Excitation of Schwarzschild black-hole quasinormal modes

Nils Andersson
Department of Physics and Astronomy, University of Wales College of Cardiff, Cardiff CF2 3YB, United Kingdom
(Received 16 June 1994)

The evolution of a scalar test field from initial data in a space-time that contains a Schwarzschild black hole is studied. The investigation involves a Green's function representation of the solution, and general formulas determining the excitation of the quasinormal modes are discussed. We use the semianalytic phase-integral method to evaluate these formulas approximately. As an example that can be studied analytically we use Gaussian initial data. For intermediate times, when the quasinormal ringing dominates the radiation, the approximate results are shown to agree perfectly with the results of numerical evolutions of the same initial data. Our approximate analysis also reveals that the slowest damped mode for a certain radiating multipole l is maximally excited when the Gaussian has a half-width $12.24M(l+1/2)^{-1}$ (M is the mass of the black hole in geometrized units). For broader pulses the fundamental mode is exponentially suppressed. Most of the presented results remain valid in the physically more interesting case of a gravitationally perturbed black hole.

PACS number(s): 97.60.Lf, 02.60.Cb, 02.70.Rw, 04.30.Db

I. INTRODUCTION

Since the beginning of the 1970s the quasinormal modes of relativistic stars and black holes have attained much interest. One of the main motivations is that we have as yet not been able to observe black holes directly. All evidence of the actual existence of such objects in our Universe is indirect and relies upon "guesstimates" of, for example, the mass of the visible companion in an x-ray binary [1]. In contrast with this, the quasinormal-mode spectrum is unique to the black hole itself, i.e., depends only on its three parameters (mass, angular momentum, and electric charge). Furthermore, the modes are supposedly excited in any dynamical process involving a black hole. These two facts suggest that the modes provide a way to identify holes. The only problem is that the electromagnetic radiation that is expected to be generated (from a perturbed $10M_\odot$ black hole, say) is of too low frequency to propagate far in the interstellar plasma (of the order of a few kHz). The hope that we will ever be able to study black holes directly therefore relies upon the future identification of gravitational waves. With the large-scale interferometers that are under construction today [2,3] it seems possible that gravitational waves will be detected within something like ten years. There is no obvious reason why these detectors should not also enable us to identify gravitational waves that carry the unique quasinormal-mode signature of a black hole.

The quasinormal-mode spectrum is also of relevance for black-hole physics at the theoretical level. One way of proving that black holes are stable against small pertubations, as they must if they are astrophysical entities, is to convincingly show that no unstable modes exist [4,5]. There has also been suggestions that the modes can be of importance in black-hole thermodynamics [6]. Furthermore, it seems reasonable that the modes will play some role in model problems where waves are scattered off black holes (see [7] for an exhaustive discussion of black-hole scattering). Specifically, one may wonder whether the modes are in some sense similar to the resonances of a quantum system [8].

Many papers have been written on the subject of quasinormal modes, but remarkably few of them discuss the actual excitation of the modes. Most studies have focused on techniques for determining the characteristic frequencies (see [9] or [10] for an extensive list of references). This may be because the calculation of highly damped quasinormal-mode frequencies is a far from trivial problem. If one cannot determine their frequencies, it will be difficult to assess whether or not these modes will be relevant in astrophysical processes.

There are, however, a few papers where the excitation of quasinormal modes in reasonable model situations is discussed. In 1977 Detweiler discussed resonant oscillations of a rotating black hole [11]. After identifying the quasinormal modes as "resonance peaks" in the emitted spectrum, he made clear that the modes formally correspond to poles of a Green's function to the inhomogeneous Teukolsky equation [12]. This idea was later put on a more rigorous mathematical footing by Leaver [13]. In his exhaustive discussion (where he considers a more general initial-value problem than Detweiler did) he extracts the quasinormal-mode contribution to the emitted radiation as a sum over residues. This sum arises when the inversion contour of the Laplace transform, which was used to separate the dependence on the spatial variables from that on time, is continued analytically in the complex frequency plane. In this way the contribution from the quasinormal modes, the singularities of the Green's function that constitutes the integrand, can be accounted for.

The way that quasinormal modes are excited by given Cauchy data has also been discussed in some detail by

0556-2821/95/51(2)/353(11)/$06.00

Sun and Price [14,15]. Their demonstrations rely to some extent on the numerical results obtained by Leaver [13,16]. Recently, the general problem was addressed by Nollert and Schmidt [17]. They used the definition of quasinormal modes as singularities of a Green's function as foundation for a powerful technique for obtaining the characteristic frequencies. Although they never did proceed beyond the calculation of frequencies, it seems plausible that their technique can be extended to discuss also how the modes are excited.

The described situation is somewhat unfortunate, and independent investigations of the way in which modes are excited are needed. This paper is an attempt to satisfy this need: We will try to use the phase-integral method to determine some characteristics of quasinormal-mode excitation. This seems like a natural thing to do since this method has been successfully used to determine quasinormal-mode frequencies [18–21]. Nevertheless, the present investigation will only provide a small step towards a complete semianalytic description of the spectral decomposition of quasinormal modes. We discuss the evolution of a scalar test field in the Schwarzschild background. This case may not be as physically intriguing as that of a gravitationally perturbed black hole, but because of its relative simplicity, it is a suitable testing ground for the technique that we will develop. In fact, most of our final formulas will carry over immediately to the case of gravitational perturbations.

It is worth mentioning that most of the general results (that make up a part of the first sections of this paper) have been known for almost 20 years. Only in the sections where we use the phase-integral method to determine some characteristics of quasinormal-mode excitation are we breaking new ground. These results cannot be properly understood without the general formulas of Secs. II–V, however. Hence, the repetition of what may be well-known is justified.

II. EQUATIONS GOVERNING A SCALAR TEST FIELD

We are interested in a scalar field that evolves according to the Klein-Gordon equation in the Schwarzschild background. It is straightforward to show that the field can be written

$$\sum_{l=0}^{\infty}(2l+1)\frac{\Phi_l(r_*,t)}{r}P_l(\cos\theta) \ .$$

That is, the angular dependence is the typical one for problems with spherical symmetry. The dependence on radius and time is governed by the partial differential equation

$$\left[\frac{\partial^2}{\partial r_*^2}-\frac{\partial^2}{\partial t^2}-V(r)\right]\Phi_l=0 \ , \tag{1}$$

where

$$V(r)=\left(1-\frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2M}{r^3}\right] \ . \tag{2}$$

M is the mass of the black hole and the subscript on Φ will be assumed in the following. The *tortoise coordinate* r_* was first introduced by Regge and Wheeler in 1957 [22]. In geometrized units ($c = G = 1$) it is related to the Schwarzschild radius r by

$$\frac{d}{dr_*}=\left(1-\frac{2M}{r}\right)\frac{d}{dr} \tag{3}$$

or

$$r_*=r+2M\ln\left(\frac{r}{2M}-1\right)+\text{const} \ . \tag{4}$$

The integration constant will not play any role in the present analysis, and so we will assume that it is equal to zero. It should, however, be mentioned that it can be of importance in other problems, such as that of scattering of waves from black holes [7].

The case when Eq. (1) is not homogeneous, when the field has a source, can be approached by means of a time-dependent Green's function (see, for example, Chap. 7.3 in [23]). This is essentially what Leaver did in 1986 [13]. In the present analysis we will restrict ourselves to the homogeneous case. We assume that any source has vanished, for example, fallen into the hole, leaving an initial field behind. The future evolution of this field is then governed by (1).

In order to get a well-posed problem we need to specify both initial data and boundary conditions as r_* goes to $\pm\infty$, at spatial infinity and the event horizon. In essence, we consider an eternal Schwarzschild black hole that is perturbed by some external agent. It then seems reasonable to introduce Cauchy data that have compact support on a spacelike hypersurface that intersects the future event horizon and future infinity, i.e., to consider an initial "time" $t_0 = t_0(r_*)$. But what boundary conditions should be imposed at the horizon ($r_* \to -\infty$) and spatial infinity ($r_* \to +\infty$)? Since the effective potential (2) is of short range, it is clear from (1) that the general solution at infinity will be a linear combination

$$\Phi(r_*,t)\sim f(t-r_*)+g(t+r_*)\quad \text{as } r_*\to+\infty \ . \tag{5}$$

The second of these terms, the one that corresponds to a wave front propagating inwards from infinity, cannot be allowed. We do not expect waves to be backscattered from infinity. The potential (2) also vanishes close to the horizon, $r = 2M$. The argument that no communication with the interior of the hole should be possible implies that only motion inwards is allowed. Hence, the solution must be

$$\Phi(r_*,t)\sim h(t+r_*)\qquad \text{as } r_*\to-\infty \ . \tag{6}$$

III. INTEGRAL TRANSFORM FOR THE TIME DEPENDENCE

In the standard approach to partial differential equations one seeks a way to separate all the independent variables. A separation of variables will usually leave a set of ordinary differential equations, the solutions of which are more attainable than a direct solution of the original equation. In many previous investigations of the black-hole problem, especially those where the computation of frequencies was the main concern, it was implied that a Fourier transform was used to deal with the time dependence. Alternatively, one simply assumed that the perturbation behaved on time as $\exp(-i\omega t)$. In the present context, however, it is important to remember that the standard Fourier transform can only be used as long as [assuming that $\Phi(r_*,t)$ vanishes for $t < t_0$ so that causality is not violated]

$$\int_{t_0}^{+\infty} |\Phi(r_*,t)|^2 dt$$

remains finite. At this point in the discussion of the black-hole problem we cannot say whether this is the case or not. One would, of course, expect the hole to be stable against a small perturbation. If that is the case, the above quantity should remain finite. Fortunately, Kay and Wald [24] have proved that, for a scalar field that has compact support on Cauchy surfaces of the Kruskal extension, there will always exist a constant C such that

$$\left|\frac{\Phi(r_*,t)}{r}\right| < C \ , \tag{7}$$

for all points $[r_*,t]$ in the exterior of the hole. It is therefore clear that we can find a constant c such that

$$\Phi(r_*,t)e^{-ct} \to e^{-\epsilon t} \quad (\epsilon > 0) \quad \text{as } t \to +\infty \ . \tag{8}$$

Hence, the integral transform

$$\hat{\Phi}(r_*,\omega) = \int_{t_0}^{+\infty} e^{i\omega t}\Phi(r_*,t)dt \tag{9}$$

is well-defined as long as $\text{Im}\,\omega \geq c$. The inversion formula for this transform is

$$\Phi(r_*,t) = \frac{1}{2\pi}\int_{-\infty+ic}^{+\infty+ic} e^{-i\omega t}\hat{\Phi}(r_*,\omega)d\omega \ . \tag{10}$$

It follows that $\hat{\Phi}(r_*,\omega)$ is a well-defined function in the upper half of the complex ω plane.

Note that a change $s = -i\omega$ makes the above transform equal to the standard Laplace transform that was used by Leaver [13], Sun and Price [14,15], and Nollert and Schmidt [17]. Here we prefer to work with (9) and (10) since a time dependence $\exp(-i\omega t)$ was assumed in our previous investigations of the problem [9,10,18–21].

Using (9) we get the ordinary differential equation

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right]\hat{\Phi}(r_*,\omega) = I(r_*,\omega) \ , \tag{11}$$

where

$$I(r_*,\omega) = e^{i\omega t_0}\left[i\omega\Phi(r_*,t) - \frac{\partial\Phi(r_*,t)}{\partial t}\right]_{t=t_0} \ . \tag{12}$$

IV. FORMAL GREEN'S FUNCTION SOLUTION

A solution to the inhomogeneous equation (11) can, at least formally, be determined by means of a Green's function $G(r_*,r'_*)$ that solves

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right]G(r_*,r'_*) = \delta(r_* - r'_*) \ . \tag{13}$$

The solution to the original equation can then be written (see, for example, Chap. 7 in [23])

$$\hat{\Phi}(r_*,\omega) = \int_{-\infty}^{\infty} I(r'_*,\omega)G(r'_*,r_*)dr'_* + \left[\hat{\Phi}(r'_*,\omega)\frac{dG(r'_*,r_*)}{dr'_*} - G(r'_*,r_*)\frac{d\hat{\Phi}(r'_*,\omega)}{dr'_*}\right]_{r'_*=-\infty}^{r'_*=+\infty} \ . \tag{14}$$

In order to get an expression in terms of known quantities we must require that the Green's function satisfy homogeneous boundary conditions. If it does, the surface terms that include $d\hat{\Phi}/dr_*$ will disappear. Hence, we require that

$$G(r_*,r'_*) \sim \begin{cases} e^{+i\omega r_*} & \text{as } r_* \to +\infty \ , \\ e^{-i\omega r_*} & \text{as } r_* \to -\infty \ , \end{cases} \tag{15}$$

for ω in the upper half of the ω plane. Furthermore, it follows from (5) and (6) that similar boundary conditions should be imposed on $\hat{\Phi}(r_*,\omega)$. Consequently, we get

$$\hat{\Phi}(r_*,\omega) = \int_{-\infty}^{\infty} I(r'_*,\omega)G(r'_*,r_*)dr'_* \ . \tag{16}$$

Since the inhomogeneity in (13) vanishes for $r_* \neq r'_*$, the Green's function can be expressed in terms of two solutions to the corresponding homogeneous equation. These solutions are characterized by their asymptotic behavior [11]. One of them is defined by

$$\hat{\Phi}^- \sim \begin{cases} e^{-i\omega r_*} \ , & r_* \to -\infty \ , \\ A_{\text{out}}e^{+i\omega r_*} + A_{\text{in}}e^{-i\omega r_*} \ , & r_* \to +\infty \ ; \end{cases} \tag{17}$$

i.e., it corresponds to purely ingoing waves crossing the event horizon. The other solution $\hat{\Phi}^+$ behaves like $e^{+i\omega r_*}$ as $r_* \to +\infty$ and corresponds to a linear combination of out- and ingoing waves at the horizon. Then the solution (16) can be written

$$\hat{\Phi}(r_*,\omega) = \hat{\Phi}^+\int_{-\infty}^{r_*}\frac{I\hat{\Phi}^-}{W(\omega)}dr'_* + \hat{\Phi}^-\int_{r_*}^{\infty}\frac{I\hat{\Phi}^+}{W(\omega)}dr'_* \ . \tag{18}$$

It is straightforward to show that the Wronskian $W(\omega)$ of the two linearly independent solutions $\hat{\Phi}^-$ and $\hat{\Phi}^+$ must be a constant. Specifically, we have

$$W(\omega) = \hat{\Phi}^- \frac{d\hat{\Phi}^+}{dr_*} - \hat{\Phi}^+ \frac{d\hat{\Phi}^-}{dr_*} = 2i\omega A_{\rm in} \ . \tag{19}$$

For r_* very large, i.e., an observer situated far away from the black hole, it seems reasonable to approximate the solution (18) by

$$\hat{\Phi}(r_*,\omega) = \frac{J(\omega)}{2i\omega A_{\rm in}(\omega)} e^{+i\omega r_*} \ , \tag{20}$$

where we have introduced the simplifying notation

$$J(\omega) = \int_{-\infty}^{\infty} I\hat{\Phi}^- dr'_* \ . \tag{21}$$

V. RECOVERING THE TIME DOMAIN

From the inversion formula (10) for the integral transform and the final formula (20) of the previous section it is clear that a solution to the wave equation (1), valid far away from the black hole, can be written

$$\Phi(r_*,t) = \frac{1}{4\pi i}\int_C \frac{e^{-i\omega(t-r_*)}}{\omega A_{\rm in}} J(\omega) d\omega \ , \tag{22}$$

with the integration contour C as in Fig. 1. We find that when $A_{\rm in}(\omega) = 0$ the integrand, essentially the Green's function, is singular. For such a frequency—let us denote it ω_n—the two functions $\hat{\Phi}^-$ and $\hat{\Phi}^+$ are no longer linearly independent according to (19). The corresponding solution to (1) therefore satisfies boundary conditions of purely outgoing waves at spatial infinity *and* purely ingoing waves crossing the horizon. These boundary conditions are exactly those commonly used to identify the quasinormal modes of the black hole; see, for example, [10]. In the Schwarzschild case the characteristic frequency of each of these modes depends only on the mass of the black hole. For each value of the angular harmonic index $l \geq 0$ there exists an infinite number of modes; see [25,26] for further discussion. The characteristic frequencies all lie in the lower half of the ω plane. That is, the modes correspond to damped oscillations according to (22). Hence, the black hole is stable and can be considered as astrophysically realistic .

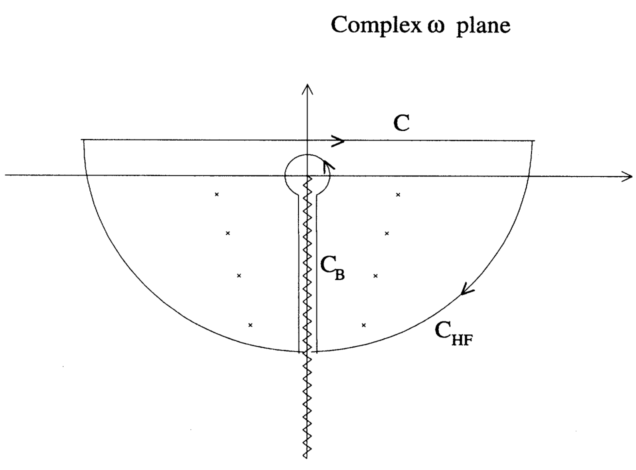

FIG. 1. Integration contours C_i in the complex ω plane used to invert the integral transform for the time dependence and account for the quasinormal-mode contribution to the emitted radiation. The quasinormal-mode frequencies are indicated by crosses. The branch cut that must be introduced from $\omega = 0$ [13] is a wavy line.

In the analysis above the integral transform (9) was defined only in the upper half of the ω plane. For Im $\omega_n < 0$ the boundary conditions (15) defining the Green's function will correspond to a function that grows exponentially as r_* approaches $\pm\infty$ along the real coordinate axis. This divergence makes the identification of the quasinormal modes difficult; see [27] for discussion. But if we disregard this difficulty and assume that the integrand in (22) can be continued analytically into the lower half of the ω plane, we can extract the contribution of the quasinormal modes to the radiated wave [13];

$$\Phi_Q(r_*,t) = -\frac{1}{4\pi i}\int_{C+C_{HF}+C_B} \frac{e^{-i\omega(t-r_*)}}{\omega A_{\rm in}} J(\omega) d\omega \ , \tag{23}$$

with integration contours as in Fig. 1. Close to the characteristic frequency ω_n we can use the linear approximation

$$A_{\rm in}(\omega) \approx (\omega-\omega_n) \left.\frac{dA_{\rm in}}{d\omega}\right|_{\omega=\omega_n} = (\omega-\omega_n)\alpha_n \ . \tag{24}$$

According to Cauchy's residue theorem we then get the mode contribution

$$\Phi_Q(r_*,t) = -\frac{1}{2}\sum_{\text{all poles}} \frac{e^{-i\omega_n(t-r_*)}}{\omega_n\alpha_n} J(\omega_n) \ . \tag{25}$$

It turns out (see [16]) that the singularities lie symmetrically distributed with respect to the imaginary ω axis. Modes with frequencies ω_n and $-\omega_n^\star$ (where the star denotes omplex conjugation) are in one-to-one correspondence. That this should be the case can be seen from the differential equation (11). In fact, it follows that

$$\hat{\Phi}^\star(r_*,\omega) = \hat{\Phi}(r_*,-\omega^\star) \ . \tag{26}$$

Therefore, we can infer that

$$\Phi_Q(r_*,t) = -\text{Re}\left[\sum_{n=0}^{\infty} \frac{e^{-i\omega_n(t-r_*)}}{\omega_n\alpha_n} J(\omega_n)\right] \ , \tag{27}$$

where we sum over all poles in the fourth quadrant.

Although formally correct, the above expression will clearly be meaningless if we cannot evaluate the coefficients α_n and the integrals $J(\omega_n)$ for a given set of initial data. Leaver's study [13] is the only previous one where values for α_n are obtained. He investigated several model problems and found that the polesum (27) accurately describes the excitation of quasinormal modes whenever the integral $J(\omega_n)$ can be computed. This was also the conclusion of Sun and Price [14,15]. Moreover, they devised alternative schemes to describe the excitation in situations where the integral could not be evaluated directly. Our present aim is to achieve a, possibly very crude, approximation to the relevant excitation quantities by means of the phase-integral method. This approach makes sense since that method has proved reliable for computing frequencies (see [18–21]). Moreover, because of the methods semianalytic nature, it seems likely that this will allow us to discuss the excitation of modes qualitatively in a way that would be impossible in a purely numerical study.

VI. APPROXIMATE QUASINORMAL-MODE SOLUTION

Although the tortoise coordinate r_* is useful in a general description of black-hole perturbations (such as that above), it is preferable to use the Schwarzschild radial coordinate r when trying to construct explicit solutions to, for example, the differential equation (11). After defining a new dependent variable (as in [10])

$$\hat{\Phi}(r_*,\omega) = \left(1-\frac{2M}{r}\right)^{-1/2}\psi(r,\omega) \ , \tag{28}$$

we have the homogeneous differential equation

$$\left[\frac{d^2}{dr^2}+R(r)\right]\psi = 0 \ , \tag{29}$$

where $R(r)$ is an analytic function given by

$$R(r) = \left(1-\frac{2M}{r}\right)^{-2}\left[\omega^2 - V(r) + \frac{2M}{r^3} - \frac{3M^2}{r^4}\right] . \tag{30}$$

The asymptotic behavior that defines the solution $\hat{\Phi}^-(r_*,\omega)$ implies that we should have

$$\psi^- \sim \left(\frac{r}{2M}-1\right)^{+1/2-2iM\omega} e^{-i\omega r} \ , \tag{31}$$

as r goes to $2M$, i.e., approaches the event horizon. In the upper half of the ω plane this solution is regular. At the same time, the asymptotic behavior at spatial infinity will be

$$\psi^- \sim A_{\rm out}\left(\frac{r}{2M}\right)^{2iM\omega} e^{+i\omega r} + A_{\rm in}\left(\frac{r}{2M}\right)^{-2iM\omega} e^{-i\omega r}$$

$$\text{as } r \to +\infty \ . \tag{32}$$

We will now use an approximate solution to (29) obtained within the phase-integral method (see [10] for a suitable introduction). For the sake of clarity, we will restrict the analysis to the lowest order of approximation. Then the general solution to (29) is represented by a combination of the two linearly independent functions

$$f_{1,2}(r,t_j) = Q^{-1/2}(r)\exp\left[\pm i\int_{t_j}^{r} Q(r')dr'\right] . \tag{33}$$

That is, the phase-integral solution to (29) is generated from a function $Q(r)$. This means that the approximation could, but need not, be equal to the approximation $Q = R^{1/2}$ that is prescribed by standard WKB analysis [28]. In the black-hole case one can argue (see [10]) that one should use

$$Q^2 = R - \frac{1}{4(r-2M)^2} \ , \tag{34}$$

in order to make the approximation accurate in the region close to the event horizon.

Because of the so-called Stokes phenomenon, a given linear combination of f_1 and f_2 remains a valid approximation to the exact solution of (29) only in a certain sector of the complex r plane. This sector is limited by what is known as the Stokes lines. On these the quantity Qdr is purely imaginary. From each zero of Q^2, the transition points t_n of the problem, emerge three such lines. This threefold symmetry in the vicinity of a transition point is apparent in Fig. 2. An approximate solution to (29) must consist of different linear combinations of f_1 and f_2 on opposite sides of each Stokes line. Specifically, the coefficient of the subdominant (exponentially small) phase-integral function (f_1 or f_2) changes smoothly as the Stokes line is crossed [29]. At the same time, the coefficient of the dominant (exponentially large) function remains unchanged.

In a phase-integral analysis the contours where Qdr is purely real, the anti-Stokes lines, are also important. On these the functions f_1 and f_2 have a wavelike behavior. Hence, it is natural to impose boundary conditions of outgoing waves (such as those in the quasinormal-mode problem) on these contours rather than on the real r axis

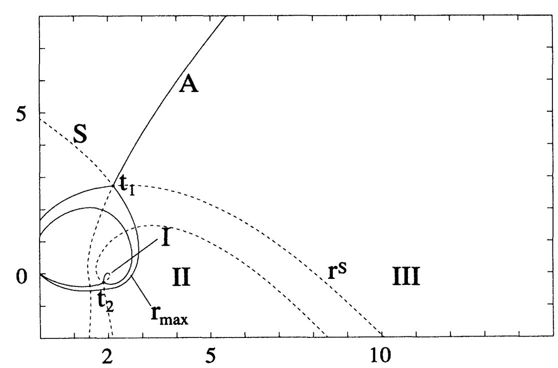

FIG. 2. The Stokes (dashed) and anti-Stokes (solid) lines in the complex r/M plane for $\omega M = 0.11 - 0.10i$ and $l = 0$. The situation remains basically the same for all slowly damped quasinormal modes. The two relevant zeros of Q^2, the transition points, are t_1 and t_2. Cuts that must be introduced to keep Q single valued are not shown but are assumed to run from t_1 to $+i\infty$ and from t_2 to $-i\infty$. The three regions where the phase-integral approximation to the quasinormal-mode solution takes different forms are I, II, and III.

(see [10]). In fact, in the black-hole case one can demonstrate that this is equivalent to posing the condition on the real r axis [30]. But there are other reasons why it is natural to focus on the anti-Stokes lines when discussing the solution to (29). Only on these contours are the two phase-integral functions f_1 and f_2 of the same order of magnitude. One of them will be subdominant at any other point in the complex r plane. Hence, a linear combination of the two functions only makes mathematical sense on the anti-Stokes lines.

For the slowest damped quasinormal modes one gets a pattern of Stokes and anti-Stokes lines similar to that shown in Fig. 2. Then it follows from, for example, the analysis in [10] that an approximate solution to (29) is

$$\psi^-(r,\omega) \approx \begin{cases} Bf_2(r,t_2)\ , \quad r \in \mathrm{I}, \\ B[-if_1(r,t_2)+f_2(r,t_2)] = B[-ie^{i\gamma_{21}}f_1(r,t_1)+e^{-i\gamma_{21}}f_2(r,t_1)]\ , \quad r\in \mathrm{II}, \\ B[-ie^{i\gamma_{21}}f_1(r,t_1)+\left[e^{i\gamma_{21}}+e^{-i\gamma_{21}}\right]f_2(r,t_1)]\ , \quad r\in\mathrm{III}\ , \end{cases} \tag{35}$$

where B is an unspecified normalization constant. We have chosen the phase of $Q(r)$ such that $f_1(r,t_1)$ corresponds to an "outgoing wave" as $r\to+\infty$. γ_{21} is defined by the line integral

$$\gamma_{21} = \int_{t_2}^{t_1} Q dr\ , \tag{36}$$

which means that the real part of γ_{21} is positive. In the analysis leading to this result it was assumed that the branch cuts that should be introduced from the transition points t_1 and t_2 to keep Q single valued are placed in such a way that they do not interfere with the analysis on the real r axis. That is, the necessary cuts proceed from t_1 to $+i\infty$ and from t_2 to $-i\infty$. The same situation was studied in [10,19,20].

In order to extract the coefficients $A_{\rm in}$ and $A_{\rm out}$ we need to study the asymptotic behavior of the phase-integral solution (35). As $r\to+\infty$ we have [with Q according to (34)]

$$\int Q(r)dr \sim \omega r + 2M\omega\ \ln\left(\frac{r}{2M}-1\right)\ . \tag{37}$$

For very large r we therefore get

$$f_{1,2}(r,t_1) \approx \frac{1}{\sqrt{\omega}}\left(\frac{r}{2M}\right)^{\pm 2iM\omega} \exp\left[\pm i(\omega r+\eta)\right]\ , \tag{38}$$

where we have defined a constant asymptotic phase [31,32]

$$\eta = \int_{t_1}^{+\infty}\left[Q(r')-\omega\left(1-\frac{2M}{r'}\right)^{-1}\right]dr' - \omega\left[t_1+2M\ \ln\left(\frac{t_1}{2M}-1\right)\right]\ . \tag{39}$$

In Table I we list the values of η for the first four quasinormal-mode frequencies and $l=0-2$.

It follows that our approximate solution (35) behaves as

$$\psi^-(r,\omega)\approx \frac{B}{\sqrt{\omega}}\left\{-ie^{i(\gamma_{21}+\eta)}\left(\frac{r}{2M}\right)^{2iM\omega}e^{i\omega r} + \left[e^{i\gamma_{21}}+e^{-i\gamma_{21}}\right]e^{-i\eta}\left(\frac{r}{2M}\right)^{-2iM\omega}e^{-i\omega r}\right\} \tag{40}$$

TABLE I. Relevant quantities for the excitation of Schwarzschild black hole quasinormal modes by a scalar test field. All results are obtained using the first-order phase-integral approximation. The approximate results are not believed to be reliable to the accuracy given in the table. In order to give an idea of how accurate they really are exact frequencies (as obtained in [9]) are also included.

l	n	$\omega_n^{\rm exact}$	ω_n	$\eta(\omega_n)$	$[\partial\gamma_{21}/\partial\omega]_{\omega=\omega_n}$
0	0	$0.1105-0.1049i$	$0.1170-0.1071i$	$-1.2196-0.2636i$	$-1.6307+13.6806i$
	1	$0.0861-0.3481i$	$0.0953-0.3520i$	$-2.5639-0.2069i$	$-0.6427+12.3914i$
	2	$0.0757-0.6011i$	$0.0867-0.6052i$	$-4.0765-0.1590i$	$-0.2796+12.4166i$
	3	$0.0704-0.8537i$	$0.0824-0.8576i$	$-5.6206-0.1325i$	$-0.1597+12.4609i$
1	0	$0.2929-0.0977i$	$0.2950-0.0980i$	$-2.5430-0.5099i$	$-1.3082+15.7262i$
	1	$0.2644-0.3063i$	$0.2667-0.3072i$	$-3.5586-0.7195i$	$-2.2133+13.7770i$
	2	$0.2295-0.5401i$	$0.2321-0.5415i$	$-4.8152-0.6679i$	$-1.6114+12.6820i$
	3	$0.2033-0.7883i$	$0.2061-0.7899i$	$-6.2246-0.5873i$	$-1.0537+12.4004i$
2	0	$0.4836-0.0968i$	$0.4849-0.0969i$	$-3.9380-0.6346i$	$-0.8600+16.0928i$
	1	$0.4639-0.2956i$	$0.4651-0.2959i$	$-4.8562-1.0811i$	$-2.0557+15.0417i$
	2	$0.4305-0.5086i$	$0.4319-0.5091i$	$-5.9243-1.1830i$	$-2.2765+13.7942i$
	3	$0.3939-0.7381i$	$0.3954-0.7388i$	$-7.1444-1.1429i$	$-1.9319+12.9765i$

for large r, i.e., in region III of Fig. 2. By comparing this expression with (32) we can identify

$$A_{\rm out} = -i\frac{B}{\sqrt{\omega}}e^{i(\gamma_{21}+\eta)} \tag{41}$$

and

$$A_{\rm in} = \frac{B}{\sqrt{\omega}}e^{-i\eta}\left[e^{i\gamma_{21}} + e^{-i\gamma_{21}}\right] . \tag{42}$$

As already mentioned, a quasinormal mode corresponds to $A_{\rm in} = 0$. Hence, it follows from (42) that these solutions are determined by

$$e^{-2i\gamma_{21}} = -1 . \tag{43}$$

After taking the logarithm we get the familiar Bohr-Sommerfeld formula (discussed in for example [10,18,21,30])

$$\gamma_{21} = \left(n + \frac{1}{2}\right)\pi . \tag{44}$$

Since the real part of γ_{21} will be positive with our choice of phase for Q, n is a non-negative integer in (44). In the present study we have used (44) to determine the first four quasinormal modes for the three lowest values of l. The numerical results are listed in Table I. In order to give an idea of the accuracy of these results we also list results obtained by a purely numerical analysis in [9]. Similar results for scalar field quasinormal-mode frequencies can be found in [18,33]. It is clear that the present results are not very accurate for $l = 0$, but the accuracy increases rapidly for higher l. If we compare the results in Table I with those of Fröman *et al.* [18] we see that the use of higher-order phase-integral approximations will not improve the results much for $l = 0$. In principle, higher orders will only be useful if the lowest order of approximation is already relatively accurate.

VII. APPROXIMATE FORMULA FOR EXCITATION

In the previous section we obtained approximate expressions for the two asymptotic amplitudes $A_{\rm out}$ and $A_{\rm in}$. But if we want to quantify the contribution that each quasinormal mode gives to the radiation generated by certain initial data, we must proceed further. We need to evaluate the coefficient α_n, the derivative of $A_{\rm in}$ with respect to ω in the region surrounding the mode frequency ω_n. If we take the derivative of (42), we get

$$\alpha_n = 2i\frac{B}{\sqrt{\omega}}e^{i(\gamma_{21}-\eta)}\left[\frac{\partial\gamma_{21}}{\partial\omega}\right]_{\omega=\omega_n} , \tag{45}$$

in the vicinity of ω_n. [Remember that (43) is satisfied for $\omega = \omega_n$.] Then (36) implies that

$$\frac{\partial\gamma_{21}}{\partial\omega} = \int_{t_2}^{t_1}\frac{\partial Q}{\partial\omega}dr , \tag{46}$$

where

$$\frac{\partial Q}{\partial\omega} = \frac{1}{2Q}\frac{\partial R}{\partial\omega} , \tag{47}$$

according to (34). Hence, we get the formula

$$\frac{\partial\gamma_{21}}{\partial\omega} = \int_{t_2}^{t_1}\frac{\omega r^2}{(r-2M)^2}\frac{1}{Q}dr . \tag{48}$$

This formula can be used to compute the values of $\partial\gamma_{21}/\partial\omega$ that are listed in Table I. An alternative, which is perhaps simpler, is to extract a numerical estimate of the derivative of γ_{21} as one iterates the condition (44) for the quasinormal-mode frequency. The estimates achieved in this way are often surprisingly accurate. In fact, the two approaches give results that agree to all digits given in Table I.

We now want to assess the reliability of (45). Unfortunately, Leaver never did calculations for the case of a scalar field [13]. But it is straightforward to extend our calculations to, for example, gravitational perturbations. All our formulas remain valid; we only have to replace the second term $2M/r^3$ in the potential (2) by $-6M/r^3$. The result of this calculation is shown in Table II. It can be seen that our results are in good agreement with Leaver's, but also that one of the two studies lead to the wrong overall sign in some cases. As can be seen in Table I, all the factors that enter into (45) in our analysis change in an orderly fashion as n increases. This can be seen as an argument in favor of the present approach in the cases when the entries in Table II differ in sign.

If we collect the results that we have obtained so far, we get an approximate expression for the excitation of quasinormal modes. After some straightforward algebra (27) can be written

TABLE II. Comparing the phase-integral quantities for gravitational perturbations of a Schwarzschild black hole with those obtained by Leaver [13].

l	n	ω_n	$A_{\rm out}/2\omega_n\alpha_n$ Phase integral	$A_{\rm out}/2\omega_n\alpha_n$ Leaver
2	0	$0.3759 - 0.0899i$	$0.1234 + 0.0498i$	$0.12690 + 0.02032i$
	1	$0.3482 - 0.2771i$	$0.0282 - 0.2356i$	$0.04768 - 0.22376i$
	2	$0.3009 - 0.4855i$	$-0.1967 + 0.0331i$	$-0.19028 + 0.01575i$
	3	$0.2510 - 0.7202i$	$0.0944 + 0.0714i$	$0.08087 + 0.07961i$
3	0	$0.6004 - 0.0928i$	$-0.0913 - 0.0398i$	$-0.09390 - 0.04919i$
	1	$0.5836 - 0.2817i$	$-0.1349 + 0.2785i$	$-0.15113 + 0.26977i$
	2	$0.5525 - 0.4798i$	$0.4284 + 0.1253i$	$0.41504 + 0.14101i$
	3	$0.5127 - 0.6912i$	$-0.0562 - 0.4200i$	$0.04338 + 0.41272i$
4	0	$0.8099 - 0.0942i$	$0.0642 + 0.0578i$	$-0.06535 - 0.06524i$
	1	$0.7973 - 0.2845i$	$0.2473 - 0.2608i$	$-0.26147 + 0.25152i$
	2	$0.7734 - 0.4802i$	$-0.5697 - 0.4180i$	$0.54926 + 0.43531i$
	3	$0.7406 - 0.6843i$	$-0.2982 + 0.8599i$	$0.31688 - 0.83788i$

$$\Phi_Q(r_*,t) = \mathrm{Re}\left\{ \sum_{n=0}^{\infty} K_n \frac{1}{2\sqrt{\omega_n}} e^{i\eta} \left[\frac{\partial \gamma_{21}}{\partial \omega}\right]^{-1}_{\omega=\omega_n} e^{-i\omega_n(t-t_0-r_*)} \right\} , \tag{49}$$

where

$$K_n = \int_{2M}^{+\infty} \left(1-\frac{2M}{r}\right)^{-3/2} \bar{\psi}^- \left[i\omega_n \Phi(r_*,t) - \frac{\partial \Phi(r_*,t)}{\partial t} \right]_{t=t_0} dr \ . \tag{50}$$

Here we have used the fact that (43) is satisfied and introduced [see (35)]:

$$\bar{\psi}^- = \begin{cases} -if_2(r,t_1) \ , & r \in \mathrm{I}, \\ f_1(r,t_1) - if_2(r,t_1) \ , & r \in \mathrm{II} \ , \\ f_1(r,t_1) \ , & r \in \mathrm{III}. \end{cases} \tag{51}$$

We have already described how many of the quantities needed for an evaluation of (49) are computed. In the following sections we will discuss various approaches to the integral K_n.

VIII. GENERAL COMMENTS ON THE EXCITATION INTEGRAL

Before we attempt to approximate the integral K_n it may be worthwhile to make some general remarks. Direct evaluation of the integral along the real r axis is, of course, a possibility. This may not be that simple, however, since the quasinormal-mode solution ($\bar{\psi}^-$ in our analysis) grows exponentially towards both spatial infinity and the event horizon. Leaver [13] and Sun and Price [14] have studied this problem in some detail. They conclude that whenever the integral can be computed directly it leads to a good estimate of the excitation of modes.

Sun and Price [14] also point out that this description of mode excitation is counterintuitive in an unfortunate way: Suppose that the initial data consists of a large "bump" close to the black hole and a tiny "bump" very far away towards infinity. Then the fact that the eigenfunction ($\bar{\psi}^-$) grows exponentially leads to the result that the "tiny" bump gives a very large contribution to the integral (K_n) compared to which the impact of the larger "bump" is negligible. Although this is unfortunate, it is not wrong. The solution to the problem is timing [14].

From the analogous situation of potential scattering in quantum mechanics, one would expect the quasinormal modes to be excited when a perturbation enters the region of the maximum of the potential barrier (2). This is also the consensus of numerical simulations in relativity [34]. Moreover, one can interpret the phase-integral solution (51) in a way that agrees well with this idea. As can be seen in Fig. 2, anti-Stokes lines almost connect $r_{\max}$ (which represents the maximum of the potential barrier) with the two transition points t_1 and t_2. This means that the phase-integral functions f_1 and f_2 are of roughly the same order of magnitude at $r_{\max}$, and it makes sense to talk of (51) as representing a standing wave at that point. The situation changes as one moves away from this point along the real r axis since the subdominant function rapidly loses all its influence. Consequently, one can argue that a quasinormal mode is, in a sense, a resonance that is trapped in the vicinity of the top of the potential barrier.

For our previous example this means that the small "bump" (initially at $r_* = x$, say) in the initial data must first propagate to the maximum of the potential barrier and then the quasinormal-mode signal must reach the distant observer. That is, it should roughly take a time

$$t - t_0 \approx r_* + x - 2r_*^{\max} \ , \tag{52}$$

if the observer is at r_*. Meanwhile, it will take a considerably longer time for the impact of a "bump" further away from the hole to reach the same observer. The exponential damping with time that is inherent in (49) will then compensate for the formally large excitation due to the small and distant "bump." Nevertheless, the effect is undesirable and Sun and Price suggest that one should perhaps "hand-shape" the initial data in such a way that different parts of the initial data are analyzed separately.

It is clear that, if the integral (K_n) is computed for complex r, one can force it to converge also in situations where it may otherwise not do so [13,14]. From the phase-integral analysis it is easy to see why this is the case. On the anti-Stokes line A that emerges from t_1 towards infinity in Fig. 2 our approximate solution is $\bar{\psi}^- = f_1(r,t_1)$. Given the time dependence $\exp(-i\omega t)$, this is an outgoing wave. The same solution is exponentially receding away from t_1 along the Stokes line S, or upwards in the complex r plane in region III in general. This fact can be used to deal with difficulties that arise for large values of r in the evaluation of K_n. Since the integral will probably converge quickly if continued analytically upwards in region III of the complex r plane, one can argue in favor of "hand-shaping" the initial data. Only data from the region close to $r_{\max}$ (where the anti-Stokes lines from t_1 and t_2 cross the real r axis in Fig. 2) will then contribute to the integral.

However, one would not expect the approximation (49) to give more accurate results than the numerical studies of Leaver [13] or Sun and Price [14,15]. Hence, it may not be worthwhile to evaluate the integral K_n numerically for a sample of initial data here. We already know that the result should be a relatively good representation of the true excitation of modes. Instead we have chosen to proceed analytically and approximate K_n in a specific example.

IX. EXAMPLE: GAUSSIAN INITIAL DATA

Let us assume that our initial data at $t = t_0$ correspond to a static Gaussian centered at $r_* = x$: i.e.,

$$\Phi(r_*, t_0) = ae^{-b(r_*-x)^2} \tag{53}$$

and

$$\left.\frac{\partial\Phi(r_*, t)}{\partial t}\right|_{t=t_0} = 0 \ . \tag{54}$$

This kind of initial data was used in Vishveshwara's classic paper where quasinormal-mode ringing was first discussed [35].

It does not seem like a restriction to suppose that the initial Gaussian is located far away from the black hole, i.e., that $r_*^{\max} < 3M \ll x$. The main contribution to the integral K_n should then come from region III in our phase-integral analysis (see Fig. 2). In that region we know that the approximate quasinormal-mode solution $\tilde{\psi}^-$ is given by $f_1(r, t_1)$. But for very large r it seems reasonable to replace the phase-integral function by the asymptotic behavior (38). Assuming that we can neglect the contribution to K_n from regions I and II, which is reasonable since our initial data vanish rapidly away from x, we get the approximate expression

$$K_n \approx i\sqrt{\omega_n}ae^{i\eta}\int_{r_*^S}^{+\infty} e^{-b(r_*-x)^2+i\omega r_*}dr_* \ , \tag{55}$$

where $r_*^{\max} \ll r_*^S \ll x$ The natural step now is to assume that we would introduce a negligible error by extending the lower limit of integration to $-\infty$. In this way we get the following analytic formula for K_n:

$$K_n \approx iae^{i\eta}\sqrt{\frac{\omega_n\pi}{b}}\exp\left[i\omega_n x - \frac{\omega_n^2}{4b}\right] \ . \tag{56}$$

Finally, from (49) we get the approximate quasinormal-mode contribution to the black-hole response to the Gaussian initial data

$$\Phi_Q(r_*, t) \approx \frac{a}{2}\sqrt{\frac{\pi}{b}}\mathrm{Re}\left\{\sum_{n=0}^{\infty} ie^{2i\eta}e^{-\omega_n^2/4b}\left[\frac{\partial\gamma_{21}}{\partial\omega}\right]^{-1}_{\omega=\omega_n} e^{-i\omega_n(t-t_0-x-r_*)}\right\} \ . \tag{57}$$

One might think that this must be a very crude approximation, the sole merit of which is that it was obtained analytically rather than numerically. To test (57) we have compared it to a numerical evolution of (1) for the same initial data. It turns out that (57) is in excellent agreement with the corresponding numerical result for intermediate times when the quasinormal-mode ringing dominates the emitted radiation. Examples of this are given in Figs. 3(a)–(c). For the approximation to be good, we must of course have the observer situated relatively far away from the black hole [otherwise (20) is not valid] and x must be large. If these restrictions are relaxed, (57) will not be that reliable, for obvious reasons. Nevertheless, tests suggest that it still predicts the amplitude of the quasinormal-mode ringing surprisingly well. But the approximate expression is no longer in phase with the exact response.

It may be worthwhile to say a few words about the numerical simulations: We used a standard second-order finite difference scheme as a quick (and rather dirty) way to solve (1) for a given set of initial data. The results obtained using this code are certainly reliable to the accuracy needed for comparisons with the approximate results discussed in this paper. But that the code has deficiencies can be seen in Figs. 3(b) and 3(c). At rather late times, just before the quasinormal-mode ringing drowns in the expected power-law tail [13,36], the signal is slightly irregular [this happens for t around 180 in Fig. 3(b) and 220 in Fig. 3(c)]. We have established that this is due to a tiny reflection from the boundary at the event horizon. It indicates that we should perhaps not trust our numerical solution after that time. On the other hand, it can be seen that the solution is dominated by a power-law falloff for later times. Moreover, the corresponding power-law exponents are in reasonable agreement with the theoretically anticipated values [36]. But, as is clear from Figs. 3(a)–(c), the possible deficiency of our numerical approach has no effect whatsoever on the conclusions of the present investigation. A more accurate numerical study of the problem was recently given by Gundlach *et al.* [36]. Our Figs. 3(a)–(c) can be compared with their Fig. 4.

In his numerical study of the black-hole response to Gaussian initial data Vishveshwara [35] concluded that for small values of b, when the initial Gaussian is extremely broad, no quasinormal-mode ringing could be seen in the reflected radiation. When the Gaussian was made thinner (b increases), ringing suddenly occurred, and the mode excitation reached a limit for large b. It is incredibly rewarding to find that our approximate formula supports these general conclusions well. It is clear that the absolute value of each term in (57) is proportional to

$$\frac{1}{\sqrt{b}}e^{-\mathrm{Re}(\omega_n^2)/4b} \ .$$

This means that, if we vary b, a certain mode will be maximally excited for $\mathrm{Re}(\omega_n^2) = 2b$. Since b is necessarily positive, only modes for which $|\mathrm{Re}\ \omega_n| > |\mathrm{Im}\ \omega_n|$ will have such maxima. According to Table I, this is the case

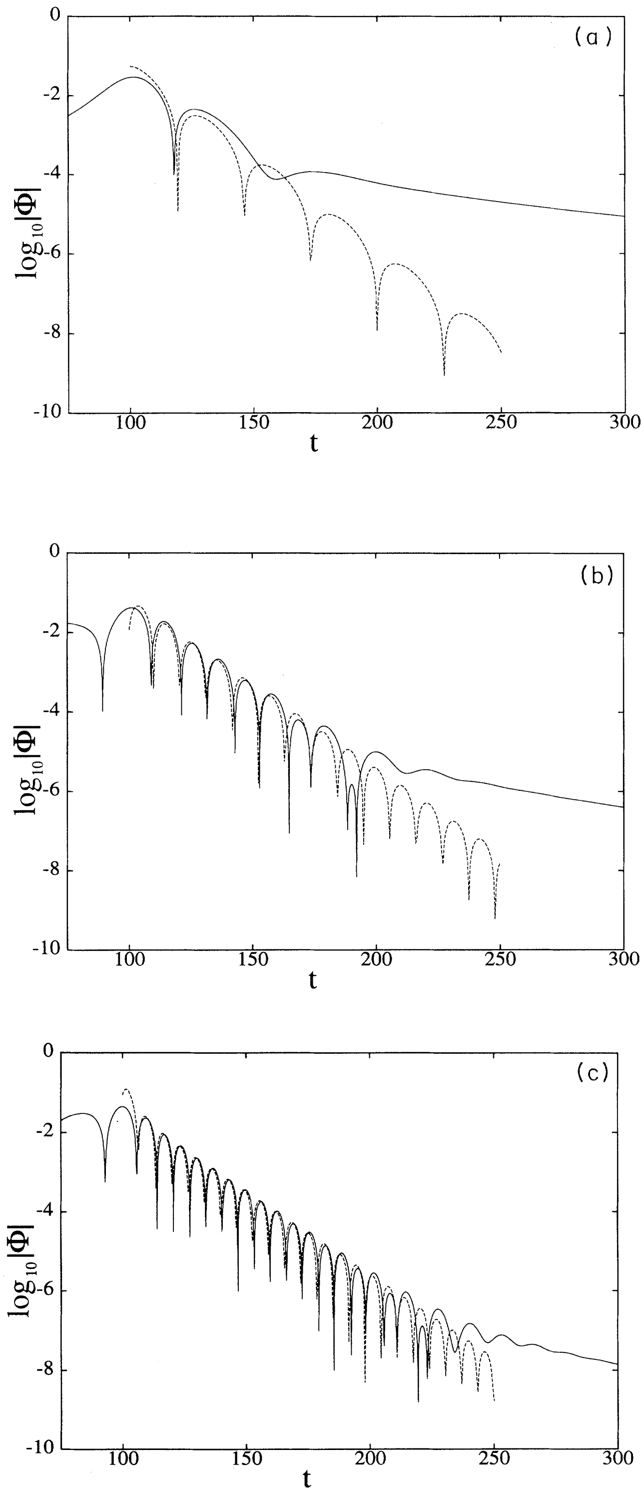

FIG. 3. Comparing the approximate expression (57) for Gaussian initial data (dashed line) to the result of a numerical evolution of the same data (solid line). The chosen Gaussian has $a = 0.15$, $b = 0.05M^{-2}$ and was centered at $x = 50M$ at $t_0 = 0$. The observer is at $r_* = 50M$. This means that the response of the black hole should not reach the observer before $t \approx 100M$. The three cases (a)–(c) correspond to $l = 0, 1$, and 2, respectively.

only for the lowest mode(s) for each value of l. Let us now recall that the quasinormal modes can be interpreted as resonances trapped in the region of the unstable photon orbit at $r = 3M$ (roughly the maximum of the potential barrier). Then the real part of the characteristic frequency for the slowest damped mode will be related to l by

$$\text{Re}\ \omega_0 \approx \left(l + \frac{1}{2}\right) \frac{1}{3\sqrt{3}M} \tag{58}$$

(see, for example, [37]). We use this approximation, which is quite accurate for $l > 0$, and assume that the square of the imaginary part of ω_n is negligible compared to the square of the real part. Then we find that, in order to excite the fundamental mode for a certain l maximally, one should use a Gaussian with half-width roughly equal to

$$12.24M \left(l + \frac{1}{2}\right)^{-1} .$$

For Gaussians wider than this the lowest mode will be exponentially suppressed. Then one would expect the first mode for which $|\text{Re}\ \omega_n| < |\text{Im}\ \omega_n|$ to dominate the ringing. But numerical simulations suggest that this mode is usually too rapidly damped to have a noticeable effect on the directly reflected part of the initial Gaussian.

X. CONCLUDING REMARKS

We have studied how the quasinormal modes of a Schwarzschild black hole are excited by a given set of Cauchy data. General formulas that describe the quasinormal-mode ringing, which is expected to dominate the response of a black hole at relatively late times, have been discussed in some detail. We have also shown how these formulas can be approximated within the phase-integral method. The resultant approximations have been proved to be in good agreement with previous results obtained by Leaver [13]. Hence, it seems likely that the final phase-integral formula (49) can be useful in future studies of more general kinds of initial data than those discussed here.

We have shown how the approximate formula (49) can be evaluated analytically in the specific case of Gaussian initial data. The approximation achieved in this way is remarkably accurate when compared to the result of a purely numerical simulation. It seems likely that we can learn a lot about the qualitative features of quasinormal-mode excitation from this approximate study. As an example of this, the formula suggests that the slowest damped quasinormal mode for a given multipole l is maximally excited for a Gaussian with half-width roughly equal to $12.24M(l+1/2)^{-1}$. For data that vary slower than this the mode should be exponentially suppressed. This is a very surprising result, but that it is, indeed, correct can be shown by numerical simulations where hardly any ringing at all is seen for broad

Gaussians. In fact, we have managed to explain a feature that was observed by Vishveshwara almost 25 years ago [35].

The results of the present investigation, especially the simple calculation for Gaussian initial data, are very encouraging. Nevertheless, it must be remembered that the present study has only provided a few small steps towards a full understanding of the quasinormal-mode phenomenon. We have, for example, learned nothing at all about *how* the modes are actually excited. In order to understand this one must clearly study the region close to the barrier top in some detail. Moreover, it is straightforward to point out several directions in which the present study could be continued. First of all, it may be worthwhile to extend the analysis leading to the approximate formula (45) for the coefficients α_n to higher orders of the phase-integral approximation. Then one can really hope to put the values that Leaver [13] obtained to the test. It is, however, unlikely that a high-order analysis will be much more accurate for the lowest value of l. In order to obtain accurate phase-integral results for $l = 0$ one must probably use uniform approximations [20]. It would also be interesting to see whether other methods that have been used to compute accurate quasinormal-mode frequencies can be extended in this direction. This does seem very likely. The numerical method in [9] is formally similar to the phase-integral method that was used above, and Nollert and Schmidt [17] actually generate the necessary Green's function in their study.

ACKNOWLEDGMENTS

It is a great pleasure to acknowledge C.V. Vishveshwara for several illuminating discussions of black-hole problems. I would also like to thank G.D. Allen for invaluable help with the numerical simulations.

[1] L. Stella, in *General Relativity and Gravitation 1992*, Proceedings of the Thirteenth International Conference, Cordoba, Argentina, 1992, edited by R.J. Gleiser, C.N. Kozameh, and O.M. Moreschi (IOP Publishing, Bristol, 1993), pp. 263–286.

[2] A. Abramovici, W.E. Althouse, R.W.P. Drever, Y. Gursel, S. Kawamura, F.J. Raab, D. Shoemaker, L. Sievers, R.E. Spero, K.S. Thorne, R.E. Vogt, R. Weiss, S.E. Whitcomb, and M.E. Zucker, Science **256**, 325 (1992).

[3] C. Bradaschia, R. DelFabbro, A. DiVirgilio, A. Giazotto, H. Kautzky, V. Montelatici, D. Passuello, A. Brillet, O. Cregut, P. Hello, C.N. Man, P.T. Manh, A. Marraud, D. Shoemaker, J.Y. Vinet, F. Barone , L. Difiore, L. Milano, G. Russo, J.M. Aguirregabiria, H. Bel, J.P. Duruisseau, G. Ledenmat, P. Tourrenc, M. Capozzi, M. Longo, M. Lops, I. Pinto, G. Rotoli, T. Damour, S. Bonazzola, J.A. Marck, E. Gourghoulon, L.E. Holloway, F. Fuligni, V. Iafolla, and G. Natale, Nucl. Instrum. Methods Phys. A **289**, 518 (1990).

[4] B.F. Whiting, in *Fifth Marcel Grossman Meeting on General Relativity*, Proceedings, Perth, Australia, 1988, edited by D.G. Blair and M.J. Buckingham (World Scientific, Singapore, 1989), pp. 1207–1216.

[5] B.F. Whiting, J. Math. Phys. **30**, 1301 (1989).

[6] J.W. York, Jr., Phys. Rev. D. **28**, 2929 (1983).

[7] J.A.H. Futterman, F.A. Handler, and R.A. Matzner, *Scattering from Black Holes* (Cambridge University Press, Cambridge, England, 1988) .

[8] R.A. Matzner and M.P. Ryan, Jr., Phys. Rev. D **16**, 1636 (1977).

[9] N. Andersson, Proc. R. Soc. London **A439**, 47 (1992).

[10] N. Andersson, M.E. Araújo, and B.F. Schutz, Class. Quantum Grav. **10**, 735 (1993).

[11] S.L. Detweiler, Proc. R. Soc. London **A352**, 381 (1977).

[12] S.A. Teukolsky, Astrophys. J. **185**, 635 (1973) .

[13] E.W. Leaver, Phys. Rev. D **34**, 384 (1986).

[14] Y. Sun and R.H. Price, Phys. Rev. D **38**, 1040 (1988).

[15] Y. Sun and R.H. Price, Phys. Rev. D **41**, 2492 (1990).

[16] E.W. Leaver, Proc. R. Soc. London **A402**, 285 (1985).

[17] H-P. Nollert and B.G. Schmidt, Phys. Rev. D **45**, 2617 (1992).

[18] N. Fröman, P.O. Fröman, N. Andersson, and A. Hökback, Phys. Rev. D **45**, 2909 (1992).

[19] N. Andersson, M.E. Araújo, and B.F. Schutz, Class. Quantum Grav. **10**, 757 (1993).

[20] N. Andersson and S. Linnæus, Phys. Rev. **D46**, 4179 (1992).

[21] N. Andersson, M.E. Araújo, and B.F. Schutz, Phys. Rev. D **49**, 2703 (1994).

[22] T. Regge and J.A. Wheeler, Phys. Rev. **108**, 1063 (1957).

[23] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).

[24] B.S. Kay and R.M. Wald, Class. Quantum Grav. **4**, 893 (1987).

[25] H-P. Nollert, Phys. Rev. D **47**, 5253 (1993).

[26] N. Andersson, Class. Quantum Grav. **10**, L61 (1993).

[27] H-P. Nollert, in *Fourth Marcel Grossman Meeting on General Relativity*, Proceedings, Rome, Italy, 1985, edited by R. Ruffini (North-Holland, Amsterdam, 1986), pp. 759–767.

[28] J. Mathews and R.L. Walker, *Mathematical Methods of Physics* (Addison-Wesley Publishing Company, Redwood City, CA, 1970).

[29] M.V. Berry, Proc. R. Soc. London **A422**, 7 (1989).

[30] M.E. Araújo, D. Nicholson, and B.F. Schutz, Class. Quantum Grav. **10**, 1127 (1993).

[31] N. Andersson, Phys. Rev. D **48**, 4771 (1993) .

[32] N. Andersson, "Scattering of scalar waves by a Schwarzschild black hole," report (unpublished).

[33] S. Iyer, Phys. Rev. D **35**, 3632 (1987).

[34] E. Seidel, Phys. Rev. D **44**, 950 (1991).

[35] C.V. Vishveshwara, Nature (London) **227**, 936 (1970).

[36] C. Gundlach, R.H. Price, and J. Pullin, Phys. Rev. D **49**, 883 (1994).

[37] K.S. Thorne, in *Theoretical Principles of Astrophysics and Relativity*, edited by N.R. Lebowitz, W.H. Reid, and P.O. Vandervoort (University of Chicago Press, Chicago, 1978), pp. 149–216.