliminate $(K+H)$ from Eqs. (30) and obtain an equation for the quantity $M=r(r-2m^*)H$:

$$(d^2M/dr^2)-[r^{-1}+(r-2m^*)^{-1}](dM/dr) \\ -(L-1)(L+2)M/r(r-2m^*)=0. \tag{34}$$

The general solution of this equation can be expressed in terms of the associated Legendre functions:

$$M=\alpha P_L{}^{(2)}(1-r/m^*)+\beta Q_L{}^{(2)}(1-r/m^*), \tag{35}$$

where

$$P_2{}^{(2)}(x)=3(1-x^2);\quad P_3{}^{(2)}(x)=15x(1-x^2);\quad\cdots;$$
$$Q_2{}^{(2)}(x)=(1-x^2)(d^2/dx^2)[\tfrac{1}{2}(3x^2-1)\,\mathrm{arc\,tanh}x-(3x/2)],$$
$$Q_3{}^{(2)}(x)=(1-x^2)(d^2/dx^2)[\tfrac{1}{2}(5x^3-3x)\,\mathrm{arc\,tanh}x \\ -(5x^2/2)+(2/3)],$$
$$\cdots.$$

By way of comparison with the solution (32), consider the expression,

$$V=Q/r+\sum(\alpha_{LM}r^L+\beta_{LM}r^{-L-1})Y_{LM}(\theta,\varphi), \tag{36}$$

for electrostatic potential in flat space. Here the α terms represent the asymmetries in the potential due to remote sources and the β terms are due to asymmetries in the distribution of internal sources. The origin of the two kinds of terms in (35) is a little more subtle. We interpret them *both* as asymmetries in the metric arising from remote masses, as follows: (1) masses in the present region of space, and (2) masses in the region of space which unfolds from the other end of the wormhole. Naturally motions of both kinds of sources will be demanded by the field equations themselves. In contrast, the field equations of electrostatics are linear and are not sufficient in and by themselves to give the equations of motion. On this account, in the expression (36) for the electrostatic potential one is content to consider the α's and β's as constants. In contrast, one might well ask how the α and β in (35) can be constant if they represent the effect of remote masses that will inevitably be set in motion. However, we deal always with the metric *interior* to these remote masses. Therefore there is no obvious reason for a change with time in these coefficients. Moreover, the equations themselves say that the solution (35) is static.

Static solutions of odd type also exist. To analyze these solutions, we annul the frequency parameter k in Eqs. (22), note that the radial factor h_1 must vanish, and find for the other radial factor the second-order equation

$$d^2h_0/dr^2+(1-2m^*/r)^{-1}(4m^*/r^3-L(L+1)/r^2)h_0=0. \quad (37)$$

Here again the general solution is a linear combination of two terms which reduce at large r to the form r^{-L} and r^{L+1}. These terms are interpreted as before as the effect of asymmetrically distributed masses which are very remote and located on the one side or the other of the wormhole. That solution which behaves asymptotically as r^{L+1} allows itself to be expressed as a hypergeometric function that reduces to a polynomial in $z=r/2m^*$:

$$\begin{aligned} h_0&=z^2F(L+2,\,1-L,\,4,\,z)\\ &=z^2+[(L+2)(1-L)/4]z^3\\ &\quad+[(L+2)(L+3)(1-L)(2-L)/2!4\times5]z^4+\cdots\\ &\quad+[(L+2)(L+3)\cdots(2L)\cdot(1-L)(2-L)\cdots\\ &\qquad\times(-1)/(L-1)!4\times5\cdots(L+2)]z^{L+1}. \end{aligned} \quad (38)$$

We shall not investigate further the behavior of static perturbations. We interpret them as the effect of distant masses, producing inhomogeneous external gravitational fields, and thereby deforming the mouth of the wormhole that would otherwise be spherically symmetric. These perturbations have nothing directly to do with the dynamics or stability of the Schwarzschild metric itself.

III. BOUNDARY CONDITIONS AND STABILITY OF THE SCHWARZSCHILD SINGULARITY

Regularity Conditions; Odd Waves Analyzed Via the Concept of Effective Potential

A physically acceptable dynamic variation away from the background metric must be represented by functions which have at the starting instant a reasonably regular behavior both at the point $r=2m^*$ and at infinity.

It is simplest to analyze first the odd dynamic disturbances, as they are governed by the single relatively simple differential Eq. (24). The general solution behaves so:

$$\begin{aligned} Q&\sim c_1e^{i\delta}(r/2m^*-1)^{2ikm^*}+c_1e^{-i\delta}(r/2m^*-1)^{-2ikm^*} && \text{for } r\to2m^*,\\ Q&\sim c_2\sin(kr+\eta) && \text{for } r\to\infty. \end{aligned} \quad (39)$$

Between these two extreme values of r the behavior of Q can be found by interpreting k_{eff}^2 in (24) in terms of an effective potential or effective refractive index encountered by gravitational waves. There is little difference in equations between the analysis here and that given previously for electromagnetic waves[6] and for neutrinos[7] moving under the influence of a spherically symmetric metric of the form (26b):

$$k_{\text{eff}}^2(\text{neutrinos})=k^2-\kappa^2e^\nu/r^2\pm(\kappa r^{-2}e^{\nu-\frac{1}{2}\lambda}-\kappa r^{-1}e^{\frac{1}{2}(\nu-\lambda)}de^{\frac{1}{2}\nu}/dr), \quad (40a)$$

$$k_{\text{eff}}^2(\text{light})=k^2-L(L+1)e^\nu/r^2, \quad (40b)$$

$$k_{\text{eff}}^2(\text{gravitational waves})=k^2-L(L+1)e^\nu/r^2+6m^*e^\nu/r^3. \quad (40c)$$

The k^2 factor on the right-hand side of each expression plays the part of the energy in the Schrödinger wave equation, and the quantity subtracted from k^2 can be given the name of effective potential.

The effective potential starts from 0 at the Schwarzschild radius, rises to a maximum and then falls off again to zero at very large r. Therefore there are three regimes to be considered. In case 1, the quantity k^2 exceeds the height of the barrier and the solution is everywhere oscillatory. In case 2, the quantity k^2 is still positive, but less than the height of the barrier. The critical barrier height, k_{crit}^2, is given by the equation

$$\begin{aligned} 2m^*r_{\text{crit}}&=\tfrac{3}{2}+\tfrac{1}{2}(L+\tfrac{1}{2})^{-2}+\cdots;\\ k_{\text{crit}}&=(2.598\times2m^*)^{-1}[(L+\tfrac{1}{2})-(9/8)(L+\tfrac{1}{2})^{-1}+\cdots], \end{aligned} \quad (41)$$

when L is large. This case 2 recalls the problem of binding photons in the gravitational field of a geon. The interesting wave either falls off or rises exponentially in the barrier region. The one case, case 2a, corresponds to gravitational waves which never escape to large r. Such waves are of interest for the theory of gravitational geons: geons which derive all their mass and energy from gravitational waves trapped in the metric. Case 2b corresponds to waves which are large outside the barrier. These solutions like the case 1 solutions represent freely running gravitational waves disturbed to a greater or lesser extent by the gravitational field of the mass that they encounter.

[6] J. A. Wheeler, Phys. Rev. **97**, 511 (1955).
[7] D. Brill and J. A. Wheeler, Revs. Modern Phys. **29**, 465(1957).

Waves that Go through the Wormhole

The behavior of the waves will be affected by the conditions they encounter at small r. The metric acting on the gravitational wave in the present problem is multiply connected, whereas the metric acting on the neutrinos and electromagnetic waves in the published examples was topologically equivalent to Euclidean space. In those examples there was a definite origin at which the wave amplitude had to go to zero. By that condition the phase of the wave was uniquely determined, as showed up most readily in the JWKB approximate representation of the solution. The contrary will be the case for the present metric. The phase is not determined as it was for electromagnetic and neutrino geons, and as it would be for a gravitational geon. The ambiguity in phase has a clear physical significance. The ingoing wave need not equal in strength the outgoing wave. The Schwarzschild space is to be visualized as not inwardly bounded.[2] Instead, it can be considered as one mouth of a wormhole, the other mouth of which emerges elsewhere. Gravitational waves must be able to propagate through this tunnel as they propagate anywhere in space. The undetermined phase of this wave gives one the freedom that one can and must demand for a complete description of such "waves through the wormhole."

Space Behavior Unacceptable for Waves of Imaginary Frequency

Regime 3 corresponds to negative values of k^2. The analysis of the space dependence of such solutions shows that the radial part is uniquely determined by the requirement that it shall not go to infinity at large r. The solution that is acceptable because it falls off for large r proves, however, to fall off also at the Schwarzschild radius. Therefore there is no possibility to join it smoothly to a solution "in the other half of the tunnel" which will be acceptable. Consequently we conclude that there are no unstable solutions for odd waves.

There remains the discussion for the even waves. Here we have to examine the system (27b,c,d) supplemented by the condition (28). Unfortunately, owing to the complication of the equations involved, we were not able to establish a convenient "effective potential" picture. However, as far as stability is concerned there is no difficulty in recognizing that the same arguments used for odd waves are still valid.

We cannot avoid pointing out the essential importance of condition (28) in this analysis. It eliminates spurious solutions which do not have the correct behavior on the singular sphere. This can be clearly shown by a power series analysis near $r=2m^*$. Indeed to have the correct "wave through the tunnel" behavior it proves essential that the radial factor K be small for $r\sim 2m^*$ as compared with the other unknowns. This is plainly a consequence of (28). Curiously enough the same equation (28) also demands that the same factor K has to be negligible for large r, insuring the correct behavior at ∞. The discussion is then the same as for the odd waves. Consequently we conclude that Schwarzschild's solution of the gravitational field equations is stable.

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