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R. A. Breuer, M. P. Ryan and S. Waller

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Some properties of spin-weighted spheroidal harmonics

BY R. A. BREUER,† M. P. RYAN JR‡ AND S. WALLER‡

† *Max-Planck-Institut für Physik und Astrophysik, Föhringer Ring* 6, 8000 *München* 40, *W-Germany*

‡ *Centro de Estudios Nucleares, U.N.A.M., Circuito Exterior, C.U. Mexico* 20, *D.F., Mexico*

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We analyse the angular eigenfunctions – spin-weighted spheroidal harmonics – and eigenvalues of Teukolsky's equation. This equation describes infinitesimal scalar, electromagnetic and gravitational perturbations of rotating (Kerr) black holes. We derive analytic expressions for the eigenvalues up to sixth order in the expansion parameter for low frequencies and an analogous expansion in the high-frequency limit. Spin-weighted spheroidal harmonics form a complete and orthonormal set of functions on a prolate spheroid. They are, however, not the eigenfunctions of the natural Laplace operator on a spheroid and thus do not allow an obvious geometrical interpretation as the corresponding spin-weighted spherical harmonics.

1. INTRODUCTION

Spin-weighted spheroidal harmonics were first defined by Teukolsky (1973) and were used by Press & Teukolsky (1973) and Teukolsky & Press (1974) to study scalar, electromagnetic and gravitational wave perturbations of Kerr black holes. These functions are extensions of ordinary spheroidal harmonics (see for example, Morse & Feshbach 1963) in the same sense that spin-weighted spherical harmonics (Gel'fand, Minlos & Shapiro 1963; Newman & Penrose 1966; Goldberg *et al.* 1966) are extensions of ordinary spherical harmonics. Since these functions may be very useful in investigating fields of spin greater than zero, we elaborate in this paper some of their more important properties: the behaviour of the eigenfunctions and eigenvalues of the spheroidal wave equation for small and large frequencies. This may provide useful analytical expressions of interest for certain astrophysical situations involving rotating black holes and radiation. Some parts of this work have previously appeared in Breuer (1975).

In the second section we define spin-weighted spherical and spheroidal harmonics and state some of their elementary properties. In the third section we study them for small values of the frequency parameter that appears in the equation and give a recursion relation for the expansion of the functions in terms of Jacobi polynomials. In the fourth section we discuss the asymptotic expansion for large real and imaginary frequencies. In the fifth section we discuss the numerical behaviour of the eigenvalues.

2. Spin-weighted harmonics

In this section we review some work on spin-weighted spherical harmonics ${}_sY_{lm}$, spin-weighted spheroidal harmonics ${}_sZ_{lm}$ and the differential operator $\eth$ ('edth').

Spherical harmonics

The differential operators ${}_s\eth$ and ${}_s\eth'$ acting on a quantity Q of spin-weight s are defined by (Newman & Penrose 1966; Goldberg *et al.* 1967; Hansen *et al.* 1976)

$$\left.\begin{aligned} {}_s\eth Q &= -\left[\partial_\theta + \frac{i}{\sin\theta}\partial_\phi - s\cot\theta\right]Q, \\ {}_s\eth' Q &= -\left[\partial_\theta - \frac{i}{\sin\theta}\partial_\phi + s\cot\theta\right]Q, \end{aligned}\right\} \tag{2.1}$$

If Q is of spin-weight s, then ${}_s\eth Q$, ${}_s\eth' Q$ are of spin-weight $(s+1)$, $(s-1)$ respectively. Hence ${}_s\eth$ and ${}_s\eth'$ are raising and lowering operators with respect to the 'helicity' s. In order to agree with convention, the index 's' in these operators will be dropped from now on.

The commutation properties of $\eth$, $\eth'$ follow directly from (2.1):

$$\left.\begin{aligned} &[\eth'\eth - \eth\eth']\,Q = 2sQ, \\ &[\eth'^p\eth^q - \eth^q\eth'^p]\,Q = 0 \quad \text{for} \quad p-q = 2s, \\ &\int \mathrm{d}\Omega[Q_1(\eth Q_2) + Q_2(\eth Q_1)] = 0 \quad \text{for} \quad s_1+s_2 = -1, \end{aligned}\right\} \tag{2.2}$$

where $\mathrm{d}\Omega$ is the area element of the unit sphere.

By using the operators $\eth$ and $\eth'$, the ${}_sY_{lm}$ for $s = 1, 2, \ldots$ and $s = -1, -2, \ldots$ can be generated from the ordinary spherical harmonics Y_{lm} by

$${}_sY_{lm} = \left\{\begin{aligned} &[(l-s)!/(l+s)!]^{\frac{1}{2}}\,\eth^s Y_{lm} \quad (0 \leqslant s \leqslant 1), \\ &[(l+s)!/(l-s)!]^{\frac{1}{2}}\,(-\eth')^s Y_{lm} \quad (-1 \leqslant s \leqslant 0). \end{aligned}\right\} \tag{2.3}$$

Since $\eth$ annihilates ${}_lY_{lm}$ and $\eth'$ annihilates ${}_{-l}Y_{lm}$, the ${}_sY_{lm}$ are not defined for $|s| > l$. Further properties of ${}_sY_{lm}$ are

$${}_0Y_{lm} \equiv Y_{lm}, \quad {}_s\overline{Y}_{lm} = (-)^{s+m}\,{}_{-s}Y_{l,-m}, \tag{2.4a}$$

$$\eth\,{}_sY_{lm} = [(l-s)\,(l+s+1)]^{\frac{1}{2}}\,{}_{s+1}Y_{lm}, \tag{2.4b}$$

$$\eth'\,{}_sY_{lm} = -[(l+s)\,(l-s+1)]^{\frac{1}{2}}\,{}_{s-1}Y_{lm}, \tag{2.4c}$$

$$\eth'\eth\,{}_sY_{lm} = -(l-s)\,(l+s+1)\,{}_sY_{lm}, \tag{2.4d}$$

$$\eth\eth'\,{}_sY_{lm} = -(l+s)\,(l-s+1)\,{}_sY_{lm}, \tag{2.4e}$$

$$\eth'^p\eth^p\,{}_sY_{lm} = (-)^p\frac{(l-s)!\,(l+s+p)!}{(l+s)!\,(l-s-p)!}\,{}_sY_{lm}. \tag{2.4f}$$

Equation (2.4e) is written explicitly as

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right) - \frac{(m+s\cos\theta)^2}{\sin^2\theta} + s\right]{}_sP_{lm} = -{}_sA_{lm}\,{}_sP_{lm}, \tag{2.5}$$

where we have put

$$ {}_sY_{lm}(\theta,\phi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}} {}_sP_{lm}(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}, \qquad (2.6)$$

$$ {}_sA_{lm} = (l-s)(l+s+1).$$

The quantum numbers s, l, m can assume integer or half-integer values with $|m| \leqslant l$. The boundary conditions are such that the ${}_sP_{lm}$ are finite for $\cos\theta = \pm 1$. It follows from equation (2.4*d*) that the ${}_sY_{lm}$ are solutions of a self-adjoint one-parameter eigenvalue problem. It is easily seen that, for given s, they form a complete and orthonormal set of functions on S^2, i.e.

$$\int \mathrm{d}\Omega\, {}_s\overline{Y}_{l'm'}\, {}_sY_{lm} = \delta_{l'l}\,\delta_{m'm},$$

$$\sum_{l,m} {}_sY_{lm}(\theta',\phi')\, {}_sY_{lm}(\theta,\phi) = \delta(\cos\theta - \cos\theta')\,\delta(\phi-\phi'). \qquad (2.7)$$

Goldberg *et al.* (1967) established the relation of ${}_sY_{lm}$ to the representation matrix $D^l_{-sm}(\theta,\phi,0)$ of the rotation group, i.e.

$$\begin{aligned} {}_sY_{lm}(\theta.\phi) &= [(2l+1)/4\pi]^{\frac{1}{2}}\, D^l_{-sm}(\phi,\theta,0) \\ &= \left[\frac{2l+1}{4\pi}\frac{(l+m)!}{(l+s)!}\frac{(l-m)!}{(l-s)!}\right]^{\frac{1}{2}} (\sin\tfrac{1}{2}\theta)^{2l}\,\mathrm{e}^{\mathrm{i}m\phi} \\ &\quad \times \sum_n \binom{l-s}{n}\binom{l+s}{n+s-m}(-1)^{l-s-n}(\cot\tfrac{1}{2}\theta)^{2n+s-m}. \end{aligned} \qquad (2.8)$$

The function ${}_sP_{lm}$ defined in equation (2.5) may be called *spin-weighted associated Legendre* functions. For $s = 0$ equation (2.5) has as solutions just the associated Legendre polynomials $P_{lm} \equiv {}_0P_{lm}$.

Spheroidal harmonics

Spin-weighted spheroidal harmonics ${}_sZ_{lm}$ were defined by Teukolsky (1973) as the angular part of the perturbation wave equation in the Kerr geometry, i.e. by the solutions of

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right) - \frac{(m+s\cos\theta)^2}{\sin^2\theta} + s + a^2\omega^2\cos^2\theta - 2sa\omega\cos\theta\right]{}_sS_{lm} = -{}_sA_{lm}\,{}_sS_{lm}, \qquad (2.9)$$

where

$$ {}_sZ_{lm}(\theta,\phi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}} {}_sS_{lm}(-a\omega,\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}. \qquad (2.10)$$

Equation (2.9) is a two parameter eigenvalue equation together with the boundary condition that ${}_sS_{lm}$ is regular for $-1 \leqslant \cos\theta \leqslant 1$, the parameters being ${}_sA_{lm}$ and $a\omega$. Equation (2.9) has two regular singular points at $\cos\theta = \pm 1$. The analytic structure of equation (2.9) was investigated in detail by Hartle & Wilkins (1974). The orthonormal set of eigenfunctions ${}_sZ_{lm}$ is strongly complete if ω is real, but for complex ω, Stewart (1975) could only prove weak completeness. For $a\omega = 0$, equation (2.9) reduces to equation (2.5) and ${}_sS_{lm}(0,\cos\theta) \equiv {}_sP_{lm}(\cos\theta)$.

It is only noted here, that equation (2.9) relates to the operators $\eth_{\mathrm{G.H.P.}}$, which arise from the version of Newman–Penrose calculus due to Geroch, Held & Penrose (1973) (see also operators $\mathscr{L}_s$ defined by Press & Teukolsky 1973). For spin-weighted quantities one has

$$\left.\begin{aligned} {}_s\eth_{\mathrm{G.H.P.}} &= -\left[\partial_\theta + \frac{\mathrm{i}}{\sin\theta}\partial_\phi - s\cot\theta - a\omega\sin\phi\right], \\ {}_s\eth'_{\mathrm{G.H.P.}} &= -\left[\partial_\theta - \frac{\mathrm{i}}{\sin\theta}\partial_\phi + s\cot\theta + a\omega\sin\theta\right]. \end{aligned}\right\} \tag{2.11}$$

In terms of these operators equation (2.9) becomes (Breuer 1975; Chandrasekhar 1976)

$$[{}_{s+1}\eth'_{\mathrm{G.H.P.}}\,{}_s\eth_{\mathrm{G.H.P.}} - (2s+1)\,2a\omega\cos\theta]\,{}_sS = -\lambda\,{}_sS, \tag{2.12}$$

where $$\lambda = {}_sA + a^2\omega^2 - 2a\omega m.$$

In addition to the differential definition of spin-weighted spheroidal harmonics given in equation (2.9) one would wish to characterize these functions also in a purely geometrical way. Unlike in the case of spherical harmonics which can be interpreted as eigenfunctions of the Laplace operator on the unit sphere (Staruszkiewicz 1967; Goldberg *et al.* 1967), a similar interpretation for spheroidal harmonics is not known yet. On the contrary, the angular functions defined in equation (2.9) are *not* eigenfunctions of the Laplace operator on a spheroid.

3. Spin-weighted spheroidal harmonics for $a\omega \ll 1$

In this section the eigenfunctions and eigenvalues of the spheroidal wave equation are examined for the case $a\omega \ll 1$. Letting $x = \cos\theta$, equation (2.16) for the eigenfunctions ${}_sS_{lm}(-a\omega, x)$ and the eigenvalues ${}_sA_{lm}(\omega)$ becomes

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} + a^2\omega^2x^2 - 2a\omega sx - \frac{(m+sx)^2}{1-x^2} + s\right]{}_sS_{lm} = -{}_sA_{lm}\,{}_sS_{lm}. \tag{3.1}$$

For $a\omega = 0$, equation (3.1) reduces to the equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} - \frac{(m+sx)^2}{1-x^2} + s\right]{}_sP_{lm} = -(l-s)(l+s+1)\,{}_sP_{lm}, \tag{3.2}$$

whose solutions are what can be called spin-weighted associated Legendre-polynomials ${}_sS_{lm}(0,x) \equiv {}_sP_{lm}(x)$. The left hand side of equations (3.1) and (3.2) differ by the term $a^2\omega^2x^2 - 2a\omega sx$. For small values of $a\omega(\ll 1)$ the influence of this additional term in (3.1) on the solutions and eigenvalues of equation (3.2) can be treated as a perturbation. Since the coefficients in equation (3.1) are analytic functions of $a\omega$ and ${}_sA$, the solution of equation (3.1) which is finite at $x = \pm 1$ is an analytic function of both parameters, and ${}_sA$ is an analytic function of $a\omega$ (see, for example, Kato 1966; Stewart 1975).

Standard perturbation methods lead to the following expansion for the eigenvalues up to second order:

$$
{}_sA_{lm} = \begin{cases} (l-s)(l+s+1) - 2a\omega \dfrac{s^2 m}{l(l+1)} + O(a^2\omega^2) & (s \neq 0), \\ l(l+1) - \frac{1}{2}a^2\omega^2 \left[1 + \dfrac{(2m-1)(2m+1)}{(2l-1)(2l+3)}\right] + O(a^4\omega^4) & (s = 0). \end{cases} \tag{3.3}
$$

We can extend the perturbation method in principle to any order in $a\omega$. In fact for $s \neq 0$, Starobinskii & Churilov (1973) have computed the $a^2\omega^2$ term using second order perturbation theory. It is less difficult than it might seem at first glance to extend this method to even higher order terms in $a\omega$. To any order the expansion for ${}_sA_{lm}$ just involves traces of matrix products of the matrix $(\mathscr{H}_1)_{l'l}$ where

$$
(\mathscr{H}_1)_{l'l} = \int \mathrm{d}\Omega\, {}_s\bar{Y}^m_{l'}\, {}_sY^m_l \mathscr{H}_1 \quad \text{and} \quad \mathscr{H}_1 = a^2\omega^2 \cos^2\theta - 2a\omega s \cos\theta.
$$

From Press & Teukolsky (1973) we have that

$$
(\mathscr{H}_1)_{l'l} = a^2\omega^2 \left[\tfrac{1}{3}\delta_{l'l} + \tfrac{2}{3}\left(\frac{2l+1}{2l'+1}\right)^{\frac{1}{2}} \langle l2m0|l'm\rangle \langle l2-s0|l'-s\rangle\right]
$$
$$
- 2a\omega s \left[\frac{2l+1}{2l+1} \langle l1m0|l'm\rangle\langle l1-s0|l'-s\rangle\right],
$$

where $\langle j_1\, j_2\, m_1\, m_2|JM\rangle$ is a Clebsch–Gordon coefficient. From the structure of the Clebsch–Gordon coefficients $(\mathscr{H}_1)_{l'l}$ one has only five non-zero terms for each $l-2 \leqslant l' \leqslant l+2$, hence the algebra involved in computing ${}_sA_{lm}$ is not impossible. To order $(a\omega)^5$, we obtain

$$
\begin{aligned}
{}_sA_{lm} = {} & (l-s)(l+s+1) + 2a\omega s B_0(l) \\
& + a^2\omega^2 \left\{ -A_0(l) + \frac{2s^2}{l}[B_{-1}(l)]^2 - \frac{2s^2}{l+1}[B_1(l)]^2 \right\} \\
& + a^3\omega^3 \left\{ \frac{(l^2-1)(l^2-s^2)}{ml^3}[A_{-1}(l)]^2 - \frac{l(l+2)}{m(l+1)}[A_1(l)]^2 - \frac{[A_1(l)]^2}{ml(l+1)^3(l+2)} \right\} \\
& + a^4\omega^4 \left\{ \frac{(l-2s^2)^2}{2l^2(2l-1)}[A_{-2}(l)]^2 + \frac{[A_{-1}(l)]^2}{2l} - \frac{[A_1(l)]^2}{2(l+1)} - \frac{2l+1+4s^2+(l+2s^2)^2}{2(l+1)^2(2l+3)}[A_2(l)]^2 \right. \\
& + \frac{2s^2}{l(l+1)}[B_{-1}(l)\,B_1(l)]^2 + \frac{2s^2}{3l^2}[B_{-1}(l)]^2 + \frac{s^2}{3l^3}[3A_0(l)-1]\,B_{-1}(l) + \frac{s^2}{3l^3(l-1)}[A_{-1}(l)]^2 \\
& - \frac{s^2}{3(l+1)^2}[B_1(l)]^2 + \frac{s^2}{(l+1)^2}[B_1(l)]^2[A_0(l) - A_0(l+1)] - \frac{s^2}{l^2}[B_{-1}(l)]^2 A_0(l-1) \\
& - \frac{s^2 l}{(l+1)^3}[A_1(l)]^2 + \frac{s^2[B_1(l)]^2}{3(l+1)^2} - \frac{s^2(l+1)}{l^3}[A_{-1}(l)]^2 + \frac{s^2(l+2)}{(l+1)^3}[A_1(l)]^2 \\
& + \frac{2s^4[A_{-1}(l)]^2}{l^5} - \frac{2s^4[B_{-1}(l)]^4}{l^3} + \frac{2s^4(l-1)^2(2l-1)(2l-3)}{l^2(l+1)^3(2l+1)(2l+3)}[A_{-2}(l)]^2 \\
& \left. + \frac{s^4 l(l-1)}{(l+1)^5}[A_1(l)]^2 - \frac{2s^4[B_1(l)\,B_{-1}(l)]^2}{l(l+1)^2} + \frac{2s^4[B_1(l)]^4}{(l+1)^3} \right\} + O(a^5\omega^5),
\end{aligned} \tag{3.4}
$$

where the coefficients $A_p(n)$ and $B_p(n)$ are given in table 1. The next-order term in $a^5\omega^5$ has also been calculated; but because of its length it cannot be printed here. On request it can be obtained from the authors.

TABLE 1. COEFFICIENTS $A_p(n)$ AND $B_p(n)$

$$A_0(n) = \frac{1}{3}+\frac{2}{3}\frac{[3m^2-n(n+1)][3s^2-n(n+1)]}{(2n-1)\,n(n+1)\,(2n+3)}$$

$$A_1(n) = -2ms\left[\frac{(n-m+1)\,(n+m+1)\,(n-s+1)\,(n+s+1)}{(2n+1)\,(2n+3)\,n^2(n+1)^2\,(n+2)^2}\right]^{\frac{1}{2}}$$

$$A_{-1}(n) = -2ms\left[\frac{(n^2-m^2)\,(n^2-s^2)}{n^2(4n^2-1)\,(n^2-1)^2}\right]^{\frac{1}{2}}$$

$$A_2(n) = \left[\frac{(n-m+2)\,(n-m+1)\,(n+m+1)\,(n+m+2)\,(n-s+1)\,(n-s+2)\,(n+s+1)\,(n+s+2)}{(2n+1)\,(2n+5)\,(n+1)^2\,(n+2)^2\,(2n+3)^2}\right]^{\frac{1}{2}}$$

$$A_{-2}(n) = \left[\frac{(n^2-m^2)\,(n^2-s^2)\,(n-m-1)\,(n+m-1)\,(n-s-1)\,(n+s-1)}{(n-1)^2\,(2n-1)^2\,(2n+1)\,n^2(2n-3)}\right]^{\frac{1}{2}}$$

$$B_0(n) = -\frac{ms}{n(n+1)},\quad B_1(n) = \left[\frac{(n-m+1)\,(n+m+1)\,(n-s+1)\,(n+s+1)}{(2n+1)\,(2n+3)\,(n+1)^2}\right]^{\frac{1}{2}}$$

$$B_{-1}(n) = \left[\frac{(n^2-m^2)\,(n^2-s^2)}{n^2(4n^2-1)}\right]^{\frac{1}{2}}.$$

To calculate ${}_sA_{lm}$ to even higher order in $a\omega$, the perturbation method may be replaced by a continuation method yielding differential equations in $a\omega$, which can be integrated numerically as was done by Press & Teukolsky (1973) and Teukolsky & Press (1974), to obtain polynomial expansion for ${}_{\pm1}A_{lm}$ and ${}_{\pm2}A_{lm}$ for $l \leqslant 6$ and all allowed m up to order $a^7\omega^7$ [in their notation ${}_sA_{lm} = {}_sE_{lm}-s(s+1)$].

We may use the expansion constants of ${}_sZ_{lm}$ in terms of ${}_sY_{lm}$ in the perturbation method to express ${}_sZ_{lm}$ as a linear combination of ${}_sY_{lm}$. For $s = 0$ we may think of this as spheroids which are obtained by a superposition of spheres whose centre lie on the rotation axis of the spheroids. We therefore want an expansion of the spheroidal eigenfunctions ${}_sS_{lm}$ in terms of the spherical eigenfunctions ${}_sP_{lm}$. Equation (3.2) has three regular singular points $\pm 1, \infty$ and can therefore be solved by means of hypergeometric functions. The Riemann symbol for equation (3.2) is given by

$$\left\{\begin{matrix} +1 & -1 & \infty & \\ \lambda & \lambda & \nu & x \\ \lambda' & \mu' & \nu' & \end{matrix}\right\}. \tag{3.5}$$

The elements occurring in this symbol satisfy the algebraic relations

$$\left.\begin{gathered} \lambda+\lambda' = 0,\quad \mu+\mu' = 0,\quad \nu+\nu' = 1, \\ \frac{2\lambda\lambda'}{(x-1)^2(x+1)}-\frac{2\mu\mu'}{(x-1)\,(x+1)^2}+\frac{\nu\nu'}{(x-1)\,(x+1)} = -\frac{(m+sx)^2}{(x-1)^2(x+1)^2}+\frac{l(l+1)-s^2}{1-x^2}, \end{gathered}\right\} \tag{3.6}$$

which have the solutions

$$\left.\begin{array}{lll}\lambda = -\frac{1}{2}(s+m), & \mu = -\frac{1}{2}(s-m), & \nu = -1,\\ \lambda' = \frac{1}{2}(s+m), & \mu' = \frac{1}{2}(s-m), & \nu' = l+1.\end{array}\right\} \tag{3.7}$$

One would like to find solutions in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ that are solutions to equation (3.2) and regular at $x = \pm 1$. The only hypergeometric functions that have such a form and are finite at both $x = \pm 1$ are (cf. Magnus, Oberhettinger & Soni (1966))

$$_s\alpha_l^m\,[(1-x)/(1+x)]^{\frac{1}{2}(s+m)}(1+x)^l F\{-(l-m),\,-(l-s);\; 1+(s+m);\,(1-x)/(1+x)\}, \tag{3.8a}$$

$$_s\alpha_l^m\,[(1+x)/(1-x)]^{\frac{1}{2}(s-m)}(1-x)^l F\{-(l+m),\,-(l-s);\; 1+(s+m);\,(1+x)/(1-x)\}, \tag{3.8b}$$

where $_s\alpha_{lm}$ is determined by normalization and the limit $a\omega \to 0$. The corresponding solutions of equation (3.2) for $|m| \geqslant |s|$ are given by

$$_sP_{lm}(x) = {_s\alpha_{lm}}(1-x)^{\frac{1}{2}(m+s)}(1+x)^{\frac{1}{2}(m-s)}\,P_{l-m}^{(m+s,\,m-s)}(x) \quad (m \geqslant 0), \tag{3.9a}$$

$$_sP_{lm}(x) = {_s\alpha_{lm}}(1+x)^{\frac{1}{2}(s-m)}(1-x)^{-\frac{1}{2}(s+m)}\,P_{l+m}^{(-s-m,\,s-m)}(x) \quad (m \leqslant 0). \tag{3.9b}$$

For $|m| < |s|$ one similarly obtains solutions of equation (3.2) of the form

$$_sP_{lm}(x) = {_s\alpha_{lm}}(1-x)^{\frac{1}{2}(s+m)}(1+x)^{\frac{1}{2}(s-m)}\,P_{l-s}^{(s+m,\,s-m)}(x) \quad (s > 0), \tag{3.10a}$$

$$_sP_{lm}(x) = {_s\alpha_{lm}}(1-x)^{-\frac{1}{2}(s+m)}(1+x)^{-\frac{1}{2}(s-m)}\,P_{l+s}^{(-s-m,\,-s+m)}(x) \quad (s < 0). \tag{3.10b}$$

From our normalization of $_sY_{lm}$ it follows that

$$\int_{-1}^{1} \mathrm{d}x\,{_sP_{lm}}(x)\,{_sP_{lm}}(x) = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}.$$

Equating this integral to the right hand sides of (3.9) and (3.10) and using

$$\int_{-1}^{1} \mathrm{d}x(1-x)^{\alpha}(1+x)^{\beta}\,[P_n^{(\alpha,\beta)}(x)]^2 = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!(\alpha+\beta+1+2n)\Gamma(\alpha+\beta+n+1)}$$

(Gradshtein & Ryzhik 1965), we find that

$$_s\alpha_{lm} = \frac{(-1)^{s+\frac{1}{2}(m+|m|)}\,(l-m)!}{2^{|m|}[(l+s)!\,(l-s)!]^{\frac{1}{2}}} \quad (|m| \geqslant |s|), \tag{3.11a}$$

$$_s\alpha_{lm} = \frac{(-1)^{m+\frac{1}{2}(s+|s|)}2^{|s|}\,[(l-s)!\,(l+s)!]^{\frac{1}{2}}}{(l+m)!} \quad (|m| < |s|). \tag{3.11b}$$

One now assumes that for $|m| \geqslant |s|$ the functions $_sS_{lm}$ can be expanded in terms of the $_sP_{lm}$ as given in equation (3.9). Thus

$$_sS_{lm}(x) = (1-x)^{\frac{1}{2}(m+s)}(1+x)^{\frac{1}{2}(m-s)}\sum_{n=m}^{\infty} d_{lm}^n\,{_s\alpha_{nm}}\,P_{n-m}^{(m+s,\,m-s)} \quad (m \geqslant 0), \tag{3.12a}$$

$$_sS_{lm}(x) = (1+x)^{\frac{1}{2}(s-m)}(1-x)^{-\frac{1}{2}(s-m)}\sum_{n=-m}^{\infty} d_{lm}^n\,{_s\alpha_{nm}}\,P_{n+m}^{(-s-m,\,s-m)}(x) \quad (m < 0), \tag{3.12b}$$

for coefficients d^n_{lm} which satisfy a five term recursion relation:

$$\begin{aligned} & a^2\omega^2 \frac{[(n+s+2)(n-s+2)(n+s+1)(n-s+1)]^{\frac{1}{2}}(n-m+2)(n-m+1)}{(n+1)(n+2)(2n+3)(2n+5)} d^{n+2}_{lm} \\ & -2\frac{[(n+s+1)(n-s+1)]^{\frac{1}{2}}(n-m+1)}{(n+1)(2n+3)}\left[a\omega s+\frac{a^2\omega^2 sm}{n(n+2)}\right] d^{n+1}_{lm} \\ & +\left\{{}_sA_{lm}+(n-s)(n+s+1)+a^2\omega^2\left[\frac{s^2m^2}{n^2(n+1)^2}+\frac{(n-s)(n+s)(n-m)(n+m)}{n^2(2n-1)(2n+1)}\right.\right. \\ & \left.\left. +\frac{(n-m+1)(n+m+1)(n-s+1)(n+s+1)}{(2n+1)(n+1)^3(2n+3)}\right]+2a\omega\frac{ms^2}{n(n+1)}\right\} d^n_{lm} \\ & -\frac{2(n+m)[(n+s)(n-s)]^{\frac{1}{2}}}{n(2n-1)}\left[a\omega s+a^2\omega^2\frac{sm}{(n-1)(n+1)}\right] d^{n-1}_{lm} \\ & +a^2\omega^2\frac{(n+m+1)(n+m)[(n+s)(n+s-1)(n-s)(n-s-1)]^{\frac{1}{2}}}{n(n-1)(2n-3)(2n-1)} d^{n-2}_{lm}=0. \end{aligned} \tag{3.13}$$

For $|m| < |s|$ the corresponding expansions in terms of the functions ${}_sA_{lm}$ given in (3.10) are

$$ {}_sS_{lm}(x) = (x+1)^{\frac{1}{2}(s+m)}(x-1)^{\frac{1}{2}(s-m)} \sum_{n=s}^{\infty} \tilde{d}^n_{lm}\, {}_s\alpha_{nm} P^{(s+m),(s-m)}_{n-s}(x) \quad (s \geqslant 0). \tag{3.14a}$$

$$ {}_sS_{lm}(x) = (x+1)^{-\frac{1}{2}(s+m)}(x-1)^{-\frac{1}{2}(s-m)} \sum_{n=-s}^{\infty} \tilde{d}^n_{lm}\, {}_s\alpha_{nm} P^{(-s-m,\,-s+m)}_{n+s}(x) \quad (s \leqslant 0). \tag{3.14b}$$

Here the expansion coefficients $\tilde{d}^n_{lm}$ again satisfy a five-term recursion relation, namely

$$\begin{aligned} & a^2\omega^2\frac{[(n-s+2)(n-s+1)(n+s+2)(n+s+1)]^{\frac{1}{2}}(n-m+1)(n-m+2)}{(n+1)(n+2)(2n+3)(2n+5)} \tilde{d}^{n+2}_{lm} \\ & -\frac{2[(n-s+1)(n+s+1)]^{\frac{1}{2}}(n-m+1)}{(n+1)(2n+3)}\left[a\omega s+a^2\omega^2\frac{sm}{n(n+1)}\right] \tilde{d}^{n+1}_{lm} \\ & +\left\{{}_sA_{lm}+(n-s)(n+s+1)+a^2\omega^2\left[\frac{s^2m^2}{n^2(n+1)^2}+\frac{(s-1)(s+1)(n-m)(n+m)}{n^2(2n-1)(2n+1)}\right.\right. \\ & \left.\left. +\frac{(n-s+1)(n+s+1)(n-m+1)(n+m+1)}{(2n+1)(2n+3)(n+1)^2}\right]+2a\omega\frac{ms^2}{n(n+1)}\right\} \tilde{d}^n_{lm} \\ & -\frac{2[(n-s)(n+s)]^{\frac{1}{2}}}{n(2n-1)(n+m)}\left[2a\omega s+a^2\omega^2\frac{sm}{(n-1)(n+2)}\right] \tilde{d}^{n-1}_{lm} \\ & +a^2\omega^2\frac{[(n-s-1)(n+s-1)(n-s)(n+s)]^{\frac{1}{2}}}{(n-1)n(2n-3)(2n-1)(n+m)(n+m-1)} \tilde{d}^{n-2}_{lm}=0. \end{aligned} \tag{3.15}$$

Here and in equation (3.13) set $d^{-2}_{lm} = d^{-1}_{lm} = \tilde{d}^{-2}_{lm} = \tilde{d}^{-1}_{lm} = 0$. The recursion relations (3.13) and (15) may in principle also be employed to find the eigenvalue expansion of equation (3.4) as a power series in $a\omega$ up to any order desired.

In the special case $s = 0$, this is possible as then equation (3.13) reduces to a three-term recursion relation, which can be solved by the following standard method (cf. Morse & Feshbach 1953; Meixner & Schäfke 1954). The pair of eigenvalues (${}_0A_{lm}$, $a^2\omega^2$) are solutions of a certain equation of continued fractions derived from the recursion relation. For small values of $a\omega$ they can be replaced by power series. Then one inserts ${}_0A_{lm} = l(l+1)+O(a^2\omega^2)$ into the inverted continuous fraction and obtains higher order terms by successive iteration. This method leads to the following low frequency expansion for the eigenvalues ${}_0A_{lm}$ (Bouwkamp 1950):

$$
\begin{aligned}
{}_0A_{lm} ={}& l(l+1) - \frac{a^2\omega^2}{2}\left[1+\frac{(2m-1)(2m+1)}{(2l-1)(2l+3)}\right] \\
&+\frac{a^4\omega^4}{2}\left[\frac{(l-m-1)(l-m)(l+m-1)(l+m)}{(2l-3)(2l-1)^3(2l+1)} - \frac{(l-m+1)(l-m+2)(l+m+1)(l+m+2)}{(2l+1)(2l+3)^3(2l+5)}\right] \\
&-a^6\omega^6(4m^2-1)\left[\frac{(l-m-1)(l-m)(l+m-1)(l+m)}{(2l-5)(2l-3)(2l-1)^5(2l+1)(2l+3)} - \frac{(l-m+1)(l-m+2)(l+m+1)(l+m+2)}{(2l-1)(2l+1)(2l+3)^5(2l+7)}\right] \\
&+a^8\omega^8\left\{2(4m^2-1)\left[\frac{(l-m-1)(l-m)(l+m-1)(l+m)}{(2l-5)^2(2l-3)(2l-1)^7(2l+1)(2l+3)^2} - \frac{(l-m+1)(l-m+2)(l+m+1)(l+m+2)}{(2l-1)^2(2l+1)(2l+3)^7(2l+5)(2l+7)^2}\right]\right. \\
&+\frac{1}{16}\left[\frac{(l-m-3)(l-m-2)(l-m-1)(l-m)(l+m-3)(l+m-2)(l+m-1)(l+m)}{(2l-7)(2l-5)^2(2l-3)^3(2l-1)^4(2l+1)}\right. \\
&\left.\quad-\frac{(l-m+1)(l-m+2)(l-m+3)(l-m+4)(l+m+1)(l+m+2)(l+m+3)(l+m+4)}{(2l+1)(2l+3)^4(2l+5)^3(2l+7)^2(2l+9)}\right] \\
&+\frac{1}{8}\left[\frac{(l-m+1)^2(l-m+2)^2(l+m+1)^2(l+m+2)^2}{(2l+1)^2(2l+3)^7(2l+5)^2} - \frac{(l-m-1)^2(l-m)^2(l+m-1)^2(l+m)^2}{(2l-3)^2(2l-1)^7(2l+1)^2}\right] \\
&\left.+\frac{1}{2}\,\frac{(l-m-1)(l-m)(l-m+1)(l-m+2)(l+m-1)(l+m)(l+m+1)(l+m+2)}{(2l-3)(2l-1)^4(2l+1)^2(2l+3)^4(2l+5)}\right\} \\
&+O(a^{10}\omega^{10}).
\end{aligned}
\tag{3.16}
$$

However, for $s \neq 0$ a suitably extended procedure would have to be developed to solve the general five-term recursion relations. To the authors' knowledge neither four- nor five-term recursion relations have been treated (for the above purpose) in the literature. Lacking a solution for the recursion relation we must rely on the perturbation method as it was used to find equation (3.4) for ${}_sA_{lm}(a\omega)$. Note that up to $O(a^5\omega^5)$, equation (3.4) reduces to (3.16) in the case $s = 0$.

4. Spin-weighted spheroidal harmonics for $a\omega \gg 1$

We now want to consider solutions of equation (3.1) for the limit $a^2\omega^2 \to \pm\infty$. Here the plus sign corresponds to real frequencies, the minus to imaginary frequencies. The method used is analogous to that given in Sips (1949), Erdelýi (1956), and Flammer (1957). One usually thinks of the real frequency case as the one which is most important in physical problems. In the purely imaginary case, however, our transformations do not give us a real expansions for the quantities of interest, but complex ones. Therefore we will not present these results here and consider only the case of real frequencies.

The aim is to transform equation (3.1) in such a way that in the limit $a^2\omega^2 \to +\infty$ it can be compared with a differential equation whose solutions are already known.

Case $m \geqslant s$

For $s > 0$, $m > 0$, $m \geqslant s$ and $x > 0$ a suitable transformation is

$$ {}_sS(x) = (1-x)^{\frac{1}{2}(m+s)}(1+x)^{\frac{1}{2}(m-s)}g(x), \tag{4.1}$$

which changes equation (3.1) into

$$(1-x^2)\,g''(x) - 2(s+mx+x)\,g'(x) + (a^2\omega^2x^2 - 2a\omega sx)\,g(x) \\ + [s(s+1) - m(m+1) + {}_sA]\,g(x) = 0. \tag{4.2}$$

A change in variable from x to $u = 2a\omega(1-x)$ with $a\omega > 0$ yields furthermore

$$ug''(u) + (m+s+1)\,g'(u) - \tfrac{1}{4}(u - \alpha + 2s - {}_sA^*)\,g(u) \\ - \frac{1}{4a\omega}[u^2g''(u) + 2(m+1)\,ug'(u) - (\tfrac{1}{4}u^2 + su)\,g(u)] = 0. \tag{4.2a}$$

Here we have defined A^* and α by

$${}_sA^* = \frac{1}{4a\omega}[\mathrm{s}(s+1) - m(m+1) + a^2\omega^2 - \alpha a\omega + {}_sA]. \tag{4.2b}$$

and no assumptions made about $\alpha \in \mathbb{R}$.

For $x < 0$ we apply the same transformation (4.1) to equation (3.1) with

$$u^* = 2a\omega(1+x)$$

instead of u we arrive at a similar equation for $g(u^*)$. The transformation of variables in (4.1) changes the regular singular points $x = \pm 1$ to u, $u^* = 0, \infty$ in the limit $a\omega \to \infty$. If we let $a\omega \to \infty$ we get for both cases

$$\left.\begin{aligned} ug''(u) + (s+m+1)\,g'(u) - \tfrac{1}{4}(u - \alpha + 2s - {}_sA^*)\,g(u) = 0 \quad (x > 0), \\ u^*g''(u^*) + (-s+m+1)\,g'(u^*) - \tfrac{1}{4}(u^* - \alpha - 2s - {}_sA^*)\,g(u^*) = 0 \quad (x < 0). \end{aligned}\right\} \tag{4.3}$$

Equations (4.3) have solutions in terms of confluent hypergeometric functions ${}_1F_1(a,b,c)$,

$$g = \begin{cases} \mathrm{e}^{-\frac{1}{2}u}\,{}_1F_1(-\tfrac{1}{4}\alpha + \mathrm{s} + \tfrac{1}{2}(m+1), s+m+1, u) & (x > 0), \\ \mathrm{e}^{-\frac{1}{2}u^*}\,{}_1F_1(-\tfrac{1}{4}\alpha - s + \tfrac{1}{2}(m+1), -s+m+1, u^*) & (x < 0). \end{cases} \tag{4.4}$$

Case $s < m$

In this case we take the transformation

$$\begin{aligned} {}_sS(x) &= (1-x)^{\frac{1}{2}(s+m)}(1+x)^{\frac{1}{2}(s-m)}g, \\ u &= 2a\omega(1-x) \quad (x > 0), \\ u^* &= 2a\omega(1+x) \quad (x < 0), \end{aligned} \tag{4.5}$$

which transforms equation (3.1) into

$$ug''(u) + (m+s+1)\,g'(u) - ({}_sA^* + \tfrac{1}{4}ug(u)) - \frac{1}{4a\omega}[u^2g''(u) + 2(s+1)\,ug'(u) - \tfrac{1}{2}u^2g(u)] = 0 \quad (x > 0), \tag{4.6}$$

where ${}_sA^*$ has the same meaning as in equation (4.2), and a similar equation is obtained for the variable u^* $(x < 0)$. In the limit $a\omega \to \infty$ these asymptotic differential equations have the solutions

$$g = \begin{cases} e^{-\frac{1}{2}u}\,{}_1F_1(-\tfrac{1}{4}\alpha + s + \tfrac{1}{2}(1+m), s+m+1, u) & (x > 0), \\ e^{-\frac{1}{2}u^*}\,{}_1F_1(-\tfrac{1}{4}\alpha + \tfrac{1}{2}(1-m), s-m+1, u^*) & (x < 0), \end{cases} \tag{4.7}$$

For $m < 0$ we can perform the appropriate transformations of ${}_sS(x)$ to $g(u)$, and we find that g has the same solution as in equation (4.7).

For the later determination of $\alpha = 2q$ we need the following theorem

THEOREM 4.1 (NUMBER OF ZEROS)

(*a*) *Every ${}_sS_{lm}(a\omega, x)$ has only real, simple zeros in $-1 < x < 1$.*

(*b*) *All zeros are regular points and analytical in $a\omega$.*

(*c*) *The number of zeros of ${}_sS_{lm}$ is independent of $a\omega$ and for $x \in (-1, 1)$ and equal to*

$$\begin{cases} (l-m) & \text{for} \quad m \geqslant s, \quad |m| \leqslant l, \\ (l-s) & \text{for} \quad m < s, \quad |m| \leqslant l, \\ 0 & \text{otherwise.} \end{cases}$$

The theorem may be obtained along the lines of a similar proof in Meixner & Schäfke (1973) and is therefore not given here. Furthermore, the eigenfunctions and eigenvalues have the symmetries (Press & Teukolsky 1973)

$$\begin{aligned} {}_{-s}S_{lm}(a\omega, x) &= {}_sS_{lm}(a\omega, -x), & {}_sS_{lm}(-a\omega, -x) &= {}_sS_{l-m}(a\omega, x), \\ {}_{-s}A_{lm}(a\omega) &= {}_sA_{lm} + 2s, & {}_sA_{lm}(a\omega) &= {}_sA_{l-m}(-a\omega). \end{aligned} \tag{4.8}$$

In confluent hypergeometric functions ${}_1F_1(a, b, u)$ in equations (4.4) and (4.7) the argument b has to be a positive and a has to be a negative integer. Solutions with non-integer and positive a and with $b = 0, -1, -2, \ldots$ have simple poles at $u = 0$ $(x = 0)$. Since in our case b is also an integer, the function reduces to a finite polynomial, the generalized Leguerre polynomials

$$L_p^{(\alpha)}(u) = [(\alpha+1)(\alpha+2)\ldots(\alpha+p-1)/p!]\,{}_1F_1(-p, \alpha+1; u).$$

Leaving the determination of $\alpha = 2q$ and p to the end of this paragraph we first perform a perturbation expansion in order to obtain higher order terms in $1/a\omega$ for the eigenfunctions and the eigenvalues. We use solutions (4.4) and (4.7) as basis functions for the expansion of $g(u)$, $g(u^*)$ and ${}_sA_{lm}$:

$m \geqslant s$:

$$g(x) = \begin{cases} \sum\limits_n a_n \mathrm{e}^{-\frac{1}{2}u}\, {}_1F_1(-\tfrac{1}{4}\alpha + s + \tfrac{1}{2}(1+m) - n, m+s+1, u) & (x > 0), \\ \sum\limits_u \tilde{a}_n \mathrm{e}^{-\frac{1}{2}u^*}\, {}_1F_1(-\tfrac{1}{4}\alpha - s + \tfrac{1}{2}(1+m) - n, m-s+1, u^*) & (x > 0); \end{cases} \tag{4.9}$$

$m < s$:

$$g(x) = \begin{cases} \sum\limits_n b_n \mathrm{e}^{-\frac{1}{2}u}\, {}_1F_1(-\tfrac{1}{4}\alpha + s + \tfrac{1}{2}(1+m) - n, s+m+1, u) & (x > 0), \\ \sum\limits_n \tilde{b}_n \mathrm{e}^{-\frac{1}{2}u^*}\, {}_1F_1(-\tfrac{1}{4}\alpha + s + \tfrac{1}{2}(1-m) - n, s-m+1, u^*) & (x < 0), \end{cases} \tag{4.10}$$

where for $n \neq 0$, $\lim\limits_{a\omega \to \infty} (a_n, \tilde{a}_n, b_n, \tilde{b}_n) = 0$.

Because of the analyticity of the coefficients of the initial differential equation (3.1) in the parameter $a\omega$, we can assume a series expansion in powers of $1/a\omega$ for ${}_sA_{lm}$ (Kato 1966), ${}_sA^* = A_0 + (1/a\omega)A_1 + \ldots$, or

$${}_sA_{lm} = -a^2\omega^2 + \alpha a\omega + A_0 + \frac{1}{a\omega}A_1 + \frac{1}{a^2\omega^2}A_2 + \frac{1}{a^3\omega^3}A_3 + \ldots. \tag{4.11}$$

Inserting ansatz (4.9) into equation (4.2) we get the following eigenvalue expansion up to $O(1/a^4\omega^4)$

$$\begin{aligned} {}_sA_{lm} = & -a^2\omega^2 + 2qa\omega - \tfrac{1}{2}[q^2 - m^2 + 2s + 1] \\ & - \frac{1}{8a\omega}[q^3 - m^2q + q - s^2(q+m)] \\ & - \frac{1}{64a^2\omega^2}[5q^4 - (6m^2 - 10)q^2 + m^4 - 2m^2 - 4s^2(q^2 - m^2 - 1) + 1] \\ & + \frac{1}{a^3\omega^3}A_3 + O(1/a^4\omega^4), \end{aligned} \tag{4.12}$$

where

$$\begin{aligned} A_3 = \tfrac{1}{512}\{\tfrac{1}{64}[& (q-m-1+2s)(q-m-1-2s)(q-m-2s-3) \\ & \qquad (q-m+2s-3)(q+m-3)^2(q+m-1)^2 \\ & -(q-m-2s+1)(q-m+2s+1)(q-m-2s+3)(q-m+2s+3) \\ & \times (q+m+1)^2(q+m+3)^2] + 2[(q-m+2s-1)(q-m-2s-1)(q-1)^2(q+m-1)^2 \\ & -(q+m+1)^2(q+1)^2(q-m-2s+1)(q-m+2s+1)] - 2A_1[(q-m+2s-1) \\ & \times (q-m-2s-1)(q+m-1)^2 + (q+m+1)^2(q-m-2s+1)(q-m+2s+1)]\}, \end{aligned}$$

and where A_1 is the coefficient of $1/a\omega$. For the $x < 0$ case in equation (4.9) the same expansion for ${}_sA_{lm}$ as in (4.12) is obtained. Equally, if we derive an analogous recursion relation for the b_n, $\tilde{b}_n$ using the expansion ansatz (4.10) we also find, that this gives us exactly the same eigenvalue expansion as in equation (4.12). Thus we can use this expansion for $a\omega > 0$, $m \gtrless 0$ and obtain the corresponding expansion for $a\omega < 0$ by the symmetry ${}_sA_l^{-m}(-a\omega) = {}_sA_l^m(a\omega)$.

The value of q in the case of $s = 0$ is obtained by comparing equations (4.3) with the one for generalized Laguerre polynomials $L_p^{(n)}(u)$ for

$$v(u) = \mathrm{e}^{-\frac{1}{2}u}\,L_p^{(n)}(u), \quad L_p^{(n)}(u) = \frac{\mathrm{e}^u u^{-n}}{p!}\frac{\mathrm{d}^p}{\mathrm{d}x^p}[\mathrm{e}^{-u}u^{n+p}].$$

This equation is
$$uv''(u) + (n+1)\,v'(u) + \left[p + \frac{n+1}{2} - \frac{u}{2}\right]v(u) = 0.$$

Thus we identify
$$\lim_{a\omega \to \infty} {}_0A^* = p + \tfrac{1}{2}(n+1) \equiv \tfrac{1}{2}q, \quad n = m, \tag{4.13}$$

where $q = 2p + m + 1$.

Since $L_p^{(m)}(u)$ has p zeros in $x \in (-1, 1)$ it follows with theorem 4.1 that

$$2p = \begin{cases} l-m & \text{for } l-m \text{ even,} \\ l-m-1 & \text{for } l-m \text{ odd,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

Finding q for $s \neq 0$ is more difficult than in the $s = 0$ case. For some ranges of q it seems that $q = l+1$ is valid for both $s = 0$ and $s \neq 0$, but for others we do not seem to be able to use the usual arguments about the number of zeros of the solution to find q. This implies that we should study the convergence of the solutions of equations (4.9) and (4.10) to the asymptotic solution, and, if it is indeed convergent, analyse more carefully the number and location of zeros. For the moment we can take our expansion of ${}_sA_{lm}$ to be correct, and leave q as a free parameter to be used in fitting the asymptotic expressions to the low-$a\omega$ expansion as we do in § 5.

5. The Behaviour of the eigenvalues

If we compare our analytic expression for ${}_sA_l^m(a\omega)$ for small $a\omega$ with that of Press & Teukolsky (1973), we find that there are some differences. Table 2 is a comparison of the values of the coefficients of our expansion with the coefficients of Press & Teukolsky for $l = 3$, $s = 2$. The differences occur because our expression is an expansion about $a\omega = 0$ while theirs is a polynomial fit in the region $a\omega < 3$. Also, the analytic expression clearly exhibits the symmetry ${}_sA_l^{-m}(-a\omega) = {}_sA_l^m(a\omega)$, while Press & Teukolsky obtain a better polynomial fit for positive $a\omega$ by relaxing this requirement. Notice that the differences between the analytic coefficients and the polynomial ones is roughly of the same order as the difference between the positive and negative m polynomial coefficients.

We will not try to compare the small $a\omega$ expressions graphically. In a graph with a reasonable scale the analytic and Press–Teukolsky eigenvalues are indistinguishable for $a\omega < 3$.

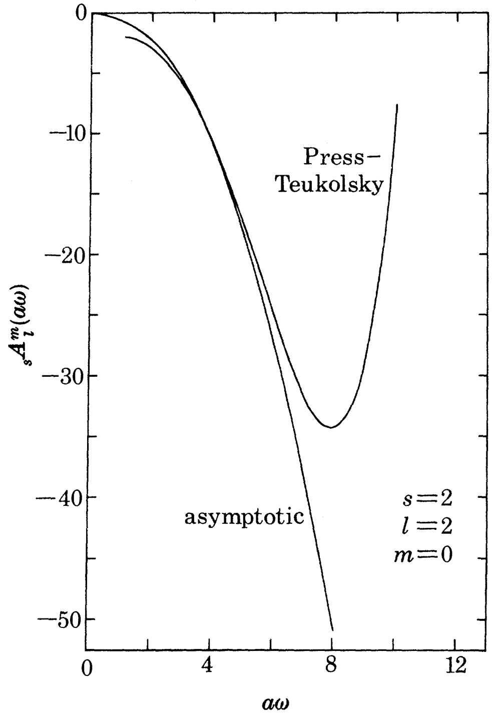

FIGURE 1. The match of the low- and high-frequency expansion for ${}_sA_{lm}\,(a\omega)$ for $s = 2$, $m = 0$, $l = 2$.

TABLE 2. COMPARISON OF ANALYTIC LOW-FREQUENCY-EXPANSION OF EIGENVALUES ${}_sA_{lm}$ WITH THE NUMERICAL FIT BY PRESS & TEUKOLSKY (1973)

$s = \pm 2$, $l = 3$ — numerical fit

$m = 3$	−1.99983	−0.501002	−0.047865	−0.0055100
$m = 2$	−1.33281	−0.410252	0.007111	−0.0134938
$m = 1$	−0.66698	−0.349910	0.006907	0.0060328
$m = 0$	−0.00078	−0.329262	−0.007370	0.0126218
$m = -1$	0.66749	−0.356215	−0.003271	−0.0054137
$m = -2$	1.33319	−0.406766	−0.002443	−0.0087279
$m = -3$	2.00012	−0.500668	0.051347	−0.0046268
	$\times (a\omega)$	$\times (a\omega)^2$	$\times (a\omega)^3$	$\times (a\omega)^4$

$s = \pm 2$, $l = 3$ — analytic

$m = 3$	−2.00000	−0.50000	−0.050000	−0.003345
$m = 2$	−1.33333	−0.407407	0.001646	−0.008725
$m = 1$	−0.666667	−0.351852	0.011316	0.001222
$m = 0$	0.00000	−0.333333	0.00000	0.006734
$m = -1$	0.666667	−0.351852	−0.011316	0.001222
$m = -2$	1.33333	−0.407407	−0.001636	−0.008725
$m = -3$	2.00000	−0.50000	0.050000	−0.003345
	$\times (a\omega)$	$\times (a\omega)^2$	$\times (a\omega)^3$	$\times (a\omega)^4$

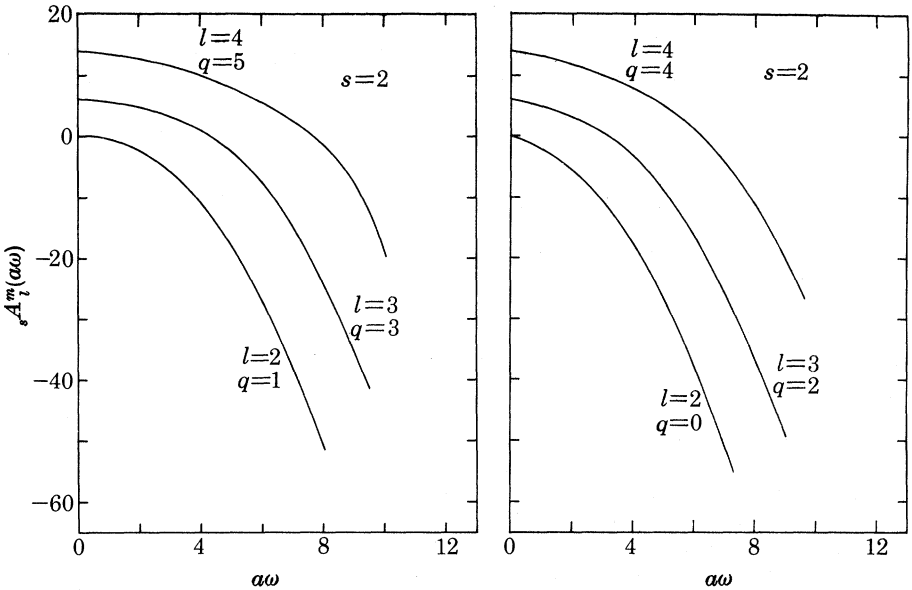

FIGURE 2. The ${}_sA_{lm}$ for $s = 2$, $m = 0$, $1 = 2, 3, 4$. The lines were drawn by matching the high- and low-frequency expansion at the point where they are closest.

FIGURE 3. The ${}_sA_{lm}$ for $s = 2$, $m = 1$, $l = 2, 3, 4$.

As was pointed out in § 4, finding q for the asymptotic expansion is difficult. We have computed q for several values of l and two values of m by requiring that the small $a\omega$ expressions and the asymptotic expressions match over some range of $a\omega$. Figure 1 shows the match of the $l = 2$, $m = 0$, $s = 2$ expansion with the Press–Teukolsky coefficients to the asymptotic expression with $q = 1$, $m = 0$, $s = 2$. The difference between the two expressions drops to 2 % or less in the region between $a\omega = 3$ and $a\omega = 4$ where the Press–Teukolsky expression should still be close to correct.

While the good fit between the high- and low-frequency values of ${}_sA_{lm}$ seems to imply that $q = 1$ is correct for $l = 2$ ($q = 0$ and $q = 2$ lie to the left and right of the low frequency curve respectively and never join) there is some problem with the finiteness of the eigenfunctions for q this small that needs to be resolved before our high frequency expression can be accepted fully.

Figures 2 and 3 show the behaviour of ${}_sA_{lm}$ respectively. They were obtained by using the Press–Teukolsky expression out to a value of $a\omega$ where it joined smoothly with the asymptotic expansion for some value of q and using the asymptotic values after that. For $l = 2$ and $l = 3$ this procedure works well, but for $l = 4$ the join occurs at large enough values of $a\omega$ that the match is not very good. For $l = 5$ the procedure breaks down entirely because there is no real match and it is impossible to tell which value of q is correct. This underlines the need for an analytic method of determining q and for an intermediate expansion for large q that will join the low and high frequency expansions.

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