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PERTURBATIONS OF A ROTATING BLACK HOLE. III. INTERACTION OF THE HOLE WITH GRAVITATIONAL AND ELECTROMAGNETIC RADIATION*

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ABSTRACT

Immersed in an external radiation field, a rotating black hole scatters radiation and may either absorb or amplify it. The process is described by the set of homogeneous solutions to the separable wave equation which was previously derived in this series. Here we describe analytic and numerical techniques for obtaining these homogeneous solutions and interpreting them physically. New technical results include an explicit local transformation between perturbation quantities of opposite spin weight, a "conserved energy" for gravitational perturbations which allows the evidence for stability of the Kerr metric to be further tightened, and an extension of the analytic results of Starobinsky and Churilov in the limit $a \approx M$. Numerical results are presented which verify that the maximum superradiant amplification for gravitational waves is 138 percent; for electromagnetic waves the maximum is 4.4 percent (and is attained in a regime not susceptible to analytic treatment). The superradiance falls sharply with decreasing rotation a of the hole, or increasing mode numbers l and m.

Subject headings: black holes — gravitation — relativity — rotation

I. INTRODUCTION

The first paper of this series (Teukolsky 1973*a*, hereafter cited as Paper I) derived ordinary second-order linear differential equations which describe the dynamical behavior of a rotating, Kerr, black hole under small external perturbations. In Paper II (Press and Teukolsky 1973) we jumped immediately into the most striking application of the equations, namely, testing the Kerr metric for dynamical instabilities. Having found no instabilities (which, had they existed, would have vitiated further use of the perturbation formalism), we are now in a position to take a more general look at the interaction of the black hole with external perturbing fields.

In this paper we investigate with some completeness the interaction of the hole with gravitational and electromagnetic *radiation* fields, i.e., sourceless perturbations which are represented as homogeneous solutions to the perturbation equations. The main physical effects which are thus treated are absorption of the waves by the hole (for some angular modes and frequencies) and superradiant amplification of the waves (for others). The numerical and analytic techniques which we describe also render tractable a number of related problems: differential scattering cross-sections of waves, changes in polarization of scattered waves, and partial polarization of unpolarized waves by scattering. Because the number of parameters describing these effects is large, and because analytic solution is not generally possible, it is not practical for us to tabulate complete solutions to these problems. But we do hope to give a complete set of "tools" which can be used by others.

Some of the techniques described here will also be used in later papers of this series to treat inhomogeneous problems (those including sources), such as the question of "floating" particle orbits and tidal-friction spindown (Press and Teukolsky 1972; Hawking 1972; Press 1972; Hawking and Hartle 1972; Hartle 1973; Teukolsky 1973*b*).

In § II we give four different representations of the fundamental perturbation equations, any one of which contains complete information about the perturbation field. The solutions to these different representations have different asymptotic forms, and we will see that it is useful to use different representations in different regimes. (In Papers I and II, only two representations were used.) Section III gives the formulae which relate the solutions of the different representations, enabling one to switch between them at will. Section IV discusses the physical interpretation of the solutions at radial infinity, and on the horizon of the black hole. Section V exhibits an exact integral conserved quantity of the perturbation equations, namely, a "conserved energy." It is somewhat surprising that such a quantity should exist for gravitational perturbations, since in general there is no unique conserved stress-energy for a gravitational wave propagating in highly curved space. This conservation law allows the tightening up of the evidence that the Kerr metric is dynamically stable. In § VI we discuss limiting cases of the perturbation equation which are soluble analytically, extending slightly the recent results of Starobinsky and Churilov (1973). Section VII gives the results of numerical calculations of the electromagnetic and gravitational absorption/amplification factors. Some further implications are discussed in § VIII.

II. REPRESENTATIONS OF THE PERTURBATION EQUATIONS

Of the numerous coordinate systems used to describe the Kerr metric, three will be particularly useful here. The first is the Boyer-Lindquist (1967) coordinate system, denoted (t, r, θ, φ), in which the metric takes the form

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of equation (I: 4.1) (we use this notation to refer to equations in Paper I). A coordinate system that is well behaved on the future event horizon, i.e., for infalling observers, is the Kerr "ingoing" coordinate system $(v, r, \theta, \tilde{\varphi})$ obtained from Boyer-Lindquist coordinates by the transformation

$$dv = dt + (r^2 + a^2)dr/\Delta\,,$$
$$d\tilde{\varphi} = d\varphi + a\,dr/\Delta\,. \tag{2.1}$$

Similarly, the Kerr "outgoing" coordinates $(u, r, \theta, \bar{\varphi})$ are regular on the past horizon and are defined by the transformation

$$\boxed{\begin{aligned} du &= dt - (r^2 + a^2)dr/\Delta\,,\\ d\bar{\varphi} &= d\varphi - a\,dr/\Delta\,. \end{aligned}} \tag{2.2}$$

The dependent variables in the perturbation equations derived in Paper I were related to certain null-tetrad components of the physical fields (Weyl or Maxwell tensor). There only one tetrad was introduced, that of Kinnersley (1969) in which the tetrad leg $\boldsymbol{l}$ is in the outgoing direction and the Newman-Penrose spin coefficient ϵ is set to zero. Kinnersley's tetrad is well behaved on the past horizon and has $[t, r, \theta, \varphi]$-components

$$l^\mu = [(r^2 + a^2)/\Delta, 1, 0, a/\Delta]\,,$$
$$n^\mu = [r^2 + a^2, -\Delta, 0, a]/2\Sigma\,,$$
$$m^\mu = [ia \sin\theta, 0, 1, i/\sin\theta]/2^{1/2}(r + ia\cos\theta)\,. \tag{2.3}$$

(As in previous papers, we have $\Delta \equiv r^2 - 2Mr + a^2, \Sigma \equiv r^2 + a^2\cos^2\theta$.) Here we will find it useful to use also a tetrad which is well behaved on the future horizon. This tetrad (distinguished by a dagger) is related to that of equation (2.3) by the transformation $t \to -t$, $\varphi \to -\varphi$ so that

$$\boldsymbol{l}^\dagger = -(2\Sigma/\Delta)\boldsymbol{n}\,,$$
$$\boldsymbol{n}^\dagger = -(\Delta/2\Sigma)\boldsymbol{l}\,,$$
$$\boldsymbol{m}^\dagger = \frac{r - ia\cos\theta}{r + ia\cos\theta}\boldsymbol{m}^*\,. \tag{2.4}$$

The master perturbation equation (I: 4.7) contained a "spin-weight" parameter s which takes on different values for different perturbing fields. Corresponding to each value of s is a dependent variable Υ_s which satisfies the perturbation equation. A scalar field has $s = 0$; for an electromagnetic field either ϕ_0 ($s = 1$) or $(r - ia\cos\theta)^2\phi_2$ ($s = -1$) can be used; for a gravitational field either ψ_0 ($s = 2$) or $(r - ia\cos\theta)^4\psi_4$ ($s = -2$) can be used. (For precise definitions of these quantities, see Paper I.) In addition to the choice between $\pm s$, there is of course the choice of which coordinate system the perturbation equation is written in, since the master perturbation equation is separable in any of the coordinates described above. However, there is *yet another* set of equivalent equations (one for every $\pm s$ and coordinate system) which are different in form from the first set but which contain the same information as the equation for the Υ_s. These can be obtained in principle by exactly repeating the derivation of (I: 4.7) but using the tetrad (2.4) instead of the tetrad (2.3). All the details of decoupling and separability must go through by symmetry, and one is left with a set of equations for field variables which we denote Ω_s, which are related to the Υ_s by

$$\Omega_s = (2/\Delta)^s\Upsilon_{-s}\,. \tag{2.5}$$

We now summarize this morass of equivalent equations [equivalent under $s \leftrightarrow -s$, $\Upsilon_s \leftrightarrow \Omega_s$ and changes of coordinates $(t, \varphi) \leftrightarrow (u, \bar{\varphi}) \leftrightarrow (v, \tilde{\varphi})$]: Separate variables in any one of the coordinates used, above, so that

$$\Upsilon_s \quad \text{or} \quad \Omega_s = \int d\omega e^{-i\omega t} \sum_{l,m} e^{im\varphi}\,{}_sS_{lm}(\theta)\,{}_sR_{\omega lm}(r) \tag{2.6a}$$

$$= \int d\omega e^{-i\omega v} \sum_{l,m} e^{im\tilde{\varphi}}\,{}_sS_{lm}(\theta)\,{}_sR_{\omega lm}(r) \tag{2.6b}$$

$$= \int d\omega e^{-i\omega u} \sum_{l,m} e^{im\bar{\varphi}}\,{}_sS_{lm}(\theta)\,{}_sR_{\omega lm}(r)\,. \tag{2.6c}$$

The angular function ${}_sS_{lm}$ for all of these separations is the regular function on $[0, \pi]$ which satisfies the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + E - s^2\right)S = 0\,, \tag{2.7}$$

for eigenvalue ${}_sE_{lm}$ and is normalized by

$$\int_0^\pi {}_sS_{lm}{}^2(\theta)\sin\theta d\theta = 1\,. \tag{2.8}$$

The radial function ${}_sR_{\omega lm}$ satisfies $\mathscr{L}R = 0$, where for equation (2.6a)

$$\mathscr{L} = \Delta\frac{d^2}{dr^2} + 2(s+1)(r-M)\frac{d}{dr} + \frac{K^2 \mp 2is(r-M)K}{\Delta} \pm 4ir\omega s - \lambda\,. \tag{2.9}$$

The upper sign refers to Υ_s and the lower to Ω_s. Here $K \equiv (r^2 + a^2)\omega - am$ and $\lambda \equiv E - 2am\omega + a^2\omega^2 - s(s+1)$. For equations (2.6b) and (2.6c) we have

$$\mathscr{L} = \Delta\frac{d^2}{dr^2} + 2[(s+1)(r-M) \mp iK]\frac{d}{dr} \mp \frac{4is(r-M)K}{\Delta} \mp 2(2s+1)i\omega r - \lambda\,, \tag{2.10}$$

where the upper sign refers to Υ_s in equation (2.6b) and the lower sign to Ω_s in equation (2.6c); or

$$\mathscr{L} = \Delta\frac{d^2}{dr^2} + 2[(s+1)(r-M) \pm iK]\frac{d}{dr} \pm 2(2s+1)i\omega r - \lambda\,, \tag{2.11}$$

where the upper sign is for Υ_s in equation (2.6c) and the lower sign is for Ω_s in equation (2.6b).

What is the use of replacing an equation by a set of 12 equations all equivalent? The answer is that the equations, while equivalent in information content, have mathematically different asymptotic solutions at radial infinity and on the event horizon. Since boundary conditions are imposed in these two asymptotic regimes, and since one is interested in calculating physical quantities such as energy flux or angular-momentum flux also in these two regimes, it is a great advantage to be able to select a form of the equation whose asymptotic solutions are adapted to the problem at hand; for example, this will be done in § VII below.

For reference, the asymptotic solutions of the equations with operators (2.9)–(2.11) are given in table 1. The quantity k is defined as $\omega - m\omega_+$, where $\omega_+ \equiv a/(2Mr_+)$ is the "angular velocity of the horizon," and r^* is defined by $dr^*/dr = (r^2 + a^2)/\Delta$.

III. RELATIONS BETWEEN THE EQUIVALENT DECOUPLED VARIABLES

To demonstrate that the four variable $\Upsilon_{\pm s}, \Omega_{\pm s}$ in any of the three coordinate systems (t, φ), $(v, \tilde{\varphi})$, $(u, \bar{\varphi})$ do contain complete information, we show here that a *local* knowledge of any one variable enables one to compute the local value of any other variable. This fact has the practical consequence that it enables one to "change horses in midstream" in numerical work and integrate whichever of the operators (2.9), (2.10), or (2.11) is locally advantageous. It will also facilitate the physical interpretation of the solutions in § VI below.

TABLE 1

ASYMPTOTIC SOLUTIONS FOR THE RADIAL FUNCTION R IN EQUATIONS (2.6)

	$r \to \infty$		$r \to r_+$	
	Outgoing Waves	Ingoing Waves	Outgoing Waves	Ingoing Waves
Υ equations:				
(t, φ)............	$e^{i\omega r^*}/r^{2s+1}$	$e^{-i\omega r^*}/r$	e^{ikr^*}	$\Delta^{-s}e^{-ikr^*}$
$(u, \bar{\varphi})$............	$r^{-(2s+1)}$	$e^{-2i\omega r^*}/r$	1	$\Delta^{-s}e^{-2ikr^*}$
$(v, \tilde{\varphi})$............	$e^{2i\omega r^*}/r^{2s+1}$	$1/r$	e^{2ikr^*}	Δ^{-s}
Ω equations:				
(t, φ)............	$e^{i\omega r^*}/r$	$e^{-i\omega r^*}/r^{2s+1}$	$\Delta^{-s}e^{ikr^*}$	e^{-ikr^*}
$(u, \bar{\varphi})$............	$1/r$	$e^{-2i\omega r^*}/r^{2s+1}$	Δ^{-s}	e^{-2ikr^*}
$(v, \tilde{\varphi})$............	$e^{2i\omega r^*}/r$	$r^{-(2s+1)}$	$\Delta^{-s}e^{2ikr^*}$	1

NOTE.—The terms "outgoing" and "ingoing" waves refer to the physical direction of propagation as seen by a local observer (e.g., ingoing waves are regular on the future event horizon and for infalling observers). The notations (t, φ), $(v, \tilde{\varphi})$, and $(u, \bar{\varphi})$ refer to Boyer-Lindquist, ingoing Kerr-Newman, and outgoing Kerr-Newman coordinates, respectively. See text for details.

Transformations among the various coordinate systems are trivially performed at any point using equations (2.6a)–(2.6c), and the explicit transformations (2.1) and (2.2). Without loss of generality, then, we can assume Boyer-Lindquist coordinates. Also, the algebraic relation between Υ_s and Ω_s at a point has already been given in equation (2.5). The nontrivial transformation left to derive is that which relates Υ_s to Υ_{-s} (or using eq. [2.5], Ω_s to Ω_{-s}), the extreme spin-weight field components of a given field on a given tetrad.

Consider first the electromagnetic field variables ϕ_0 and ϕ_2 (Newman and Penrose 1962; equations from which are hereafter cited by the notation NP). The sourceless Maxwell equations in the Kerr background in NP notation are

$$(\delta^* + \pi - 2\alpha)\phi_0 = (D - 2\rho)\phi_1 , \tag{3.1}$$

$$(\delta^* + 2\pi)\phi_1 = (D - \rho + 2\epsilon)\phi_2 , \tag{3.2}$$

$$(\delta - \tau + 2\beta)\phi_2 = (\Delta + 2\mu)\phi_1 , \tag{3.3}$$

$$(\delta - 2\tau)\phi_1 = (\Delta + \mu - 2\gamma)\phi_0 , \tag{3.4}$$

valid for any of the tetrad choices of the previous section. (Explicit expressions for the spin coefficients are given in I: 4.5.) The quantity ϕ_1 can be eliminated from equations (3.1) and (3.2) by using the following identity for the Kerr metric:

$$[\delta^* - (p+1)\alpha - (q+1)\beta^* + (m+1)\pi][D - p\epsilon - q\epsilon^* - (m+p)\rho]\psi$$
$$- [D - (p-1)\epsilon - (q+1)\epsilon^* - (m+p+1)\rho][\delta^* - p\alpha - q\beta^* + m\pi]\psi = p\psi(D + \epsilon - \epsilon^*)\pi , \tag{3.5}$$

where ψ is any NP quantity and p, q, and m are arbitrary constants. This identity is proved using (in this order) equations (NP 4.4), (NP 4.2d), (NP 4.2e), (NP 4.4), and (NP 4.5). Operate on equation (3.1) with $(\delta^* + 3\pi - \beta^* - \alpha)$ and on equation (3.2) with $(D - 3\rho + \epsilon - \epsilon^*)$, and subtract one equation from the other. The identity (3.5) with $p = q = 0$ and $m = 2$ shows that the terms in ϕ_1 cancel, yielding

$$(\delta^* + 3\pi - \beta^* - \alpha)(\delta^* + \pi - 2\alpha)\phi_0 = (D - 3\rho + \epsilon - \epsilon^*)(D - \rho + 2\epsilon)\phi_2 . \tag{3.6}$$

The interchange $\boldsymbol{l} \leftrightarrow \boldsymbol{n}$, $\boldsymbol{m} \leftrightarrow \boldsymbol{m}^*$ (cf. I: § II) gives

$$(\delta - 3\tau + \alpha^* + \beta)(\delta - \tau + 2\beta)\phi_2 = (\Delta + 3\mu + \gamma^* - \gamma)(\Delta + \mu - 2\gamma)\phi_0 . \tag{3.7}$$

(This equation could also have been derived directly from eqs. [3.3] and [3.4].)

Into equation (3.6) substitute for ϕ_0 and ϕ_2 the separated sum (2.6a), using the relations (I: table 1)

$$\Upsilon_1 = \phi_0 , \qquad \Upsilon_{-1} = (r - ia\cos\theta)^2\phi_2 . \tag{3.8}$$

Amazingly, the equation separates, giving

$$(\partial_\theta + m\,\mathrm{cosec}\,\theta - a\omega\sin\theta)(\partial_\theta + m\,\mathrm{cosec}\,\theta - a\omega\sin\theta + \cot\theta)S_1 = S_{-1} , \tag{3.9}$$

$$(\partial_r - iK/\Delta)(\partial_r - iK/\Delta)R_{-1} = \tfrac{1}{2}R_1 . \tag{3.10}$$

Here R_s is proportional to ${}_sR_{\omega lm}$, and S_s is proportional to ${}_sS_{lm}$. Angular equations like equation (3.9) and equations (3.15), (3.21), and (3.27) below were encountered previously when finding the relation between extreme spin-weight quantities asymptotically at infinity (e.g., II: Appendix B). We now see that they are in fact valid at all values of r. Starobinsky and Churilov (1973) have recently given the solutions of these equations. For completeness we outline how these solutions are obtained.

Suppose S_1 is normalized to be equal to ${}_1S_{lm}$. Then one can verify from equation (3.9) that S_{-1} satisfies equation (2.7) with $s = -1$, so $S_{-1} = B\,{}_{-1}S_{lm}$, where B is some normalizing constant. Write equation (3.9) in the form

$$\mathscr{L}_0\mathscr{L}_1\,{}_1S_{lm} = B\,{}_{-1}S_{lm} , \tag{3.11}$$

where

$$\mathscr{L}_n = \partial_\theta + m\,\mathrm{cosec}\,\theta - a\omega\sin\theta + n\cot\theta . \tag{3.12}$$

Then

$$B^2 = B^2 \int_0^\pi ({}_{-1}S_{lm})^2 \sin\theta d\theta$$

$$= \int_0^\pi (\mathscr{L}_0 \mathscr{L}_1 \, {}_1S_{lm})^2 \sin\theta d\theta \quad \text{by eq. (3.11)}$$

$$= \int_0^\pi {}_1S_{lm}(\mathscr{L}_0{}^\dagger \mathscr{L}_1{}^\dagger \mathscr{L}_0 \mathscr{L}_1 \, {}_1S_{lm}) \sin\theta d\theta \,.$$

In the last line we have used integration by parts to move the derivative operators to act only one on ${}_1S_{lm}$. Here,

$$\mathscr{L}_n{}^\dagger = \mathscr{L}_n(-\omega, -m) \,. \tag{3.13}$$

Now substitute for $d^2 \, {}_1S_{lm}/d\theta^2$ from equation (2.7) to reduce the fourth-order operator acting on ${}_1S_{lm}$ to a first-order operator. The result is that the coefficient of $d\, {}_1S_{lm}/d\theta$ is zero, while the coefficient of ${}_1S_{lm}$ is a constant. Since the integral of ${}_1S_{lm}{}^2$ is unity, we find that B^2 is equal to the constant:

$$B^2 = (E + a^2\omega^2 - 2a\omega m)^2 + 4ma\omega - 4a^2\omega^2 \,. \tag{3.14}$$

The sign of B can be found by comparing with the spherical case ($a\omega = 0$) when the angular functions reduce to the spin-weighted spherical harmonics and the $\mathscr{L}_n$ to $\eth$ operators (see Goldberg *et al.* 1967). This comparison shows that we must take the positive square root in equation (3.14).

For the radial functions, one checks similarly that if R_{-1} satisfies the radial equation (2.9) (with the upper sign and $s = -1$), then R_1 related by (3.10) will also satisfy (2.9) but with $s = 1$. Equation (3.10) then gives the *normalized* value of R_1 at any radius in terms of two derivatives of R_{-1} at that radius; but since R_{-1} satisfies a second-order equation, this is equivalent to an *algebraic* relation between R_1, R_{-1}, and dR_{-1}/dr.

A complementary set of equations to (3.9) and (3.10) is obtained by substituting into equation (3.7) instead of equation (3.6). This gives

$$\mathscr{L}_0{}^\dagger \mathscr{L}_1{}^\dagger S_{-1} = B^2 S_1 \,, \tag{3.15}$$

$$\mathscr{D}^\dagger \mathscr{D}^\dagger \Delta R_1 = 2B^2 \Delta^{-1} R_{-1} \,, \tag{3.16}$$

where

$$\mathscr{D} = \partial_r - iK/\Delta \tag{3.17}$$

and

$$\mathscr{D}^\dagger = \mathscr{D}(-\omega, -m) = \partial_r + iK/\Delta \,. \tag{3.18}$$

The factor of B^2 has been put into equations (3.15) and (3.16) to make the normalizations of the functions the same as in equations (3.9) and (3.10).

In summary, the transformation from Υ_1 to Υ_{-1} takes a solution

$$\Upsilon_1 = \phi_0 = R_1(r) \, {}_1S_{lm}(\theta) \exp(-i\omega t + im\varphi) \tag{3.19}$$

into a solution

$$\Upsilon_{-1} = (r - ia\cos\theta)^2 \phi_2 = BR_{-1}(r) \, {}_{-1}S_{lm}(\theta) \exp(-i\omega t + im\varphi) \,, \tag{3.20}$$

where B is given by equation (3.14) and the values and first radial derivatives of R_1 and R_{-1} are related by the essentially algebraic equations (3.10) and (3.16).

The analogous relation for Υ_2 and Υ_{-2} is more difficult to derive because the perturbed Einstein equations contain many perturbed NP quantities besides ψ_0 and ψ_4. The results can be guessed from the electromagnetic case and the results of (II: Appendix B), and then verified rigorously as outlined below. The equations corresponding to equations (3.9) and (3.10) are

$$\mathscr{L}_{-1} \mathscr{L}_0 \mathscr{L}_1 \mathscr{L}_2 S_2 + 12Mi\omega S_2{}^\dagger = S_{-2} \,, \tag{3.21}$$

$$\mathscr{D}\mathscr{D}\mathscr{D}\mathscr{D} R_{-2} = \tfrac{1}{4} R_2 \,. \tag{3.22}$$

If S_2 is normalized to be equal to ${}_2S_{lm}$, then $S_2{}^\dagger = {}_{-2}S_{lm}$ (cf. [II: 3.5, 3.6]) and $S_{-2} = C\,{}_{-2}S_{lm}$, where, as Starobinsky and Churilov (1973) have shown,

$$|C|^2 = (Q^2 + 4a\omega m - 4a^2\omega^2)[(Q-2)^2 + 36a\omega m - 36a^2\omega^2] + (2Q-1)(96a^2\omega^2 - 48a\omega m) + 144\omega^2(M - a^2)\,, \tag{3.23}$$

$$Q \equiv E + a^2\omega^2 - 2a\omega m\,. \tag{3.24}$$

Taking imaginary parts of equation (3.21) and using the fact that ${}_{\pm 2}S_{lm}$ is real gives

$$\operatorname{Im} C = 12M\omega\,. \tag{3.25}$$

Thus C is completely determined, with the sign of Re C fixed by comparison with the spherical case:

$$\operatorname{Re} C = +[|C|^2 - (\operatorname{Im} C)^2]^{1/2}\,. \tag{3.26}$$

The equations corresponding to equations (3.15) and (3.16) are

$$\mathscr{L}_{-1}{}^\dagger\mathscr{L}_0{}^\dagger\mathscr{L}_1{}^\dagger\mathscr{L}_2{}^\dagger S_{-2} + 12Mi\omega S_{-2}{}^\dagger = C^2 S_2\,, \tag{3.27}$$

$$\mathscr{D}^\dagger\mathscr{D}^\dagger\mathscr{D}^\dagger\mathscr{D}^\dagger\Delta^2 R_2 = 4|C|^2\Delta^{-2}R_{-2}\,. \tag{3.28}$$

To summarize: a solution

$$\Upsilon_2 = \psi_0 = R_2(r)\,{}_2S_{lm}(\theta)\exp(-i\omega t + im\varphi) \tag{3.29}$$

corresponds to a solution

$$\Upsilon_{-2} = (r - ia\cos\theta)^4\psi_4 = CR_{-2}(r)\,{}_{-2}S_{lm}(\theta)\exp(-i\omega t + im\varphi)\,, \tag{3.30}$$

where C is given by equations (3.23)–(3.26), and the values and first derivatives of $R_{\pm 2}$ are related by equations (3.22) and (3.28).

A reader wishing to check these relations without deriving them from the NP equations may do so as follows: equation (3.21) holds at infinity (II: Appendix B), so it will hold at all radii if equation (3.22) does. With some straightforward but tedious algebra one can verify that if R_{-2} satisfies the radial equation (2.9) with $s = -2$, then R_2 defined by equation (3.22) satisfies the same equation with $s = +2$. All that then remains to be checked is the overall normalization of R_2, and this can be done by using the asymptotic solution of the radial equation at infinity in equation (3.22) and comparing with (II: Appendix B). Equations (3.27) and (3.28) can be checked in a similar manner, or more simply by using the symmetry properties of the equations under the dagger operation.

IV. ENERGY AND ANGULAR-MOMENTUM FLUXES AT INFINITY AND ON THE HORIZON

In this section we give the physical interpretation of the solutions to the perturbation equation, in terms of the energy and angular momentum (component along the hole's rotation) fluxes that they carry. For definiteness we work in Boyer-Lindquist coordinates and the tetrad (2.3). By means of the transformations given above, the interpretation in any other tetrad can be effected.

For the scalar case $s = 0$, the asymptotic form of the solution at large r is

$$\Upsilon_0 \sim \exp(-i\omega t + im\varphi)S_{lm}(\theta)(Z_{\text{in}}e^{-i\omega r^*}/r + Z_{\text{out}}e^{i\omega r^*}/r)\,. \tag{4.1}$$

Since both ingoing and outgoing solutions appear in leading order $1/r$, one easily obtains

$$\frac{d^2E_{\text{out}}}{dtd\Omega} = \frac{S_{lm}{}^2(\theta)}{2\pi}\frac{\omega^2}{2}|Z_{\text{out}}|^2\,, \tag{4.2a}$$

$$\frac{d^2E_{\text{in}}}{dtd\Omega} = \frac{S_{lm}{}^2(\theta)}{2\pi}\frac{\omega^2}{2}|Z_{\text{in}}|^2\,. \tag{4.2b}$$

For the electromagnetic case $s = \pm 1$ one has at large r (table 1)

$$\Upsilon_1 = \phi_0 \sim \exp(-i\omega t + im\varphi)\,{}_1S_{lm}(\theta)(Y_{\text{in}}e^{-i\omega r^*}/r + Y_{\text{out}}e^{i\omega r^*}/r^3)\,, \tag{4.3}$$

$$\Upsilon_{-1} = (r - ia\cos\theta)^2\phi_2 \sim \exp(-i\omega t + im\varphi)\,{}_{-1}S_{lm}(\theta)(Z_{\text{in}}e^{-i\omega r^*}/r + Z_{\text{out}}re^{i\omega r^*})\,. \tag{4.4}$$

From equations (3.9) and (3.10),

$$BY_{\text{in}} = -8\omega^2 Z_{\text{in}}\,, \tag{4.5}$$

while equations (3.15) and (3.16) give

$$-2\omega^2 Y_{\text{out}} = BZ_{\text{out}} . \tag{4.6}$$

The energy fluxes are related directly to the $1/r$ parts of the fields (I: § 5), but can now be found from the corresponding $1/r^3$ part by using equations (4.5) and (4.6). Thus,

$$\frac{d^2E_{\text{out}}}{dtd\Omega} = \frac{{}_{-1}S_{lm}{}^2(\theta)}{2\pi} |Z_{\text{out}}|^2 = \frac{{}_{-1}S_{lm}{}^2(\theta)}{2\pi} \frac{4\omega^4}{B^2} |Y_{\text{out}}|^2 , \tag{4.7a}$$

$$\frac{d^2E_{\text{in}}}{dtd\Omega} = \frac{{}_1S_{lm}{}^2(\theta)}{2\pi} \tfrac{1}{4}|Y_{\text{in}}|^2 = \frac{{}_1S_{lm}{}^2(\theta)}{2\pi} \frac{16\omega^4}{B^2} |Z_{\text{in}}|^2 . \tag{4.7b}$$

For gravitational perturbations $s = \pm 2$ the corresponding equations are

$$\Upsilon_2 = \psi_0 \sim \exp(-i\omega t + im\varphi)\, {}_2S_{lm}(\theta)(Y_{\text{in}}e^{-i\omega r^*}/r + Y_{\text{out}}e^{i\omega r^*}/r^5) , \tag{4.8}$$

$$\Upsilon_{-2} = (r - ia\cos\theta)^4\psi_4 \sim \exp(-i\omega t + im\varphi)\, {}_{-2}S_{lm}(\theta)(Z_{\text{in}}e^{-i\omega r^*}/r + Z_{\text{out}}r^3e^{i\omega r^*}) , \tag{4.9}$$

$$CY_{\text{in}} = 64\omega^4 Z_{\text{in}} , \tag{4.10}$$

$$4\omega^4 Y_{\text{out}} = C^*Z_{\text{out}} , \tag{4.11}$$

$$\frac{d^2E_{\text{out}}}{dtd\Omega} = \frac{{}_{-2}S_{lm}{}^2(\theta)}{2\pi} \frac{1}{2\omega^2} |Z_{\text{out}}|^2 = \frac{{}_{-2}S_{lm}{}^2(\theta)}{2\pi} \frac{8\omega^6}{|C|^2} |Y_{\text{out}}|^2 , \tag{4.12a}$$

$$\frac{d^2E_{\text{in}}}{dtd\Omega} = \frac{{}_2S_{lm}{}^2(\theta)}{2\pi} \frac{1}{32\omega^2} |Y_{\text{in}}|^2 = \frac{{}_2S_{lm}{}^2(\theta)}{2\pi} \frac{128\omega^6}{|C|^2} |Z_{\text{in}}|^2 . \tag{4.12b}$$

From equation (4.2), (4.7), or (4.12), the total energy flux follows by integrating over all angles and using the normalization equation (2.8). Furthermore, because both t and φ are Killing coordinates of the background metric, the angular-momentum increment along the symmetry axis dJ is generally related to the energy increment by

$$dJ = \frac{m}{\omega} dE \tag{4.13}$$

(cf. Bekenstein 1973).

Turn now to the horizon, where we ask: What is the energy flux going "down the hole"? Here it is convenient to use a different tetrad, that of Hawking and Hartle (1972). It is an outgoing tetrad which is made well behaved on the future horizon by scaling $\boldsymbol{l}$ so that $l^v = 1$. The transformation from Kinnersley's tetrad (2.3) is

$$\boldsymbol{l}^{\text{HH}} = [\Delta/2(r^2 + a^2)]\boldsymbol{l} , \qquad \boldsymbol{n}^{\text{HH}} = [2(r^2 + a^2)/\Delta]\boldsymbol{n} . \tag{4.14}$$

The $[v, r, \theta, \tilde{\varphi}]$ components of the tetrad are given in equation (I: 5.8); the field quantities are

$$\Upsilon_s{}^{\text{HH}} = [\Delta/2(r^2 + a^2)]^s\Upsilon_s = (r^2 + a^2)^{-s}\Omega_{-s} . \tag{4.15}$$

We shall impose the boundary condition that there be no outgoing radiation from the horizon. The 3-surface element of the horizon, with normal vector in the inward radial direction, is

$$\begin{aligned} d^3\Sigma_\mu &= l_\mu{}^{\text{HH}}2Mr_+ \sin\theta d\theta d\tilde{\varphi} dv , \\ &= l_\mu{}^{\text{HH}}2Mr_+ \sin\theta d\theta d\varphi dt , \end{aligned} \tag{4.16}$$

where we have used the fact that the Jacobian $\partial(\tilde{\varphi}, v)/\partial(\varphi, t) = 1$. Since the Kerr metric has a time Killing vector

$$\boldsymbol{\xi}_{(t)} = \frac{\partial}{\partial t} = \frac{\partial}{\partial v} , \tag{4.17}$$

any perturbation field that has a well-defined stress-energy tensor will admit an associated conserved energy flux vector $T_{\mu\nu}\xi_{(t)}{}^\mu$. For the cases $s = 0, \pm 1$, this will give the desired formulae. The case $s = \pm 2$ requires a more sophisticated treatment given below.

If there is a flux of energy across the 2-surface element formed by the intersection of an element of the horizon with two surfaces of constant v separated by dv, then the change in energy of the hole is

$$dE_{\text{hole}} = T_\mu{}^\nu\xi^\mu{}_{(t)}d^3\Sigma_\nu . \tag{4.18}$$

Therefore from equation (4.16)

$$\frac{d^2E_{\rm hole}}{dtd\Omega} = 2Mr_+T_\mu{}^\nu\xi_{(t)}{}^\mu l_\nu{}^{\rm HH} \tag{4.19}$$

Similarly, the axial Killing vector

$$\xi_{(\varphi)} = \frac{\partial}{\partial\varphi} = \frac{\partial}{\partial\tilde{\varphi}} \tag{4.20}$$

implies the existence of a conserved angular-momentum flux $-T_{\mu\nu}\xi_{(\varphi)}{}^\mu$ (there is a minus sign because the signature of the metric is $+---$). Thus

$$\frac{d^2J_{\rm hole}}{dtd\Omega} = -2Mr_+T_\mu{}^\nu\xi_{(\varphi)}{}^\mu l_\nu{}^{\rm HH}\,. \tag{4.21}$$

But on the horizon

$$l^{\rm HH} = \xi_{(t)} + \omega_+\xi_{(\varphi)} \tag{4.22}$$

(I: 5.8), so

$$\frac{d^2E_{\rm hole}}{dtd\Omega} - \omega_+\frac{d^2J_{\rm hole}}{dtd\Omega} = 2Mr_+T^{\mu\nu}l_\mu{}^{\rm HH}l_\nu{}^{\rm HH}\,; \tag{4.23}$$

or using equation (4.13),

$$\frac{d^2E_{\rm hole}}{dtd\Omega} = \frac{\omega}{k}2Mr_+T^{\mu\nu}l_\mu{}^{\rm HH}l_\nu{}^{\rm HH}\,. \tag{4.24}$$

A scalar perturbation has the form

$$\Upsilon_0 \sim \exp(-i\omega t + im\varphi)S_{lm}(\theta)Z_{\rm hole}e^{-ikr^*} \tag{4.25}$$

on the horizon. The stress-energy tensor for scalar perturbations (I: 5.12) gives

$$\frac{d^2E_{\rm hole}}{dtd\Omega} = Mr_+\omega k\frac{S_{lm}{}^2(\theta)}{2\pi}|Z_{\rm hole}|^2\,. \tag{4.26}$$

An electromagnetic perturbation has the form on the horizon:

$$\Upsilon_1 = \phi_0 \sim \exp(-i\omega t + im\varphi)\,{}_1S_{lm}(\theta)Y_{\rm hole}\Delta^{-1}e^{-ikr^*}\,, \tag{4.27}$$

$$\Upsilon_{-1} = (r - ia\cos\theta)^2\phi_2 \sim \exp(-i\omega t + im\varphi)\,{}_{-1}S_{lm}(\theta)Z_{\rm hole}\Delta e^{-ikr^*}\,, \tag{4.28}$$

where by equations (3.9) and (3.10),

$$BY_{\rm hole} = -32ikM^2r_+{}^2(-ik + 2\epsilon)Z_{\rm hole}\,. \tag{4.29}$$

Here

$$\epsilon = (M^2 - a^2)^{1/2}/(4Mr_+)\,. \tag{4.30}$$

Since

$$\Upsilon_1{}^{\rm HH} = \Delta\Upsilon_1/2(r^2 + a^2)\,, \tag{4.31}$$

equation (4.24) and (I: 5.12), the expression for the electromagnetic stress-energy tensor, give

$$\frac{d^2E_{\rm hole}}{dtd\Omega} = \frac{{}_1S_{lm}{}^2(\theta)}{2\pi}\frac{\omega}{8Mr_+k}|Y_{\rm hole}|^2 = \frac{{}_1S_{lm}{}^2(\theta)}{2\pi}\frac{128\omega kM^3r_+{}^3(k^2 + 4\epsilon^2)}{B^2}|Z_{\rm hole}|^2\,. \tag{4.32}$$

Equation (4.24) fails for gravitational perturbations, since there is no known stress-energy tensor for this case. Instead we can use the results of Hawking and Hartle (1972) to derive an analogous result. The area of a Kerr black hole is

$$A = 8\pi[M^2 + (M^4 - J^2)^{1/2}]\,, \tag{4.33}$$

where $J = aM$. Thus

$$\frac{d^2A}{dtd\Omega} = \frac{8\pi}{(M^4 - J^2)^{1/2}}\left(2M^2r_+\frac{d^2M}{dtd\Omega} - J\frac{d^2J}{dtd\Omega}\right)\,. \tag{4.34}$$

Now $dM = dE_{\text{hole}}$, and using equation (4.13) we get

$$\frac{d^2A}{dtd\Omega} = \frac{16\pi M^2 r_+ k}{(M^4 - J^2)^{1/2}\omega} \frac{d^2E_{\text{hole}}}{dtd\Omega} . \tag{4.35}$$

Hawking and Hartle show that

$$\frac{d^2A}{dtd\Omega} = \frac{2Mr_+}{\epsilon} |\sigma^{\text{HH}}|^2 , \tag{4.36}$$

where σ^{HH} is the perturbation in the shear of the generators of the horizon. Thus

$$\frac{d^2E_{\text{hole}}}{dtd\Omega} = \frac{\omega M r_+}{2\pi k} |\sigma^{\text{HH}}|^2 . \tag{4.37}$$

The quantity σ^{HH} can be found from $\psi_0{}^{\text{HH}}$ via the following equation, valid on the horizon in the Hawking-Hartle tetrad:

$$D\sigma^{\text{HH}} = 2\epsilon\sigma^{\text{HH}} + \psi_0{}^{\text{HH}} \tag{4.38}$$

(cf. NP 4.2b and Hawking and Hartle 1972). From equation (4.22) this gives

$$\sigma^{\text{HH}} = -\psi_0{}^{\text{HH}}/(ik + 2\epsilon) . \tag{4.39}$$

The asymptotic form of the field components on the horizon is

$$\Upsilon_2 = \psi_0 \sim \exp(-i\omega t + im\varphi)\, {}_2S_{lm}(\theta) Y_{\text{hole}} \Delta^{-2} e^{-ikr^*} , \tag{4.40}$$

$$\Upsilon_{-2} = (r - ia\cos\theta)^4 \psi_4 \sim \exp(-i\omega t + im\varphi)\, {}_{-2}S_{lm}(\theta) Z_{\text{hole}} \Delta^2 e^{-ikr^*} , \tag{4.41}$$

where by equations (3.2) and (3.2)

$$C Y_{\text{hole}} = 64(2Mr_+)^4 ik(k^2 + 4\epsilon^2)(-ik + 4\epsilon) Z_{\text{hole}} . \tag{4.42}$$

Since

$$\psi_0{}^{\text{HH}} = \Delta^2\psi_0/4(r^2 + a^2)^2 , \tag{4.43}$$

we find

$$\begin{aligned} \frac{d^2E_{\text{hole}}}{dtd\Omega} &= \frac{{}_2S_{lm}{}^2(\theta)}{2\pi} \frac{\omega}{32k(k^2 + 4\epsilon^2)(2Mr_+)^3} |Y_{\text{hole}}|^2 \\ &= \frac{{}_2S_{lm}{}^2(\theta)}{2\pi} \frac{128\omega k(k^2 + 4\epsilon^2)(k^2 + 16\epsilon^2)(2Mr_+)^5}{|C|^2} |Z_{\text{hole}}|^2 . \end{aligned} \tag{4.44}$$

Notice that from equations (4.26), (4.32), and (4.44) energy flows *out* of the hole if k/ω is negative—this is what we have called "superradiant scattering," since more energy comes out at infinity than was originally sent in (cf. Misner 1972; Press and Teukolsky 1972; Zel'dovich 1971, 1972; Starobinsky 1973; Starobinsky and Churilov 1973). Since equation (4.35) is valid for all types of perturbations, and Hawking's (1972) area theorem says that dA/dt is nonnegative, we see that it is a general consequence of the area theorem that superradiant scattering must occur if k/ω is negative (cf. Bekenstein 1973).

V. CONSERVATION OF ENERGY FOR GRAVITATIONAL PERTURBATIONS AND A REMARK ON STABILITY

Consider a rotating hole which is immersed in a scalar or an electromagnetic radiation field. Since these fields admit a divergenceless stress-energy tensor $T_{\mu\nu}$, any real-frequency perturbation will have

$$\frac{dE_{\text{in}}}{dt} - \frac{dE_{\text{out}}}{dt} = \frac{dE_{\text{hole}}}{dt} , \tag{5.1}$$

a conservation law relating the net absorption of the wave at infinity to the net increase in the hole's mass. Equation (5.1), through the formulae of § IV, translates to a connection relation between the asymptotic coefficients Z_{in}, Z_{out}, and Z_{hole} for homogeneous solutions of the perturbation equations. We now show that these connection relations also follow directly from the form of the radial differential equations; further, this direct procedure gives a connection relation for gravitational perturbations as well. Although not surprising physically, this result *is* surprising mathematically because there is no known divergenceless $T_{\mu\nu}$ for a gravitational perturbation on the

Kerr background. The result proves that a gravitational wave cannot gain or lose energy to the exterior field of the hole; any deficit or surplus must appear as a wave crossing the horizon.

Write the radial equation (2.9) in the form

$$Y_{,r^*r^*} + VY = 0 \tag{5.2}$$

by letting

$$Y = \Delta^{s/2}(r^2 + a^2)^{1/2}R\,, \tag{5.3}$$

$$\begin{aligned} V = {} & [K^2 - 2isK(r - M) + \Delta(4ir\omega s - Q) - s^2(M^2 - a^2)]/(r^2 + a^2)^2 \\ & - \Delta(2Mr^3 + a^2r^2 - 4Mra^2 + a^4)/(r^2 + a^2)^4\,. \end{aligned} \tag{5.4}$$

The Wronskian of any two solutions of equation (5.2) is conserved (i.e., independent of r). For real ω, E and hence Q is real and independent of the sign of s. Thus $V(r, \omega, m, l, s, a) = V^*(r, \omega, m, l, -s, a)$, so two linearly independent solutions of equation (5.2) are $Y(s)$ and $Y^*(-s)$. (We omit parameters which are kept constant.) Thus

$$[Y(s)_{,r^*}Y^*(-s) - Y(s)Y^*(-s)_{,r^*}]_{r=r_+} = [Y(s)_{,r^*}Y^*(-s) - Y(s)Y^*(-s)_{,r^*}]_{r=\infty}\,. \tag{5.5}$$

Let us first apply equation (5.5) to scalar perturbations ($s = 0$). From equations (4.25), (4.1), and (5.3), equation (5.5) becomes

$$-2ik(2Mr_+)|Z_{\rm hole}|^2 = -2i\omega|Z_{\rm in}|^2 + 2i\omega|Z_{\rm out}|^2\,, \tag{5.6}$$

which using equations (4.2) and (4.26) is exactly the energy condition (5.1).

Next applying equation (5.5) to electromagnetic perturbations with $s = 1$, and using equations (5.3), (4.27)–(4.29), and (4.3)–(4.6), we find

$$\frac{iB|Y_{\rm hole}|^2}{8kMr_+} = \frac{iB|Y_{\rm in}|^2}{4\omega} - \frac{4i\omega^3|Y_{\rm out}|^2}{B}\,. \tag{5.7}$$

Comparison with equations (4.7) and (4.32) again gives the energy conservation law (5.1).

For gravitational perturbations with $s = 2$, equations (5.3), (4.40)–(4.42), and (4.8)–(4.11) give

$$\frac{-iC^*|Y_{\rm hole}|^2}{32k(2Mr_+)^3(k^2 + 4\epsilon^2)} = \frac{-iC^*|Y_{\rm in}|^2}{32\omega^3} + \frac{8i\omega^5|Y_{\rm out}|^2}{C}\,. \tag{5.8}$$

This is once again the energy conservation law (5.1), as comparison with equations (4.12) and (4.44) shows.

These connection formulae, incidentally, allow one to tighten up the evidence for the dynamical stability of the Kerr metric which was presented in Paper II. There it was shown that an instability corresponds to a zero of $Z_{\rm in}$ (or $Y_{\rm in}$) in the upper-half complex ω plane. Hartle and Wilkins (1974) have recently proved—this was assumed without proof in our Paper II—that a zero of $Z_{\rm in}$ can occur in the upper half-plane only by migrating smoothly from the lower half-plane as the specific angular momentum a of the hole is increased from its (zero) Schwarzschild value. Here we point out that equations (5.6), (5.7), and (5.8) imply that a zero may cross the real axis only in the finite segment between 0 and $m\omega_+$, because only in this range can the two sides of the connection relations have the same sign when $Z_{\rm in} = 0$. (This range was examined numerically in Paper II.) This result was previously obtained in the case of scalar perturbations by Detweiler and Ipser (1973); also, it shows that axisymmetric modes are all stable, as found by Friedman and Schutz (1974). The analytic results of the next section show that the hole is stable in the limits $\omega M \ll 1$ or $a \sim M$ and $\omega \sim m\omega_+$. A general analytic proof of stability is still lacking.

VI. SUMMARY OF LIMITING CASES WHICH ARE SOLUBLE ANALYTICALLY

In neighborhoods of the two boundary points of the superradiant segment, $\omega \approx 0$ and $\omega \approx m\omega_+$, some solutions of the radial perturbation equation can be given analytically.

When $\omega = 0$, for arbitrary a, the equations can be solved in terms of hypergeometric functions. These solutions can be used to treat the spindown of a rotating black hole by a stationary, nonaxisymmetric perturbation; the calculation will be published in a later paper in this series (cf. Press 1972; Hawking and Hartle 1972; Hartle 1973; Teukolsky 1973*b*).

These solutions also enable one to solve the scattering problem of waves off a Kerr black hole in the limit $\omega M \ll 1$. The calculation has been done by Starobinsky (1973) and Starobinsky and Churilov (1973). The idea is that the solution in terms of hypergeometric functions is a good approximation for small ω all the way from the horizon until one gets to fairly large r (the wave zone). There one has a solution describing traveling waves which can be matched to the inner solution. If

$$Z \equiv \frac{dE_{\rm out}/dt}{dE_{\rm in}/dt} - 1 = -\frac{dE_{\rm hole}/dt}{dE_{\rm in}/dt} \tag{6.1}$$

is the fractional gain (or loss) of energy in a scattered wave, Starobinsky and Churilov find (in our notation)

$$Z = -\frac{k}{\epsilon}[\omega(r_+ - r_-)]^{2l+1}\left[\frac{(l-s)!\,(l+s)!}{(2l)!\,(2l+1)!!}\right]^2 \prod_{n=1}^{l}\left(1 + \frac{k^2}{4\epsilon^2 n^2}\right), \tag{6.2}$$

which shows the amplification (Z positive) in the superradiant regime ($k\omega$ negative).

For $\omega = m\omega_+$, the radial equations can be solved in terms of confluent hypergeometric functions in the special case $a = M$ (maximally rotating hole). Starobinsky and Churilov perturbed away from this solution to find solutions for Z in a neighborhood $\omega \approx m\omega_+$, but with $a = M$ exactly. Here their calculation is generalized to a neighborhood $a \approx M$.

When $a = M$, there are two distinct cases for the behavior of Z, depending on the value of

$$\delta^2 \equiv a^2m^2/M^2 - \lambda - (s + \tfrac{1}{2})^2\,. \tag{6.3}$$

(Note that our definition of λ differs from that used by Starobinsky and Churilov by $-2am\omega$.) When $\delta^2 < 0$, Z passes through smoothly through zero as ω passes through $m\omega_+$. When $\delta^2 > 0$, however (which occurs for all modes with $l = m$ and for other modes with l close enough to m), Z oscillates an infinite number of times between two positive values when $\omega \to m\omega_+$ from below, while Z oscillates an infinite number of times between a negative value and -1 when $\omega \to m\omega_+$ from above. The magnitude of the oscillations is generally too small to be seen in numerical work, with one exception: for $s = l = m = 1$, $Z_{\max}/Z_{\min} = 1.44$. (For comparison, the value for $s = l = m = 2$ is 1.00002.) However, the numerical work of the next section shows no hint of an oscillation, even for $s = l = m = 1$ and $a = 0.99999M$. The reason for this is seen in studying the behavior of Z in the double limit $a \to M$, $\omega \to m\omega_+$.

Appendix A extends the calculation of Z to the case where

$$\alpha \equiv 1 - \omega/(m\omega_+) \ll 1\,, \qquad \gamma \equiv (1 - a^2/M^2)^{1/2} \ll 1\,. \tag{6.4}$$

The result is

$$\begin{aligned} Z = {} & \sinh^2 2\pi\delta \sinh \pi\kappa \exp(2\pi\hat{\omega})/[e^{-\pi\delta}\cosh^2 \pi(2\hat{\omega} - \delta)\cosh \pi(2\hat{\omega} + \delta + \kappa) \\ & + e^{\pi\delta}\cosh^2 \pi(2\hat{\omega} + \delta)\cosh \pi(2\hat{\omega} - \delta + \kappa) \\ & - 2\cos\psi \cosh \pi(2\hat{\omega} + \delta)\cosh \pi(2\hat{\omega} - \delta)\cosh^{1/2} \pi(2\hat{\omega} + \delta + \kappa)\cosh^{1/2} \pi(2\hat{\omega} - \delta + \kappa)]\,, \end{aligned} \tag{6.5}$$

where $\kappa = ma\alpha/(\gamma M)$, $\hat{\omega} = \omega r_+$, and ψ is defined in equation (A16). The result of Starobinsky and Churilov is recovered by letting $\gamma \to 0$ ($\kappa \to \infty$). For α positive, this gives

$$Z = \sinh^2 2\pi\delta/[\cosh^2 \pi(2\hat{\omega} - \delta) + \cosh^2 \pi(2\hat{\omega} + \delta) - 2\cos\psi \cosh \pi(2\hat{\omega} - \delta)\cosh \pi(2\hat{\omega} + \delta)]\,, \tag{6.6}$$

where $\psi = \text{constant} - 2\delta ln(2m^2\alpha)$. As $\alpha \to 0$, this exhibits an infinite number of oscillations between

$$Z_{\max} = \cosh^2 \pi\delta/\sinh^2 2\pi\hat{\omega} \tag{6.7}$$

and

$$Z_{\min} = \sinh^2 \pi\delta/\cosh^2 2\pi\hat{\omega}\,. \tag{6.8}$$

However, if $\alpha \to 0$ for fixed (small) γ in equation (6.5), we find $Z \to 0$ linearly with α. Thus for fixed small γ there will be a finite number of oscillations before $Z \to 0$. In the favorable case of $s = l = m = 1$, δ^2 is not positive (i.e., eq. [6.5] showing oscillations is not valid) until $a > 0.995M$. Even then, numerical evaluation of equation (6.5) shows that one must have a within 10^{-7} of M before an oscillation becomes visible. Thus for all practical purposes the oscillations can be ignored as a mathematical curiosity of the nonuniform double limit $a \to M$, $\omega \to m\omega_+$.

VII. NUMERICAL RESULTS: WAVE ABSORPTION AND SUPERRADIANT SCATTERING

Our primary task is to calculate Z (eq. [6.1]), the fractional gain or loss of energy in a scattered wave. Z is a function of the nondimensional frequency of the wave ωM, the hole's rotation a/M, the mode numbers l and m, and of course $|s|$ which tells what perturbing field we are looking at. Using the results of the preceding sections, the calculation of Z is a straightforward: choose a form of the radial perturbation equation (§ II) which makes the inward boundary condition easy to impose and (importantly!) numerically stable against contamination with the other solution when integrated outward. At some moderate radius, say $r \sim 3M$, use the relations between the solutions (§ III) to switch to a different form of the radial equation, one which makes the ingoing wave solution stable to numerical integration out to radial infinity, and the coefficient of this wave easy to determine there. Completing the integration, use the results of § IV to read off the ingoing energy fluxes at the horizon and at

TABLE 2

Polynomial Approximations for the Angular Eigenvalues $_{\pm 1}E_{lm}(a\omega)$

			$E = 2. +$			
$s = 1, l = 1, m = 1$..................	−1.00076	−.546806	−.078558	−.0106090	.0062159	−.00069360
$s = 1, l = 1, m = 0$..................	.00281	−.413370	.021476	−.0335098	.0025402	.00032399
$s = 1, l = 1, m = -1$................	1.00010	−.551012	.078143	−.0144321	.0000024	−.00005802
			$E = 6. +$			
$s = 1, l = 2, m = 2$..................	−.66595	−.326331	−.031461	−.0123611	.0008485	.00013890
$s = 1, l = 2, m = 1$..................	−.33332	−.365775	.041651	.0072663	−.0074567	.00079712
$s = 1, l = 2, m = 0$..................	−.00258	−.368784	−.019229	.0231819	−.0017607	−.00036074
$s = 1, l = 2, m = -1$................	.33287	−.363632	−.051066	.0080035	−.0014728	.00007984
$s = 1, l = 2, m = -2$................	.66674	−.323163	.038494	−.0087605	.0009191	−.00012047
			$E = 12. +$			
$s = 1, l = 3, m = 3$..................	−.50025	−.217365	−.022266	−.0015063	−.0014874	.00019079
$s = 1, l = 3, m = 2$..................	−.33397	−.334826	.010003	.0057601	−.0004161	−.00020091
$s = 1, l = 3, m = 1$..................	−.16594	−.413169	.022682	−.0028969	.0013993	−.00009824
$s = 1, l = 3, m = 0$..................	−.00014	−.432586	−.001378	.0029328	−.0004449	−.00003031
$s = 1, l = 3, m = -1$................	.16697	−.410949	−.013716	.0004978	.0009654	.00002668
$s = 1, l = 3, m = -2$................	.33333	−.337934	−.015414	.0022248	−.0006883	.00003612
$s = 1, l = 3, m = -3$................	.50001	−.218815	.019652	−.0041318	.0003965	−.00004611
			$E = 20. +$			
$s = 1, l = 4, m = 4$..................	−.40008	−.161742	−.011985	−.0014650	−.0004609	.00001977
$s = 1, l = 4, m = 3$..................	−.29974	−.293235	.007457	−.0022561	.0014528	−.00019537
$s = 1, l = 4, m = 2$..................	−.20009	−.383915	.008391	.0018676	−.0003466	.00006066
$s = 1, l = 4, m = 1$..................	−.09996	−.440165	.006963	.0007045	.0000180	−.00002478
$s = 1, l = 4, m = 0$..................	−.00009	−.457980	−.000882	.0018381	−.0003362	.00006200
$s = 1, l = 4, m = -1$................	.10006	−.440261	−.005854	.0005216	.0004136	−.00005030
$s = 1, l = 4, m = -2$................	.19992	−.383943	−.010139	.0018465	−.0003547	.00008283
$s = 1, l = 4, m = -3$................	.30001	−.291873	−.004481	.0001673	−.0002414	.00000829
$s = 1, l = 4, m = -4$................	.40000	−.162187	.011175	−.0021499	.0001792	−.00001903
			$E = 30. +$			
$s = 1, l = 5, m = 5$..................	−.33334	−.127452	−.006992	−.0011934	−.0001097	−.00001338
$s = 1, l = 5, m = 4$..................	−.26659	−.251848	.001845	−.0008714	.0003781	−.00002175
$s = 1, l = 5, m = 3$..................	−.20001	−.347774	.004576	.0004021	.0000377	.00000433
$s = 1, l = 5, m = 2$..................	−.13333	−.416750	.005056	.0005387	−.0000101	−.00000489
$s = 1, l = 5, m = 1$..................	−.06668	−.457941	.002885	.0008766	−.0001153	.00001301
$s = 1, l = 5, m = 0$..................	.00000	−.471816	.000036	.0007487	.0000088	.00000159
$s = 1, l = 5, m = -1$................	.06666	−.457983	−.003110	.0007918	.0000400	.00000164
$s = 1, l = 5, m = -2$................	.13334	−.416756	−.004886	.0005390	.0000641	−.00000067
$s = 1, l = 5, m = -3$................	.19997	−.347711	−.004957	.0005506	−.0001646	.00002860
$s = 1, l = 5, m = -4$................	.26667	−.251485	−.001010	−.0003356	−.0000905	.00000107
$s = 1, l = 5, m = -5$................	.33333	−.127490	.006915	−.0012284	.0000891	−.00000879
			$E = 42. +$			
$s = 1, l = 6, m = 6$..................	−.28571	−.104394	−.004526	−.0008073	−.0000324	−.00001174
$s = 1, l = 6, m = 5$..................	−.23809	−.219067	−.000069	−.0003862	.0000579	.00000914
$s = 1, l = 6, m = 4$..................	−.19047	−.312879	.002374	−.0000634	.0000634	.00000069
$s = 1, l = 6, m = 3$..................	−.14285	−.385851	.003204	.0002062	.0000197	−.00000229
$s = 1, l = 6, m = 2$..................	−.09524	−.437941	.002757	.0004615	−.0000351	.00000256
$s = 1, l = 6, m = 1$..................	−.04762	−.469219	.001574	.0005773	−.0000330	.00000346
$s = 1, l = 6, m = 0$..................	−.00000	−.479650	−.000007	.0006036	−.0000026	.00000164
$s = 1, l = 6, m = -1$................	.04762	−.469231	−.001590	.0005514	.0000266	−.00000033
$s = 1, l = 6, m = -2$................	.09524	−.437954	−.002788	.0004398	.0000243	.00000104
$s = 1, l = 6, m = -3$................	.14286	−.385829	−.003180	.0002554	−.0000117	.00000524
$s = 1, l = 6, m = -4$................	.19047	−.312825	−.002411	.0000342	−.0000738	.00001022
$s = 1, l = 6, m = -5$................	.23810	−.219063	.000148	−.0004088	−.0000365	−.00000058
$s = 1, l = 6, m = -6$................	.28571	−.104370	.004562	−.0007555	.0000482	−.00000447
	$\times (a\omega)^1$	$\times (a\omega)^2$	$\times (a\omega)^3$	$\times (a\omega)^4$	$\times (a\omega)^5$	$\times (a\omega)^6$

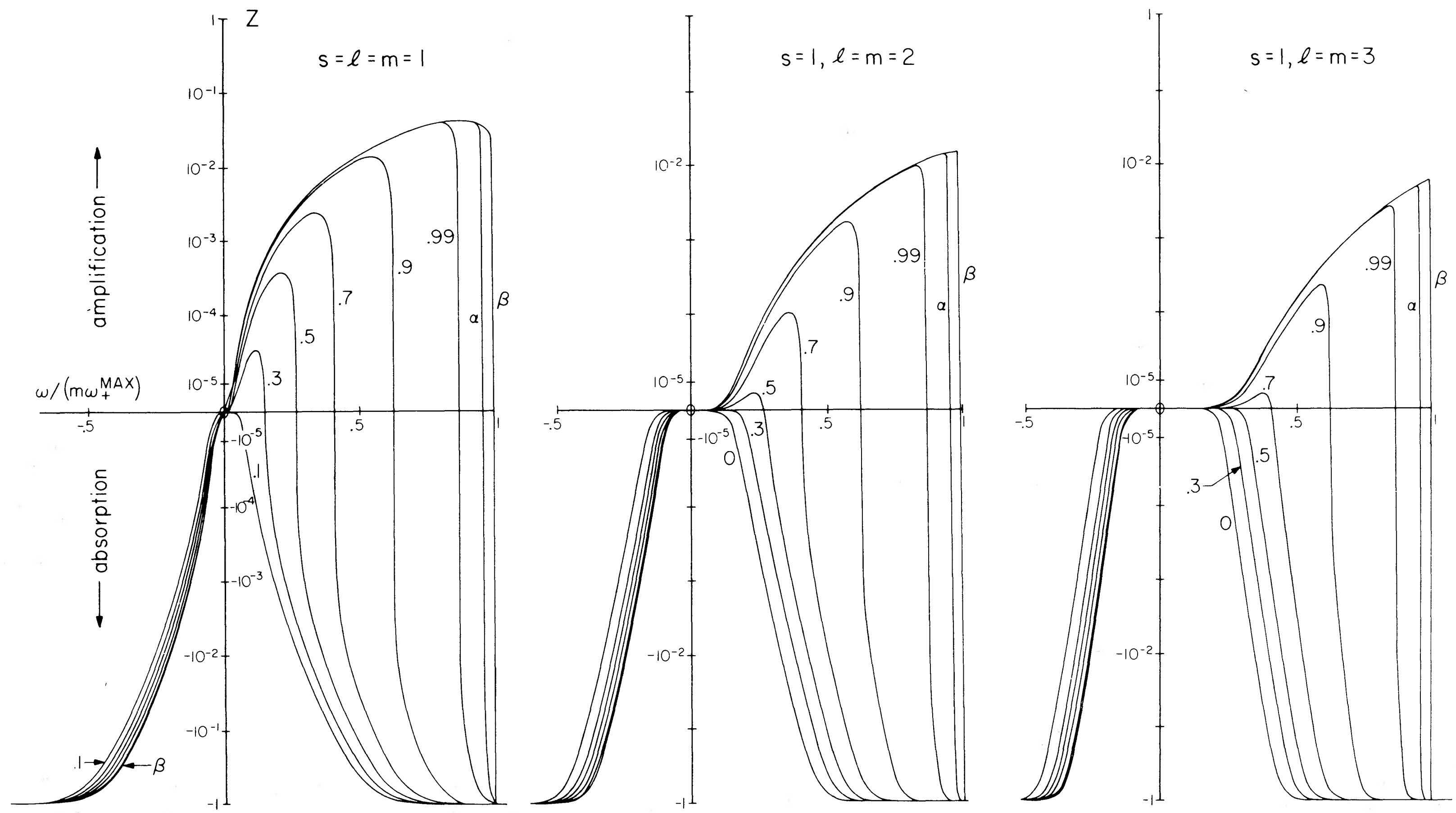

FIG. 1.—Amplification and absorption of electromagnetic ($s = 1$) waves by a rotating black hole. The fractional energy amplification (positive Z) or absorption (negative Z) of the three most strongly amplified spheroidal-wave modes is shown as a function of the wave's frequency ω (in units of m times the hole's maximum rotation frequency $\omega_+{}^{\max} = 2/M$). The different curves parametrize the specific angular momentum a of the hole in units of its mass M, ranging from $a/M = 0$ (Schwarzschild) to $a/M \approx 1$ (extreme Kerr). The curves labeled α and β indicate values $a/M = 0.999$ and 0.99999, respectively. To show five logarithmic decades of both positive and negative values, and a linear region where the zero crossings occur, the ordinate has been plotted proportional to $\sinh^{-1}(10^5 Z)$. The significance of $Z \to -1$ as $|\omega|$ becomes large is that the hole becomes "black," with all incident energy absorbed. The significance of a range of ω with Z positive is that waves in this range extract energy from the rotating hole, and come out "superradiantly amplified."

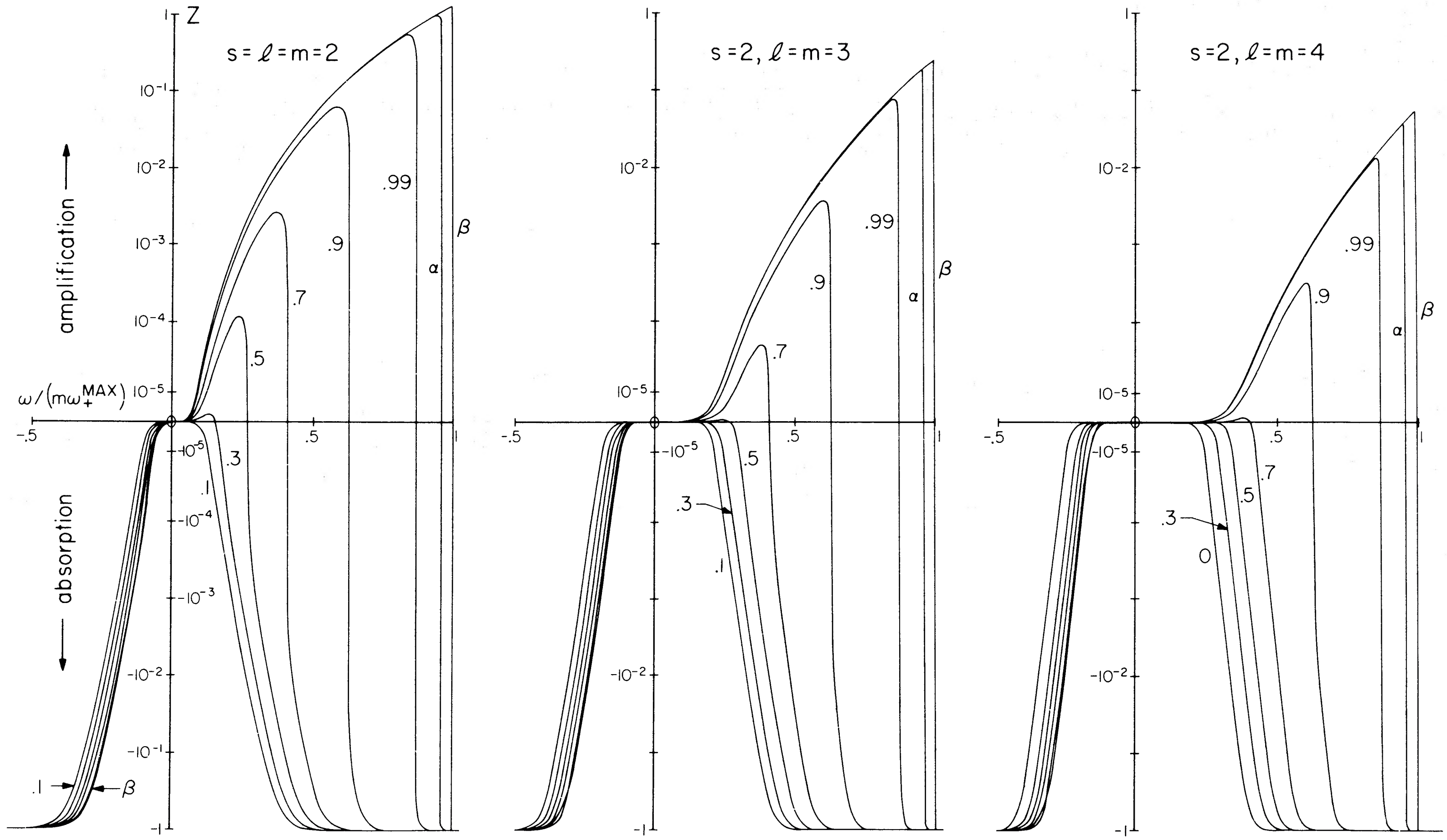

FIG. 2.—Amplification and absorption of gravitational ($s = 2$) waves by a rotating black hole. Shown are the three most strongly amplified modes. As in fig. 1, the different curves are for different values of the hole's specific angular momentum a/M, with α and β indicating the values 0.999 and 0.99999, respectively. The ordinate is plotted proportional to $\sinh^{-1}(10^5 Z)$.

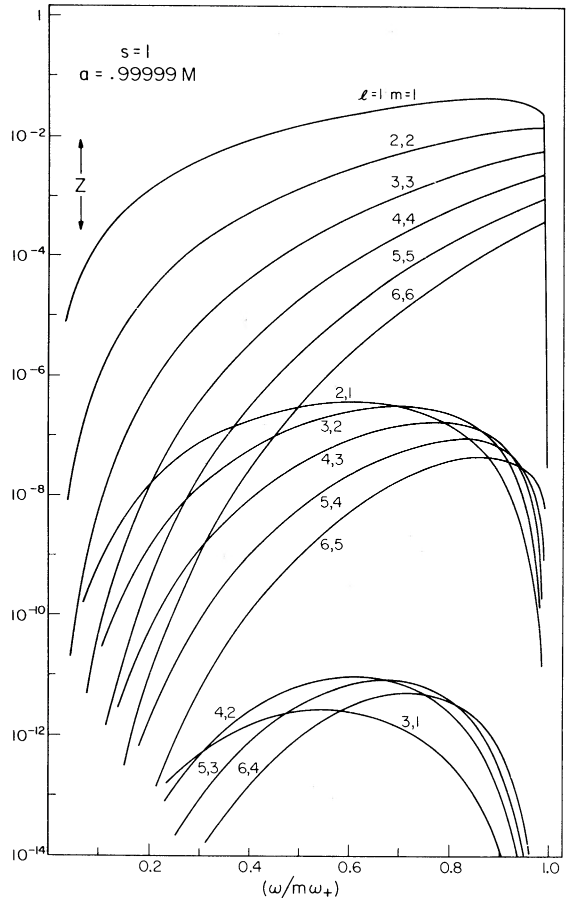

FIG. 3.—Amplification of electromagnetic ($s = 1$) waves by a near-maximally rotating black hole. Shown here are the 15 most strongly amplified modes. The fractional energy amplification Z (on a logarithmic scale) is plotted against the wave frequency ω, normalized to make the superradiant regime (0, 1). One sees here that the superradiant amplification is larger than 10^{-6} only for modes with $l = m$. The largest amplification occurs for $l = m = 1$ at $\omega \approx 0.88\omega_+$ and is 4.4 percent. These results would be only insensibly different for an extreme Kerr hole, $a = M$.

infinity, and use the energy conservation result of § V to determine the outgoing flux at infinity (whose numerical accuracy thus need not have been maintained in the outward integration). (By using a different radial equation, the numerical roles of outgoing and ingoing solutions at infinity can be exchanged.)

The eigenvalue of the angular equation ("spin-weighted spheroidal harmonics"), ${}_sE^m{}_l(a\omega)$, appears as a parameter in all forms of the radial perturbation equation. Paper II, which was concerned with gravitational stability, tabulated numerical values of this function for the case $s = \pm 2$. Here we are also interested in electromagnetic perturbations, so we give the eigenvalues for $s = \pm 1$ in table 2 (in the form of polynomial fits for the range $0 \leq a\omega \lesssim 3$; see [II: Appendix D] for further description).

One other remark concerning numerical methods ought to be made. In some cases one can avoid the change of representations of the equation in mid-integration, relying instead on a mathematical transformation of the equation to stabilize the solutions in the two asymptotic regimes $r \to r_+$ and $r \to \infty$. The following transformation has been used by J. M. Bardeen (private communication) with considerable success: Start with the radial equation (2.11) and use the nondimensionalization of Appendix A, equation (A2), to change this to the form of equation (A3). The solutions near the horizon $x \to 0$ are

$$R_{\text{ingoing}} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots ,$$

$$R_{\text{outgoing}} = \exp(2ikr_*) x^{-s}(1 + b_1 x + \cdots) , \tag{7.1}$$

where the a's and b's are straightforward to compute. At infinity the solution is

$$R \sim Z_{\text{in}} x^{-(2s+1)} + Z_{\text{out}} \exp(2i\omega r_*)/x , \tag{7.2}$$

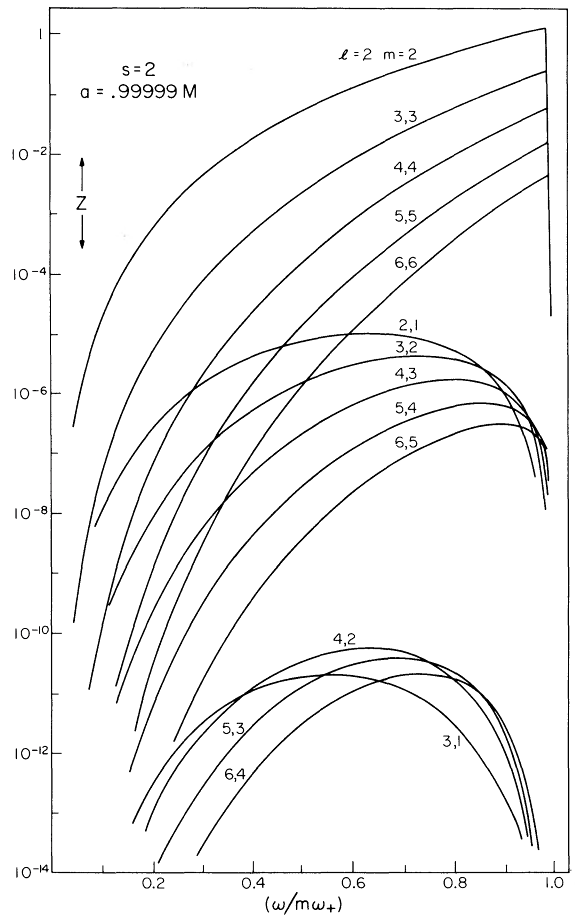

FIG. 4.—Amplification of gravitational ($s = 2$) waves by a near-maximally rotating black hole. As in fig. 3, only modes with $l = m$ achieve significant amplification. The maximum amplification $l = m = 2$ occurs at $\omega = 2\omega_+$ and is 138 percent.

where Z_{in} is related to inward energy flux by equations (A17). Choosing the negative choice for s makes Z_{in} stable and easily determined by an outward integration. However, for such an integration R_{outgoing} is unstable against contaminating the correct solution (only R_{ingoing} present). To counteract this, let

$$
\begin{aligned}
Y &= R - (1 + a_1 x)\,, & s &= -1\,, \\
&= R - (1 + a_1 x + a_2 x^2)\,, & s &= -2\,.
\end{aligned} \tag{7.3}
$$

Then, if equation (A3) is written as $\mathscr{L}R = 0$, Y satisfies the equation

$$\mathscr{L}Y = f\,, \tag{7.4}$$

where

$$
\begin{aligned}
f &= 2(\sigma - 4i\tau)a_2 x\,, & s &= -1 \\
&= 3(\sigma - 4i\tau)a_3 x^2 - 2i\hat{\omega}a_2 x^3\,, & s &= -2\,,
\end{aligned} \tag{7.5}
$$

and σ and τ are defined in equation (A2).

Equation (7.4) is now stably integrated through both asymptotic regimes, i.e., from $x = 0$ to $x = \infty$. Many numerical results which we present here have been independently obtained by Bardeen, agreeing with ours to the accuracy of the calculations, about 4 significant figures.

Our results are shown in figures 1–4, which plot the superradiance function Z for a number of modes of both gravitational and electromagnetic perturbations. The most interesting (because most amplified) modes are the lowest ones with $l = m$, and the behavior of these over a range of values of a is also shown. The superradiance is always greatest for extreme values of a, $a \to M$. The low modes achieve their maximum very near the limiting superradiant frequency $\omega \to m\omega_+$ (exactly at this frequency for $s = 2$ in the limit $a \to M$). However, there are some notable exceptions to this, e.g., the lowest electromagnetic mode $s = l = m = 1$ whose maximum of 4.4 percent amplification is achieved at $\omega \approx 0.88\omega_+$ for $a \to M$. Since there is never a positive superradiance for $m = 0$ modes (cf. § V), these modes do not appear in the figures. Their behavior for all a would be qualitatively like the $a = 0$ (Schwarzschild) modes which are plotted: perfect absorption $Z \to -1$ in the limit $|\omega| \to \infty$, and perfect reflection $Z \to 0$ in the limit $|\omega| \to 0$, where the long-wavelength wave diffracts around the "small" hole with vanishing attenuation.

VIII. CONCLUSIONS AND IMPLICATIONS

A most striking result is the dependence of the maximum superradiant amplification on the spin of the perturbing wave: 0.3 percent for scalar fields (Press and Teukolsky 1972), 4.4 percent for electromagnetic, and 138 percent for gravitational. In a classical context, the large value for gravitational waves is relevant to the question of "floating orbits" which will be treated in a later paper of this series.

There is also an interesting quantum-mechanical implication: Since a rotating black hole is a classical amplifier, it is also a spontaneous emitter of photons and gravitons (Zel'dovich 1971, 1972; Starobinsky 1973). Further, for boson fields, the details of the quantum emission can be quantitatively computed in terms of the classical superradiance function Z tabulated here. The fact that the classical amplification is of order unity then means that the emission goes as rapidly as its phase space will allow (Page 1974). We have not dealt with fermion fields (e.g., $s = \frac{1}{2}$) in this paper; for these there is no classical superradiance, and a treatment of the quantum process requires a second quantization procedure whose details are not yet generally agreed upon (Unruh 1973, 1974; [Hawking 1974 has reached some striking conclusions about the second-quantized *boson* field, but his treatment is also not known to be unique]).

Again in a classical context, the electromagnetic superradiance might possibly be important astrophysically. The effects of surrounding a black hole by a spherical conducting mirror have already been noted (Press and Teukolsky 1972). D. M. Eardley (private communication) has suggested the possibility that some more complicated —and more realistic—configuration of conducting plasma could also "impedance match" a growing mode and spontaneously extract energy from the hole. For example, coupling electromagnetic modes of the hole to the conducting plane of a hydrodynamical accretion disk might lead to a variety of interesting phenomena. Any energy extracted must ultimately go toward hydrodynamic bulk motions, or to thermal heating of the surrounding region, because the typical superradiant frequencies are far too low to propagate in the interstellar medium.

We can rule out the possibility that a cluster of black holes could be dense enough to lead to a "chain reaction" of gravitational wave emission. Superradiance of order unity means at best a (negative!) absorption cross-section of about the physical area of the hole, M^2. The condition that a cluster of size R containing N holes be supercritical thus depends on cross-section, radius, and number density in the usual way:

$$(M^2)R(N/R^3) \gtrsim 1 . \tag{8.1}$$

But this implies

$$(MN)/R \gtrsim N^{1/2} , \tag{8.2}$$

which says that the cluster is inside its own gravitational radius, an impossibility.

We are grateful to J. M. Bardeen and to A. A. Starobinsky for useful discussions.

APPENDIX A

SOLUTIONS OF THE RADIAL EQUATION FOR $a \to M, \omega \to m\omega_+$

Consider the radial equation (2.11) corresponding to the separation (2.6b) for Ω_s:

$$\Delta \frac{d^2R}{dr^2} + 2[(s + 1)(r - M) - iK]\frac{dR}{dr} - [2(2s + 1)i\omega r + \lambda]R = 0 . \tag{A1}$$

Introduce dimensionless variables

$$x = (r - r_+)/r_+ , \qquad \sigma = (r_+ - r_-)/r_+ , \qquad \tau = Mk , \qquad \hat{\omega} = \omega r_+ . \tag{A2}$$

Equation (A1) becomes

$$x(x + \sigma)R_{,xx} - \{2i\hat{\omega}x^2 + x[4i\hat{\omega} - 2(s + 1)] - (s + 1)\sigma + 4i\tau\}R_{,x} - [2(2s + 1)i\hat{\omega}(x + 1) + \lambda]R = 0 . \tag{A3}$$

The limits $a \to M$, $\omega \to m\omega_+$ correspond to $\sigma \to 0$, $\tau \to 0$. When $x \gg \max(\sigma, \tau)$, equation (A3) is approximately

$$x^2R_{,xx} - \{2i\hat{\omega}x^2 + x[4i\hat{\omega} - 2(s + 1)]\}R_{,x} - [2(2s + 1)i\hat{\omega}(x + 1) + \lambda]R = 0 . \tag{A4}$$

The solution of equation (A4) is

$$R = Ax^{-s-1/2+2i\omega+i\delta}\,{}_1F_1(\tfrac{1}{2} + s + 2i\hat{\omega} + i\delta, 1 + 2i\delta, 2i\hat{\omega}x) + B(\delta \to -\delta)\,, \tag{A5}$$

where A and B are constants and the notation $(\delta \to -\delta)$ means replace δ by $-\delta$ in the preceding term. Here

$$\delta^2 = 4\hat{\omega}^2 - (s + \tfrac{1}{2})^2 - \lambda\,. \tag{A6}$$

Using the asymptotic form of the confluent hypergeometric function for large x (e.g., Erdélyi *et al.* 1953, p. 278, eq. [2]), we find from equation (A5) that the coefficient of $x^{-(1+2s)}$ is

$$Z_{\rm in} = A\,\frac{\Gamma(1 + 2i\delta)}{\Gamma(\frac{1}{2} + i\delta - s - 2i\hat{\omega})}\left(\frac{e^{i\pi}}{2i\hat{\omega}}\right)^{1/2+s+2i\hat{\omega}+i\delta} + B\,(\delta \to -\delta)\,. \tag{A7}$$

When $x \ll 1$, equation (A3) becomes approximately

$$x(x + \sigma)R_{,xx} - \{x[4i\hat{\omega} - 2(s + 1)] - (s + 1)\sigma + 4i\tau\}R_{,x} - [2(2s + 1)i\hat{\omega} + \lambda]R = 0\,. \tag{A8}$$

The solution of equation (A8) is

$$R = {}_2F_1(-2i\hat{\omega} + s + \tfrac{1}{2} + i\delta, -2i\hat{\omega} + s + \tfrac{1}{2} - i\delta; 1 + s + i\kappa; -x/\sigma)\,. \tag{A9}$$

Here

$$\kappa = -4\tau/\sigma\,. \tag{A10}$$

The solution (A9) has been chosen to correspond to ingoing waves on the horizon ($R \sim$ constant at $x = 0$) with $Z_{\rm hole} = 1$.

The solutions (A5) and (A9) can be matched in the region of overlap, $\max(\sigma, \tau) \ll x \ll 1$. When $x \to 0$ in equation (A5),

$$R \to Ax^{-s-1/2+2i\omega+i\delta} + B\,(\delta \to -\delta)\,. \tag{A11}$$

When $x \to \infty$ in equation (A9) (e.g., Erdélyi *et al.* 1953, p. 108, eq. [2]),

$$R \to \frac{\Gamma(1 + s + i\kappa)\Gamma(2i\delta)}{\Gamma(-2i\hat{\omega} + s + \frac{1}{2} + i\delta)\Gamma(\frac{1}{2} + 2i\hat{\omega} + i\delta + i\kappa)}\left(\frac{x}{\sigma}\right)^{2i\hat{\omega}-s-1/2+i\delta} + (\delta \to -\delta)\,. \tag{A12}$$

Matching coefficients of $x^{-s-1/2+2i\hat{\omega}\pm i\delta}$ in equations (A11) and (A12) gives A and B, and then equation (A7) gives $Z_{\rm in}$. Let

$$\Gamma(1 + 2i\delta) = r_1 \exp(i\phi_1)\,, \qquad \Gamma(\tfrac{1}{2} + s - i\delta + 2i\hat{\omega}) = r_2 \exp(i\phi_2)\,,$$
$$\Gamma(\tfrac{1}{2} + s - i\delta - 2i\hat{\omega}) = r_3 \exp(i\phi_3)\,, \qquad \Gamma(\tfrac{1}{2} - i\delta - 2i\hat{\omega} - i\kappa) = r_4 \exp(i\phi_4)\,,$$
$$\Gamma(\tfrac{1}{2} - i\delta + 2i\hat{\omega} + i\kappa) = r_5 \exp(i\phi_5)\,. \tag{A13}$$

Assume $\delta^2 > 0$, i.e., δ real. Then find

$$|Z_{\rm in}|^2 = G(s)/Z \tag{A14}$$

where

$$G(s) = \kappa\left(\frac{\sigma}{2\hat{\omega}}\right)^{2s+1}(s!)^2\prod_{n=1}^{s}\left[1 + \left(\frac{\kappa}{n}\right)^2\right], \qquad s \geq 0\,,$$
$$= \left\{\left(\frac{\sigma}{2\hat{\omega}}\right)^{2s+1}\prod_{n=1}^{|s|-1}\left[1 + \left(\frac{\kappa}{n}\right)^2\right]^{-1}\right\}\Big/\{\kappa[(|s| - 1)!]^2\}\,, \qquad s < 0\,. \tag{A15}$$

Here Z is given by equation (6.5) and ψ in equation (6.5) is

$$\psi = 4\phi_1 + 2\phi_2 + 2\phi_3 + \phi_4 + \phi_5 - 2\delta \ln 2\hat{\omega}\sigma\,. \tag{A16}$$

The factors in $G(s)$ are exactly those found in § IV to convert $|Z_{\rm in}|^2$ to Z, with $Z_{\rm hole}$ normalized to unity and the appropriate transformations made from Kinnersley's tetrad. Explicitly, these formulae are

$$Z = \frac{-2k}{\hat{\omega}|Z_{\rm in}|^2} \qquad \text{(scalar)} ,$$

$$Z = \frac{-\hat{\omega}}{2k|Z_{\rm in}|^2} \qquad \text{(e.m., } s = -1\text{)} ,$$

$$Z = -\frac{\hat{\omega}^3}{8k(k^2 + 4\epsilon^2)|Z_{\rm in}|^2} \qquad \text{(grav., } s = -2\text{)} . \tag{A17}$$

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