Mode stability of the Kerr black hole

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Separate differential and integral transformations are introduced for the individual radial and angular equations governing the (infinitesimally) gauge invariant Newman–Penrose quantities which represent massless perturbations of the Kerr black hole. Using these new transformations it is shown, without need for numerical investigation or reference to the analytic behavior of the separation constant, that no unstable mode perturbations exist for any physical value of the spin of massless fields on the rotating black hole background.

I. INTRODUCTION

In its most general form, which has been studied at great length over the last 30 years, the problem of black hole stability arises as an initial value problem for the stability of linearized gravitational perturbations on a black hole space-time background. Frequently, the methods used are also applicable to other massless fields of lower spin, so that often a general analysis is able to provide, simultaneously, results for all perturbations of physical or theoretical interest. The most recent results can be categorized[1] as (i) a transition away from mode analysis to obtain pointwise bounds for perturbation of the Schwarzschild space-time; (ii) a demonstration of mode stability for general massless perturbations of the Kerr black hole; and (iii) a specification of the criteria for stability in the boundary value problem for spherical geometries, which arises in the comparatively new context of gravitational thermodynamics. Although result (iii) will not be considered further here, it is perhaps useful to note that in the thermodynamic regime dominated by black hole geometries, the condition for gravitational stability exactly ensures that the various criteria are satisfied for the independently defined question of thermodynamical stability.[2]

With regard to the initial value problem for perturbations on spherical black hole space-times, whether eternal or forming from collapse, the best available results beyond mode stability[3,4] are found in the recent work of Kay and Wald.[5] Kay and Wald[5] have established that the evolution of regular (i.e., smooth and bounded), compact initial data will remain pointwise bounded in time throughout the entire domain of outer communication, including the boundary (horizon). An important point of physical relevance is that the result of Kay and Wald holds even for fields that do not vanish initially on the horizon (nor, in particular, on the bifurcation two-sphere of the global, Kruskal extension of the exterior Schwarzschild space-time).

The other new result pertinent to the initial value problem for black hole stability is the subject of this paper, viz. mode stability for perturbations of the Kerr (rotating) black hole. Although of a different nature than the most recent results for the Schwarzschild space-time, the work reported here represents a major step forward in the study of black hole stability. The present work is also of great astrophysical significance since almost all astronomical bodies rotate, including those that might eventually undergo gravitational collapse. Thus stability of rotating black holes is really a more pressing concern than is the stability of a static, spherically symmetric black hole.

In the remainder of this paper, an introduction to the Kerr metric is first given, after which the difficult nature of the stability problem that it presents is indicated. Then, from a tractable form of the perturbation equations, differential and integral transformations of the equations and their solutions are developed. Finally, the construction of a positive definite "energy" integral is given, permitting the proof of mode stability to be completed. The Appendix gives some mathematical details. The proof given here does not adhere closely to any previous work on this problem.

II. THE KERR BLACK HOLE

The Kerr metric, representing an axisymmetric, black hole solution to the source-free Einstein equations, was discovered by Kerr[6] almost 50 years after the spherically symmetric solution was first written. In subsequent work, Carter was able to establish,[7] rather unexpectedly, that the Hamilton–Jacobi equation for a free particle and the Klein–Gordon equation for a scalar field were separable. (Carter also studied the relationship between this result and the *form* of the Kerr metric[8]). Then, following a method used by Bardeen and Press[9] for (the gauge and tetrad invariant) perturbations of the Weyl tensor in the Schwarzschild geometry, Teukolsky demonstrated[10] that analogous perturbations in the case of the Kerr black hole also obeyed a separable equation. Similar perturbation equations have since been found[11,12] for all fields of physical interest in a background Kerr geometry.

III. PERTURBATION EQUATIONS

Massless Klein–Gordon,[7] Dirac (neutrino),[11] Maxwell,[10] Rarita–Schwinger,[12] and linearized Einstein[10] equations, when written in a decoupled form for the corresponding tetrad and gauge invariant Newman–Penrose[13] (NP) quantities in the Kerr background (and including the Schwarzschild limit), can all be represented in a single (separable) master equation of Teukolsky, which in Boyer–Lindquist coordinates[14] may be written as

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$$\left\{\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} - \frac{1}{\Delta}\left\{(r^2+a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial\phi} - (r-M)s\right\}^2 - 4s(r+ia\cos\theta)\frac{\partial}{\partial t}\right.$$
$$\left. + \frac{\partial}{\partial\cos\theta}\sin^2\theta\frac{\partial}{\partial\cos\theta} + \frac{1}{\sin^2\theta}\left\{a\sin^2\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial\phi} + i\cos\theta\cdot s\right\}^2\right\}\psi_{*s} = 0,$$

where $\Delta = r^2 - 2Mr + a^2$ (ψ_{*s} can be given suitably in terms of the relevant NP quantities). In the case of spin-zero fields on the Schwarzschild black hole, this equation immediately admits a conserved "energy" integral with a positive definite integrand, establishing that there are no modes that have an unbounded time derivative, i.e., there are no unstable modes. The radial part of the equation is complex when acting on an individual mode of higher spin, but by a transformation of the radial function for obtaining the Regge–Wheeler[3] (or corresponding[15]) equation, a similar proof of stability can again be given[4] for massless perturbations of spin-2 (and spin-1) in the Schwarzschild space-time.

Except for axisymmetric scalar perturbations, no analogous consideration has previously yielded even this limited proof of stability in the case of Kerr black holes for the following two reasons.

(i) For nonaxisymmetric scalar perturbations the coefficient $(1/\sin^2\theta - a^2/\Delta)$ of $|\partial\psi_{*s}/\partial\phi|^2$ in the energy integrand is only positive outside the ergosphere, indicative of the fact that there is no Killing vector which is timelike everywhere within the region exterior to the event horizon.

(ii) Prior to this work, although transformations were known[16] that mapped Kerr radial functions to solutions of equations that reduced to Regge–Wheeler-type equations in the Schwarzschild limit, for the rotating black hole these other equations are quite unlike the Regge–Wheeler equation[3] in that they depend on the separation constant (i.e., the unknown angular eigenvalue) in a highly nonlinear way.

Procedures that remedy the above difficulties are given below.

Physical considerations[11] lead one to regard as unstable modes those perturbation solutions that are purely ingoing on the horizon and purely outgoing at (null) infinity. Consequently, in the exterior region, unstable modes have, asymptotically, support only on the *future* horizon and at *future* null infinity. Unstable modes have characteristic frequencies with positive imaginary parts; thus on a spacelike section, they can be regarded as radial eigenfunctions, which become unboundedly large to the future.

In the development of a proof of the mode stability for perturbations of a Kerr black hole, new progress has become possible through a generalization of certain previously known results; these results are detailed as follows.

(i) There are Teukolsky–Starobinsky[17] ordinary differential relations that can be used to change helicity from s to $-s$ for the radial and angular dependence of the separated solutions of Teukolsky's master equation.[10]

(ii) The kernel of an integral equation can be written[18] for radial functions of spin-zero fields in the Schwarzschild background.

In order to proceed, it will be convenient to introduce a notation for exploiting the similarity between the r and $\cos\theta$ dependence of the Teukolsky equation.[10]

The separated radial and angular equations can be written in the form [for $\psi_{*s} = e^{-i\omega t}e^{im\phi}R_s(r)S_s(\theta)$]

$$\left\{\partial_{xx} - \alpha^2 + \frac{\alpha\kappa + \lambda + \frac{1}{2}\kappa^2}{x} + \frac{\frac{1}{4}-\beta^2}{x^2} + \frac{\alpha\kappa - \lambda - \frac{1}{2}\kappa^2}{x-1} + \frac{\frac{1}{4}-\gamma^2}{(x-1)^2}\right\}\sqrt{x(x-1)}\,u = 0,$$

where for the angular equation, $x = (\cos\theta + 1)/2$, $u = S_s$,

$$\alpha = 2a\omega, \quad \kappa = s,$$
$$\beta = (s-m)/2, \quad \gamma = (s+m)/2,$$
$$\lambda = \tfrac{1}{2} + \alpha(\gamma-\beta) - \tfrac{1}{2}(\gamma-\beta)^2 + (\lambda_T + s)$$

(with α, β, γ given above)

and for the radial equation, $x = (r - r_-)/(r_+ - r_-)$, $u = R_s$,

$$\alpha = 2iM\omega\epsilon_0, \quad \kappa = s - 2iM\omega,$$
$$\beta = (s/2 + iM\omega) - (i/\epsilon_0)(M\omega - am/2M),$$
$$\gamma = (s/2 + iM\omega) + (i/\epsilon_0)(M\omega - am/2M),$$
$$\lambda = \tfrac{1}{2} + \alpha(\gamma-\beta) - \tfrac{1}{2}(\gamma-\beta)^2 + (\lambda_T + s),$$

(with α, β, γ given above).

The separation constant λ_T appears in Teukolsky's radial equation[10]; $(\lambda_T + s)$ is invariant under $s \to -s$. The relations $\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$ and $\epsilon_0 = (r_+ - r_-)/(r_+ + r_-)$ have been used.

λ_T+s=λ_C-|s|

$$\left.\begin{aligned}&\text{Near } x = 0,\ u \sim x^{\delta'\beta},\\ &\text{near } x = 1,\ u \sim (x-1)^{\delta''\gamma},\\ &\text{near } x = \infty,\ u \sim e^{\delta\alpha x}x^{-\delta\kappa - 1},\end{aligned}\right\}\quad \delta, \delta', \delta'' = \pm 1.$$

We are particularly interested in transformations of the above equations which leave the singular points (number and type) and $(\lambda_T + s)$ dependence unchanged.

IV. DIFFERENTIAL TRANSFORMATIONS

A general structure[19] can be shown to underlie the contiguous relations for special functions; familiar examples would be the differential operators which change angular momentum for the spherical Bessel and associated Legendre functions. By examining the Teukolsky–Starobinsky relations[17] in this context one can view them as a particular consequence of the following. For those values of ϵ, ϵ', ϵ'' $(= \pm 1)$ that allow

$$n = \epsilon\kappa + \epsilon'\beta + \epsilon''\gamma$$

to be a positive integer (there can be four such at most), then with

$$\tilde{\alpha} = -\epsilon\alpha, \quad \tilde{\kappa} = \epsilon'\beta + \epsilon''\gamma,$$
$$\tilde{\beta} = n/2 - \epsilon'\beta, \quad \tilde{\lambda} = \lambda,$$
$$\tilde{\gamma} = n/2 - \epsilon''\gamma$$

the function given by

$$\tilde{u} = e^{\tilde{\alpha}x}x^{\tilde{\beta}}(x-1)^{\tilde{\gamma}}\left(\frac{\partial}{\partial x}\right)^n e^{\epsilon\alpha x}x^{\epsilon'\beta}(x-1)^{\epsilon''\gamma}u$$

satisfies a similar equation to that satisfied by u, where the new parameters are given by the overtilde quantities above. The set of equations that can be obtained by repeated application of this result is finite. A certain degeneracy occurs when the derivatives act on the polynomials that they annihilate. There is an inverse transformation operator

$$e^{\epsilon\alpha x}x^{\epsilon'\beta}(x-1)^{\epsilon''\gamma}\left(\frac{\partial}{\partial x}\right)^n e^{\tilde{\alpha}x}x^{\tilde{\beta}}(x-1)^{\tilde{\gamma}}$$

which maps solutions for $\tilde{u}$ onto solutions for u, with a similar statement as above applying in the case of degeneracy.

The Teukolsky–Starobinsky (helicity flipping) relations[17] are a known example of this kind of transformation, with $n = |2s|$ in the radial and angular transformations. For the radial functions, no other integer values of n are possible for a *general* value of the frequency except on a Schwarzschild space-time, in which case a new expression of known results is obtained.[20] For the angular functions, $n = |s-m|$ and $n = |s+m|$ are always additionally possible; they map angular eigenfunctions to eigenfunctions of two new operators which are related to one another by $x \to 1-x$, i.e., by $\cos\theta \to \cos(\pi-\theta)$. A solution of one of these new operators is given by

$$_mT_{ls} = (\sin\theta)^{|s-m|}\left(\frac{\epsilon\partial}{\partial\cos\theta} + a\omega + \frac{(s+m\cos\theta)}{\sin^2\theta}\right)^{|s-m|} {}_sS_{lm}(\theta),$$

where $s-m = \epsilon|s-m|$. This new angular function T will be used in a proof of stability: For nonzero $\alpha = 2a\omega$, its construction can never be degenerate.

V. INTEGRAL TRANSFORMATIONS

The known integral equation[18] for scalar wave functions in the Schwarzschild space-time was of the Laplace type, where the "center" of the kernel [i.e., that part of the integral kernel given by the nonseparable functions $H(x,y)$ which occur in the expressions below] is of the form e^{-axy}. For all the radial and angular functions arising from perturbations of the Kerr black hole, I have recently found kernels of integral equations which they satisfy, where these kernels are now more complicated functions of xy: Similarly, Euler-type kernels depending on $(x+y-1)$, etc., have also been found. Moreover, the conditions that disallow the existence of integral kernels simply depending on the functions xy, $x+y-1$ (or their variants) are precisely those conditions that permit the construction of integral transformations[21] to the solutions of *new* equations of the same general type as we need to consider. In the radial case, for the spin-2 (gravitational) perturbations on the Schwarzschild background, one of these new equations turns out to be the Regge–Wheeler equation[3] previously related to the NP perturbations[13] only by a differential transformation: Its simple generalization in the Kerr case will again be useful in a proof of stability.

These integral transforms can be described as follows. Under suitable conditions, the function

$$\tilde{u} = \int_A^B \mathscr{K}(x,y)u(y)dy$$

will satisfy an equation of our given form provided that ϵ, ϵ', $\epsilon''(=\pm 1)$ can be chosen so that $y(y-1) \times W(u(y),\mathscr{K}(x,y))|_A^B$ vanishes identically. (Here W is the Wronskian.) The function $\mathscr{K}(x,y)$ has the general form

$$e^{\tilde{\alpha}x}x^{\tilde{\beta}}(x-1)^{\gamma}H(x,y)e^{\epsilon\alpha y}y^{\epsilon'\beta}(y-1)^{\epsilon''\gamma}$$

and a number of different usable "centers" $H(x,y)$ have been identified, e.g., $e^{-2\epsilon\alpha xy}$, $(x+y-1)^{-\nu-1}$, etc. (where $\nu = \epsilon\kappa + \epsilon'\beta + \epsilon''\gamma$ whether or not it is an integer or real). For the first of these "centers," the quantities

$$\tilde{\alpha} = \epsilon\alpha, \quad \tilde{\kappa} = \epsilon'\beta - \epsilon''\gamma,$$
$$\tilde{\beta} = \tfrac{1}{2}(\epsilon\kappa + \epsilon'\beta + \epsilon''\gamma), \quad \tilde{\lambda} = \lambda,$$
$$\tilde{\gamma} = \tfrac{1}{2}(-\epsilon\kappa + \epsilon'\beta + \epsilon''\gamma),$$

give the parameters in the equation for $\tilde{u}$. (Note that λ is again unchanged.)

We will choose a bounded new radial function $_mK_{ls}$ given by an integral transform of $_{-|s|}R_{lm}(r)$ over the range (r_+,∞), with ϵ,ϵ', $\epsilon'' = -1$. Since the "center" is of the Laplace type, this integral transformation will never be degenerate. In what follows, stability for negative helicity modes will also assure stability for positive helicity modes because, via the Teukolsky–Starobinsky relations,[17] the modes can be independently transformed into one another. (Only for the algebraically special perturbations are these transformations singular,[22] but then the boundary conditions given above for unstable modes are not satisfied.)

VI. PROOF OF STABILITY

The transformations that have been chosen ensure that the Kerr angular and radial functions corresponding to an unstable mode will map to bounded solutions of the new operators. In addition, this construction implies that the function

$$\Phi_s = e^{-i\omega t}e^{im\phi}K_s(r)T_s(\theta)$$

will satisfy the equation given below, where

$$f(r)\left\{\frac{(r-r_-)^2}{(r-r_+)^2} - \frac{a^2}{M^2}\frac{(r-M)^2}{\Delta}\right\}\frac{\Delta}{\epsilon_0^2}(>0 \text{ for } r>r_+):$$

$$\left[\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - (f(r)+a^2\cos^2\theta)\frac{\partial^2}{\partial t^2} - 2a\left(\cos\theta + \frac{r-M}{\epsilon_0 M}\right)\frac{\partial^2}{\partial t\,\partial\phi} - s^2\left(\frac{1-\cos\theta}{1+\cos\theta} + \frac{r-r_+}{r-r_-}\right)\right]\Phi_s = 0.$$

Here the operator is totally independent of Teukolsky's separation constant λ_T.[10] (Note that in the metric that can be derived from this equation, $\partial/\partial t$ is globally null.) The conserved quantity that follows from the operator has a positive integrand:

$$\frac{\partial}{\partial t}\frac{1}{2}\int dr\,d\theta\,d\phi\,\sin\theta\left\{(f(r)+a^2\cos^2\theta)\left|\frac{\partial\Phi}{\partial t}\right|^2+\Delta\left|\frac{\partial\Phi}{\partial r}\right|^2+\left|\frac{\partial\Phi}{\partial\theta}\right|^2+s^2\left(\frac{1-\cos\theta}{1+\cos\theta}+\frac{r-r_+}{r-r_-}\right)|\Phi|^2\right\}=0.$$

Since the leading radial dependence of Φ_s near the horizon is $(r-r_+)^{-2iM\omega}$ and near infinity it is $e^{i\omega r}$ and the leading angular dependence near the south pole is $(\sin\theta)^s$, every term in the above integrand is integrable for unstable modes. Hence the value of the conserved "energy" bounds the integral of the time derivative terms, which consequently cannot grow exponentially. Thus there can be no unstable modes for Kerr angular and radial functions since we have now ruled out the solutions of the above equation to which they would be mapped.

Parameters in the transformations depended explicitly on the mode decomposition. Consequently, it seems that any stronger result for the Kerr black hole would require additional understanding concerning mode completeness since the particular methods of Kay and Wald,[5] which might circumvent this, are not directly applicable and no appropriate generalization is known at present.

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APPENDIX

Properties of the transformations referred to in this paper may be sufficiently unfamiliar to some that it would be useful here to sketch a proof of the most important relations. A simple derivation of the differential transformation can be obtained via the use of the following two elementary results:

Result (i):

$$e^{\epsilon\alpha x}x^{\epsilon'\beta}(x-1)^{\epsilon''\gamma}\left\{\partial_{xx}-\alpha^2+\frac{\alpha\kappa+\Lambda}{x}+\frac{\frac{1}{4}-\beta^2}{x^2}+\frac{\alpha\kappa-\Lambda}{x-1}+\frac{\frac{1}{4}-\gamma^2}{(x-1)^2}\right\}\sqrt{x(x-1)}$$
$$=\frac{1}{\sqrt{x(x-1)}}\{x(x-1)\partial_{xx}+(ax^2+bx+c)\partial_x+dx+e\}e^{\epsilon\alpha x}x^{\epsilon'\beta}(x-1)^{\epsilon''\gamma},$$

where

$$a=-2\epsilon\alpha,\quad b=2(\epsilon\alpha-\epsilon'\beta-\epsilon''\gamma+1),\quad c=2\epsilon'\beta-1,$$
$$d=2\epsilon\alpha(\epsilon\kappa+\epsilon'\beta+\epsilon''\gamma-1),\quad e=-\alpha\kappa-\Lambda-(2\epsilon'\beta-1)(\epsilon\alpha+\tfrac{1}{2}-\epsilon''\gamma).$$

Result (ii):

$$\partial_x^N\{x(x-1)\partial_{xx}+(ax^2+bx+c)\partial_x-a(N-1)x+e\}$$
$$=\{x(x-1)\partial_{xx}+(ax^2+(b+2N)x+c-N)\partial_x+a(N+1)x+e+N(b+N-1)\}\partial_x^N,$$

which accounts for the necessity that N be a positive integer.

In the case of the integral transformations, explicit demonstration of the results depends on the form of the integral "center" $H(x,y)$, e.g., for $H(x,y)=e^{-axy}$ we have

$$[x(x-1)\partial_{xx}+(ax^2+bx+c)d_x-avx+e]e^{-axy}=[y(y-1)\partial_{yy}+(ay^2+by+v)\partial_y-acy+e]e^{-axy}.$$

When $c=v$ it is thus possible to construct an integral equation for $u(x)$.

Similarly, for $H(x,y)=(x-y)^v$, say, we can obtain

$$[x(x-1)\partial_{xx}+(ax^2+bx+c)\partial_x-avx+e](x-y)^v$$
$$=[y(y-1)\partial_{yy}+\{ay^2+(b+2(v+1))y+c-(v+1)\}\partial_y+a(v+2)x+e+(v+1)(b+v)]^\dagger(x-y)^v,$$

where we have indicated by the superscript † the adjoint operator which arises under the integral sign in a verification of the transform properties. Note the similarity of the transformed operator here to the operator arising for the differential transform (with $N\to v+1$). The transformation properties for other similar "centers," e.g., for those depending on $(x-1)(y-1)$ or $(x+y-1)$, can be constructed directly or obtained by the substitution $x\to 1-x$, etc.

Kernels for integral equations can be produced in abundance as a result of the elementary (but nontrivial) observations

$$\{x(x-1)\partial_{xx}+(ax^2+bx+c)\partial_x+dx+e\}-\{y(y-1)\partial_{yy}+(ay^2+by+c)\partial_y+dy+e\}$$
$$=(x-y)[(u\partial_{uu}+(au-c)\partial_u+d+\mu)-(v\partial_{vv}+(-av+a+b+c)\partial_v+\mu)]$$
$$=(x-y)\{(f\partial_{ff}+(af+a+b)\partial_f+d+\sigma/f)-[(1-g)/f]$$
$$\times[g(1-g)\partial_{gg}+(-(2+c)g+a+b+c)\partial_g+\sigma/(1-g)]\},$$

where $u = xy$, $v = (x-1)(y-1)$, $f = x + y - 1$, and $g = (x-1)(y-1)/xy$. Here, μ and σ would be new constants of separation.

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