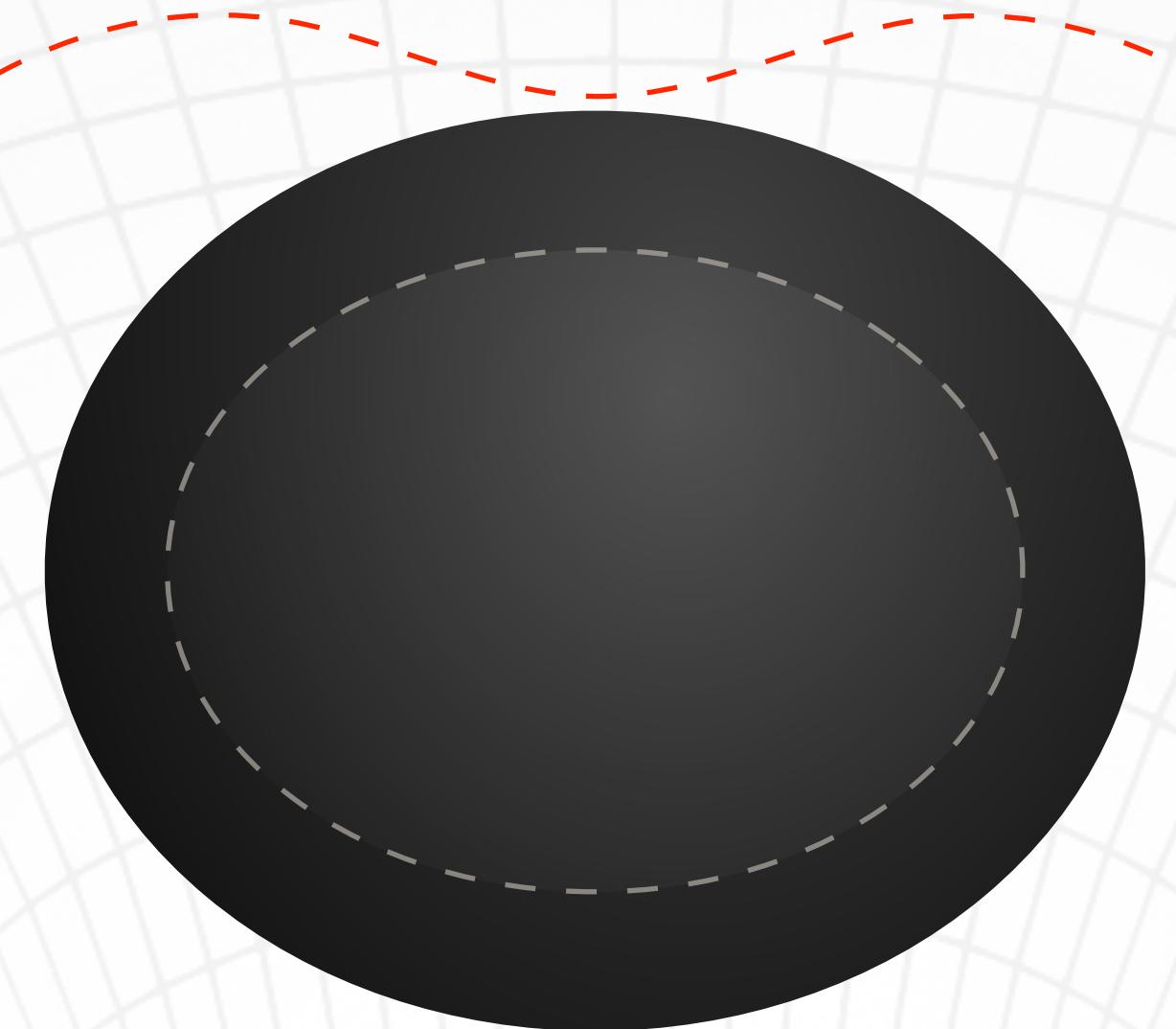




Resources



Lionel London

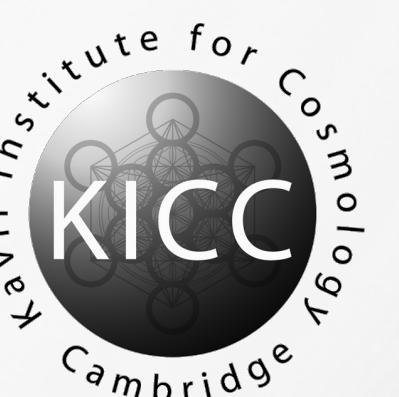
An introduction to black hole perturbation theory

Kavli-Villum Summer School

September 2023



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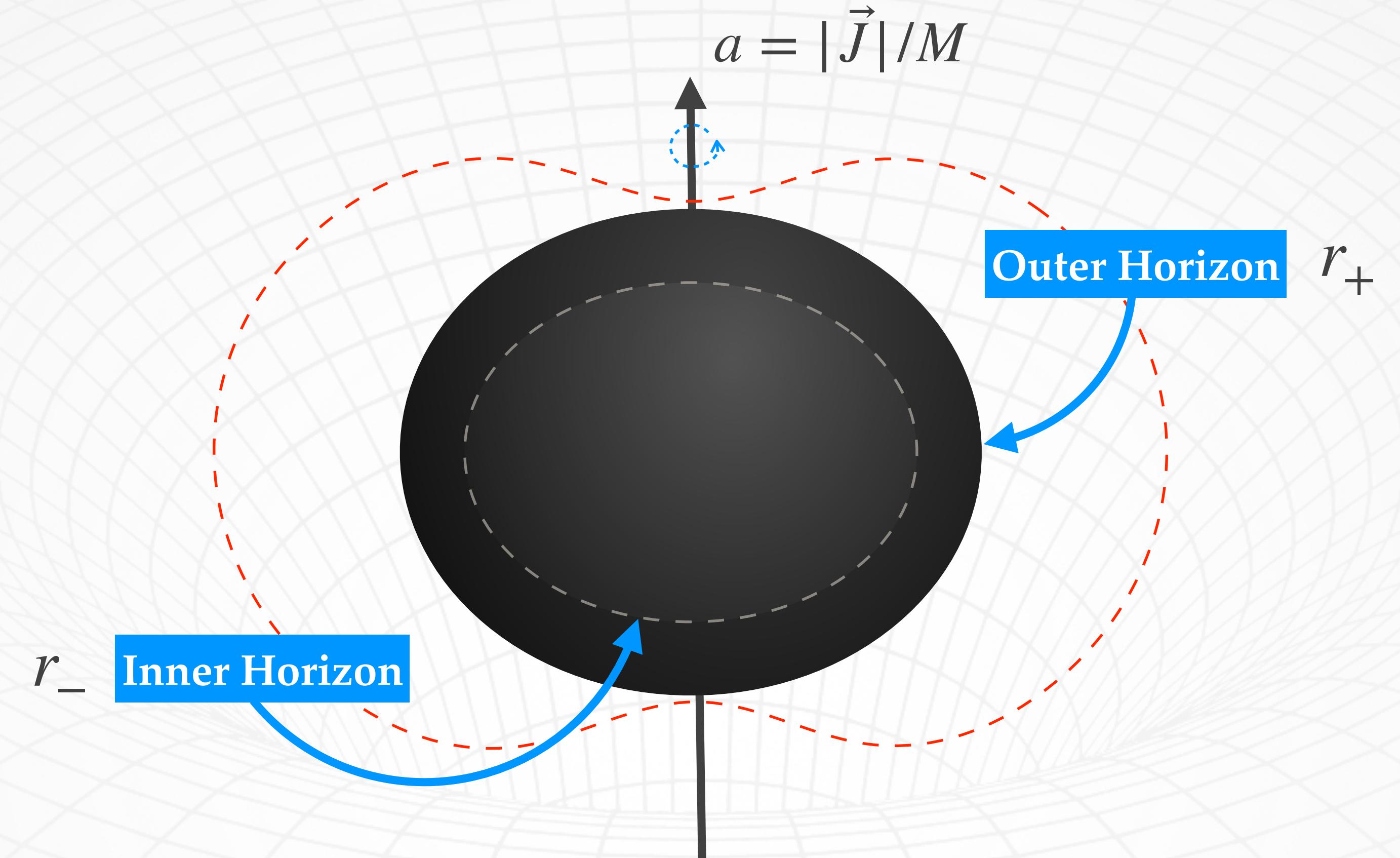


Corfu





Resources

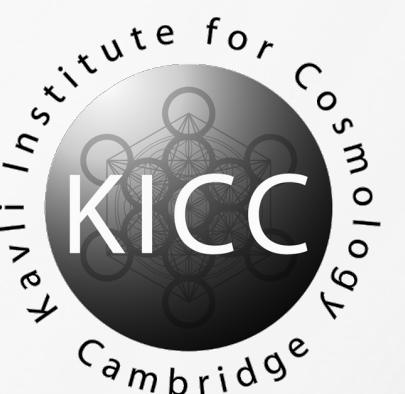


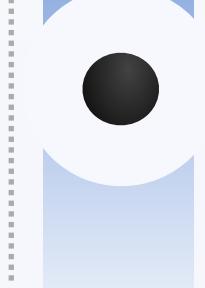
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An introduction to black hole perturbation theory



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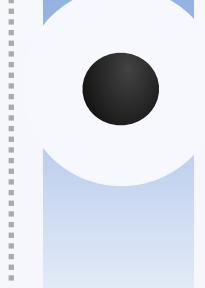




Outline Introduction to BH perturbation theory 2

Today's topics

- ❖ Teukolsky's equations, recap and overview
- ❖ The road towards QNM solutions for Kerr
 - Teukolsky's angular equations, and analytic methods of solution
 - Teukolsky's radial equation
 - A refined (spatially compactified) coordinate choice
 - Derivation of asymptotic boundary conditions
 - A basic series solution
- ❖ Computing QNM frequencies by combining radial and angular recursion relations
- ❖ A primer on QNM orthogonality
- ❖ Questions / Hands on in Mathematica to review points above in more detail

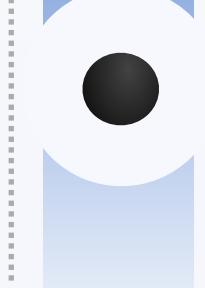


Einstein's equations linearized around Kerr

$$\begin{aligned}
 & \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] \frac{\partial^2 \psi}{\partial t^2} + \frac{4Mar}{\Delta} \frac{\partial^2 \psi}{\partial t \partial \varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right] \frac{\partial^2 \psi}{\partial \varphi^2} \\
 & - \Delta^{-s} \frac{\partial}{\partial r} \left(\Delta^{s+1} \frac{\partial \psi}{\partial r} \right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) - 2s \left[\frac{a(r - M)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right] \frac{\partial \psi}{\partial \varphi} \\
 & - 2s \left[\frac{M(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right] \frac{\partial \psi}{\partial t} + (s^2 \cot^2 \theta - s)\psi = 4\pi \Sigma T.
 \end{aligned}$$

Teukolsky's “master” equation

Separate with ansatz

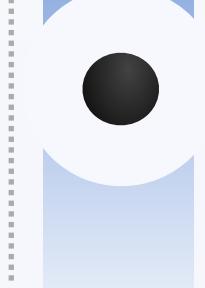


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 \end{aligned}$$

Teukolsky's “master” equation

Separate with ansatz



Einstein's equations linearized around Kerr

BH Spin parameter = $|J|/M$

$$\begin{aligned}
 & \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] \frac{\partial^2 \psi}{\partial t^2} + \frac{4Mar}{\Delta} \frac{\partial^2 \psi}{\partial t \partial \varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right] \frac{\partial^2 \psi}{\partial \varphi^2} \\
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 \end{aligned}$$

BH Spin parameter = $|J|/M$

$\xrightarrow{(r-r_m)(r-r_p)}$

Teukolsky's "master" equation

Separate with ansatz

Einstein's equations linearized around Kerr

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 \end{aligned}$$

BH Spin parameter = $|J|/M$

$(r-r_m)(r-r_p)$ → (r-rm)(r-rp)

spin weight → spin weight

Teukolsky's “master” equation

Separate with ansatz

Einstein's equations linearized around Kerr

1970s

Teukolsky's equation has far reaching applications

“spin coefficient”
 $\rho = -1/(r - i a \cos \theta)$

Teukolsky 1971	
ψ	s
Φ	0
$\frac{\chi_0}{\rho^{-1}} \chi_1$	$-\frac{1}{2}$
$\frac{\phi_0}{\rho^{-2}} \phi_2$	-1
$\frac{\psi_0^B}{\rho^{-4}} \psi_4^B$	-2

scalar fields

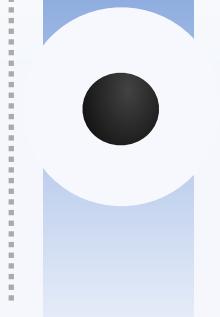
neutrinos

electromagnetic fields

gravitational fields

Teukolsky's “master” equation

Separate with ansatz



Towards QNM Solutions

The full form of a **single** QNM

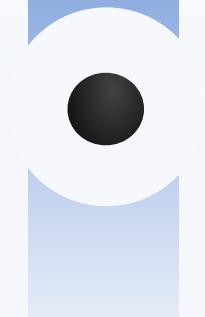
$$\psi \propto R(r) S(\theta) e^{-i\tilde{w}t} e^{-im\phi}$$

In more general scenarios, one should perform an integral transform.

ukolsky's "master" equation

Separate with ansatz

Teukolsky's Equat



Towards QNM Solutions

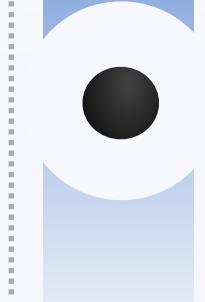
$$\mathcal{L}_u S(u) = -A \overset{u = \cos(\theta)}{\downarrow} S(u)$$
$$\mathcal{L}_r R(r) = +A R(r)$$

The QNMs' radial and angular functions are special functions that, when solved simultaneously, fully constrain a Kerr QNM

Separate with ansatz

Teukolsky's Equations

The Angular Equation



Towards QNM Solutions

$$\mathcal{L}_u = V_S(u) + \partial_u(1 - u^2)\partial_u$$

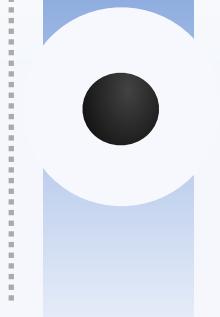
Having chosen u rather than theta, the angular equation is explicitly in “**Sturm-Liouville form**”.

- ❖ The eigenfunctions of Sturm-Liouville problems are known to be **orthogonal** and **complete** (Functional analysis creeping in)
- ❖ The problem’s potential is *almost* of the Legendre type

Leukolsky’s Equations

The Angular Equation

Spheroidal Harmonic



Towards QNM Solutions

$$\mathcal{L}_u = V_S(u) + \partial_u(1 - u^2)\partial_u$$

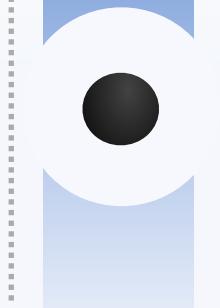
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Leukolsky’s Equations

The Angular Equation

Spheroidal Harmonic



Towards QNM Solutions

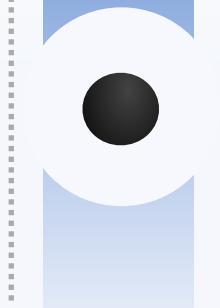
$$V_S(u) = s + u a \tilde{w} (u a \tilde{w} - 2s) - \frac{(m + su)^2}{1 - u^2}$$

- ❖ The problem's potential is *almost* of the Legendre type
- ❖ The Legendre-like part is the rational function
- ❖ The additional quadratic dependence on the domain variable add an **irregular singular point at $u \sim \text{infinity}$**
- ❖ The frequency parameter also appears quadratically

Teukolsky's Equations

The Angular Equation

Spheroidal Harmonic



Towards QNM Solutions

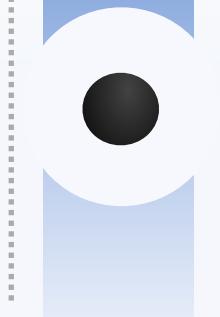
$$V_S(u) = s + ua\tilde{w}(ua\tilde{w} - 2s) - \frac{(m + su)^2}{1 - u^2}$$

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Teukolsky's Equations

The Angular Equation

Spheroidal Harmonic



Towards QNM Solutions

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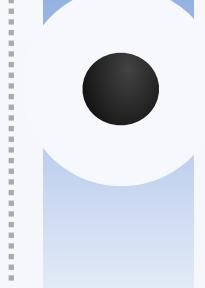
$$V_S(u) = s + ua\tilde{w}(ua\tilde{w} - 2s) - \frac{(m + su)^2}{1 - u^2}$$

- ❖ The angular problem has two regular singular points, and one irregular one, making it a “confluent Heun” equation
- ❖ A hypergeometric equation (e.g. the Legendre equation) has at most three regular singular points (“confluence” reduces this number)

Leukolsky's Equations

The Angular Equation

Spheroidal Harmonic



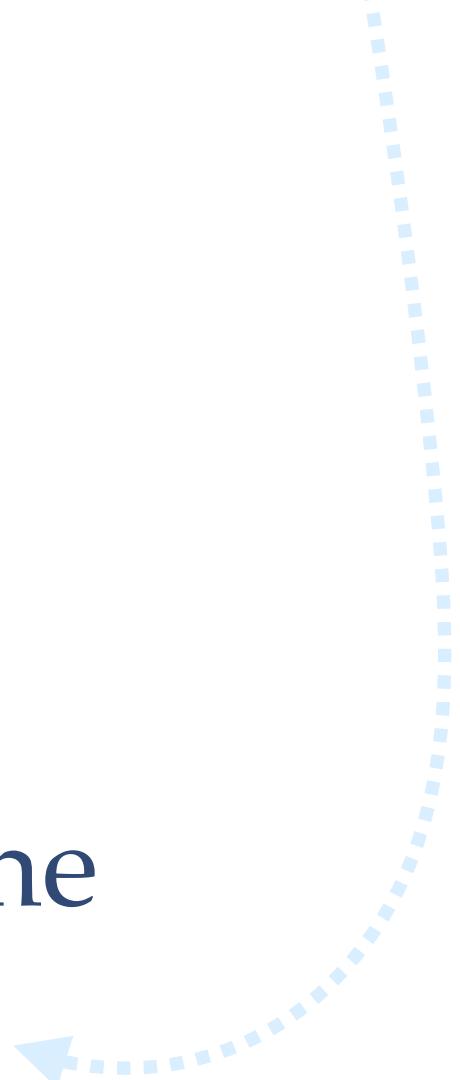
Towards QNM Solutions

$$u = \cos(\theta) \leq 1 \ll \infty$$

$$\mathcal{L}_u = V_S(u) + \partial_u(1 - u^2)\partial_u$$

$$V_S(u) = s + ua\tilde{w}(ua\tilde{w} - 2s) - \frac{(m + su)^2}{1 - u^2}$$

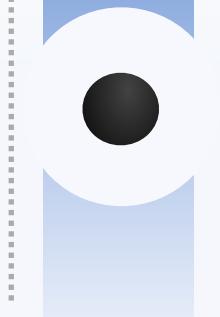
- ❖ The angular equation is effectively hypergeometric because the confluent singular point is outside of the physical domain.
- ❖ As a result, we can construct analytic solutions in *almost* the same way that we can for e.g. the Legendre polynomials



Teukolsky's Equations

The Angular Equation

Spheroidal Harmonic



Towards QNM Solutions

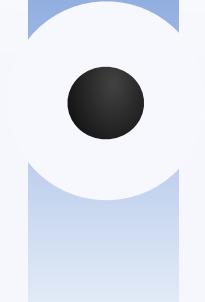
$$\mathcal{L}_u S(u) = -A S(u)$$

- ❖ The **spheroidal harmonics** are eigenfunctions of the angular differential operator
- ❖ As a result, we can construct analytic solutions in *almost* the same way that we can for e.g. the Legendre polynomials

The Angular Equation

Spheroidal Harmonics

The Radial Equation



Towards QNM Solutions

Leaver (1985), “Analytic representation for the modes of Kerr black holes”

$$S(u) = e^{\kappa_\infty u} (1-u)^{\kappa_-} (1+u)^{\kappa_+}$$

Frobenious series solution

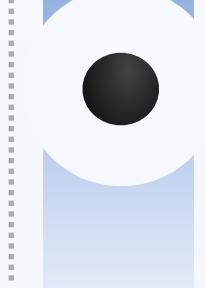
$$\sum_{k=0}^{\infty} a_k (1+u)^k$$

- ❖ A “Frobenious” expansion is sufficient to form a global solution
- ❖ The prefactor function represents a “**similarity**”, “**conformal**” or “**holomorphic**” transformation on the original problem
- ❖ Eigenvalues are preserved

The Angular Equation

Spheroidal Harmonics

The Radial Equation



Towards QNM Solutions

Leaver (1985), “Analytic representation for the modes of Kerr black holes”

$$S(u) = e^{\kappa_\infty u} (1 - u)^{\kappa_-} (1 + u)^{\kappa_+} \sum_{k=0}^{\infty} a_k (1 + u)^k$$

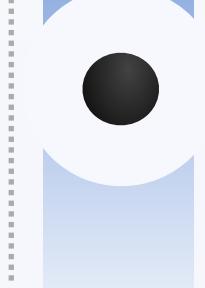
“conformal factor”

- ❖ A “Frobenious” expansion is sufficient to form a global solution
- ❖ The prefactor function represents a “**similarity**”, “**conformal**” or “**holomorphic**” transformation on the original problem
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The Angular Equation

Spheroidal Harmonics

The Radial Equation



Towards QNM Solutions

Leaver (1985), “Analytic representation for the modes of Kerr black holes”

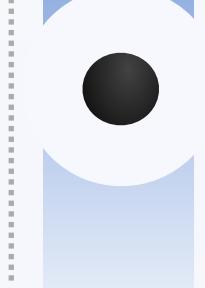
$$S(u) = e^{\kappa_\infty u} (1-u)^{\kappa_-} (1+u)^{\kappa_+} \sum_{k=0}^{\infty} a_k (1+u)^k$$

- ❖ The constants, $\{\kappa_-, \kappa_+, \kappa_\infty\}$, can be determined by **regularity conditions** and/or “asymptotic” boundary conditions
- ❖ Regularity conditions: enforcing that the potential is at most linear
- ❖ Asymptotic boundary conditions: physical requirements on functional form near boundaries. Here: only that the functions are finite at $u=+1$

The Angular Equation

Spheroidal Harmonics

The Radial Equation



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

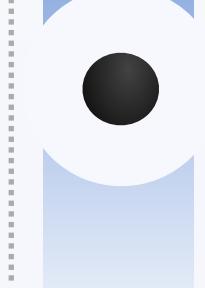
$$S(u) = e^{\kappa_\infty u} (1-u)^{\kappa_-} (1+u)^{\kappa_+} \sum_{k=0}^{\infty} a_k (1+u)^k$$

- ❖ The expansion coefficients, a_k , are determined by a **three-term recursion** relation that **depends on the separation constant A**
- ❖ For convergence of the infinite sum, **the separation constant can only take on special discrete values ultimately label in ℓ** ... more on this later

The Angular Equation

Spheroidal Harmonics

The Radial Equation



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\mathcal{L}_r R(r) = +A R(r)$$

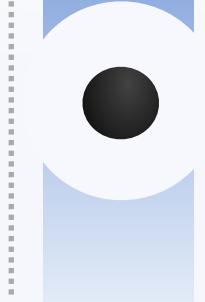
Some connections and tensions with the angular equation:

- ❖ Also 3 singular points, one confluent. So **it's also a confluent Heun equation.**
- ❖ But **the confluent singular point (at $r\sim\text{infinity}$) is inside of the domain this time!**

Spheroidal Harmonics

The Radial Equation

Homogeneous Solutions



Towards QNM Solutions

Leaver (1985), “Analytic representation for the modes of Kerr black holes”

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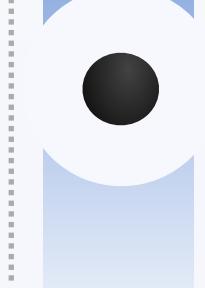
Some connections and tensions with the angular equation:

- ❖ It's **not clear** whether there is a simple coordinate choice that puts the problem in Sturm-Liouville form.
- ❖ Thus is has **not been clear** whether there the “homogeneous functions” R , are orthogonal and complete (**very recent progress**)

Spheroidal Harmonics

The Radial Equation

Homogeneous Solutions



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\begin{aligned}\mathcal{L}_r = & \left(A_0 + A_1 r + A_2 r^2 + \frac{A_3}{r - r_-} + \frac{A_4}{r - r_+} \right) \\ & + (A_5 + A_6 r) \partial_r \\ & + (r - r_-)(r - r_+) \partial_r^2\end{aligned}$$

A part of the confusion is due to coordinate choice: The physical boundary conditions are defined in the vicinity of r -plus and $r \sim \text{Infinity}$, **not at r -minus**. However, the Boyer-lindquist coordinate doesn't reflect this.

Spheroidal Harmonics

The Radial Equation

Homogeneous Solutions



Towards QNM Solutions

Leaver (1985), “Analytic representation for the modes of Kerr black holes”

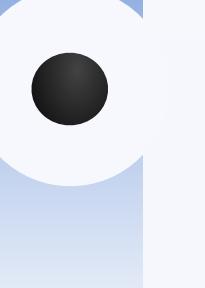
$$\xi = \frac{r - r_+}{r - r_-}$$

Taking inspiration from the quantum mechanics literature, Leaver introduced an alternative coordinate, ξ (pronounced “ksai”).

Spheroidal Harmonics

The Radial Equation

Homogeneous Solutions



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\xi = \frac{r-r_+}{r-r_-}$$

$$\begin{aligned} \mathcal{L}_\xi = & \left(B0 + \frac{B1}{\xi} + \frac{B2}{1-\xi} + \frac{B3^2}{(1-\xi)^2} + B4(1-\xi) \right) \\ & + (B5(1-\xi) + B6(1-\xi)^2) \partial_\xi + \xi(1-\xi)^2 \partial_\xi^2 \end{aligned}$$

This coordinate results in a different form of the radial operator.

Spheroidal Harmonics

The Radial Equation

Homogeneous Solutions



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\xi = \frac{r-r_+}{r-r_-}$$

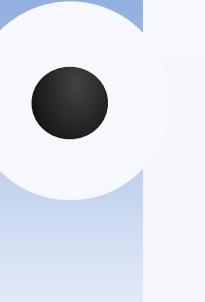
$$\begin{aligned} \mathcal{L}_\xi = & \left(B_0 + \frac{B_1}{\xi} + \frac{B_2}{1-\xi} + \frac{B_3^2}{(1-\xi)^2} + B_4(1-\xi) \right) \\ & + (B_5(1-\xi) + B_6(1-\xi)^2) \partial_\xi + \xi(1-\xi)^2 \partial_\xi^2 \end{aligned}$$

This new operator is not in Sturm-Liouville form, but its second derivative coefficient is zero at the event horizon and spatial infinity ... (more on this later)

Spheroidal Harmonics

The Radial Equation

Homogeneous Solutions



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

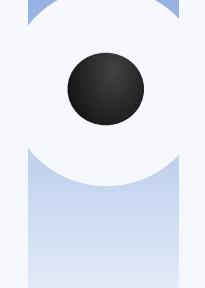
$$R(\xi) = e^{\sigma_\infty \xi} \xi^{\sigma_0} (1 - \xi)^{\sigma_1} \sum_{k=0}^{\infty} b_k \xi^k$$

In 1985, Leaver sought a series solution to the radial equation using the same concepts that were applied to the angular problem.

The Radial Equation

Homogeneous Solutions

QNM Boundary Cond



Towards QNM Solutions

Leaver (1985), “Analytic representation for the modes of Kerr black holes”

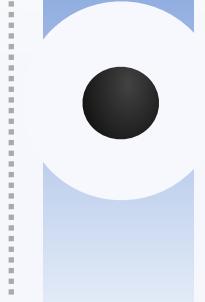
$$R(\xi) = e^{\sigma_\infty \xi} \xi^{\sigma_0} (1 - \xi)^{\sigma_1} \sum_{k=0}^{\infty} b_k \xi^k$$

As with the angular problem, the constants, $\{\sigma_-, \sigma_+, \sigma_\infty\}$, may be determined by enforcing **physical (asymptotic) boundary conditions**.

Inhomogeneous Solutions

QNM Boundary Conditions

Three-term recursio



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$R(\xi) = e^{\sigma_\infty \xi} \xi^{\sigma_0} (1 - \xi)^{\sigma_1} \sum_{k=0}^{\infty} b_k \xi^k$$

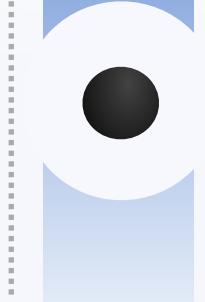
Asymptotic boundary conditions for the radial problem:

- ❖ $R(r \rightarrow \infty)$ is a **purely outgoing wave** (towards spatial infinity)
- ❖ $R(r \rightarrow 0)$ is a **purely ingoing wave** (towards the event horizon)

Inhomogeneous Solutions

QNM Boundary Conditions

Three-term recursio



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

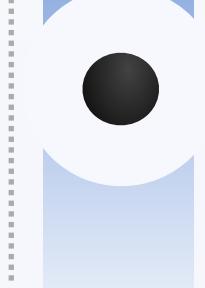
$$R(\xi) = e^{\sigma_\infty \xi} \xi^{\sigma_0} (1 - \xi)^{\sigma_1} \sum_{k=0}^{\infty} b_k \xi^k$$

Explicit function forms of the boundary conditions may be worked out via **asymptotic analysis of the differential equation**, or by **inspection of phase velocities** after allowed values of sigmas have been determined algebraically. Leaver plugged and chugged ...

Inhomogeneous Solutions

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Three-term recursio



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$R(\xi) = e^{\sigma_\infty \xi} \xi^{\sigma_0} (1 - \xi)^{\sigma_1} \sum_{k=0}^{\infty} b_k \xi^k$$

As with the angular problem, b_k , are only valid if they satisfy a special three-term recursion relation and its discrete eigenvalues are ultimately labeled in n . So we now have two such relations that must be solved simultaneously

Inhomogeneous Solutions

QNM Boundary Conditions

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Towards QNM Solutions

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$$\alpha_k^u a_{k+1} + \beta_k^u a_k + \gamma_k^u a_{k-1} = 0$$

$$\alpha_0^u a_1 + \beta_0^u a_0 = 0$$

From the angular problem

$$\alpha_k^\xi b_{k+1} + \beta_k^\xi b_k + \gamma_k^\xi b_{k-1} = 0$$

$$\alpha_0^\xi b_1 + \beta_0^\xi b_0 = 0$$

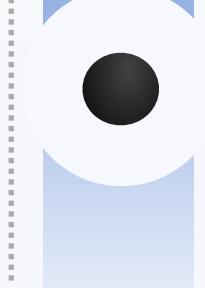
From the radial problem

- ❖ Each alpha, beta, gamma coefficient depends on **BH spin** and **frequency parameter**
- ❖ Only the beta coefficients depend on the separation constant, A

NM Boundary Conditions

Three-term recursions

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Towards QNM Solutions

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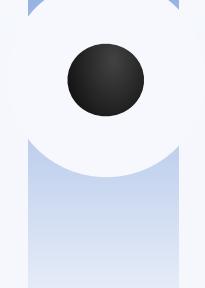
From the radial problem

Three-term recursions have two solutions: one is an absolutely convergent series, and the other is not. The absolutely convergent one is sometimes called “**minimal**” and it’s the one we want — i.e. physical series solutions converge.

NM Boundary Conditions

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$$\alpha_0^\xi b_1 + \beta_0^\xi b_0 = 0$$

From the radial problem

The requirement of convergence can be used to determine eigenvalues of the radial and angular problems separately. We can then think about them together.

NM Boundary Conditions

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Towards QNM Solutions

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From the angular problem

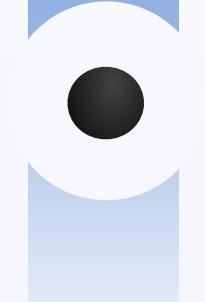
$$R_k^u = \frac{a_k}{a_{k-1}}$$

For the angular problem, we may **construct a continued fraction by considering the ratio of adjacent terms**. An easy way: Define the ratio, and then plug in a_{k-1}

NM Boundary Conditions

Three-term recursions

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Towards QNM Solutions

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From the angular problem

$$R_k^u = -\frac{a_k \gamma_k^u}{a_k \beta_k^u + a_{k+1} \alpha_k^u}$$

For the angular problem, we may **construct a continued fraction by considering the ration of adjunct terms**. An easy way: Define the ratio, and then **plug in $a(k-1)$**

NM Boundary Conditions

Three-term recursions

QNM Eigenvalues



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\begin{aligned}\alpha_k^u a_{k+1} + \beta_k^u a_k + \gamma_k^u a_{k-1} &= 0 \\ \alpha_0^u a_1 + \beta_0^u a_0 &= 0\end{aligned}$$

From the angular problem

$$R_k^u = -\frac{a_k \gamma_k^u}{a_k \beta_k^u + a_{k+1} \alpha_k^u}$$

And then divide the numerator and denominator by $a(k)$

NM Boundary Conditions

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$$\begin{aligned}\alpha_k^u a_{k+1} + \beta_k^u a_k + \gamma_k^u a_{k-1} &= 0 \\ \alpha_0^u a_1 + \beta_0^u a_0 &= 0\end{aligned}$$

From the angular problem

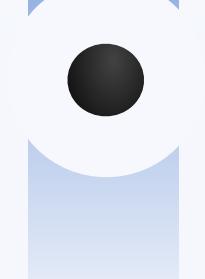
$$R_k^u = - \frac{\gamma_k^u}{\beta_k^u + R_{k+1}^u \alpha_k^u}$$

From here, one can see that **iteratively substituting in R_k^u** creates a fraction within a fraction within a fraction ... i.e. **a continued fraction**.

NM Boundary Conditions

Three-term recursions

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Towards QNM Solutions

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From the angular problem

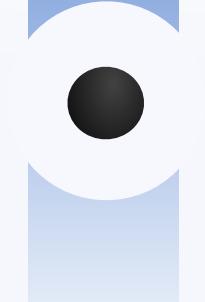
$$R_1^u = -\frac{a_1}{a_0}$$

In turn, this means that the recursion's boundary condition at zero can be framed to explicitly include all recursions ...

NM Boundary Conditions

Three-term recursions

QNM Eigenvalues



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\begin{aligned}\alpha_k^u a_{k+1} + \beta_k^u a_k + \gamma_k^u a_{k-1} &= 0 \\ \alpha_0^u a_1 + \beta_0^u a_0 &= 0\end{aligned}$$

From the angular problem

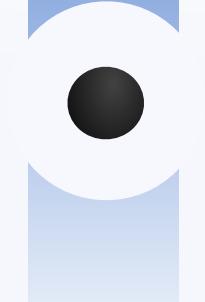
$$R_1^u = \frac{a_1}{a_0} = - \frac{\gamma_1^u}{\beta_1^u + R_2^u \alpha_1^u}$$

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NM Boundary Conditions

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Towards QNM Solutions

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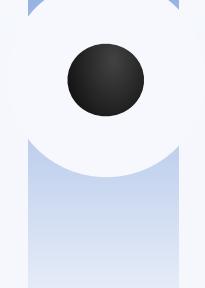
$$\left| a_1 + a_0 \frac{\gamma_1^u}{\beta_1^u + R_2^u \alpha_1^u} \right| = 0$$

This allows for the definition of a kind of generalized “**characteristic equation**”. Keep in mind that R_2 is to be continued to include all higher R_k .

NM Boundary Conditions

Three-term recursions

QNM Eigenvalues



Towards QNM Solutions

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From the angular problem

$$\left| a_1 + a_0 \frac{\gamma_1^u}{\beta_1^u + R_2^u \alpha_1^u} \right| = 0$$

The characteristic equation can be thought of a polynomial with **discrete roots that only occurs at special values of the separation constant A.**

NM Boundary Conditions

Three-term recursions

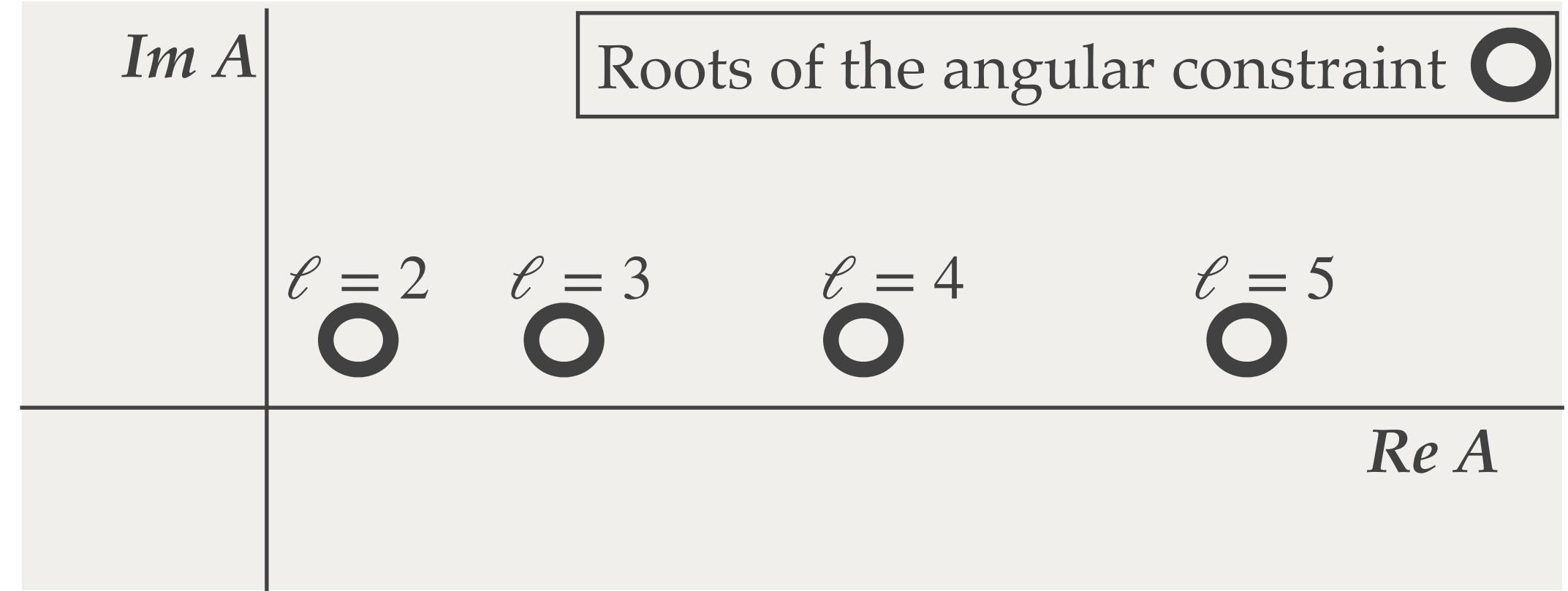
QNM Eigenvalues



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

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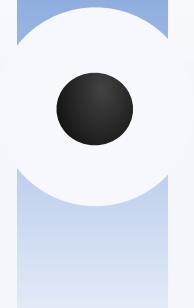


The characteristic equation can be thought of a polynomial with **discrete roots that only occurs at special values of the separation constant A .**

NM Boundary Conditions

Three-term recursions

QNM Eigenvalues



Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

Angular problem

$$\left| a_1 + a_0 \frac{\gamma_1^u}{\beta_1^u + R_2^u \alpha_1^u} \right| = 0$$

Radial problem

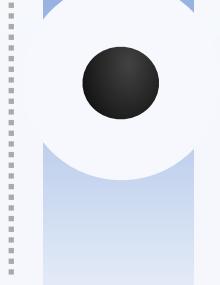
$$\left| b_1 + b_0 \frac{\gamma_1^\xi}{\beta_1^\xi + R_2^\xi \alpha_1^\xi} \right| = 0$$

Recall that we also have a three-term recursion for the **radial problem**.
 This has its own separate continued fraction.

NM Boundary Conditions

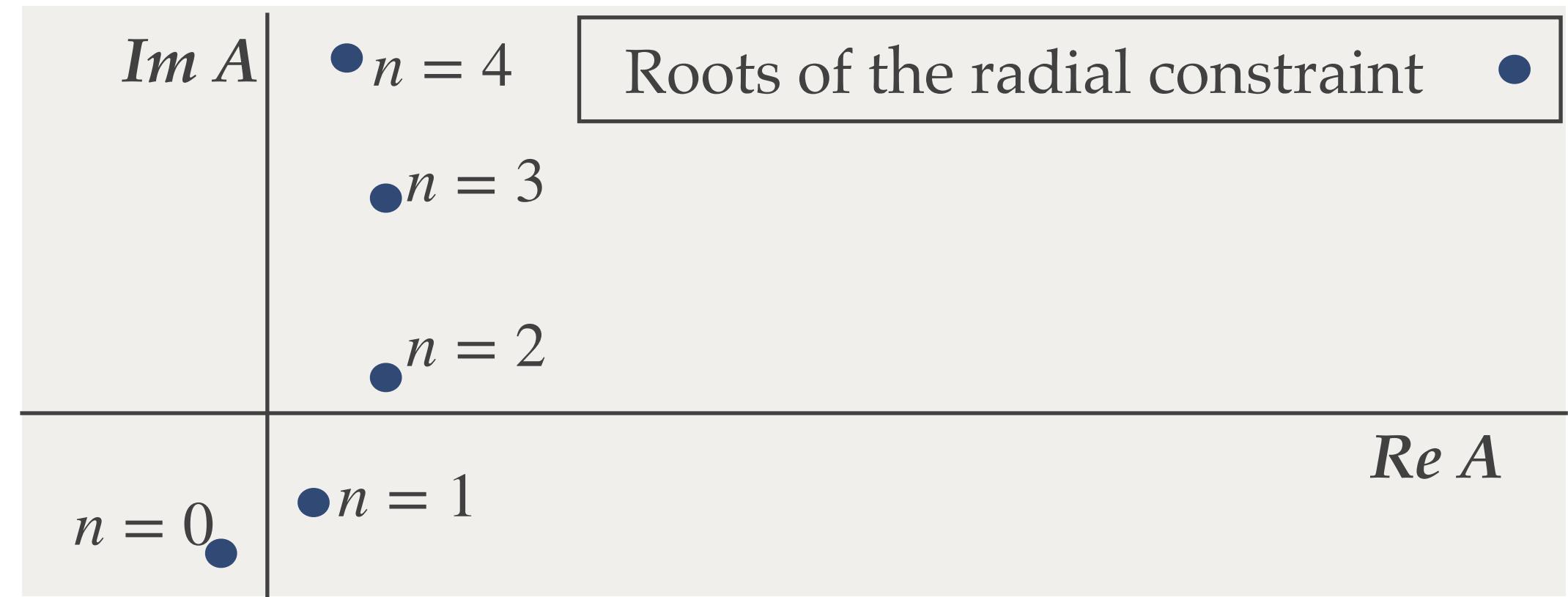
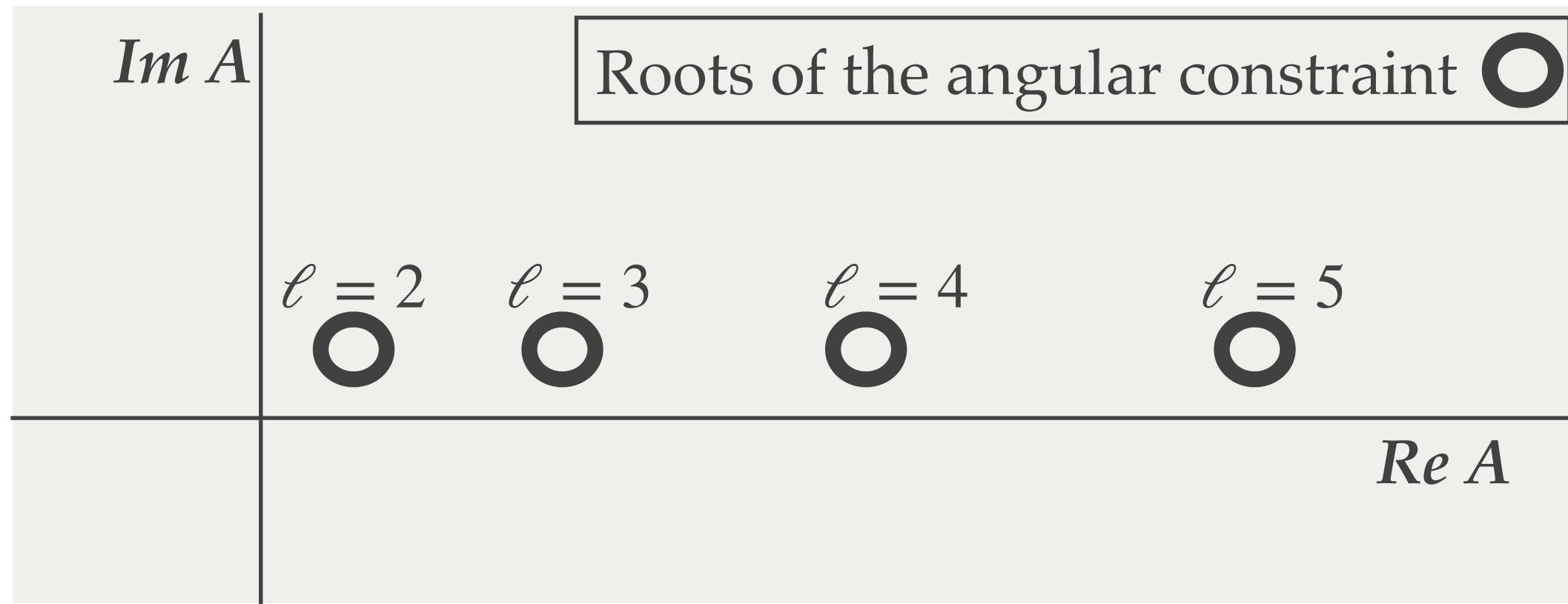
Three-term recursions

QNM Eigenvalues



Towards QNM Solutions

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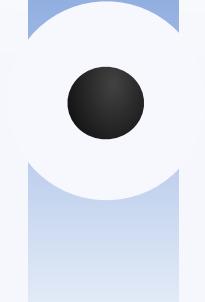


The characteristic equation can be thought of a polynomial with **discrete roots that only occur at special values of the separation constant A** .

Three-term recursions

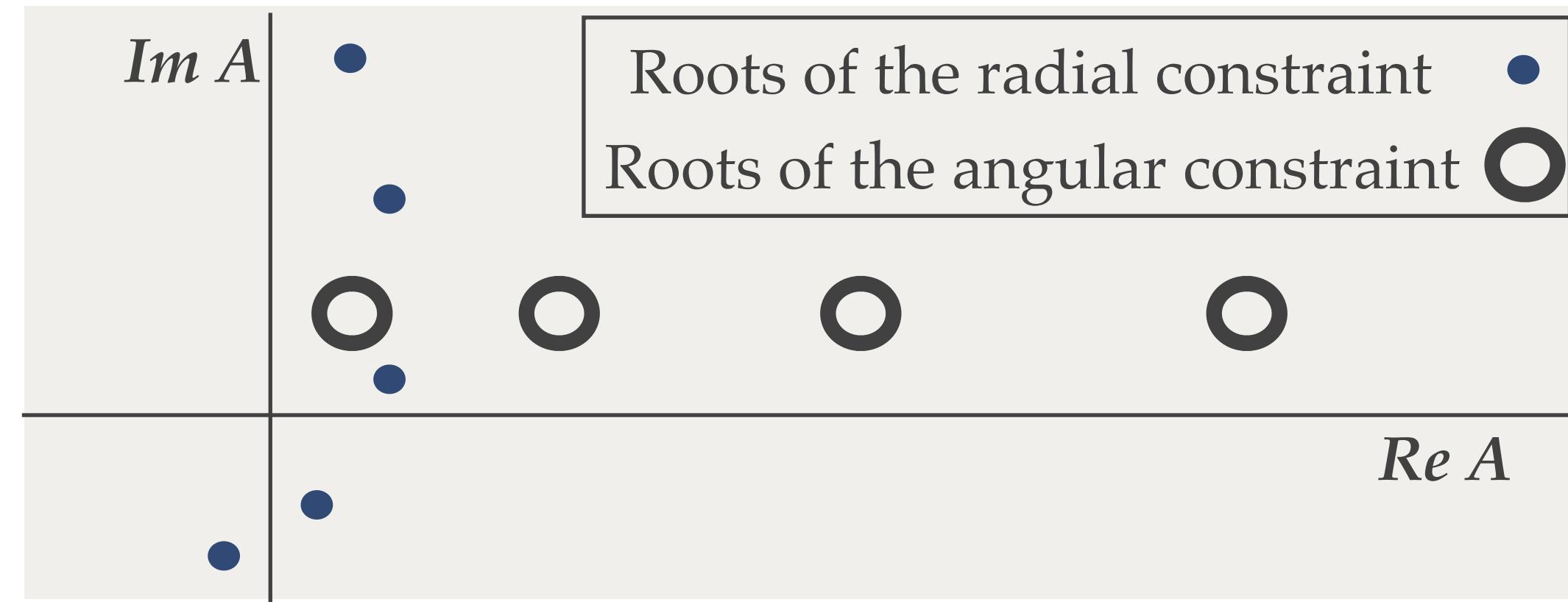
QNM Eigenvalues

The QNM Frequencies



Towards QNM Solutions

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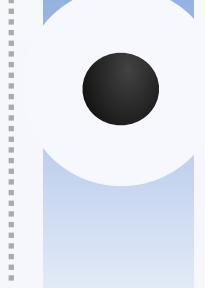


For arbitrary frequencies, the two eigenvalue distributions will not necessarily coincide. That is, we may think of the separation constants as: $A=A(a, \mathbf{w}, l, m, n)$

Three-term recursions

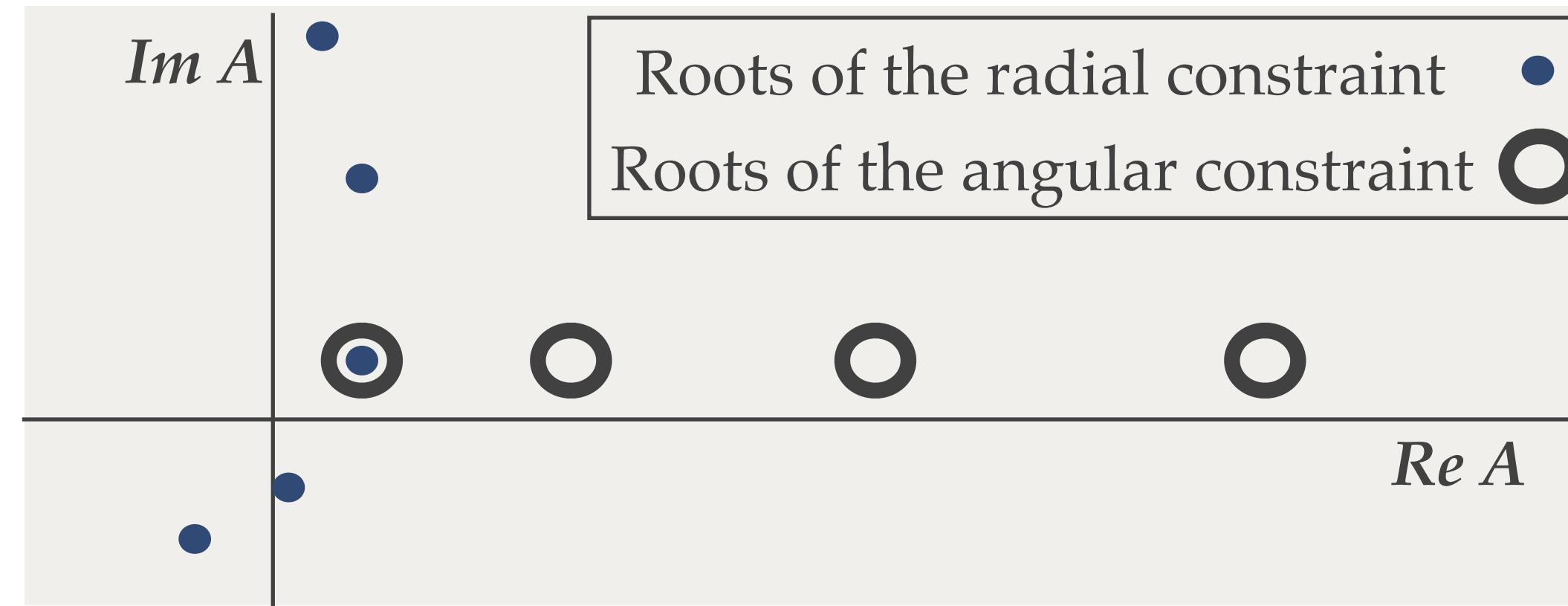
QNM Eigenvalues

The QNM Frequencies



Towards QNM Solutions

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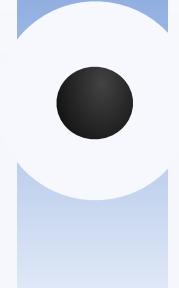


Since the radial and angular values of A must be identical for physical solutions, w may only take on special discrete values — these are the complex valued QNM frequencies.

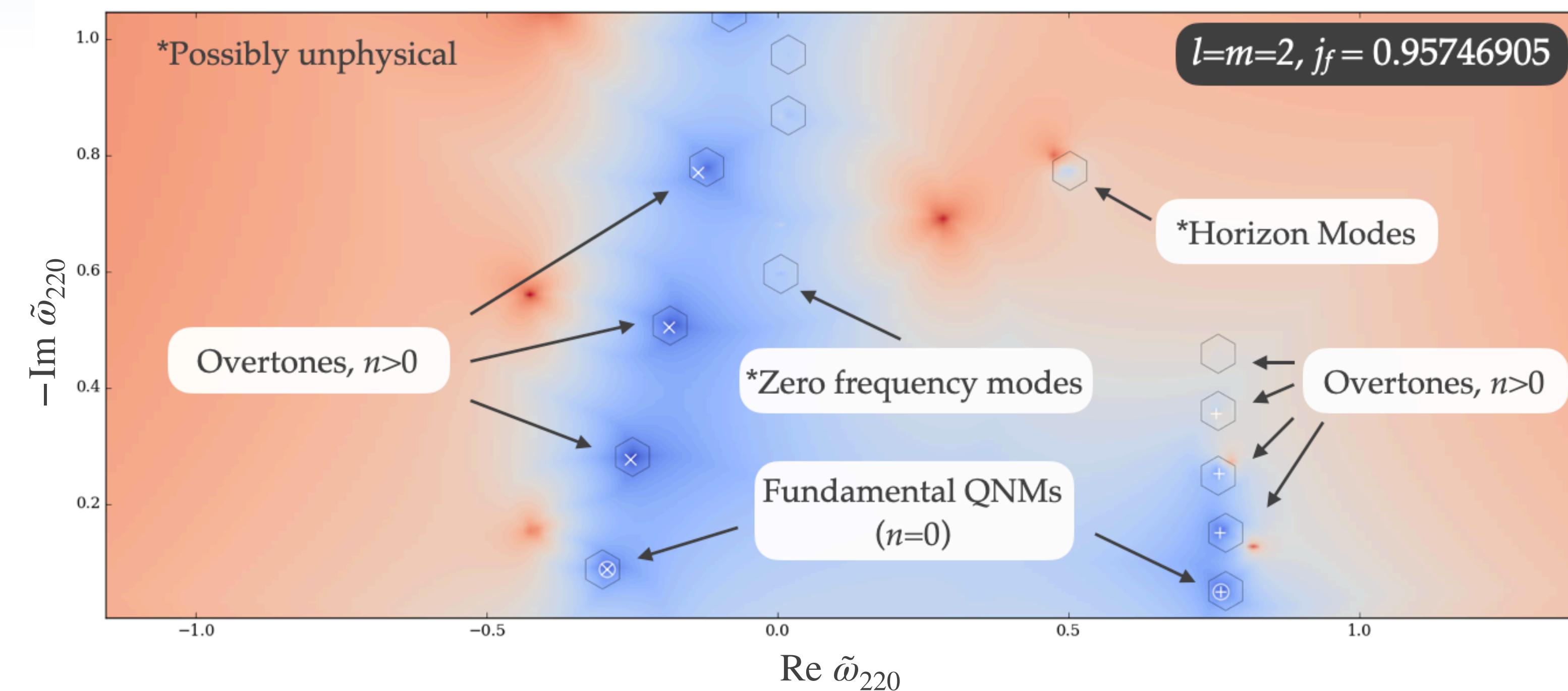
Three-term recursions

QNM Eigenvalues

The QNM Frequencies



Black hole specific special functions

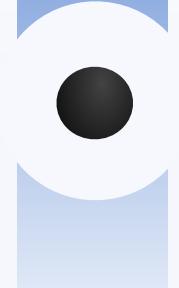


Example: **Blue** (Low work function values), **Red** (High). Using Berti's approximation for the separation constants allows 2D visualisation of QNM solution space

QNM Eigenvalues

The QNM Frequencies

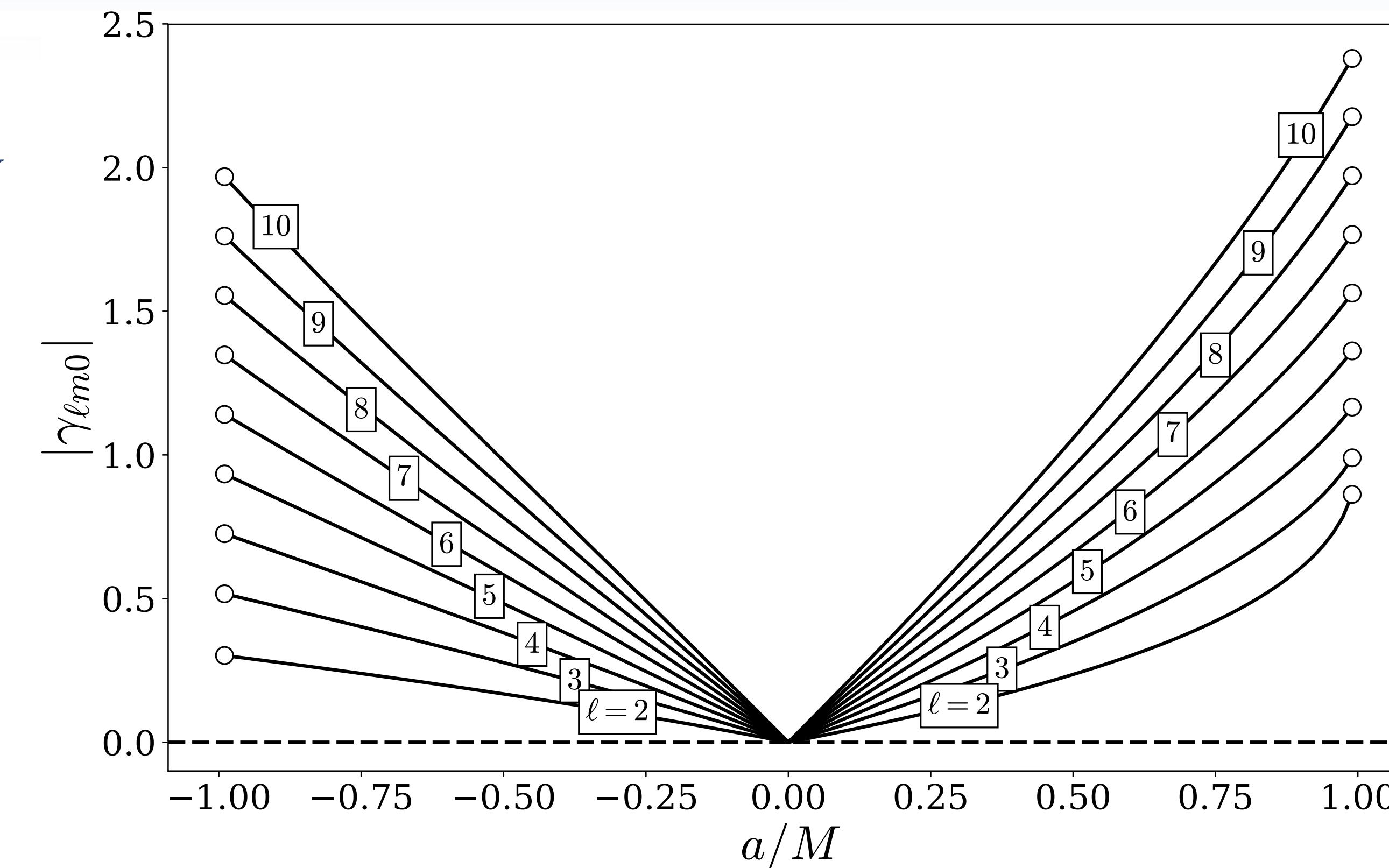
QNM Orthogonalit



Black hole specific special functions

The QNM frequencies are typically unique functions of BH spin, and the retrograde frequencies are sometimes thought of as corresponding to negative spin.

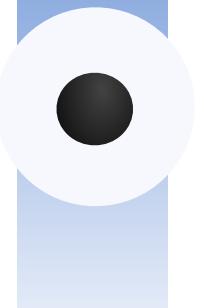
This is not quite right, but can be practical. (Right: $\gamma_{\ell mn} = a \tilde{\omega}_{\ell mn}$)



QNM Eigenvalues

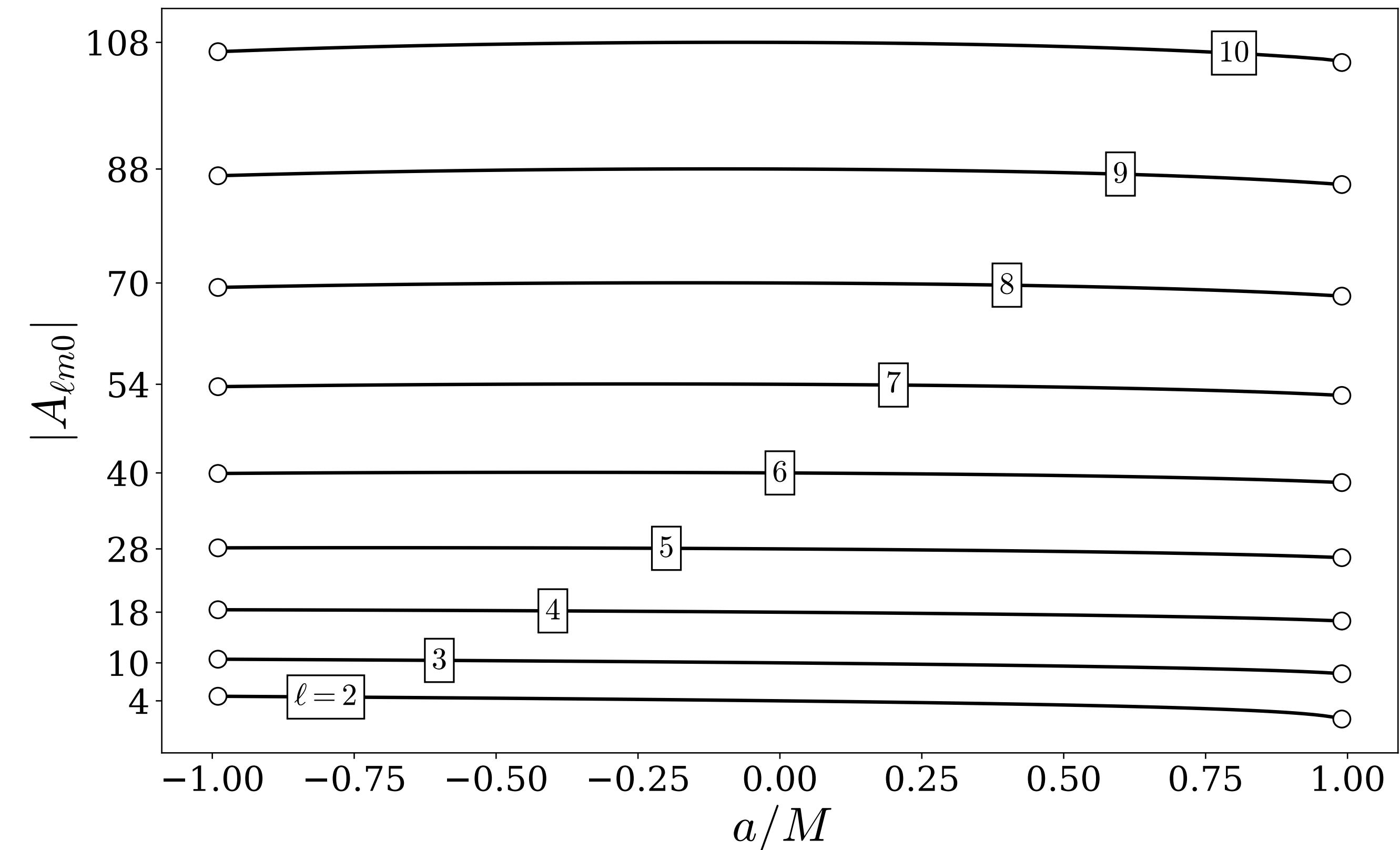
The QNM Frequencies

QNM Orthogonality



Black hole specific special functions

The QNM separation constants are typically unique. Recall from linear algebra that this implies that the QNM functions themselves are linearly independent, and perhaps orthogonal (in some way).



he QNM Frequencies

QNM Orthogonality Primer

Black hole specific special functions

Recent developments on QNM orthogonality and applications:

- London “Bi-orthogonal harmonics for ...” ([arXiv:2006.11449](#))
- Green+ “Conserved currents” ([arXiv:2210.15935](#))
- London “A radial scalar product for Kerr QNMs” ([arXiv:2312.17678](#))
- London & Gurevich “Natural polynomials for Kerr QNMs”
([arXiv:2312.17680](#))

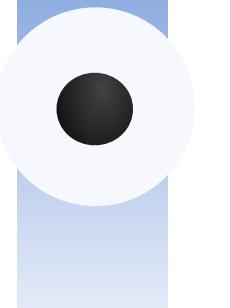
Green and London perspectives on QNM orthogonality are mathematically different, but functionally equivalent. Currently, only the London+ approach allows for potential statements about completeness, but the Green+ approach has already been applied to a few non-GR scenarios ...

he QNM Frequencies

QNM Orthogonality Primer

Black hole specific special functions

1980s-2000s



A small part of the story goes like this ...

he QNM Frequencies

QNM Orthogonality Primer

Black hole specific special functions

The hard parts

- ❖ $\mathcal{D}_u(\omega_{\ell mn}) = (ua\omega_{\ell mn} - 2s)ua\omega_{\ell mn} - \frac{(m + su)^2}{1 - u^2} + \partial_u(1 - u^2)\partial_u$
- ❖ $\mathcal{D}_u(\omega_{\ell mn})^\dagger = \mathcal{D}_u(\omega_{\ell mn})^*$

Their solutions

- ❖
$$\mathcal{T} = \sum_{\ell=2}^{\infty} \sum_{\ell'=2}^{\infty} |Y_{\ell m}\rangle\langle Y_{\ell m}|S_{\ell' m}\rangle\langle Y_{\ell' m}|$$
- ❖ $|S_\ell\rangle = \mathcal{T}|Y_{\ell m}\rangle, \text{ and } |\tilde{S}_\ell\rangle = \mathcal{T}^{\dagger -1}|Y_\ell\rangle$

↑
new “adjoint-spheroidals”

Isomorphism between spherical and spheroidal harmonics

Black hole specific special functions

The hard parts

- ❖ $\mathcal{D}_u(\omega_{\ell mn}) = (ua\omega_{\ell mn} - 2s)ua\omega_{\ell mn} - \frac{(m + su)^2}{1 - u^2} + \partial_u(1 - u^2)\partial_u$

- ❖ $\mathcal{D}_u(\omega_{\ell mn})^\dagger = \mathcal{D}_u(\omega_{\ell mn})^*$

Their solutions

bi-orthogonality

- ❖ $\langle \tilde{S}_\ell | S_{\ell'} \rangle = \int_\Omega {}_s \tilde{S}_\ell^*(\theta; \tilde{\omega}_{\ell mn}) {}_s S_{\ell'}(\theta; \tilde{\omega}_{\ell' mn}) d\Omega = \delta_{\ell\ell'} / 2\pi$

- ❖ $\sum_{\ell=2}^{\infty} \sum_{\ell'=2}^{\infty} |S_\ell\rangle \langle \tilde{S}_{\ell'}| = \hat{\mathbb{I}}$

completeness: the ability to represent arbitrary GW signals

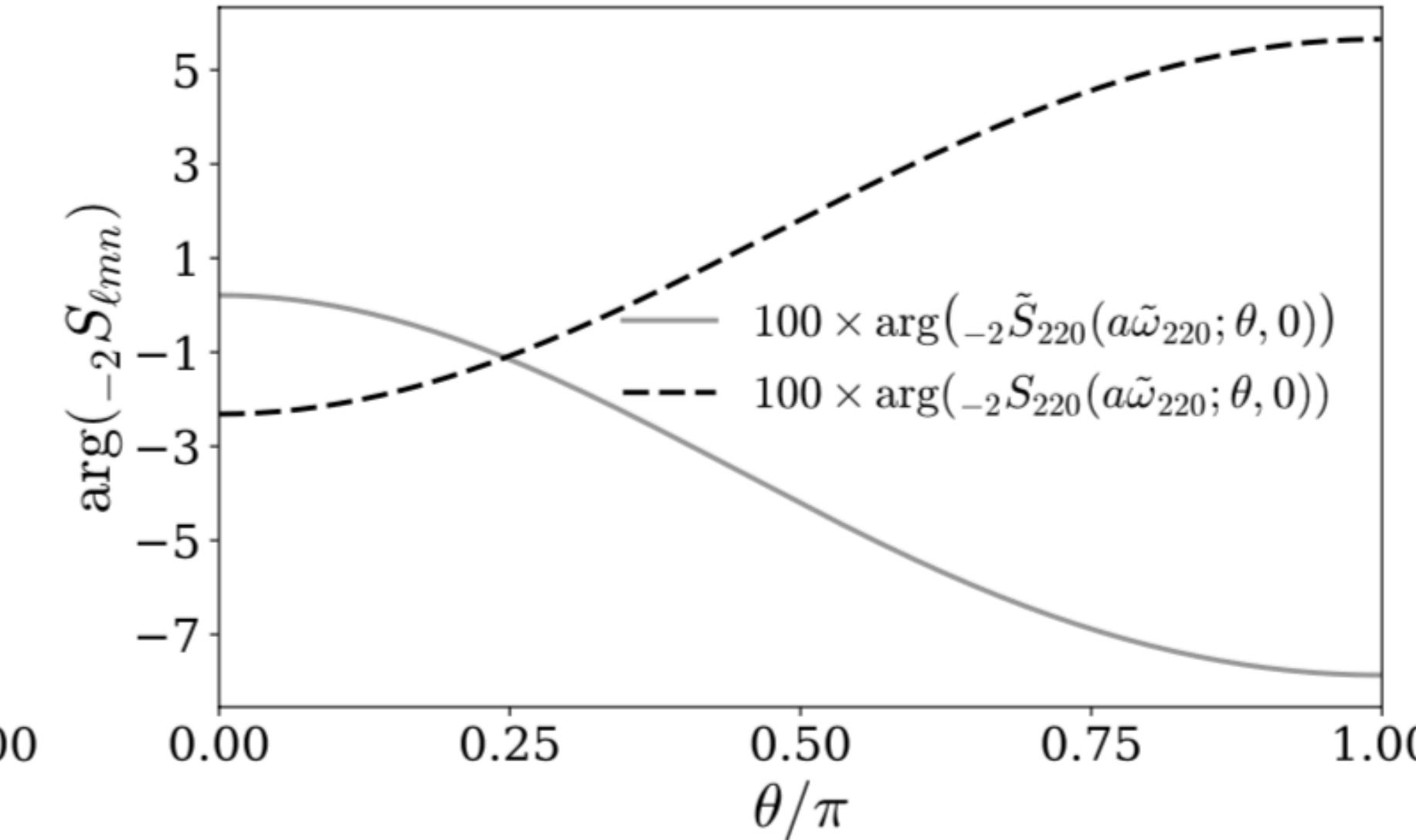
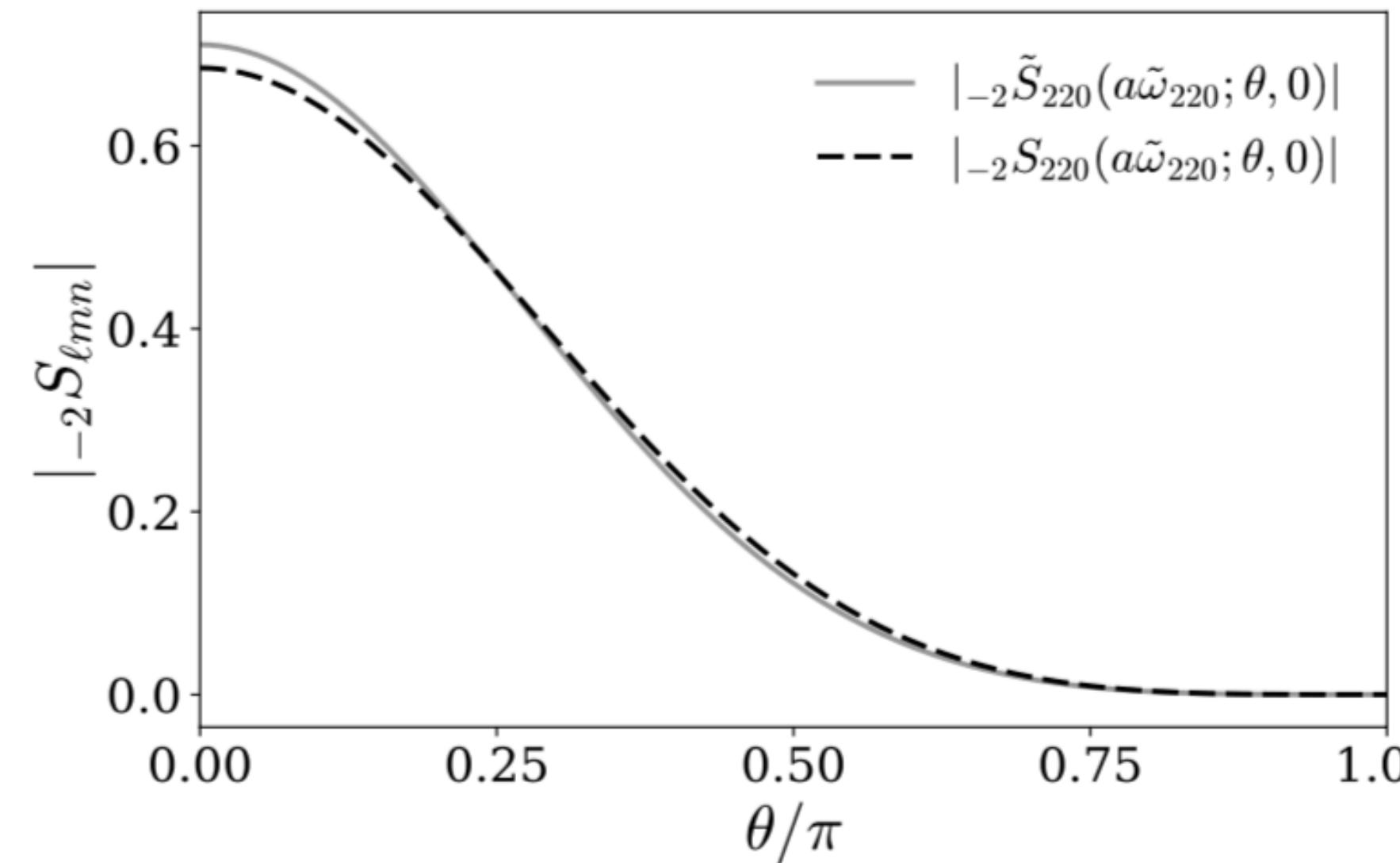
Black hole specific special functions

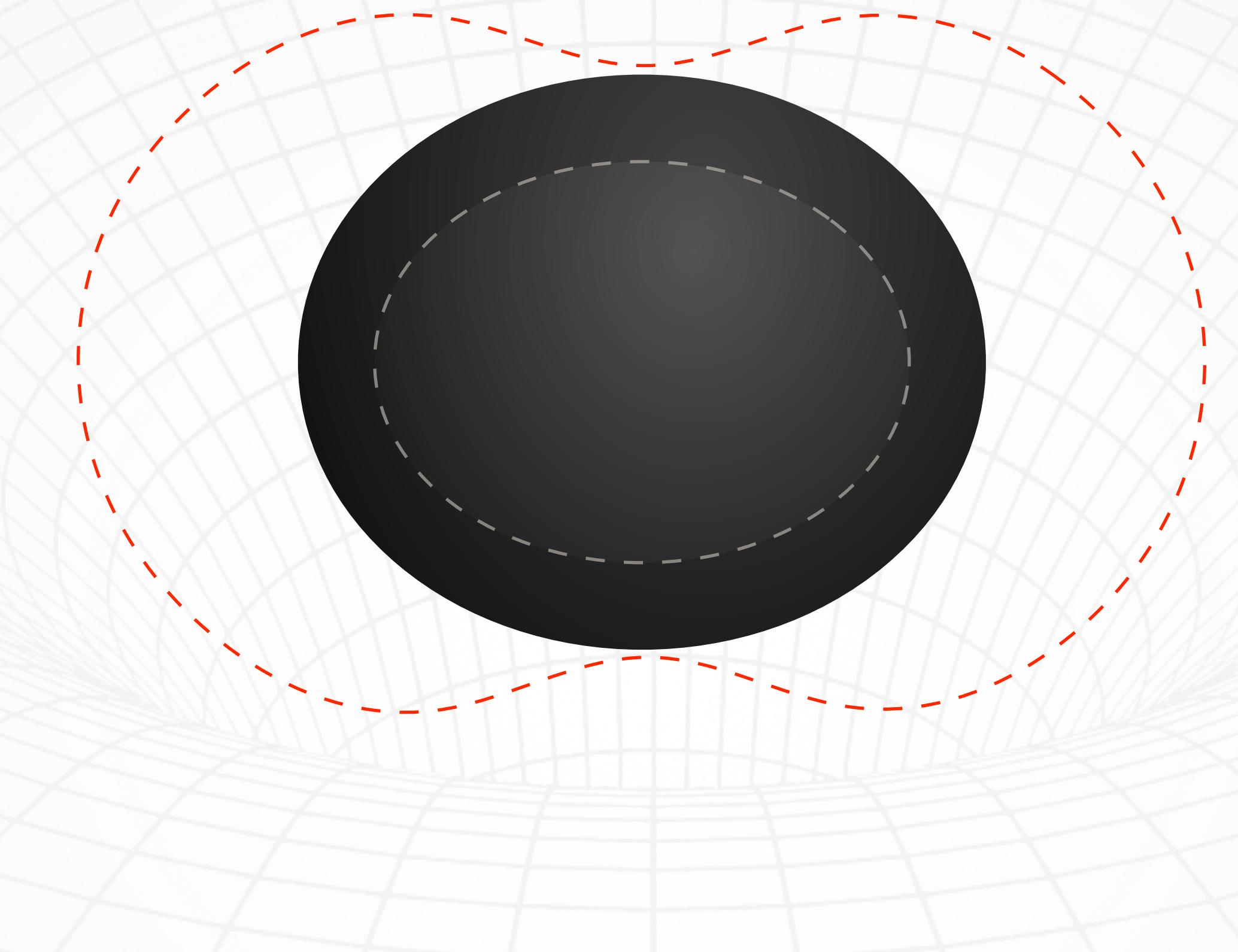
1980s-2000s

Example “small spin” expansion for **term-by-term comparison**:

$$S_{\ell mn} \approx Y_{\ell m} + a\tilde{\omega}_{\ell mn} c_{\ell}^{\ell-1} Y_{\ell-1,m} + a\tilde{\omega}_{\ell mn} c_{\ell}^{\ell+1} Y_{\ell+1,m}$$

$$\tilde{S}_{\ell mn} \approx -Y_{\ell m} + a\tilde{\omega}_{\ell-1,m,n}^* c_{\ell-1}^{\ell} Y_{\ell-1,m} + a\tilde{\omega}_{\ell+1,m,n}^* c_{\ell+1}^{\ell} Y_{\ell+1,m}$$





Let's stop here for questions ...