PROCEEDINGS OF THE ROYAL SOCIETY A | MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the Equations Governing the Perturbations of the Schwarzschild Black Hole

S. Chandrasekhar

Proc. R. Soc. Lond. A 1975 **343**, 289-298
doi: 10.1098/rspa.1975.0066

Proc. R. Soc. Lond. A. **343**, 289–298 (1975)
Printed in Great Britain

On the equations governing the perturbations of the Schwarzschild black hole

By S. Chandrasekhar, F.R.S.
University of Chicago, Chicago, Illinois, 60637

(*Received* 7 *October* 1974)

A coherent self-contained account of the equations governing the perturbations of the Schwarzschild black hole is given. In particular, the relations between the equations of Bardeen & Press, of Zerilli and of Regge & Wheeler are explicitly established.

1. Introduction

The equations governing the perturbations of the vacuum Schwarzschild metric – the Schwarzschild black hole – have been the subject of many investigations (Regge & Wheeler 1957; Vishveshwara 1970; Edelstein & Vishveshwara 1970; Zerilli 1970*a, b*; Fackerell 1971; Bardeen & Press 1972; Friedman 1973). Nevertheless, there continues to be some elements of mystery shrouding the subject. Thus, Zerilli (1970*a*) showed that the equations governing the perturbation, properly analysed into spherical harmonics (belonging to the different l values) and with a time dependence $e^{i\sigma t}$, can be reduced to a one dimensional Schrödinger equation of the form

$$\mathrm{d}^2 Z/\mathrm{d}r_*^2 + (\sigma^2 - V_Z)\,Z = 0 \quad (+\infty > r_* > -\infty), \tag{1}$$

where

$$V_Z = \frac{2n^2(n+1)\,r^3 + 6n^2mr^2 + 18nm^2r + 18m^3}{r^3(nr+3m)^2}\left(1-\frac{2m}{r}\right), \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}r_*} = \left(1-\frac{2m}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r} \quad \text{and} \quad n = \tfrac{1}{2}(l-1)\,(l+2). \tag{3}$$

(Here, units are used in which $c = G = 1$.) Accordingly, the reflexion and the transmission coefficients for incident plane waves of various assigned wavenumbers, will clearly suffice to determine the evolution of any initial perturbation of the Schwarzschild black hole. This reduction of the general perturbation problem to the elementary one of determining the reflexion and the transmission coefficients of a one dimensional potential barrier is, of course, a remarkable simplification. But what is the origin of the particular form of the potential V_Z?

Again, considering a particular component of the Riemann tensor – the perturbed Newman–Penrose $\delta\Psi_0$ – Bardeen & Press (1973) showed that *all* the physical results relative to the perturbation of the Schwarzschild black hole can equally be derived from the solutions of the equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r_*^2} + \left[\sigma^2 - 4\mathrm{i}\sigma\frac{r-3m}{r^2} - \frac{l(l+1)\,(r-2m)+2m}{r^3}\right]\phi = 0, \tag{4}$$

with the boundary conditions,

$$\phi \to e^{i\sigma r_*}/(r-2m) \quad \text{as} \quad r_* \to -\infty \quad \text{and} \quad \phi \to e^{-i\sigma r_*} r^{-2} \quad \text{as} \quad r_* \to +\infty, \tag{5}$$

appropriate for ingoing waves at the horizon ($r_* \to -\infty$) and outgoing waves at infinity ($r_* \to +\infty$). If, as claimed, the physical contents of equations (1) and (4) are entirely the same, then there must be a simple relation between the functions Z and ϕ which makes this equivalence manifest. But what is this relation?

In this paper we shall give a coherent, self-contained account of the theory of the perturbations of the Schwarzschild black hole while resolving at the same time some of the unanswered questions, in particular, the relation between the Zerilli and the Bardeen–Press equations.

2. The equations governing the perturbations of the Schwarzschild black hole

The vacuum metric of Schwarzschild written in its standard form is

$$ds^2 = -\left(1-\frac{2m}{r}\right)(dt)^2 + r^2 \sin^2\theta(d\varphi)^2 + \frac{(dr)^2}{1-2m/r} + r^2(d\theta)^2. \tag{6}$$

Because of this spherical symmetry, first order perturbations of this metric can be analysed into spherical harmonics belonging to the different l and m values. Perturbations belonging to a particular l and m will be axisymmetric about some axis. Accordingly, there will be no loss of generality in restricting ourselves to axisymmetric perturbations from the outset.

As has been shown (Chandrasekhar & Friedman 1972) a form of the metric adequate for treating time-dependent axisymmetric systems in general relativity is

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}(d\varphi - \omega\, dt - q_{2,0}\, dx^2 - q_{3,0}\, dx^3)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2, \tag{7}$$

where ν, ψ, μ_2, μ_3, ω, q_2, and q_3 are all functions of t and the spatial variables x^2 and x^3 but independent of the azimuthal angle φ.

With the identifications

$$x^2 = r, \quad x^3 = \theta, \quad \text{and} \quad \mu_2 = \lambda \quad \text{(say)}, \tag{8}$$

the Schwarzschild metric corresponds to the various metric functions having the values

$$e^{2\nu} = e^{-2\lambda} = 1 - 2m/r, \quad e^{\psi} = r\sin\theta, \quad e^{\mu_3} = r, \tag{9}$$

and

$$\omega = q_2 = q_3 = 0.$$

When the metric is perturbed, we shall write

$$\nu + \delta\nu, \quad \lambda + \delta\lambda, \quad \psi + \delta\psi, \quad \text{and} \quad \mu_3 + \delta\mu_3, \tag{10}$$

for the corresponding metric functions while ω, q_2, and q_3 will be considered as quantities of the first order of smallness like $\delta\nu$, $\delta\lambda$, $\delta\psi$, and $\delta\mu_3$.

The equations governing the various quantities describing the perturbations can be readily written down by transcribing in the present context the various equations given by Chandrasekhar & Friedman (1972; see Part III of this paper). Thus, from the (1, 2)- and the (1, 3)-components of the field equations we find that

$$Q = r^2\left(1-\frac{2m}{r}\right)(q_{2,3}-q_{3,2})\sin^3\theta \tag{11a}$$

satisfies the equation

$$\frac{\partial}{\partial r}\left(\frac{r-2m}{r^3\sin^3\theta}\frac{\partial Q}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{r^4\sin^3\theta}\frac{\partial Q}{\partial\theta}\right)+\frac{\sigma^2}{r(r-2m)\sin^3\theta}Q = 0. \tag{11b}$$

This equation follows from Chandrasekhar & Friedman (1972; equation (153)) by substituting for ψ, ν, etc., their present values. We observe that (11b) is independent of the other field variables: it governs the 'so-called' odd-parity perturbations. Equation (11b) is considered in the appendix.

Of the remaining field equations, it will suffice to consider the (0, 2)-, (0, 3)-, (2,3)-, and (2, 2)-components. These equations are

$$(\delta\psi+\delta\mu_3)_{,r}+(1/r-\nu_{,r})(\delta\psi+\delta\mu_3)-(2/r)\,\delta\lambda = 0 \quad (\delta R_{02}=0), \tag{12}†$$

$$(\delta\psi+\delta\lambda)_{,\theta}+(\delta\psi-\delta\mu_3)\cot\theta = 0 \quad (\delta R_{03}=0), \tag{13}$$

$$(\delta\psi+\delta\nu)_{,r\theta}+(\delta\psi-\delta\mu_3)_{,r}\cot\theta+\left(\nu_{,r}-\frac{1}{r}\right)\delta\nu_{,\theta}-\left(\nu_{,r}+\frac{1}{r}\right)\delta\lambda_{,\theta} = 0 \quad (\delta R_{23}=0), \tag{14}$$

and

$$\begin{aligned}&e^{-2\lambda}\left[\frac{2}{r}\delta\nu_{,r}+\left(\frac{1}{r}+\nu_{,r}\right)(\delta\psi+\delta\mu_3)_{,r}-2\delta\lambda\left(\frac{1}{r^2}+2\frac{\nu_{,r}}{r}\right)\right]\\&\quad+\frac{1}{r^2}[(\delta\psi+\delta\nu)_{,\theta\theta}+(2\delta\psi+\delta\nu-\delta\mu_3)_{,\theta}\cot\theta+2\delta\mu_3]\\&\quad-e^{-2\nu}(\delta\psi+\delta\mu_3)_{,00} = 0 \quad (\delta G_{22}=0).\end{aligned} \tag{15}$$

In addition, we shall find the following equation, expressing the condition $\delta R_{11}=0$, useful:

$$\begin{aligned}&e^{+2\nu}\left[\delta\psi_{,rr}+2\left(\frac{1}{r}+\nu_{,r}\right)\delta\psi_{,r}+\frac{1}{r}(\delta\psi+\delta\nu+\delta\mu_3-\delta\lambda)_{,r}-2\frac{\delta\lambda}{r}\left(\frac{1}{r}+2\nu_{,r}\right)\right]\\&\quad+\frac{1}{r^2}[\delta\psi_{,\theta\theta}+\delta\psi_{,\theta}\cot\theta+(\delta\psi+\delta\nu-\delta\mu_3+\delta\lambda)_{,\theta}\cot\theta+2\,\delta\mu_3]-e^{-2\nu}\,\delta\psi_{,00} = 0.\end{aligned} \tag{16}$$

As Friedman (1973) has shown, the variables r and θ in the foregoing equations can be separated by the substitutions

$$\begin{aligned}\delta\nu &= N(r)\,P_l(\cos\theta)\,e^{i\sigma t},\\ \delta\lambda &= \delta\mu_2 = L(r)\,P_l(\cos\theta)\,e^{i\sigma t},\\ \delta\mu_3 &= [T(r)\,P_l+V(r)\,P_{l,\theta\theta}]\,e^{i\sigma t},\end{aligned}$$

and

$$\delta\psi = [T(r)\,P_l+V(r)\,P_{l,\theta}\cot\theta]\,e^{i\sigma t}, \tag{17}$$

where we have further assumed a time dependence expressed by the factor $e^{i\sigma t}$.

With the foregoing substitutions, equation (13) at once gives

$$T-V+L = 0. \tag{18}$$

† Commas, in subscripts, signify differentiations with respect to the variable (or variables) that follow.

Accordingly, only three of the four radial functions we have defined are linearly independent; and we shall choose N, L, and V as the independent functions.

The (0, 2)- and the (2,3)-components of the field equations – equations (12) and (14) – give

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}+\left(\frac{1}{r}-\nu_{,r}\right)\right][2T-l(l+1)\,V]-\frac{2}{r}L=0 \tag{19}$$

and

$$(T-V+N)_{,r}-\left(\frac{1}{r}-\nu_{,r}\right)N-\left(\frac{1}{r}+\nu_{,r}\right)L=0, \tag{20}$$

or, after the elimination of T with the aid of equation (18), we obtain

$$N_{,r}-L_{,r}=\left(\frac{1}{r}-\nu_{,r}\right)N+\left(\frac{1}{r}+\nu_{,r}\right)L \tag{21}$$

and

$$L_{,r}+\left(\frac{2}{r}-\nu_{,r}\right)L=-n\left[V_{,r}+\left(\frac{1}{r}-\nu_{,r}\right)V\right]. \tag{22}$$

Similarly, equations (15) and (16) give

$$\frac{2}{r}N_{,r}+\left(\frac{1}{r}+\nu_{,r}\right)[2T-l(l+1)\,V]_{,r}-\frac{2}{r}\left(\frac{1}{r}+2\nu_{,r}\right)L$$
$$-l(l+1)\frac{\mathrm{e}^{-2\nu}}{r^2}N-2n\frac{\mathrm{e}^{-2\nu}}{r^2}T+\sigma^2\mathrm{e}^{-4\nu}[2T-l(l+1)\,V]=0 \tag{23}$$

and

$$V_{,rr}+2\left(\frac{1}{r}+\nu_{,r}\right)V_{,r}+\frac{\mathrm{e}^{-2\nu}}{r^2}(N+L)+\sigma^2\mathrm{e}^{-4\nu}V=0. \tag{24}$$

Equation (23), after the elimination of T, takes the form

$$\frac{2}{r}N_{,r}-l(l+1)\frac{\mathrm{e}^{-2\nu}}{r^2}N-\frac{2}{r}\left(\frac{1}{r}+2\nu_{,r}\right)L-2\left(\frac{1}{r}+\nu_{,r}\right)(L+nV)_{,r}$$
$$-2n\frac{\mathrm{e}^{-2\nu}}{r^2}(V-L)-2\sigma^2\mathrm{e}^{-4\nu}(L+nV)=0, \tag{25}$$

where

$$n=\tfrac{1}{2}(l-1)(l+2). \tag{26}$$

It will be observed that equations (21), (22), and (25) provide three linear first order equations for the three radial functions L, N, and V. By suitably combining them, we can express the derivative of each of them as linear combinations of L, N, and V. Thus,

$$N_{,r}=aN+bL+cX, \tag{27}$$

$$L_{,r}=\left(a-\frac{1}{r}+\nu_{,r}\right)N+\left(b-\frac{1}{r}-\nu_{,r}\right)L+cX \tag{28}$$

and

$$X_{,r}=-\left(a-\frac{1}{r}+\nu_{,r}\right)N-\left(b+\frac{1}{r}-2\nu_{,r}\right)L-\left(c+\frac{1}{r}-\nu_{,r}\right)X, \tag{29}$$

where, for the sake of brevity, we have written

$$X = nV = \tfrac{1}{2}(l-1)(l+2)V, \tag{30}$$

$$\left.\begin{aligned} a &= \frac{n+1}{r-2m}, \quad \nu_{,r} = \frac{m}{r(r-2m)}, \\ b &= -\frac{1}{r} - \frac{n}{(r-2m)} + \frac{m}{r(r-2m)} + \frac{m^2}{r(r-2m)^2} + \sigma^2 \frac{r^3}{(r-2m)^2} \\ \text{and} \qquad c &= -\frac{1}{r} + \frac{1}{(r-2m)} + \frac{m^2}{r(r-2m)^2} + \sigma^2 \frac{r^3}{(r-2m)^2}. \end{aligned}\right\} \tag{31}$$

Equations (27), (28), and (29) provide the basic equations of the theory. *All the remaining field equations, including equation* (24), *are verifiable consequences of these equations*†

For later use, we may note here the following equation obtained by adding equations (28) and (29)

$$(L+X)_{,r} = -\left(\frac{2}{r} - \nu_{,r}\right)L - \left(\frac{1}{r} - \nu_{,r}\right)X. \tag{32}$$

3. Zerilli's equation

We shall verify that by virtue of equations (27)–(29), the function,

$$Z = \frac{r^2}{nr+3m}\left(\frac{3m}{r}V - L\right), \tag{33}$$

satisfies Zerilli's equation (1).

First, we observe that an equivalent expression for Z is

$$Z = rV - \frac{r^2}{nr+3m}(L+X). \tag{34}$$

Differentiating this expression with respect to the variable

$$r_* = r + 2m \lg\left(\frac{r}{2m} - 1\right) \quad (r > 2m), \tag{35}$$

† In Regge & Wheeler's discussion (Regge & Wheeler 1957) of this same problem, in a gauge different from the one adopted in this paper, the corresponding situation is somewhat curious. They obtain a set of six equations for three scalar functions. Three of these equations are of the first order while the remaining three are of the second order. They say 'one will expect that the three first order equations will suffice to determine the solution apart from the boundary conditions. Actually, the second order equations contain an additional piece of information...specifically, a rather elaborate investigation shows that [the second order equations] can all be deduced from [the first order equations]' provided a linear combination, F (say), of the three scalar functions vanishes. At the same time Vishveshwara (1970) says that if $F_{,r}$ is evaluated with the aid of the first order equations, $F_{,r} = -\nu_{,r}F$ and that therefore 'if $F = 0$ for some r, it is zero everywhere'. All this is somewhat confusing. In any event, when working in the gauge adopted in this paper, *all* the second order equations are *identically* satisfied by virtue of the first order equations (27)–(29) and no 'additional piece of information' is provided.

and making use of equation (32), we obtain

$$Z_{,r_*} = \left(1-\frac{2m}{r}\right) Z_{,r}$$
$$= (r-2m)\,V_{,r} + \frac{3m(r-2m)}{r(nr+3m)}\,V + \frac{nr^2-3nmr-3m^2}{(nr+3m)^2}\,(L+X). \tag{36}$$

Differentiating this last expression, once again, with respect to r_* and simplifying with the aid of equations (24), (29), and (34), we find after some considerable reductions that, remarkably, we are left with

$$\mathrm{d}^2Z/\mathrm{d}r_*^2 + (\sigma^2 - V_Z)\,Z = 0, \tag{37}$$

where V_Z is given by equation (2). Thus, Zerilli's equation, while it is a directly verifiable consequence of equations (27)–(29), the manner in which it emerges is shrouded in (apparently) miraculous cancellations.

An interesting integral property of Zerilli's potential V_Z may be noted here:

$$2m\int_{-\infty}^{+\infty} V_Z\,\mathrm{d}r_* = 2n+\tfrac{1}{2}. \tag{38}$$

4. The expression of $\delta\Psi_0$ in terms of Zerilli's function

In another connexion, Friedman (1973; §3 of this paper) has already considered the perturbed components of the Riemann tensor; and the component $\delta\Psi_0$ to which the Bardeen–Press equation (4) refers is

$$\delta\Psi_0 = = \tfrac{1}{4}\mathrm{e}^{-2\nu}[\delta R_{(t)(\theta)(t)(\theta)} + \delta R_{(r)(\theta)(r)(\theta)} + 2\,\delta R_{(t)(\theta)(r)(\theta)}$$
$$- \delta R_{(t)(\varphi)(t)(\varphi)} - \delta R_{(r)(\varphi)(r)(\varphi)} - 2\,\delta R_{(t)(\varphi)(r)(\varphi)}]. \tag{39}$$

Expressions for the components of the Riemann tensor (in the tetrad-frame in which they are referred) have been given by Chandrasekhar & Friedman (1972; equations (16)) under general time-dependent axisymmetric conditions. Specializing them to the case on hand, we find that in accordance with equation (39) (cf. Friedman 1973, equation (56))

$$\delta\Psi_0 = -\tfrac{1}{4}\mathrm{e}^{-2\nu}\left\{\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} - \cot\theta\right)(\delta\nu - \delta\lambda)\right.$$
$$\left. + \left[\mathrm{e}^{-4\nu}\frac{\partial^2}{\partial t^2} + 2\frac{\partial}{\partial t}\left(\frac{\partial}{\partial r} + \frac{1}{r} - \nu_{,r}\right) + \mathrm{e}^{2\nu}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\right](\delta\psi - \delta\mu_3)\right\}. \tag{40}$$

Separating the variables in accordance with equations (17) and making use of equation (24), we obtain

$$\delta\Psi_0 = \tfrac{1}{2}\left\{\mathrm{i}\sigma\,\mathrm{e}^{-2\nu}\left[V_{,r} + \left(\frac{1}{r} - \nu_{,r}\right)V\right] - \left(\nu_{,r}V_{,r} + \sigma^2\,\mathrm{e}^{-4\nu}\,V + \frac{\mathrm{e}^{\cdot 2\nu}}{r^2}N\right)\right\}\mathrm{e}^{\mathrm{i}\sigma t}(P_{l,\theta\theta} - P_{l,\theta}\cot\theta). \tag{41}$$

In the further use of this expression, we shall suppress the angular dependence $(P_{l,\theta\theta} - P_{l,\theta}\cot\theta)$ and the factor $\frac{1}{2}\,\mathrm{e}^{\mathrm{i}\sigma t}$ as well. With this understanding, we can rewrite the expression for $\delta\Psi_0$ in the following form after eliminating $V_{,r}$ in the second term in parenthesis on the right hand side with the aid of equation (29):

$$\delta\Psi_0 = \mathrm{i}\sigma\frac{r}{r-2m}\left[V_{,r}+\frac{r-3m}{r(r-2m)}V\right]+\sigma^2\frac{r^2}{(r-2m)^2}\left[\frac{m}{n(r-2m)}(L+X)-V\right]$$
$$-\frac{nr^2-3nmr-3m^2}{nr^2(r-2m)^2}N-\frac{m[nr^2-mr(2n-1)-3m^2]}{nr^2(r-2m)^3}L+\frac{m(r^2-3mr+3m^2)}{r^2(r-2m)^3}V. \tag{42}$$

Making use of equations (34) and (36) giving Z and $Z_{,r_*}$, we find that we can express $\delta\Psi_0$ in terms of these quantities; we find

$$\delta\Psi_0 = \frac{1}{2r(r-2m)}\left\{\frac{2n^2(n+1)\,r^3+6n^2mr^2+18nm^2r+18m^3}{r^2(nr+3m)^2}Z\right.$$
$$\left.+\left[2\mathrm{i}\sigma\frac{r^2}{r-2m}+2\frac{nr^2-3nmr-3m^2}{(r-2m)\,(nr+3m)}\right](Z_{,r_*}+\mathrm{i}\sigma Z)\right\}. \tag{43}$$

We observe that in the first term in braces on the right hand side of equation (43), the coefficient of Z is, apart from a factor $r^2/(r-2m)$ the potential V_Z in Zerilli's equation. In making use of this fact, we shall find it convenient to introduce the operators

$$\Lambda_\pm = \mathrm{d}/\mathrm{d}r_* \pm \mathrm{i}\sigma \quad \text{and} \quad \Lambda^2 = \Lambda_+\Lambda_- = \Lambda_-\Lambda_+ = \mathrm{d}^2/\mathrm{d}r_*^2+\sigma^2 \tag{44}$$

and write Zerilli's equation in the form

$$\Lambda^2 Z = V_Z Z. \tag{45}$$

We now rewrite the expression (43) for $\delta\Psi_0$ in the manner

$$\delta\Psi_0 = \frac{1}{2r(r-2m)}\left[\frac{r^2}{r-2m}(\Lambda^2 Z+2\mathrm{i}\sigma\Lambda_+ Z)+2\frac{nr^2-3nmr-3m^2}{(r-2m)\,(nr+3m)}\Lambda_+ Z\right]$$
$$= \frac{r}{2(r-2m)^2}\left[\Lambda_+(\Lambda_-+2\mathrm{i}\sigma)\,Z+2\frac{nr^2-3nmr-3m^2}{r^2(nr+3m),}\Lambda_+ Z\right], \tag{46}$$

or finally,
$$\delta\Psi_0 = \frac{r}{2(r-2m)^2}\left[\Lambda_+ + 2\frac{nr^2-3nmr-3m^2}{r^2(nr+3m)}\right]\Lambda_+ Z. \tag{47}$$

Equation (47) suggests that we define the function

$$Y = 2\frac{(r-2m)^2}{r}\,\delta\Psi_0; \tag{48}$$

in terms of Y, equation (47) takes the simple form

$$Y = \Lambda_+\Lambda_+ Z + W\Lambda_+ Z, \tag{49}$$

where
$$W = 2\frac{nr^2-3nmr-3m^2}{r^2(nr+3m)}. \tag{50}$$

An equivalent form of equation (49) is

$$Y = V_Z Z + (W+2\mathrm{i}\sigma)\,\Lambda_+ Z. \tag{51}$$

It is noteworthy that in this last equation, expressing Y in terms of Z and $\Lambda_+ Z$, V_Z appears as the coefficient of Z.

5. The derivation of the Bardeen–Press equation from the Zerilli equation and conversely

Applying the operator Λ_- to equation (49) and making use of Zerilli's equation satisfied by Z, we obtain

$$\Lambda_- Y = \left(\frac{\mathrm{d}V_Z}{\mathrm{d}r_*} + WV_Z\right) Z + \left(V_Z + \frac{\mathrm{d}W}{\mathrm{d}r_*}\right) \Lambda_+ Z. \tag{52}$$

It can now be directly verified that with the functions V_Z and W as defined

$$\frac{\mathrm{d}V_Z}{\mathrm{d}r_*} + WV_Z = -6m\frac{(r-2m)^2}{r^6} \tag{53}$$

and

$$\frac{\mathrm{d}W}{\mathrm{d}r_*} + V_Z = 2\left(1-\frac{2m}{r}\right)\frac{nr+3m}{r^3}. \tag{54}$$

Accordingly, equation (52) takes the form

$$\Lambda_- Y = -6m\frac{(r-2m)^2}{r^6} Z + 2\left(1-\frac{2m}{r}\right)\frac{nr+3m}{r^3}\Lambda_+ Z. \tag{55}$$

Equations (51) and (55) can now be solved for Z and $\Lambda_+ Z$ to give

$$[\tfrac{2}{3}n(n+1)+2m\mathrm{i}\sigma]\, Z = \frac{r^2(nr+3m)}{3(r-2m)} Y - \frac{r^6}{6(r-2m)^2}(W+2\mathrm{i}\sigma)\Lambda_- Y \tag{56}$$

and

$$[\tfrac{2}{3}n(n+1)+2m\mathrm{i}\sigma]\, \Lambda_+ Z = mY + \frac{r^6}{6(r-2m)^2} V_Z \Lambda_- Y. \tag{57}$$

In deriving the foregoing solutions, we have made use of the surprisingly simple relation,

$$\frac{r^2}{r-2m}(nr+3m)\, V_Z + 3mW = 2n(n+1), \tag{58}$$

which obtains between V_Z and W.

We may parenthetically remark here that the fact that the functions V_Z and W satisfy three simple relations such as equations (53), (54), and (58), to some extent, dispels the 'mystery' about them.

We can now eliminate Z between equations (56) and (57) to obtain an equation for Y. Thus, from equation (57) it follows that

$$\begin{aligned}\Lambda_-\left[\frac{r^6}{6(r-2m)^2} V_Z \Lambda_- Y + mY\right] &= [\tfrac{2}{3}n(n+1)+2m\mathrm{i}\sigma]\, \Lambda^2 Z \\ &= [\tfrac{2}{3}n(n+1)+2m\mathrm{i}\sigma]\, V_Z Z;\end{aligned} \tag{59}$$

while, in accordance with equation (56), we may conclude

$$\Lambda_-\left[\frac{r^6}{6(r-2m)^2} V_Z \Lambda_- Y + mY\right] = V_Z\left[\frac{r^2(nr+3m)}{3(r-2m)} Y - \frac{r^6}{6(r-2m)^2}(W+2m\mathrm{i}\sigma)\Lambda_- Y\right]. \tag{60}$$

On simplifying this last equation, we are left with

$$\Lambda^2 Y + 4\frac{r-3m}{r^2}\Lambda_- Y - 2\left(1-\frac{2m}{r}\right)\frac{nr+3m}{r^3}Y = 0. \tag{61}$$

With the substitution

$$Y = \frac{r-2m}{r^3}\phi, \tag{62}$$

it can now be readily verified that we recover equation (4) of Bardeen & Press.

Even as equations (56) and (57), together with Zerilli's equation, enabled us to derive the Bardeen–Press equation, we may, conversely, derive Zerilli's equation from equations (51) and (55) together with the Bardeen–Press equation in the form (61).

6. Concluding remarks

In bringing together in a common framework the different equations which govern the perturbations of the Schwarzschild black hole and clarifying their inner relation, the present paper raises the question whether one may not be able to simplify some, at least, of the aspects of the perturbation problem as it pertains to the Kerr black hole. Thus, Teukolsky's equation (Teukolsky 1973) governing the perturbations of the Kerr black hole reduces, when specialized to the Schwarzschild black hole, to the Bardeen–Press equation (4) and *not* to the Zerilli equation (1). The question naturally arises whether there may not be a transformed version of the Teukolsky equation, which when specialized to the Schwarzschild case becomes the Zerilli equation. Such a transformation, if it were possible, would clearly be advantageous: we shall then need to determine only the reflexion and the transmission coefficients for a one-dimensional potential barrier. In a subsequent paper (Chandrasekhar & Detweiler, in preparation) it will be shown how Teukolsky's equation, for the case of axisymmetric perturbations, can be transformed into a one-dimensional wave equation with, indeed, four possible potentials.

The research reported in this paper has in part been supported by the National Science Foundation under grant GP-34721X1 with the University of Chicago and the Louis Block Fund.

Appendix

(*Communicated* 11 *November* 1974)

In the course of an investigation of Teukolsky's equation (Chandrasekar & Detweiler, in preparation), it became clear that the Regge–Wheeler equation governing the odd-parity perturbations of the Schwarzschild metric must also be equivalent to the Bardeen–Press equation. We give a brief *ab initio* demonstration of this equivalence.

The variables r and θ in equation (11*b*), governing the odd-parity perturbations, can be separated by the substitution,

$$Q = rX(r)P_{l+2}(\cos\theta/-3),$$

where $P_{l+2}(x/-3)$ is the Gegenbauer, polynomial of order $(l+2)$ and index -3. With the foregoing substitution, equation (11b) reduces to the Regge–Wheeler equation (Regge & Wheeler 1957),

$$\Lambda^2 X = V_0 X, \tag{A 1}$$

where

$$V_0 = 2\left(1-\frac{2m}{r}\right)\frac{(n+1)r-3m}{r^3}. \tag{A 2}$$

Defining

$$W_0 = 2\frac{r-3m}{r^2}, \tag{A 3}$$

we can readily verify that the functions V_0 and W_0 satisfy the identities

$$\frac{\mathrm{d}V_0}{\mathrm{d}r_*}+W_0 V_0 = 6m\frac{(r-2m)^2}{r^6}, \tag{A 4}$$

$$\frac{\mathrm{d}W_0}{\mathrm{d}r_*}+V_0 = 2\left(1-\frac{2m}{r}\right)\frac{nr+3m}{r^3} \tag{A 5}$$

and

$$\frac{r^2}{r-2m}(nr+3m)V_0-3mW_0 = 2n(n+1). \tag{A 6}$$

These identities are entirely analogous to those satisfied in the context of Zerilli's equation (cf. equations (53), (54) and (58)).

If we now let

$$Y = \Lambda_+\Lambda_+X+W_0\Lambda_+X, \tag{A 7}$$

or equivalently

$$Y = V_0X+(W_0+2\mathrm{i}\sigma)\Lambda_+X, \tag{A8}$$

then by virtue of equations (A 4) and (A 5) (cf. equation (55))

$$\Lambda_-Y = 6m\frac{(r-2m)^2}{r^6}X+2\left(1-\frac{2m}{r}\right)\frac{nr+3m}{r^3}\Lambda_+X. \tag{A 9}$$

Solving these equations for X and Λ_+X in terms of Y and $\Lambda\;\;Y$, we find (cf. equations (56) and (57)),

$$[\tfrac{2}{3}n(n+1)-2m\mathrm{i}\sigma]X = \frac{r^2(nr+3m)}{3(r-2m)}Y-\frac{r^6}{6(r-2m)^2}(W_0+2\mathrm{i}\sigma)\Lambda_-Y \tag{A 10}$$

and

$$[\tfrac{2}{3}n(n+1)-2m\mathrm{i}\sigma]\Lambda_+X = -mY+\frac{r^6}{6(r-2m)^2}V_0\Lambda_-Y. \tag{A 11}$$

It can now be verified that *equations* (A 8), (A 9), (A 10), and (A 11) *are necessary and sufficient for the Regge–Wheeler equation to imply the Bardeen–Press equation and conversely.*

References

Bardeen, J. M. & Press, W. H. 1973 *J. Math. Phys.* **14**, 7.
Chandrasekhar, S. & Friedman, J. L. 1972 *Astrophys. J.* **175**, 379.
Edelstein, L. & Vishveshwara, C. V. 1970 *Phys. Rev.* D **1**, 3514.
Fackerell, E. D. 1971 *Astrophys. J.* **166**, 197.
Friedman, J. L. 1973 *Proc. R. Soc. Lond.* A **335**, 163.
Regge, T. & Wheeler, J. A. 1957 *Phys. Rev.* **108**, 1063.
Teukolsky, S. A. 1973 *Astrophys. J.* **185**, 635.
Vishveshwara, C. V. 1970 *Phys. Rev.* D **1**, 2870.
Zerilli, F. J. 1970a *Phys. Rev. Lett.* **24**, 737.
Zerilli, F. J. 1970b *Phys. Rev.* D **2**, 2141.