Lionel London

An introduction to black hole perturbation theory

UK Royal Society

KING'S College LONDON

Kavli Institute for Cosmology Cambridge KICC

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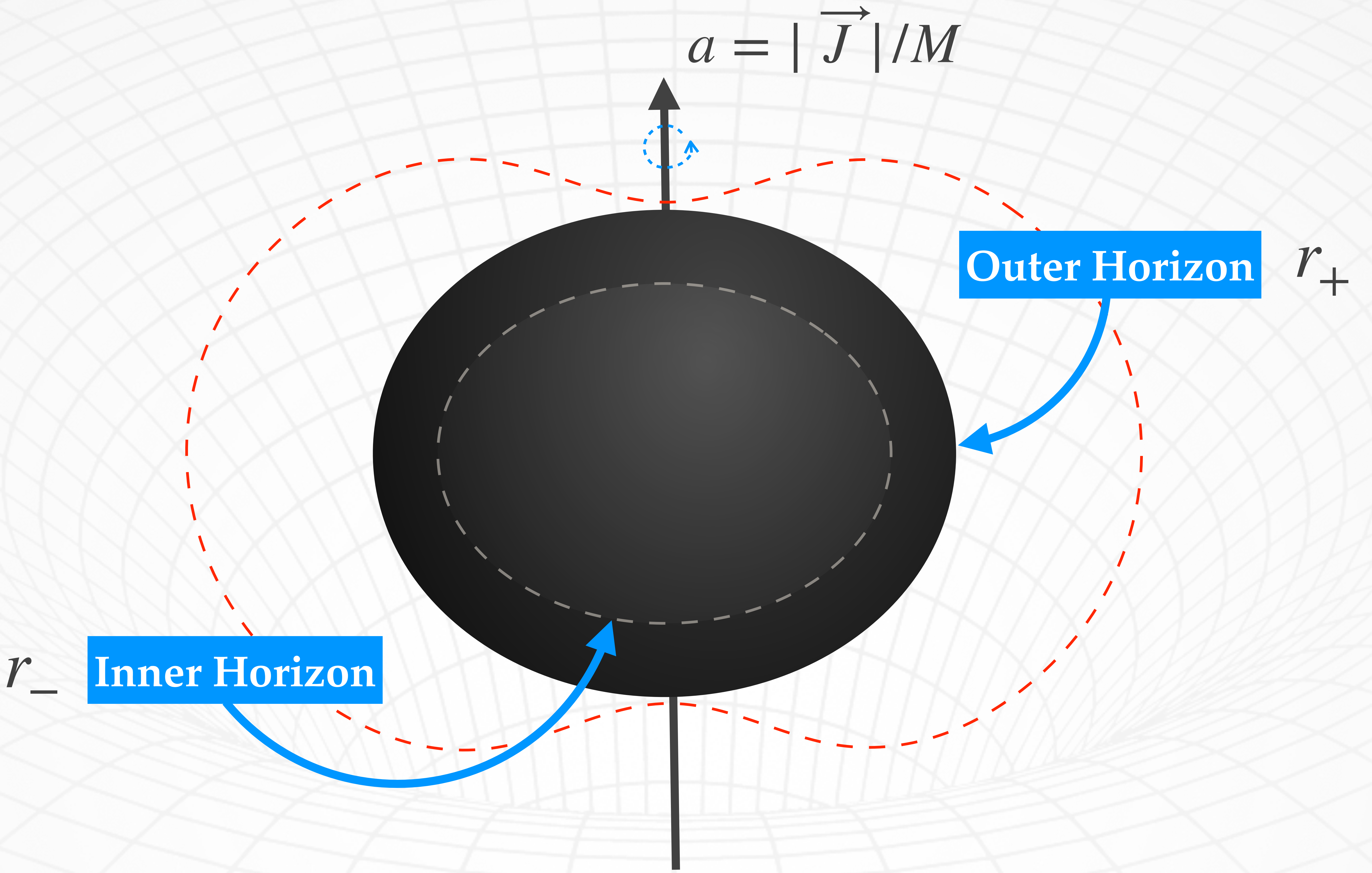
Kavli-Villum Summer School September 2023 Corfu



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Outline Introduction to BH perturbation theory 2

Today's topics

- Teukolsky's equations, recap and overview
- The road towards QNM solutions for Kerr
  - Teukolsky's angular equations, and analytic methods of solution
  - Teukolsky's radial equation
    - A refined (spatially compactified) coordinate choice
    - Derivation of asymptotic boundary conditions
    - A basic series solution
- Computing QNM frequencies by combining radial and angular recursion relations
- A primer on QNM orthogonality
- Questions / Hands on in Mathematica to review points above in more detail

Einstein's equations linearized around Kerr

1970s

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}$$

$$- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}$$

$$- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 4\pi\Sigma T\,.$$

Teukolsky's "master" equation

Separate with ansatz

Einstein's equations linearized around Kerr

1970s

$$
\left[\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\,\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}
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$$
- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 4\pi\overset{=0}{\cancel{\Sigma T}}\,.
$$

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BH Spin parameter = $|J|/M$

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}$$

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(r-rm)(r-rp)

Teukolsky's "master" equation

Separate with ansatz

Einstein’s equations linearized around Kerr

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(r-rm)(r-rp)

spin weight

Teukolsky’s “master” equation

Separate with ansatz

Einstein's equations linearized around Kerr

1970s

Teukolsky's equation has **far reaching applications**

"spin coefficient"

$$\rho = -1/(r - i\, a \cos\theta)$$

Teukolsky 1971

ψ	s	
Φ	0	scalar fields
χ_0 $\rho^{-1}\chi_1$	$\frac{1}{2}$ $-\frac{1}{2}$	neutrinos
ϕ_0 $\rho^{-2}\phi_2$	1 -1	electromagnetic fields
$\psi_0{}^{B}$ $\rho^{-4}\psi_4{}^{B}$	2 -2	gravitational fields

Teukolsky's "master" equation

Separate with ansatz

Towards QNM Solutions

1970s

The full form of a **single** QNM

$$\psi \;\propto\; R(r)\, S(\theta)\, e^{-i\tilde{\omega} t}\, e^{-im\phi}$$

In more general scenarios, one should perform an integral transform.

ukolsky's "master" equation | Separate with ansatz | Teukolsky's Equa

Towards QNM Solutions

1970s

$$u = \cos(\theta)$$

$$\mathcal{L}_u\, S(u) \;=\; -A\, S(u)$$

$$\mathcal{L}_r\, R(r) \;=\; +A\, R(r)$$

The QNMs' radial and angular functions are special functions that, when solved simultaneously, fully constrain a Kerr QNM

Separate with ansatz | **Teukolsky's Equations** | The Angular Equatio

Towards QNM Solutions

1970s

$$\mathcal{L}_u = V_S(u) + \partial_u(1 - u^2)\partial_u$$

Having chosen ***u*** rather than theta, the angular equation is explicitly in "**Sturm-Liouville form**".

- The eigenfunctions of Sturm-Liouville problems are known to be orthogonal and complete (Functional analysis creeping in)
- The problem's potential is ***almost*** of the Legendre type

Towards QNM Solutions

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Towards QNM Solutions

1970s

$$V_S(u) = s + ua\tilde{\omega}(ua\tilde{\omega} - 2s) - \frac{(m+su)^2}{1-u^2}$$

- The problem's potential is ***almost*** of the Legendre type
- The Legendre-like part is the rational function
- The additional quadratic dependence on the domain variable add an **irregular singular point at u~infinity**
- The frequency parameter also appears quadratically

Towards QNM Solutions

1970s

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Towards QNM Solutions

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$$V_S(u) \;=\; s + ua\tilde{\omega}(ua\tilde{\omega} - 2s) - \frac{(m+su)^2}{1-u^2}$$

- The angular problem has two regular singular points, and one irregular one, making it a "**confluent Heun**" equation
- **A hypergeometric equation** (e.g. the Legendre equation) has at most three regular singular points ("confluence" **reduces** this number)

Towards QNM Solutions

1970s

$u = \cos(\theta) \leq 1 \ll \infty$

$$\mathcal{L}_u = V_S(u) + \partial_u(1-u^2)\partial_u$$

$$V_S(u) \;=\; s + ua\tilde{w}(ua\tilde{w} - 2s) - \frac{(m+su)^2}{1-u^2}$$

- The angular equation is effectively hypergeometric because the confluent singular point is outside of the physical domain.
- As a result, we can construct analytic solutions in ***almost*** the same way that we can for e.g. the Legendre polynomials

Towards QNM Solutions

$$\mathcal{L}_u \, S(u) = -A \, S(u)$$

- The **spheroidal harmonics** are eigenfunctions of the angular differential operator
- As a result, we can construct analytic solutions in ***almost*** the same way that we can for e.g. the Legendre polynomials

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

Frobenious series solution

$$S(u) \;=\; e^{\kappa_\infty u}(1-u)^{\kappa_-}(1-u)^{\kappa_+}\boxed{\sum_{k=0}^{\infty} a_k\,(1+u)^k}$$

- A "Frobenious" expansion is sufficient to form a global solution
- The prefactor function represents a "similarity", "conformal" or "holomorphic" transformation on the original problem
- Eigenvalues are preserved

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

"conformal factor"

$$S(u) = \underbrace{e^{\kappa_\infty u}(1-u)^{\kappa_-}(1-u)^{\kappa_+}} \sum_{k=0}^{\infty} a_k \,(1+u)^k$$

- A "Frobenious" expansion is sufficient to form a global solution
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Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$S(u) = e^{\kappa_\infty u}(1-u)^{\kappa_-}(1-u)^{\kappa_+}\sum_{k=0}^{\infty} a_k\,(1+u)^k$$

- The constants, $\{\kappa_-, \kappa_+, \kappa_\infty\}$, can be determined by regularity conditions and/or "asymptotic" boundary conditions
- Regularity conditions: enforcing that the potential is at most linear
- Asymptotic boundary conditions: physical requirements on functional form near boundaries. Here: only that the functions are finite at u=+-1

Towards QNM Solutions

Leaver (1985), “Analytic representation for the modes of Kerr black holes”

$$S(u) \;=\; e^{\kappa_\infty u}(1-u)^{\kappa_-}(1-u)^{\kappa_+}\sum_{k=0}^{\infty} a_k \,(1+u)^k$$

- The expansion coefficients, a_k, are determined by a three-term recursion relation that depends on the separation constant *A*
- For convergence of the infinite sum, the separation constant can only take on special discrete values ultimately label in ℓ … more on this later

Towards QNM Solutions

1980s

Leaver (1985), “Analytic representation for the modes of Kerr black holes”

$$\mathcal{L}_r\, R(r) \;=\; +A\, R(r)$$

Some connections and tensions with the angular equation:

- Also 3 singular points, one confluent. So it’s also a confluent Heun equation.
- But the confluent singular point (at *r*~infinity) is inside of the domain this time!

Spheroidal Harmonics | The Radial Equation | Homogeneous Soluti

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\mathcal{L}_r \, R(r) \; = \; +A \, R(r)$$

Some connections and tensions with the angular equation:

- It's **not clear** whether there is a simple coordinate choice that puts the problem in Sturm-Liouville form.
- Thus is has **not been clear** whether there the "homogeneous functions" ***R***, are orthogonal and complete (very recent progress)

Spheroidal Harmonics | The Radial Equation | Homogeneous Soluti

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\begin{aligned}\mathcal{L}_r \;=\; & \left(\mathrm{A}_0 + \mathrm{A}_1 r + \mathrm{A}_2\, r^2 + \frac{\mathrm{A}_3}{r - r_-} + \frac{\mathrm{A}_4}{r - r_+} \right) \\ & + \;(\mathrm{A}_5 + \mathrm{A}_6 r)\, \partial_r \\ & + \;(r - r_-)(r - r_+)\, \partial_r^2\end{aligned}$$

A part of the confusion is due to coordinate choice: The physical boundary conditions are defined in the vicinity of r-plus and r~Infinity, **not at r-minus**. However, the Boyer-lindquist coordinate doesn't reflect this.

Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\xi = \frac{r - r_+}{r - r_-}$$

Taking inspiration from the quantum mechanics literature, Leaver introduced an alternative coordinate, xi.

Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$\xi = \frac{r-r_+}{r-r_-}$

$$\mathcal{L}_\xi = \left(\mathrm{B0} + \frac{\mathrm{B1}}{\xi} + \frac{\mathrm{B2}}{1-\xi} + \frac{\mathrm{B3}^2}{(1-\xi)^2} + \mathrm{B4}(1-\xi) \right) + \left(\mathrm{B5}(1-\xi) + \mathrm{B6}(1-\xi)^2 \right) \partial_\xi + \xi(1-\xi)^2 \partial_\xi^2$$

This coordinate results in a different form of the radial operator.

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\xi = \frac{r-r_+}{r-r_-}$$

$$\mathcal{L}_\xi = \left(\mathrm{B0} + \frac{\mathrm{B1}}{\xi} + \frac{\mathrm{B2}}{1-\xi} + \frac{\mathrm{B3}^2}{(1-\xi)^2} + \mathrm{B4}(1-\xi)\right) + \left(\mathrm{B5}(1-\xi) + \mathrm{B6}(1-\xi)^2\right)\partial_\xi + \xi(1-\xi)^2\partial_\xi^2$$

This new operator is not in Sturm-Liousville form, but its second derivative coefficient is zero at the event horizon and spatial infinity … (more on this later)

Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$R(\xi) \;=\; e^{\sigma_\infty \xi} \xi^{\sigma_0} (1-\xi)^{\sigma_1} \sum_{k=0}^{\infty} b_k \, \xi^k$$

In 1985, Leaver sought a series solution to the radial equation using the same concepts that were applied to the angular problem.

Towards QNM Solutions

1980s

Leaver (1985), “Analytic representation for the modes of Kerr black holes”

$$R(\xi) \;=\; e^{\sigma_\infty \xi}\,\xi^{\sigma_0}\,(1-\xi)^{\sigma_1} \sum_{k=0}^{\infty} b_k\,\xi^k$$

As with the angular problem, the constants, $\{\sigma_-, \sigma_+, \sigma_\infty\}$, may be determined by enforcing physical (asymptotic) boundary conditions.

Towards QNM Solutions

Leaver (1985), “Analytic representation for the modes of Kerr black holes”

$$R(\xi) \;=\; e^{\sigma_\infty \xi}\,\xi^{\sigma_0}(1-\xi)^{\sigma_1}\sum_{k=0}^{\infty} b_k\,\xi^k$$

Asymptotic boundary conditions for the radial problem:

- R($r \to \infty$) is a purely outgoing wave (towards spatial infinity)
- R($r \to 0$) is a purely ingoing wave (towards the event horizon)

omogeneous Solutions | QNM Boundary Conditions | Three-term recursio

Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$R(\xi) \;=\; e^{\sigma_\infty \xi}\,\xi^{\sigma_0}(1-\xi)^{\sigma_1}\sum_{k=0}^{\infty} b_k\,\xi^k$$

Explicit function forms of the boundary conditions may be worked out via asymptotic analysis of the differential equation, **or** by inspection of phase velocities after allowed values of sigmas have been determined algebraically. Leaver plugged and chugged …

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$R(\xi) \;=\; e^{\sigma_\infty \xi}\, \xi^{\sigma_0} (1-\xi)^{\sigma_1} \sum_{k=0}^{\infty} b_k\, \xi^k$$

As with the angular problem, b_k, are only valid if the satisfy a special three-term recursion reaction and its discrete eigenvalues are ultimately labeled in ***n***. **So we now have two such relations that must be solved simultaneously ….**

omogeneous Solutions | QNM Boundary Conditions | Three-term recursio

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\begin{aligned} \alpha_k^u a_{k+1} + \beta_k^u a_k + \gamma_k^u a_{k-1} &= 0 \\ \alpha_0^u a_1 + \beta_0^u a_0 &= 0 \end{aligned}$$

From the angular problem

$$\begin{aligned} \alpha_k^\xi b_{k+1} + \beta_k^\xi b_k + \gamma_k^\xi b_{k-1} &= 0 \\ \alpha_0^\xi b_1 + \beta_0^\xi b_0 &= 0 \end{aligned}$$

From the radial problem

- Each alpha, beta, gamma coefficient depends on BH spin and frequency parameter
- Only the beta coefficients depend on the separation constant, *A*

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

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From the radial problem

Three-term recursions have two solutions: one is an absolutely convergent series, and the other is not. The absolutely convergent one is sometimes called "minimal" and it's the one we want — i.e. physical series solutions converge.

Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

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From the radial problem

The requirement of convergence can be used to determine eigenvalues of the radial and angular problems separately. We can then think about them together.

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\alpha_k^u a_{k+1} + \beta_k^u a_k + \gamma_k^u a_{k-1} = 0$$
$$\alpha_0^u a_1 + \beta_0^u a_0 = 0$$

From the angular problem

$$R_k^u = \frac{a_k}{a_{k-1}}$$

For the angular problem, we may construct a continued fraction by considering the ration of adjunct terms. An easy way: Define the ratio, and then plug in a(k-1)

Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

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From the angular problem

$$R_k^u = -\frac{a_k \gamma_k^u}{a_k \beta_k^u + a_{k+1} \alpha_k^u}$$

For the angular problem, we may construct a continued fraction by considering the ration of adjunct terms. An easy way: Define the ratio, and then plug in a(k-1)

Towards QNM Solutions

1980s

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From the angular problem

$$R_k^u = -\frac{a_k \gamma_k^u}{a_k \beta_k^u + a_{k+1} \alpha_k^u}$$

And then divide the numerator and denominator by a(k)

NM Boundary Conditions | Three-term recursions | QNM Eigenvalues

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\begin{aligned} \alpha_k^u a_{k+1} + \beta_k^u a_k + \gamma_k^u a_{k-1} &= 0 \\ \alpha_0^u a_1 + \beta_0^u a_0 &= 0 \end{aligned}$$

From the angular problem

$$R_k^u = -\frac{\gamma_k^u}{\beta_k^u + R_{k+1}^u \alpha_k^u}$$

From here, one can see that iteratively substituting in Rk creates a fraction within a fraction within a fraction … i.e. a continued fraction.

NM Boundary Conditions | Three-term recursions | QNM Eigenvalues

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

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$$\alpha_0^u a_1 + \beta_0^u a_0 = 0$$

From the angular problem

$$R_1^u = -\frac{a_1}{a_0}$$

In turn, this means that the recursion's boundary condition at zero can be framed to explicitly include all recursions …

Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\begin{aligned} \alpha_k^u a_{k+1} + \beta_k^u a_k + \gamma_k^u a_{k-1} &= 0 \\ \alpha_0^u a_1 + \beta_0^u a_0 &= 0 \end{aligned}$$

From the angular problem

$$R_1^u = \frac{a_1}{a_0} = -\frac{\gamma_1^u}{\beta_1^u + R_2^u \alpha_1^u}$$

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Towards QNM Solutions

1980s

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From the angular problem

$$|\, a_1 + a_0 \frac{\gamma_1^u}{\beta_1^u + R_2^u \alpha_1^u} \,| = 0$$

This allows for the definition of a kind of generalized "characteristic equation". Keep in mind that R_2 is to be continued to include all higher R_k.

Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

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From the angular problem

$$|a_1 + a_0 \frac{\gamma_1^u}{\beta_1^u + R_2^u \alpha_1^u}| = 0$$

The characteristic equation can be thought of a polynomial with discrete roots that only occurs at special values of the separation constant A.

NM Boundary Conditions | Three-term recursions | QNM Eigenvalues

Towards QNM Solutions

Leaver (1985), “Analytic representation for the modes of Kerr black holes”

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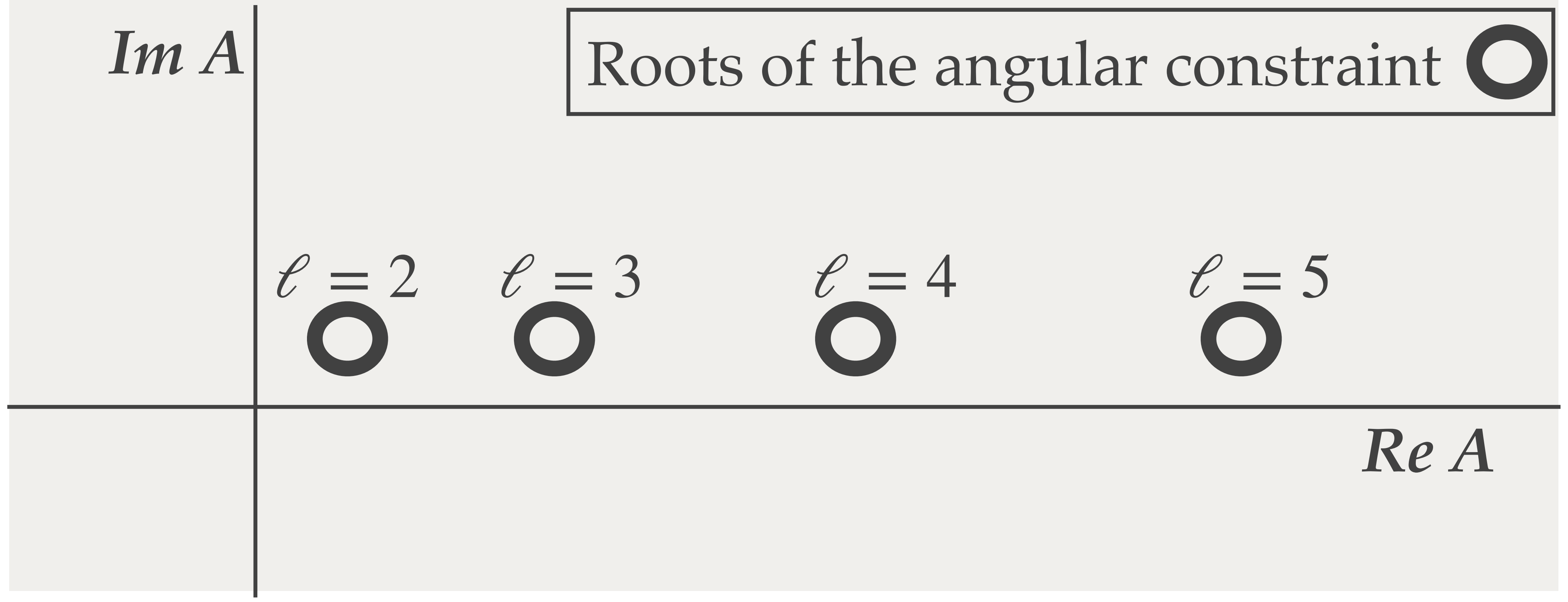

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Towards QNM Solutions

1980s

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

$$\left| a_1 + a_0 \frac{\gamma_1^u}{\beta_1^u + R_2^u \alpha_1^u} \right| = 0$$

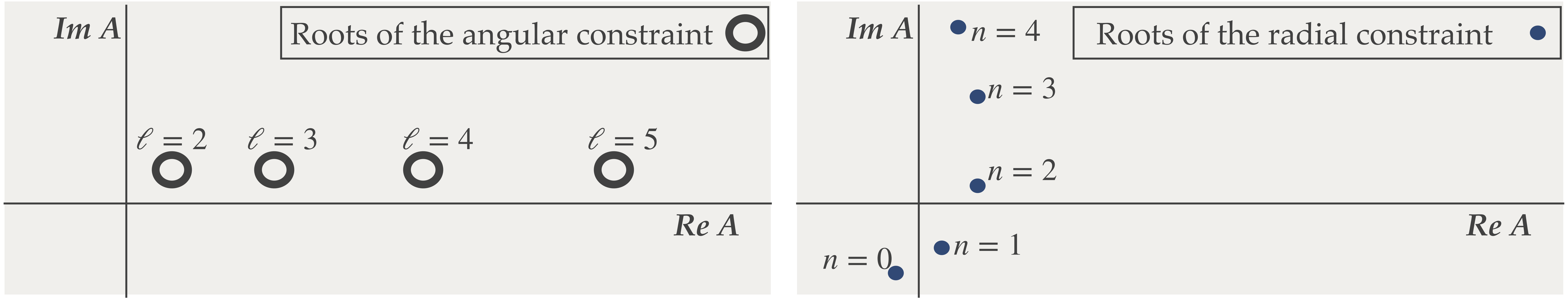
$$\left| b_1 + b_0 \frac{\gamma_1^\xi}{\beta_1^\xi + R_2^\xi \alpha_1^\xi} \right| = 0$$

Recall that we also have a three-term recursion for the radial problem.
This has its own separate continued fraction.

NM Boundary Conditions | **Three-term recursions** | QNM Eigenvalues

Towards QNM Solutions

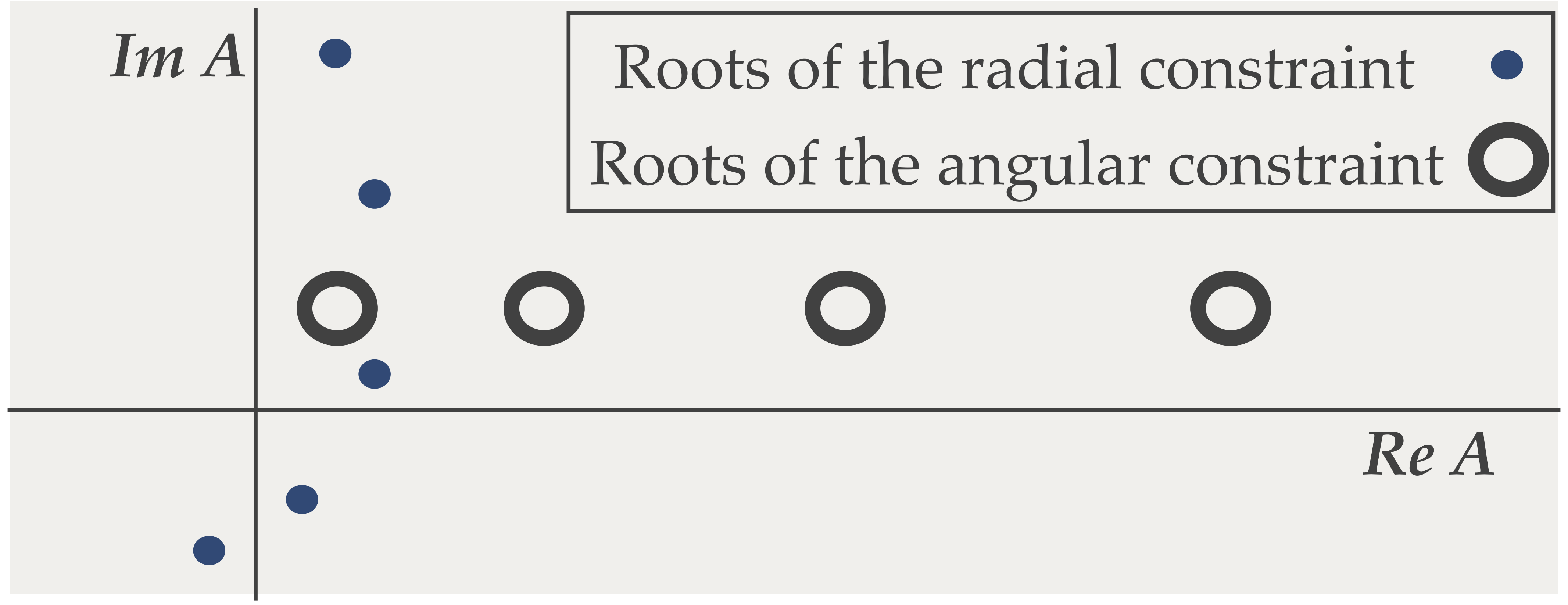
Leaver (1985), "Analytic representation for the modes of Kerr black holes"



The characteristic equation can be thought of a polynomial with discrete roots that only occur at special values of the separation constant ***A***.

Towards QNM Solutions

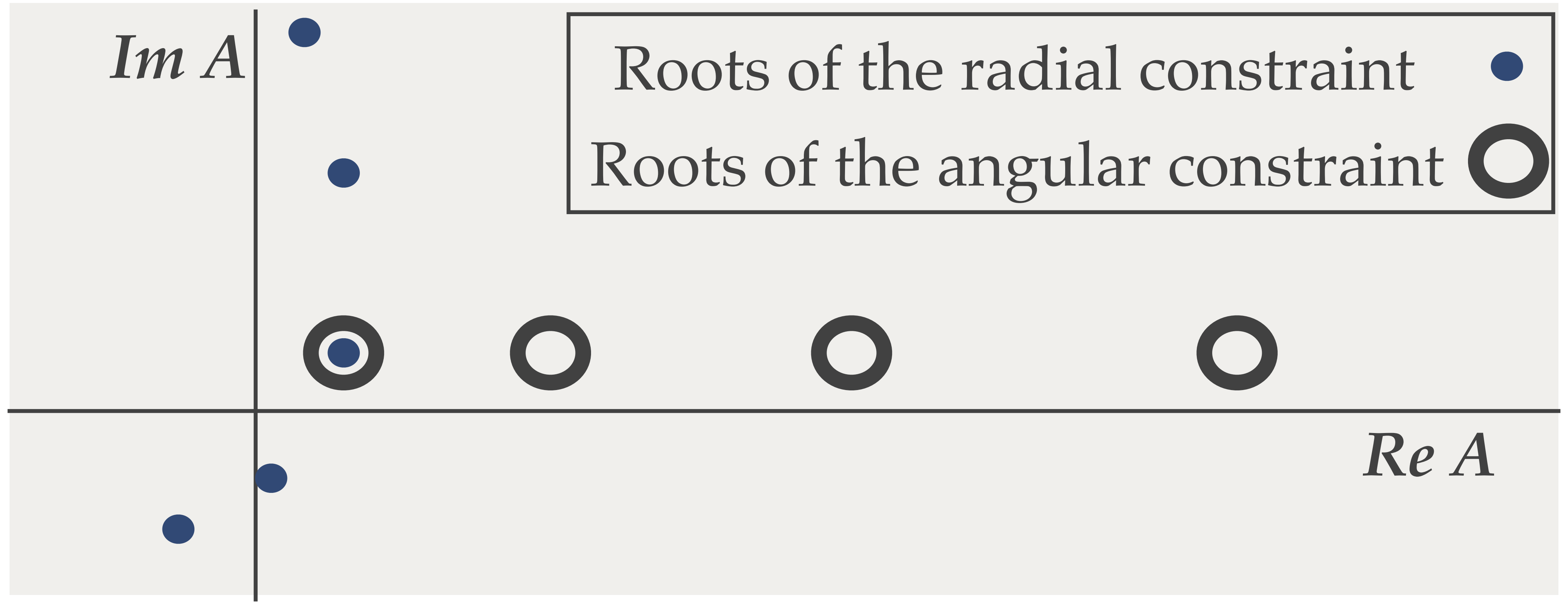
Leaver (1985), "Analytic representation for the modes of Kerr black holes"



For arbitrary frequencies, the two eigenvalue distributions will not necessarily coincide. That is, we may think of the separation constants as: $A=A(a,w,l,m,n)$

Towards QNM Solutions

Leaver (1985), "Analytic representation for the modes of Kerr black holes"

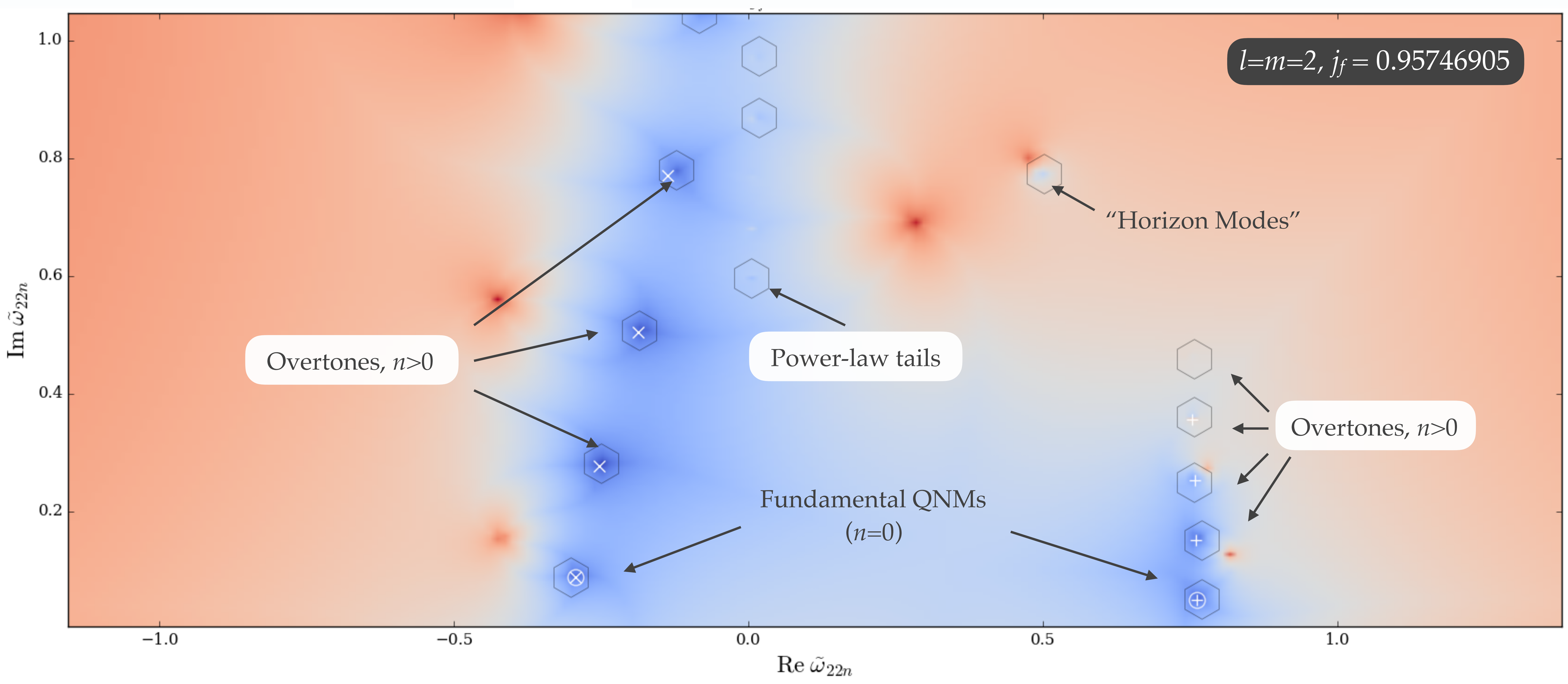


Since the radial and angular values of *A* must be identical for physical solutions, w may only take on special discrete values — these are the complex valued QNM frequencies.

Three-term recursions | QNM Eigenvalues | The QNM Frequenci

Black hole specific special functions

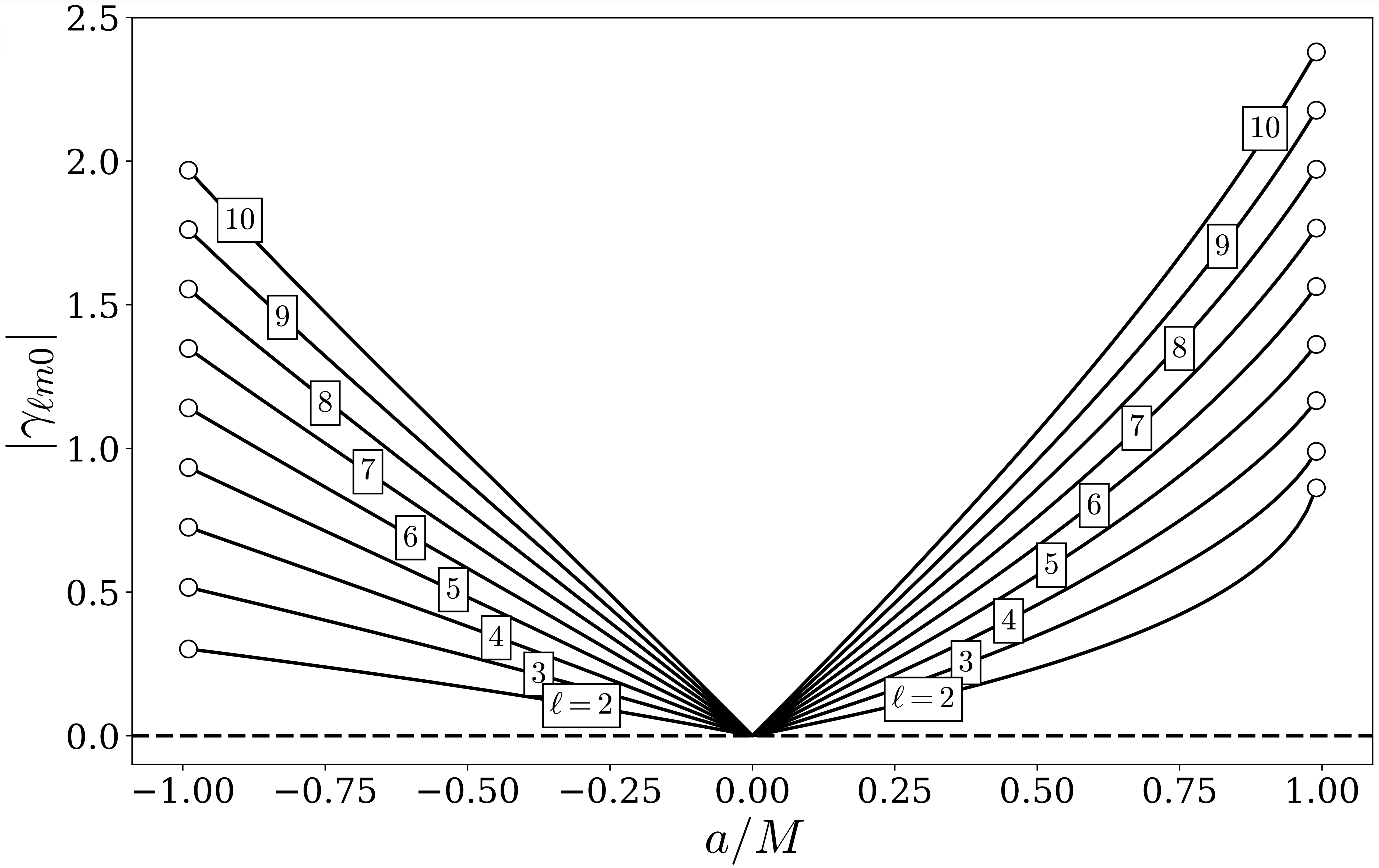
1980s-2000s



QNM Eigenvalues | **The QNM Frequencies** | QNM Orthogonalit

Black hole specific special functions

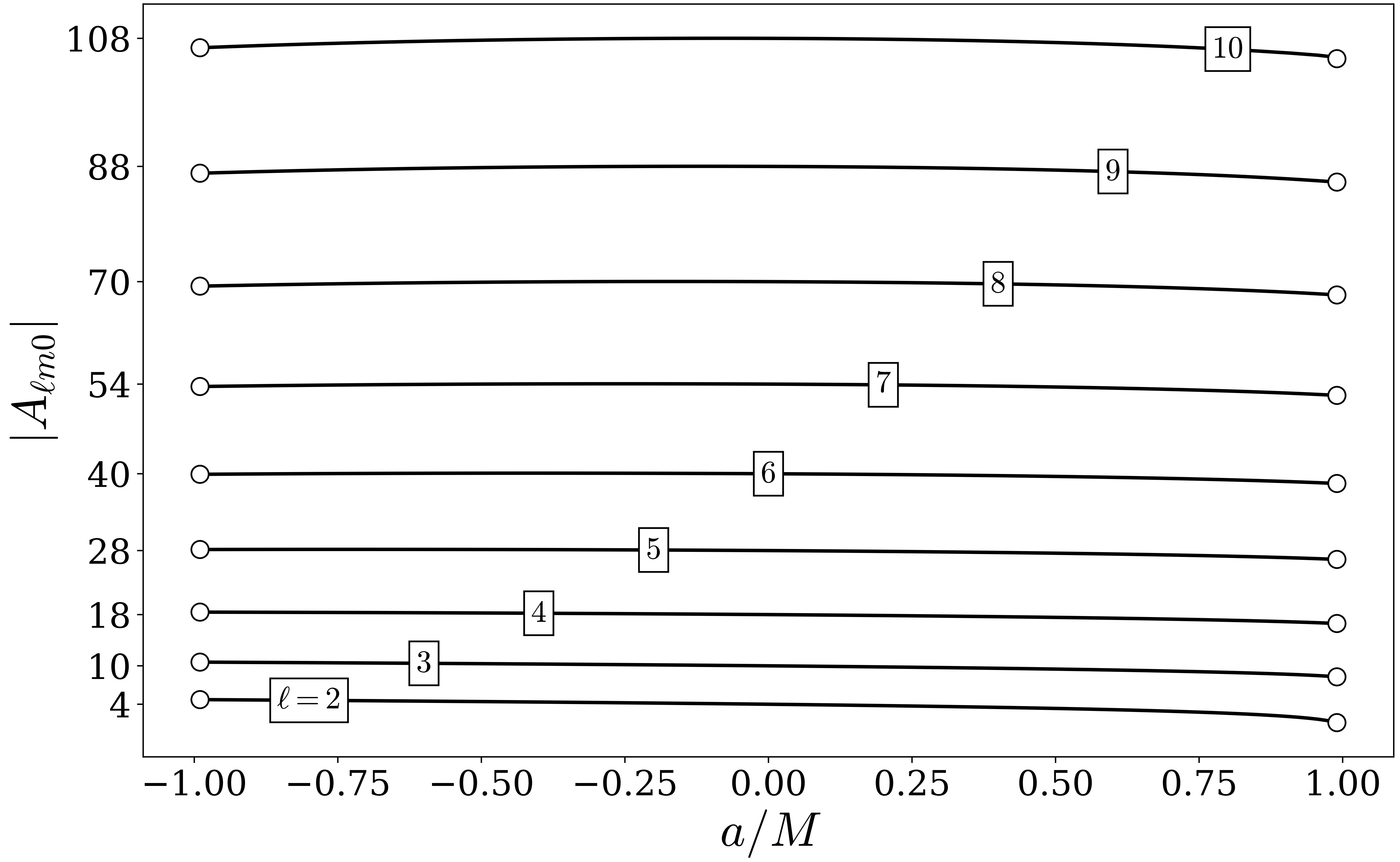
The QNM frequencies are typically unique functions of BH spin, and the retrograde frequencies are sometimes thought of as corresponding to negative spin. This is not quite right, but can be practical.



Black hole specific special functions

1980s-2000s

The QNM separation constants are typically unique. Recall from linear algebra that this implies that the QNM functions themselves are linearly independent, and perhaps orthogonal (in some way).



he QNM Frequencies

QNM Orthogonality Primer

Black hole specific special functions

1980s-2000s

Bi-Orthogonality of QNM
spheroidal harmonics

he QNM Frequencies

QNM Orthogonality Primer

Black hole specific special functions

The hard parts

- $$\mathcal{D}_u(\omega_{\ell mn}) = (ua\omega_{\ell mn} - 2s)ua\omega_{\ell mn} - \frac{(m+su)^2}{1-u^2} + \partial_u(1-u^2)\partial_u$$

- $$\mathcal{D}_u(\omega_{\ell mn})^{\dagger} \; = \; \mathcal{D}_u(\omega_{\ell mn})^{*}$$

Their solutions

- $$\mathcal{T} \; = \sum_{\ell=2}^{\infty}\sum_{\ell'=2}^{\infty} |Y_{\ell m}\rangle\langle Y_{\ell m}|S_{\ell' m}\rangle\langle Y_{\ell' m}|$$

- $$|S_\ell\rangle \; = \; \mathcal{T}\,|Y_{\ell m}\rangle\,, \quad \text{and} \quad |\tilde{S}_\ell\rangle \; = \; \mathcal{T}^{\dagger^{-1}}\,|Y_\ell\rangle$$

new "adjont-spheroidals"

Black hole specific special functions

1980s-2000s

The hard parts

- $$\mathcal{D}_u(\omega_{\ell mn}) = (ua\omega_{\ell mn} - 2s)ua\omega_{\ell mn} - \frac{(m+su)^2}{1-u^2} + \partial_u(1-u^2)\partial_u$$

- $$\mathcal{D}_u(\omega_{\ell mn})^{\dagger} = \mathcal{D}_u(\omega_{\ell mn})^{*}$$

Their solutions

bi-orthogonality

- $$\langle \tilde{S}_\ell | S_{\ell'} \rangle = \int_\Omega {}_s\tilde{S}^*_\ell(\theta; \tilde{\omega}_{\ell mn})\ {}_sS_{\ell'}(\theta; \tilde{\omega}_{\ell' mn})\ d\Omega = \delta_{\ell\ell'}/2\pi$$

- $$\sum_{\ell=2}^{\infty} \sum_{\ell'=2}^{\infty} |S_\ell\rangle\langle\tilde{S}_{\ell'}| = \hat{\mathbb{I}}$$

completeness: the ability to represent arbitrary GW signals

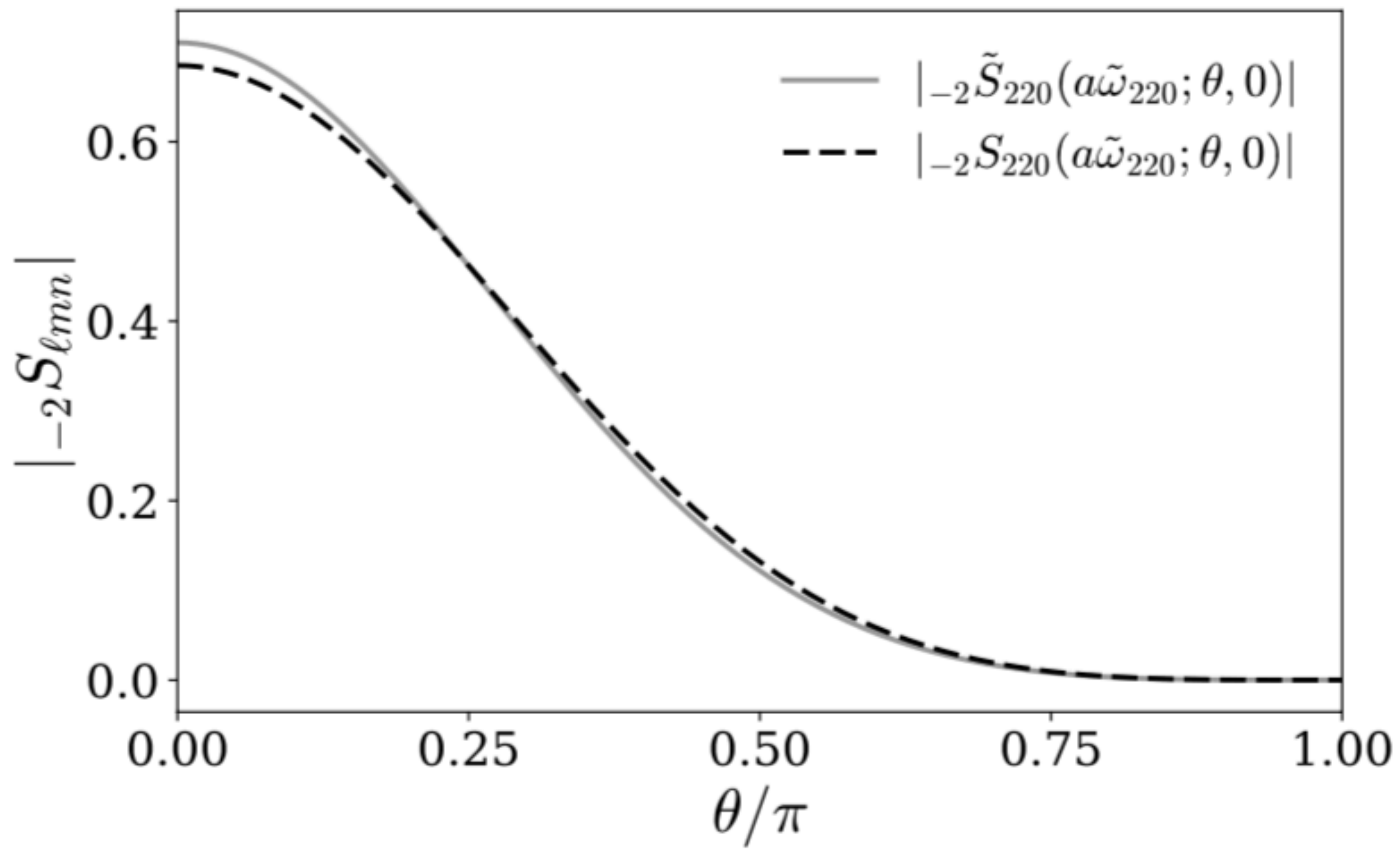
Black hole specific special functions

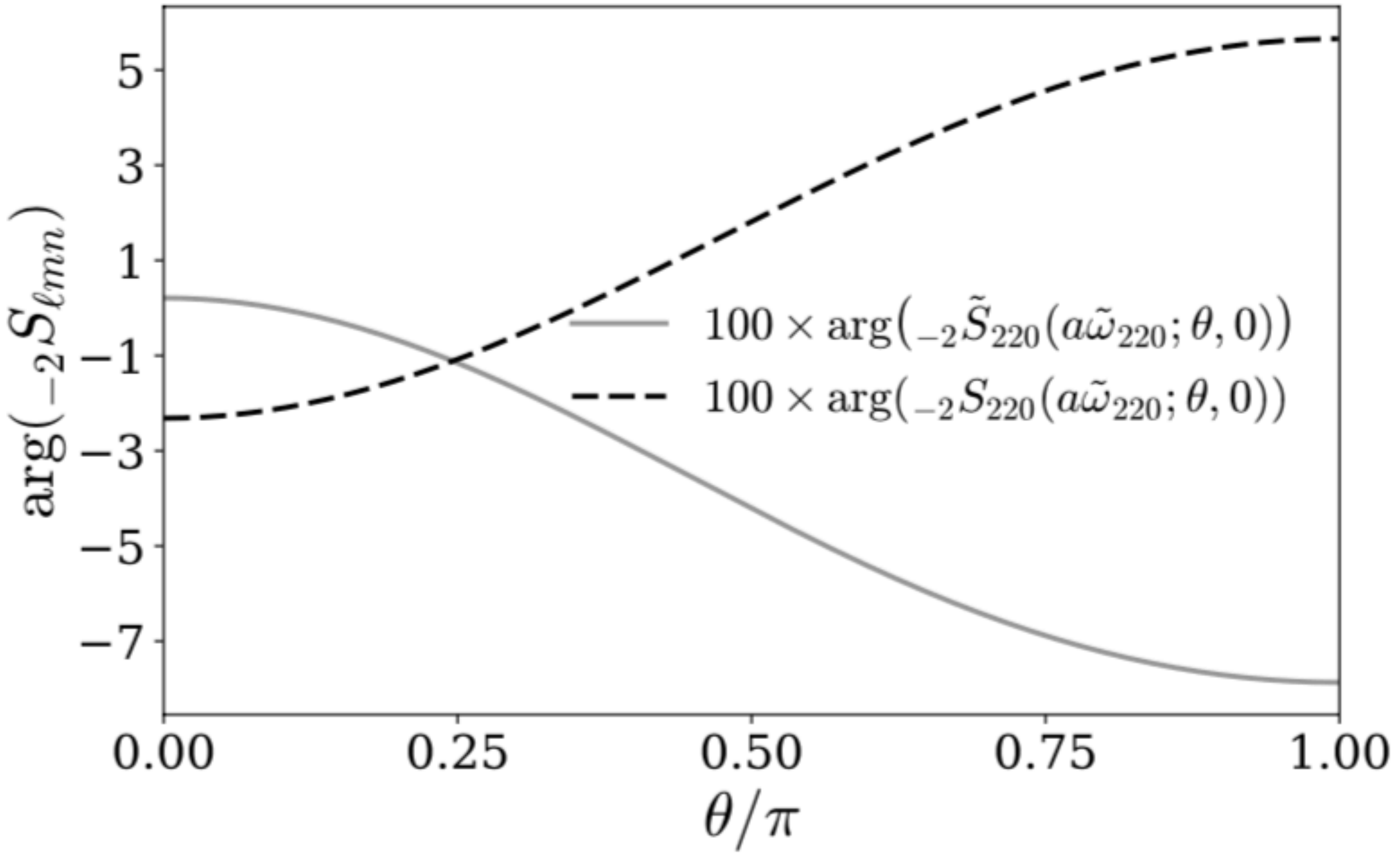
1980s-2000s

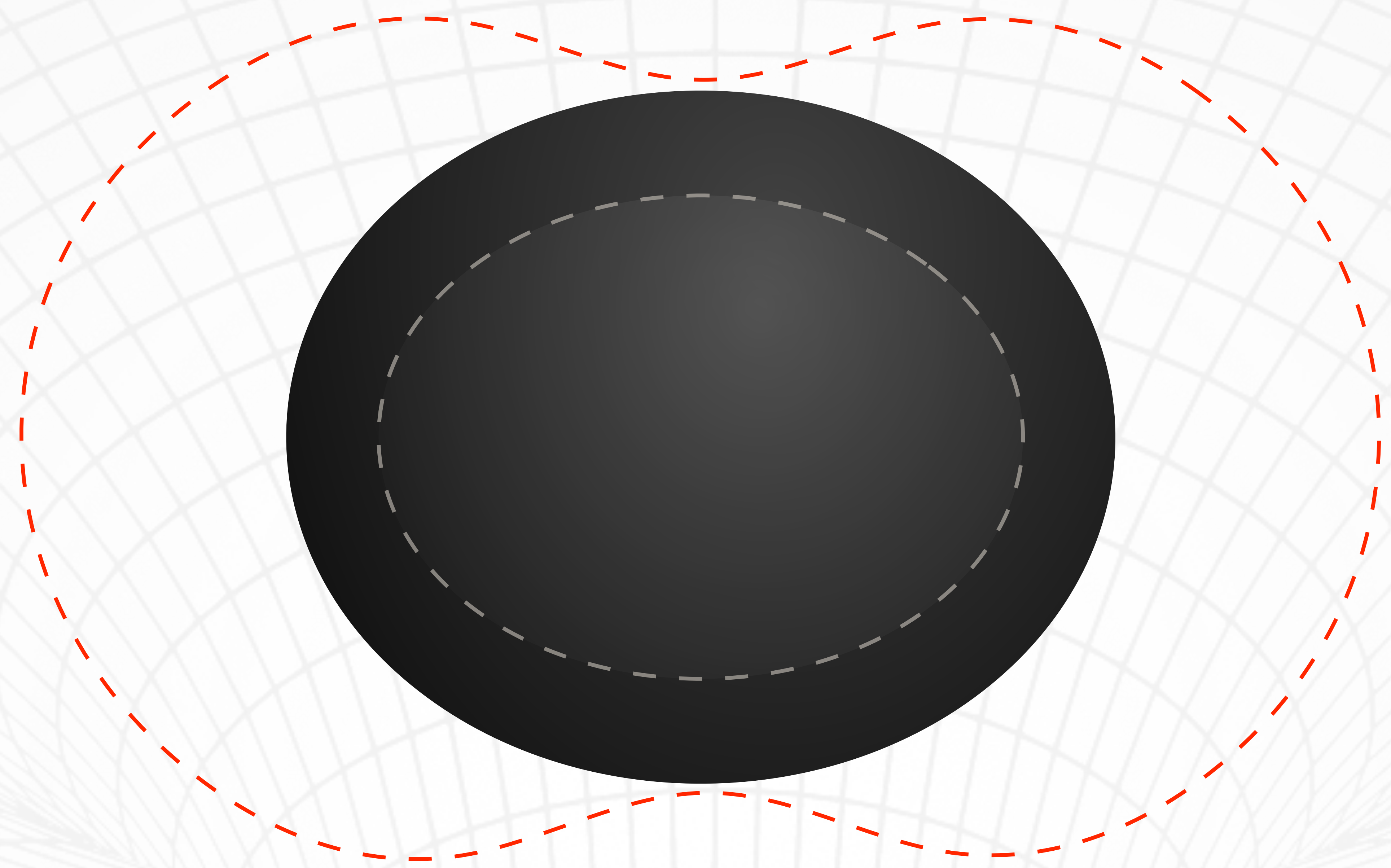
Example "small spin" expansion for term-by-term comparison:

$$S_{\ell mn} \approx \quad Y_{\ell m} + a\tilde{\omega}_{\ell mn} c_{\ell}^{\ell-1} Y_{\ell-1,m} + a\tilde{\omega}_{\ell mn} c_{\ell}^{\ell+1} Y_{\ell+1,m}$$

$$\tilde{S}_{\ell mn} \approx -Y_{\ell m} + a\tilde{\omega}^{*}_{\ell-1,m,n} c^{\ell}_{\ell-1} Y_{\ell-1,m} + a\tilde{\omega}^{*}_{\ell+1,m,n} c^{\ell}_{\ell+1} Y_{\ell+1,m}$$







Let’s stop here for questions and more details