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A New Method for Solving Perturbation Equations[1]

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Abstract

A new, generally applicable technique is described for constructing solutions of a coupled linear system of partial differential equations when a decoupled equation has been derived. This method has already yielded an extremely simple derivation of perturbation formulas given by Cohen and Kegeles and by Chrzanowski as well as new formulas for the complete solutions of the coupled Einstein-Maxwell equations describing Reissner-Nordström perturbations. It is hoped that it will prove valuable for many other applications.

Perhaps the greatest obstacle to a better understanding of classical general relativity is the difficulty of obtaining solutions of the field equations. Although some very interesting and physically relevant solutions are known, essentially all known solutions possess a high degree of symmetry. Aside from some rather vague (though quite important) conclusions that can be drawn from the singularity theorems, we really have very little idea of what occurs in generic situations.

However, it frequently happens that one wishes to describe a situation where the deviations from a known solution are expected to be small and perturbation theory can be justified as a satisfactory approximation. Prime examples are studies of the behavior of small departures from exact cosmological models and from exact black hole solutions. The equations describing the perturbations are linear and the major obstacle of dealing with nonlinear equations is thereby avoided. Nevertheless, the linearized equations describing metric perturbations form such a complicated coupled system that they are virtually intractable. Even in the simplest cases—the Friedman cosmology and the Schwarzschild

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black hole—major efforts were required to give a complete analysis of the solutions by direct means.

Fortunately, in a case of great interest—the Kerr black hole—it was possible for Teukolsky [1] to derive from the full electromagnetic perturbation equations a decoupled equation for the Newman-Penrose Maxwell tensor component ϕ_0, and from full gravitational perturbation equations a decoupled equation for the Newman-Penrose quantity ψ_0. These derivations, in fact, work for all algebraically special vacuum space-times in the gravitational case and for a somewhat wider class of space-times in the electromagnetic case. Many quantities of physical interest such as radiation fluxes can be calculated directly from ϕ_0 and ψ_0 and the derivation of decoupled equations for them represented a real breakthrough. However, ϕ_0 and ψ_0 give only partial information concerning many interesting questions concerning the behavior of the perturbed metric or electromagnetic field. A direct integration of the remaining field equations to solve for the complete perturbations in the Kerr case required a further major effort which was completed only recently by Chandrasekhar [2].

However, not long after Teukolsky's results were obtained, Cohen and Kegeles [3] gave a prescription for constructing complete solutions of the electromagnetic perturbation equations. Their prescription works in precisely those cases for which Teukolsky's derivation applies. Furthermore, in the type D case, where there are two Teukolsky equations (for ϕ_0 and ϕ_2), the Cohen-Kegeles equation used in the constructive procedure is just one of the Teukolsky equations. This strongly suggests a close relation between the Teukolsky and the Cohen-Kegeles results. However, in the more general cases where there is only one Teukolsky equation, the Cohen-Kegeles equation is *not* the Teukolsky equation and, unfortunately, the complicated nature of the Cohen-Kegeles derivation makes it difficult to obtain further insight.

Very soon after this work of Cohen and Kegeles, Chrzanowski [4] in his study of electromagnetic and gravitational perturbations of the Kerr metric made an *ad hoc* hypothesis that the full Green's functions for these equations could be expressed in a certain factorized form. This assumption led to formulas for the complete vector potential and metric perturbations of Kerr. The vector potential perturbations agreed precisely with the Cohen-Kegeles formulas. Chrzanowski partially checked his formula for the metric perturbations of Kerr by substitution in the linearized field equations. His formula passed all the tests he tried, but the algebra proved so messy that the complete field equations were not checked. Encouraged by these results, Chrzanowski conjectured a formula for the metric perturbations of an arbitrary algebraically special vacuum space-time—precisely the case where Teukolsky's derivation of a decoupled gravitational equation applies—reasoning by analogy with his Kerr formula and the general Cohen-Kegeles electromagnetic formula. Very shortly thereafter, Cohen and Kegeles [5] gave the same formula. Only very recently have Cohen and Kegeles published a proof that this formula is correct [9].

The above work raises a number of intriguing questions: What is the relation (if any) of the constructive procedures of Cohen-Kegeles and of Chrzanowski to the decoupled equation of Teukolsky? Why is the equation used in the constructive procedure the same as one of the Teukolsky equations in the type D case but different from it in other cases? How general is this constructive method; is it special to the specific equations under study (as the derivations of Cohen-Kegeles and of Chrzanowski certainly seem to indicate) or is it applicable for a large variety of equations?

The main purpose of this essay is to give a simple answer to these questions. We shall show that the existence of the constructive procedure is intimately related to the existence of a decoupled equation. We will thereby be able to give an elementary proof of the Cohen-Kegeles-Chrzanowski (CKC) formulas. New applications of the general method will also be mentioned.

The key to understanding the relation of Teukolsky's derivation and the CKC formulas is a simple observation concerning the nature of Teukolsky's derivation. Let f denote the unknown variables (i.e., either the vector potential A_μ or the metric perturbation $h_{\mu\nu}$). These variables f satisfy an equation

$$\mathcal{E} f = 0 \tag{1}$$

where $\mathcal{E}$ is a linear partial differential operator. The decoupled variable ϕ (i.e., either ϕ_0 or ψ_0) is given by a linear partial differential operation $\mathcal{T}$ on f,

$$\phi = \mathcal{T} f \tag{2}$$

Teukolsky's equation, namely,

$$\mathcal{O} \phi = 0 \tag{3}$$

where $\mathcal{O}$ is another linear partial differential operator, is derived by performing linear partial differential operations $\mathcal{S}$ on equation (1). Inspection of Teukolsky's derivation thus shows that it actually proves the operator identity

$$\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T} \tag{4}$$

(so that $\mathcal{E}f = 0$ implies $\mathcal{O}[\mathcal{T}f] = 0$).

Suppose, now, that an operator identity of the form equation (4) has been proven. Taking the adjoint of equation (4) we find [6]

$$\mathcal{E}^\dagger \mathcal{S}^\dagger = \mathcal{T}^\dagger \mathcal{O}^\dagger \tag{5}$$

Let ψ be a solution of

$$\mathcal{O}^\dagger \psi = 0 \tag{6}$$

Applying both sides of equation (5) to ψ, we find

$$\mathcal{E}^\dagger (\mathcal{S}^\dagger \psi) = 0 \tag{7}$$

Hence, if $\mathcal{E}$ is self-adjoint, as it is for the Maxwell, linearized Einstein, and other equations of physical interest, the quantity

$$f = \mathcal{S}^\dagger \psi \tag{8}$$

is a solution of equation (1). Equation (6) is, in the appropriate cases, precisely the CKC equation and equation (8) is precisely their perturbation formula!

Thus a direct connection has been obtained between the derivation of a decoupled equation and the existence of a constructive procedure for generating solutions of the full system of equations. Furthermore, an explanation of why the CKC equation reduces to the Teukolsky equation in the type D case but not otherwise can be given. In all cases, the CKC equation is the adjoint of the Teukolsky equation. However, it happens that in the type D case, the adjoint of the Teukolsky operator for ϕ_0 is just the Teukolsky operator for ϕ_2 and the adjoint of the operator for ψ_0 is the operator for ψ_4. This fact also explains the remarkable differential identities ("Starobinski-Teukolsky relations") which were known to exist between ϕ_0 and ϕ_2 and between ψ_0 and ψ_4 for Kerr perturbations. The above results prove that the operator $\mathcal{T}\mathcal{S}^\dagger$ maps solutions for the second variable into solutions for the first. As shown in more detail elsewhere [6], these Starobinski-Teukolsky relations exist for perturbations of all vacuum type D space-times.

While the above ideas give an elegant explanation of some old results, the greatest potential utility of these ideas lies in new applications. Chandrasekhar [7] has derived decoupled equations from the equations describing perturbations of a Reissner-Nordström black hole. Very recently the constructive procedure arising from this decoupling has been used to generate the full solutions of the coupled Einstein-Maxwell system [8]. It is worth emphasizing the general applicability of this method. All that is needed is the operator identity of the form equation (4), and it is not necessary, for example, that the decoupled equation be a single equation for a single variable. Thus, for example, Chandrasekhar [2] has obtained a decoupled system of four equations in four unknowns for perturbations of a Kerr-Newman black hole. Solutions of the adjoint system could be used to construct solutions to the full system of equations by this method. These and other applications may help give us more insight into the behavior of solutions which deviate only slightly from known exact solutions.

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