

Lionel London

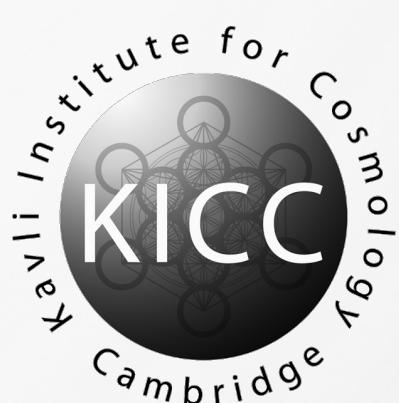
An introduction to black hole perturbation theory

Kavli-Villum Summer School

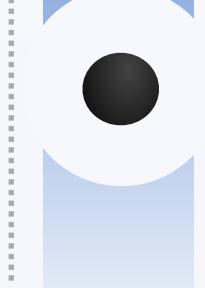
September 2023



KING'S
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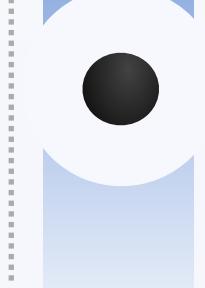


Corfu



Preface Organization and scope of lectures

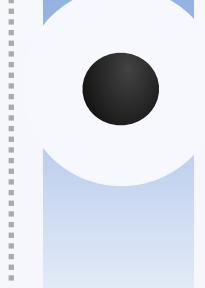
- ❖ Some knowledge of the following will be assumed:
 - Differential calculus, General Relativity, Ordinary Differential Equations
 - There are no stupid questions (almost)
- ❖ Philosophy of the lecturer
 - Black hole perturbation theory is a story with a plot, characters, twists ...
 - Let the story drive understanding of the details, not the other way around (!)
 - Concepts before calculations, forest before trees, horses before carts, the etc ...
 - The tortoise beats the hare ...
- ❖ Goal of the lectures
 - Help you formulate questions that amount to “What happens next?”
 - Provide a few tools that may help answer some of your future questions



Preface Organization and scope of lectures

Topics that will be covered

- Lecture #1
 - **1.1** “Belief in the possible presence of BHs” (Linear Stability)
 - **1.2** Einstein’s equations linearized around Kerr (Teukolsky’s Equations)
 - **1.3** Black hole specific special functions (Quasinormal Modes, QNMs)
- Lecture #2
 - **2.1** So many quasi-normal modes (Structure of the QNM solution space)
 - **2.2** Introduction to QNM Orthogonality
 - **2.3** What’s next? (Some known open questions)



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- Lecture #1
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Preface Organization and scope of lectures

Some overlapping topics mostly left to references and upcoming lectures

- Second order perturbations (nonlinear QNMs)
 - Campanelli+ 1999, London+ 2014, Lagos+Hui 2023, Cheung+ 2023, others
- Extreme mass ratio inspirals and self-force (Adam)
- Beyond GR (Lavinia)
- QNMs and GW signal modeling
 - Husa 2015, Ossokine+ 2020, Hamilton+ 2023, and many others ...
- Tests of GR with QNMs (Gregorio)

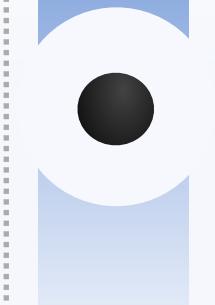
Preface Organization and scope of lectures

My favorite topical reviews

- The characteristic ‘sound’ of black holes and neutron stars (Nollert)
- Quasi-Normal Modes of Stars and Black Holes (Kokkotas+)
- Quasinormal modes of black holes and branes (Berti+)

“The discovery of pulsars and the conviction that they are neutron stars resulting from gravitational collapse have strengthened *the belief in the possible presence of Schwarzschild black holes...*”

Vishveshwara,
1970

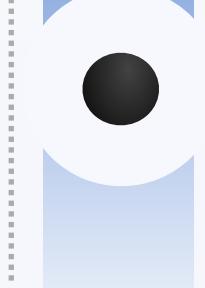


1.1 On the possible presence of black holes

“We have equilibrium, but *is it stable?*”

Regge and Wheeler, 1957

- By the 1950s, there were a number of known “black hole” solutions to Einstein’s equations. Schwarzschild’s (1915) was the first and simplest.
- But for black holes to be physical objects they must also **survive perturbations**.
- Example of an unstable object: sphere of water surrounded by a shell of mercury held together by self-gravity — *pluck it*, and mercury falls to the center.
- Regge and Wheeler studied metric perturbations of the Schwarzschild solution.



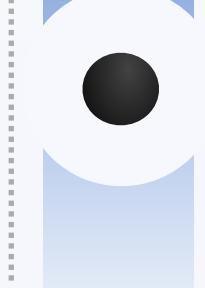
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$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \end{aligned}$$



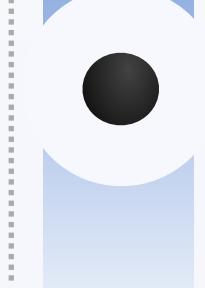
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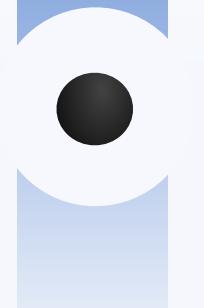
$$g'_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$$

Diagram illustrating metric perturbations:

```

graph TD
    PM[perturbed metric] --> GM[background metric]
    GM --> P[h<sub>αβ</sub>]
    P --> H[h<sub>αβ</sub>]
    H --> GM
    
```

The diagram shows the decomposition of a perturbed metric into a background metric and a perturbation. A curved arrow points from the label "perturbed metric" to the term $g'_{\alpha\beta}$. A vertical arrow points from the label "background metric" to the term $g_{\alpha\beta}$. A curved arrow points from the label "perturbation" to the term $h_{\alpha\beta}$.

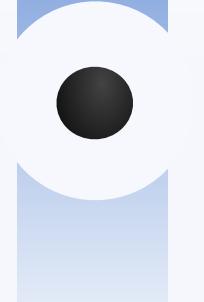


1.1 On the possible presence of black holes

“We have equilibrium, but *is it stable?*”

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.
- The expected key result is a differential equation for the metric perturbation.
- Guiding concept: If perturbation solutions grow in time, then the BH is unstable. If they shrink, then the BH is stable.

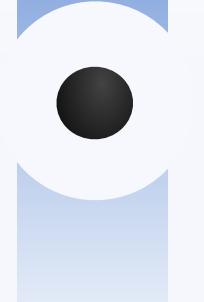


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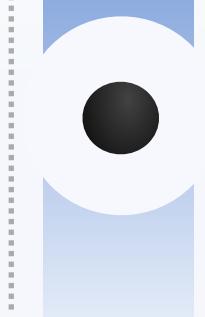


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- The expected key result is a differential equation for the metric perturbation.
- Guiding concept: If perturbation solutions grow in time, then the BH is unstable. If they shrink, then the BH is stable. (What's the other possibility?)

1.1 On the possible presence of black holes

1950s-1970s

“We have equilibrium, but *is it stable?*”

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

Notation: Perturbed quantities are primed, and $G = c = 1$.

$$g'_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$$

Diagram illustrating the decomposition of a perturbed metric:

- A box labeled "perturbed metric" contains the symbol $g'_{\alpha\beta}$.
- A box labeled "background metric" contains the symbol $g_{\alpha\beta}$.
- A box labeled "perturbation" contains the symbol $h_{\alpha\beta}$.
- A downward arrow points from the background metric box to the equation.
- Curved arrows point from each of the three boxes to their respective components in the equation: one from "perturbed metric" to $g'_{\alpha\beta}$, one from "background metric" to $g_{\alpha\beta}$, and one from "perturbation" to $h_{\alpha\beta}$.



1.1 On the possible presence of black holes

“We have equilibrium, but *is it stable?*”

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric (compute the Ricci tensor)

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

First think about the unperturbed case.



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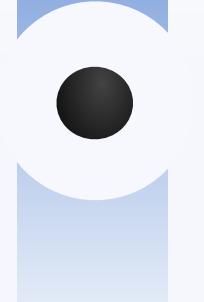
Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric (compute the Ricci tensor)

$$G_{\mu\nu} = 0$$

The right-hand-side is zero because the BH is isolated.



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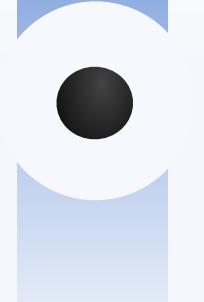
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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric (compute the Ricci tensor)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

The left-hand-side is determined by the Ricci tensor and scalar.



1.1 On the possible presence of black holes

*"We have equilibrium, but **is it stable?**"*

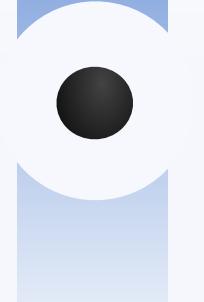
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$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\cancel{R}^{\cancel{=0}} = 0$$

R=0 for Schwarzschild.



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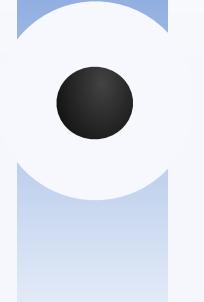
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$$R_{\mu\nu}(g_{\mu\nu}) = 0$$

Schematically, the Ricci scalar is a function of the metric.



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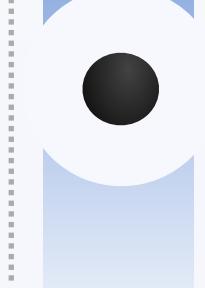
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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric (compute the Ricci tensor)

$$R_{\mu\nu}(g'_{\mu\nu}) = 0$$

The same is true for the perturbed case.



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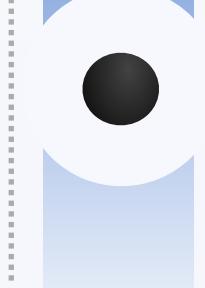
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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric

$$R_{\mu\nu}(g_{\mu\nu} + h_{\mu\nu}) = R_{\mu\nu}(g_{\mu\nu}) + \delta R_{\mu\nu}(g_{\mu\nu}, h_{\mu\nu})$$

Expanding this gives ...



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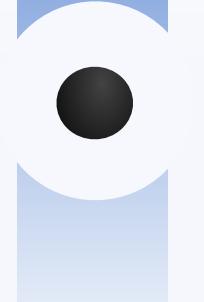
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We're only interested in the deviation.



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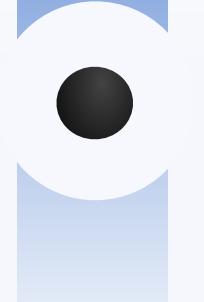
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1. Re-derive the field equations using the perturbed metric

$$\delta R_{\mu\nu} = -\delta\Gamma_{\mu\nu;\beta}^{\beta} + \delta\Gamma_{\mu\beta;\nu}^{\beta}$$

covariant derivative

This can be written only in terms of the deviation of the Christoffel symbols



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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric

$$\delta\Gamma_{\beta\gamma}^{\alpha} = \frac{1}{2}g^{\alpha\nu}(h_{\beta\nu;\gamma} + h_{\gamma\nu;\beta} - h_{\beta\gamma;\nu})$$

After turning the tensor crank, it can be shown that the deviation of the Christoffel symbols is ...

1.1 On the possible presence of black holes

1950s-1970s

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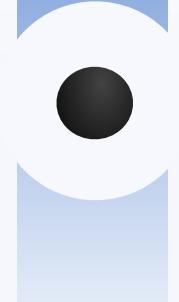
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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

1. Re-derive the field equations using the perturbed metric

$$\begin{aligned}\delta\Gamma_{\beta\gamma}^{\alpha} &= \frac{1}{2}g^{\alpha\nu}(h_{\beta\nu;\gamma} + h_{\gamma\nu;\beta} - h_{\beta\gamma;\nu}) \\ &= 0\end{aligned}$$

We have arrived at the field equations, but more strategy is needed to proceed ...



1.1 On the possible presence of black holes

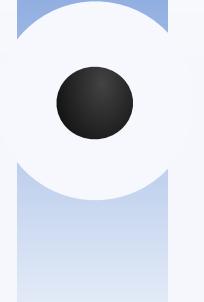
“We have equilibrium, but *is it stable?*”

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form (*a summary*)

- **Make use of symmetry**: Decompose the perturbation into tensor spherical harmonics so that the related moments are rationally invariant.
- Use a simplifying coordinate gauge, i.e. “Regge-Wheeler” gauge.
- Consider “**odd**” (even/polar) and “**even**” (odd/axial) parity terms separately



1.1 On the possible presence of black holes

*"We have equilibrium, but ***is it stable?***"*

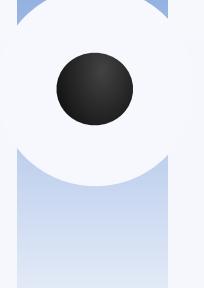
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- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form

- Make use of symmetry

$$h_{\mu\nu} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{p}^{10} C_{\ell m}^{(p)}(t, r) (Y_{\ell m}^{(p)})_{\mu\nu}(\theta, \phi)$$



1.1 On the possible presence of black holes

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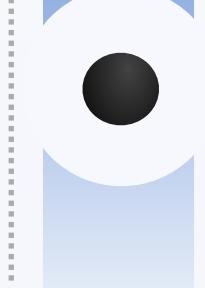
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e.g. $\frac{\partial}{\partial\theta} \frac{\partial}{\partial\phi} Y_{\ell m}(\theta, \phi)$



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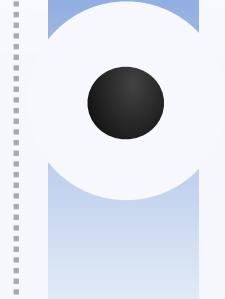
2. Put field equations into a useful form

- Handle different parity components separately

Regge-Wheeler's “odd”

= Chandrasekhar's “polar”

$$\begin{aligned}
 (Y_{\ell m}^{(\text{odd})})_{\mu\nu}(\pi - \theta, \phi + \pi) &= (Y_{\ell m}^{(\text{polar})})_{\mu\nu}(\pi - \theta, \phi + \pi) \\
 &= (-1)^\ell (Y_{\ell m}^{(\text{polar})})_{\mu\nu}(\theta, \pi)
 \end{aligned}$$



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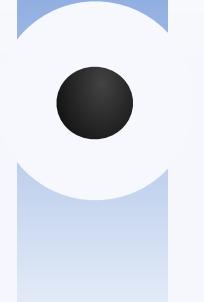
- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form

- Handle different parity components separately

Regge-Wheeler's “even”
= Chandrasekhar's “axial”

$$\begin{aligned}
 (Y_{\ell m}^{(\text{even})})_{\mu\nu}(\pi - \theta, \phi + \pi) &= (Y_{\ell m}^{(\text{axial})})_{\mu\nu}(\pi - \theta, \phi + \pi) \\
 &= (-1)^{\ell+1} (Y_{\ell m}^{(\text{axial})})_{\mu\nu}(\theta, \pi)
 \end{aligned}$$



1.1 On the possible presence of black holes

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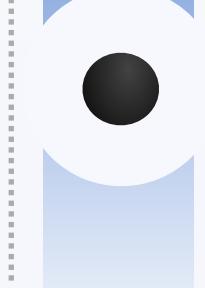
Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form

- The result is that, it became evident that perturbations can take the form

$$h_{\mu\nu} \propto \frac{1}{r} e^{-iwt} {}_{-2}Y_\ell(\cos(\theta))$$



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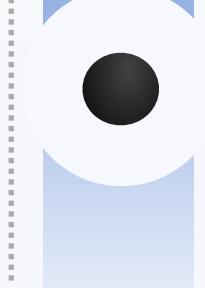
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more on this in a moment



1.1 On the possible presence of black holes

“We have equilibrium, but *is it stable?*”

Regge and Wheeler, 1957

- The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.

2. Put field equations into a useful form (the final form is a *single* equation)

- Further, the remaining r dependence is determined by a single quantity which obeys a second order linear differential equation

$$[p_0(r, w) + p_1(r) \partial_r + p_2(r) \partial_r^2] \psi(r) = 0$$

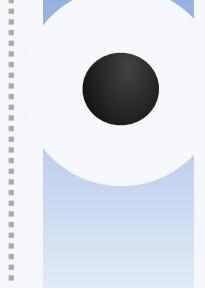


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“We have equilibrium, but *is it stable?*”

Regge and Wheeler, 1957

- ✓ • The task: Given a perturbed metric, derive the resulting field equations to linear order in the perturbation.
- ✓ • The expected key result is a differential equation for the metric perturbation.
- Guiding concept: If perturbation solutions grow in time, then the BH is unstable. If they shrink, then the BH is stable.



1.1 On the possible presence of black holes

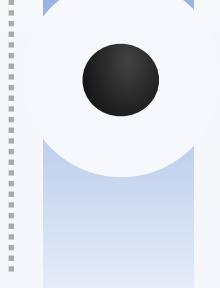
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- Guiding concept: If perturbation solutions grow in time, then the BH is unstable. If they shrink, then the BH is stable.

3. Stability analysis ...

- **Modern perspective**: Show that only damped and oscillatory time-domain are physical. (i.e. damped harmonic oscillation)
- **Prior to 1970**: Only show that purely growing but non-oscillatory time-domain modes are unphysical. (i.e. exclude pure exponential growth)



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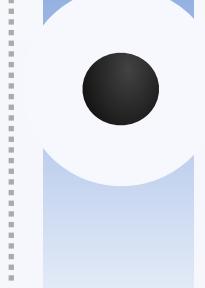
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- Guiding concept: If perturbation solutions grow in time, then the BH is unstable. If they shrink, then the BH is stable.

3. Stability analysis' key idea: w must be consistent between time and radial sectors

$$h_{\mu\nu} \propto \frac{1}{r} e^{-iwt} {}_{-2}Y_\ell(\cos(\theta))$$

$$[p_0(r, w) + p_1(r)\partial_r + p_2(r)\partial_r^2] \psi(r) = 0$$



1.1 On the possible presence of black holes

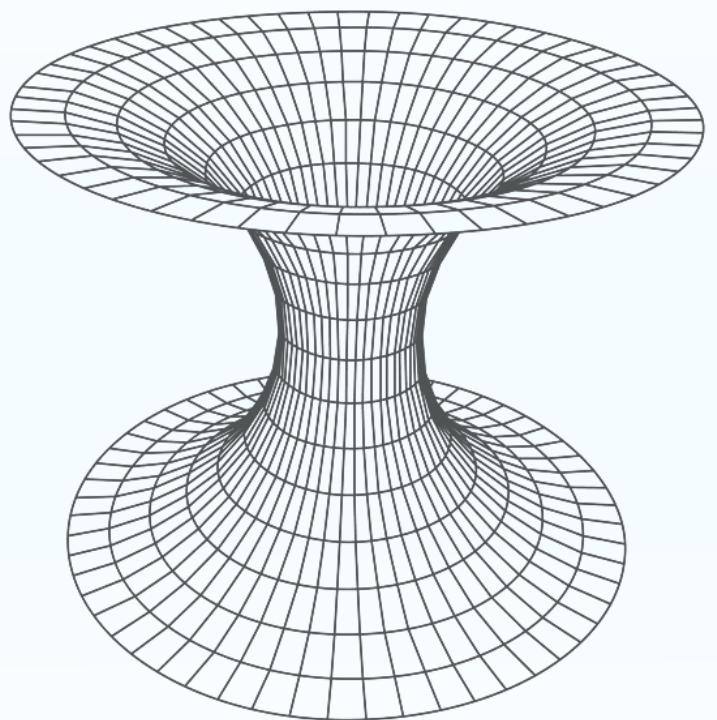
*“We have equilibrium, but ***is it stable?***”*

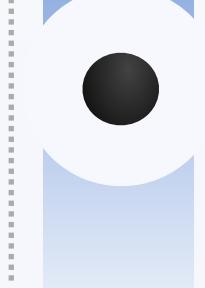
Regge and Wheeler, 1957

- Guiding concept: If perturbation solutions grow in time, then the BH is unstable. If they shrink, then the BH is stable.

3. Stability analysis ...

- **The Regge+Wheeler argument:** The physical boundary conditions are that the radiation is regular at spatial infinity, and should be thought of as entering a *wormhole* at the BH event horizon. Their wormhole picture required that one should be able to “stitch” the radiation to outgoing radiation elsewhere ...





1.1 On the possible presence of black holes

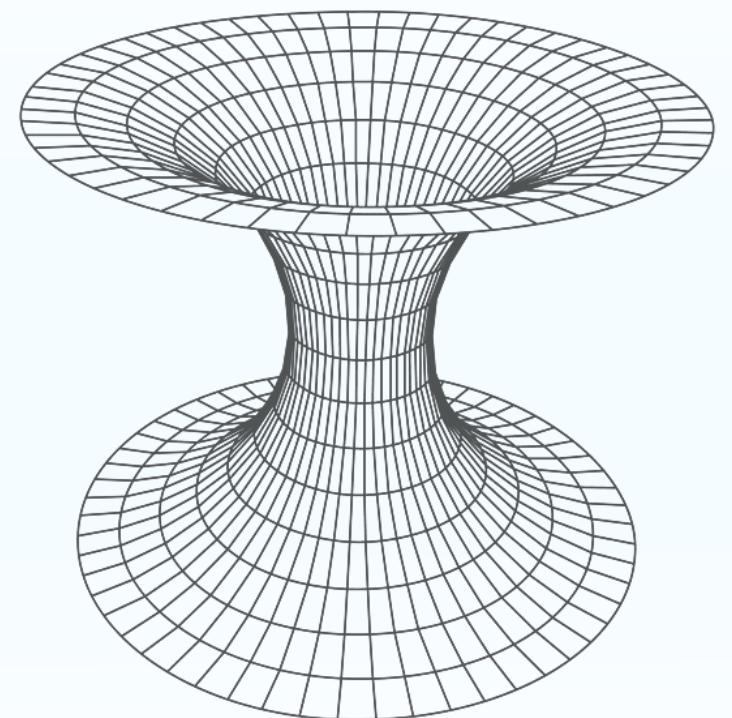
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- Regge+Wheeler claimed without explicit proof that if one assumed a purely imaginary frequency, *then* radial solutions vanish at the event horizon, meaning that the solution could not be smoothly stitched to non-zero radiation at the other end of the wormhole. ***Since such solutions are unphysical, Schwarzschild BHs are stable.***



1.1 On the possible presence of black holes

1950s-1970s

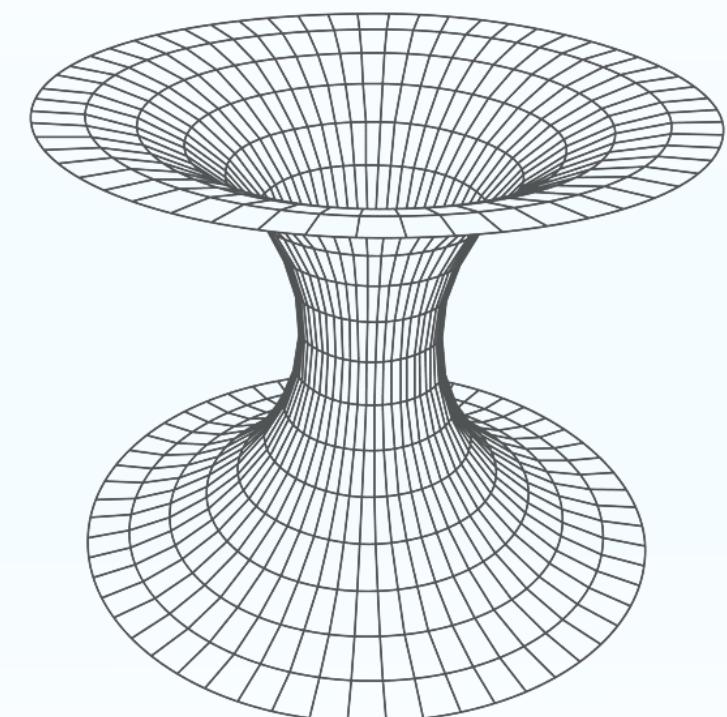
"We have equilibrium, but *is it stable?*"

Regge and Wheeler, 1957

Does this argument make
sense to us?

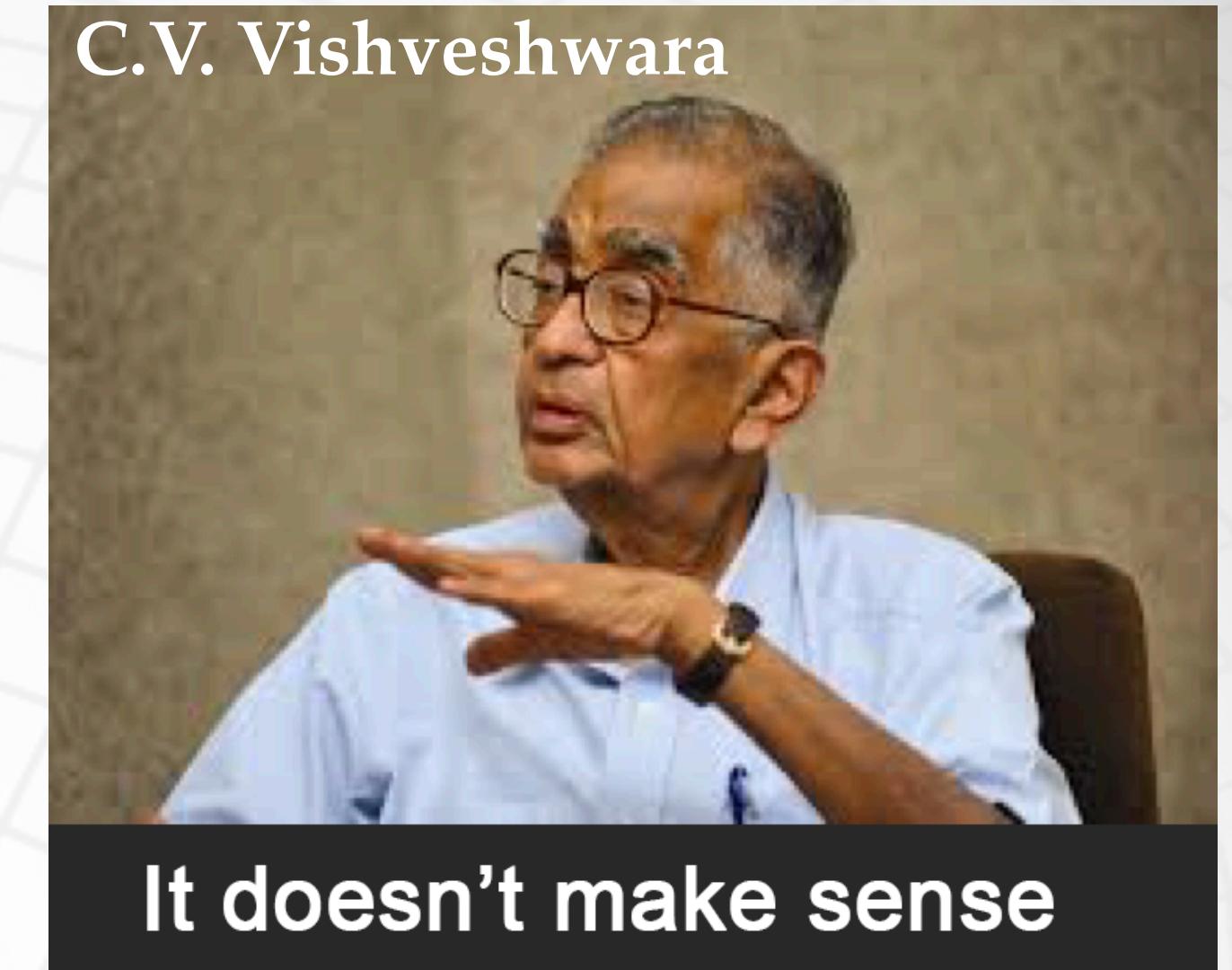
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Vishveshwara, 1968 (PhD thesis)

- Using Schwarzschild coordinates leads to **misleading and unphysical behavior at the event horizon.**
- Instead, use Kruskal coordinates so that there is no coordinate singularity at the event horizon.
- In those coordinates, imposing regularity at spatial infinity is sufficient to exclude purely imaginary modes without relying on a wormhole argument.

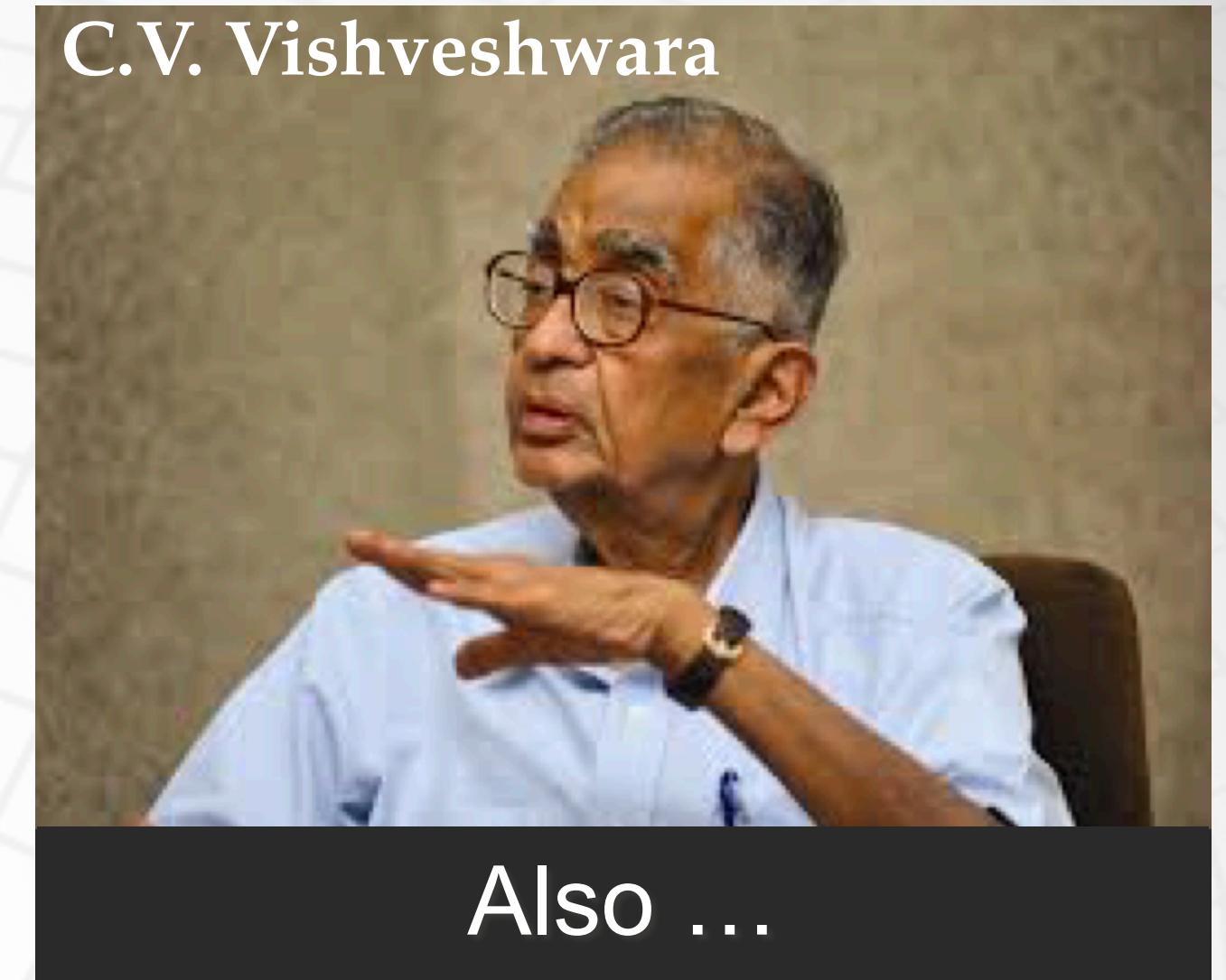


C.V. Vishveshwara

It doesn't make sense

Vishveshwara, 1970

- Even that picture is incomplete because it is still based in a *normal mode* perspective.
- Vishveshwara later showed that a “quasi-normal” mode picture is more appropriate since BH systems are dissipative.
- Also, along with Edelstein, he showed that there were various minor errors in the Regge-Wheeler derivation ...



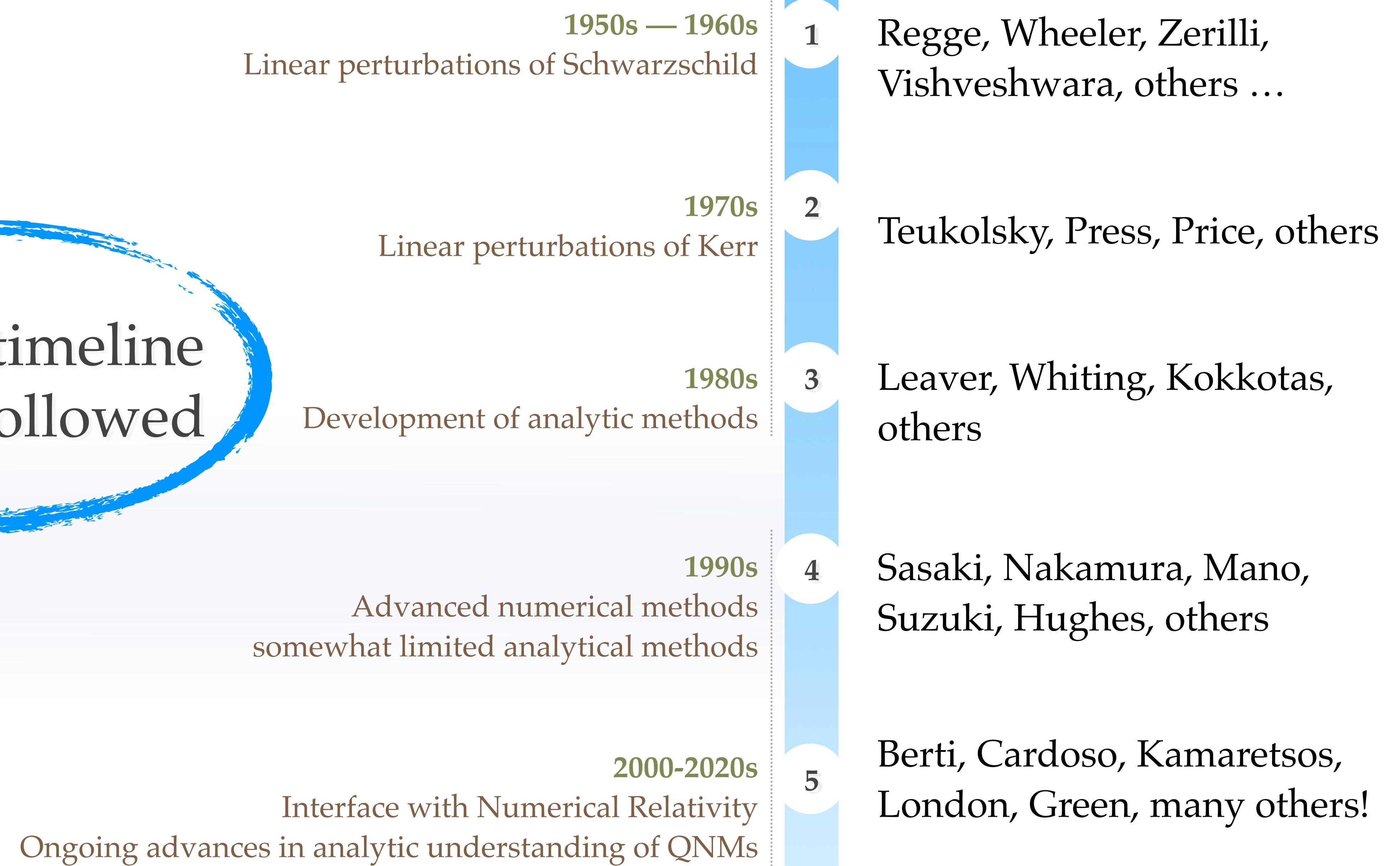


Theory is messy ✓

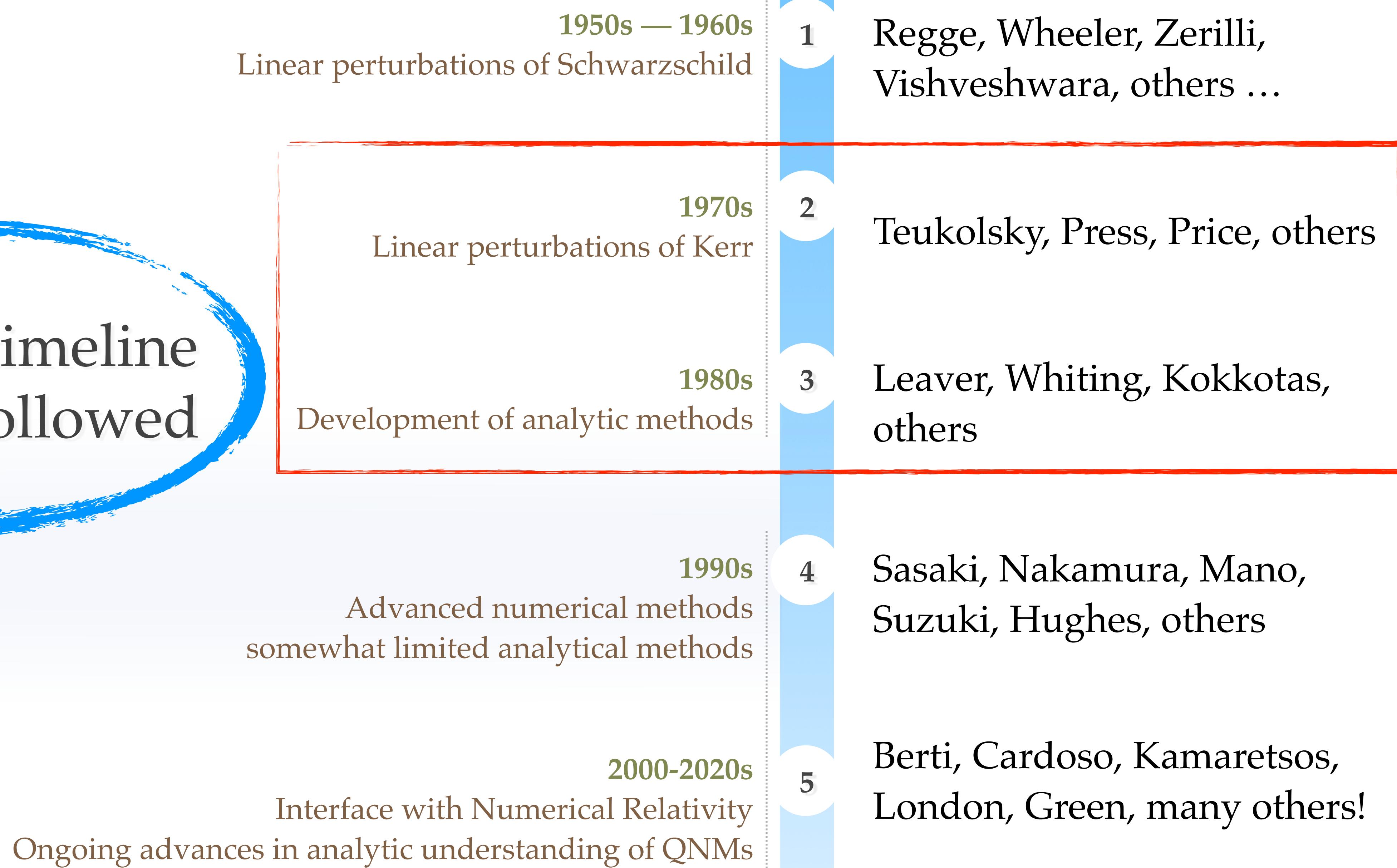
Skepticism is essential

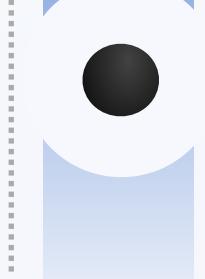


A rough timeline of what followed



A rough timeline of what followed

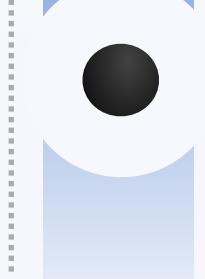




1.2 Einstein's equations linearized around Kerr

- ❖ While Regge, Wheeler and other were navigating the Schwarzschild problem ...
 - Penrose, Newman, Pirani and others were developing gauge invariant tetrad formalisms (mostly during the early to mid 1960s)
 - Teukolsky was first to apply the Newman-Penrose formalism to the linear stability of Kerr (after the confusion with Schwarzschild, that perhaps seemed a worthwhile effort)
- ❖ The resulting linearization of Einstein's equations around Kerr is (in Boyer-Lindquist coords)

$$\begin{aligned}
 & \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] \frac{\partial^2 \psi}{\partial t^2} + \frac{4Mar}{\Delta} \frac{\partial^2 \psi}{\partial t \partial \varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right] \frac{\partial^2 \psi}{\partial \varphi^2} \\
 & - \Delta^{-s} \frac{\partial}{\partial r} \left(\Delta^{s+1} \frac{\partial \psi}{\partial r} \right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) - 2s \left[\frac{a(r - M)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right] \frac{\partial \psi}{\partial \varphi} \\
 & - 2s \left[\frac{M(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right] \frac{\partial \psi}{\partial t} + (s^2 \cot^2 \theta - s)\psi = 4\pi \Sigma T.
 \end{aligned}$$

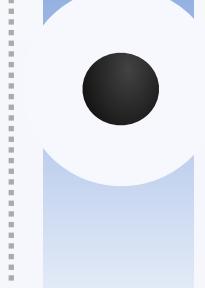


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Weyl scalar

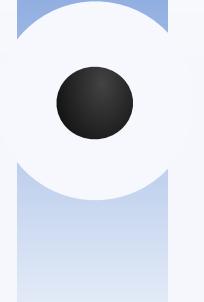


1.2 Einstein's equations linearized around Kerr

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Teukolsky's “master” equation

Separate with ansatz



1.2 Einstein's equations linearized around Kerr

The full form of a **single** QNM

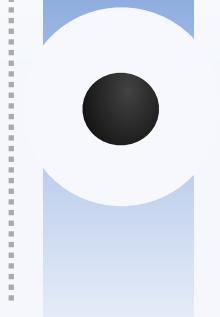
$$\psi \propto R(r) S(\theta) e^{-iwt} e^{-im\phi}$$

In more general scenarios, one should perform an integral transform.

Teukolsky's "master" equation

Separate with ansatz

Teukolsky's Equat



1.3 Black hole specific special functions

$$\mathcal{L}_u S(u) = -A S(u)$$

$$\mathcal{L}_r R(r) = +A R(r)$$

The QNMs' radial and angular functions are special functions that, when solved simultaneously, fully constrain a Kerr QNM

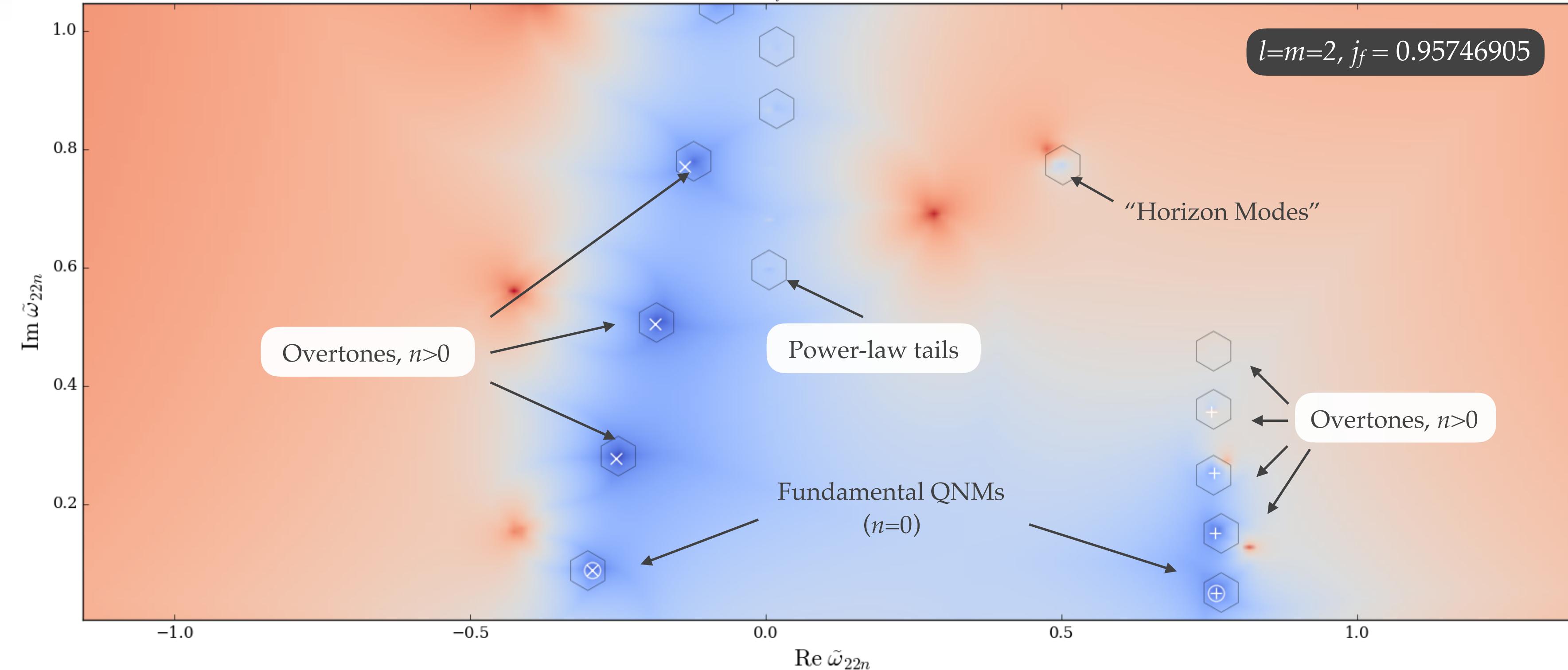
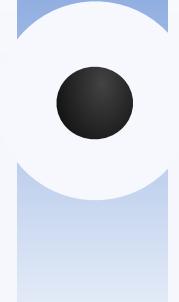
Separate with ansatz

Teukolsky's Equations

Next lecture

1.3 Black hole specific special functions

1980s-2000s

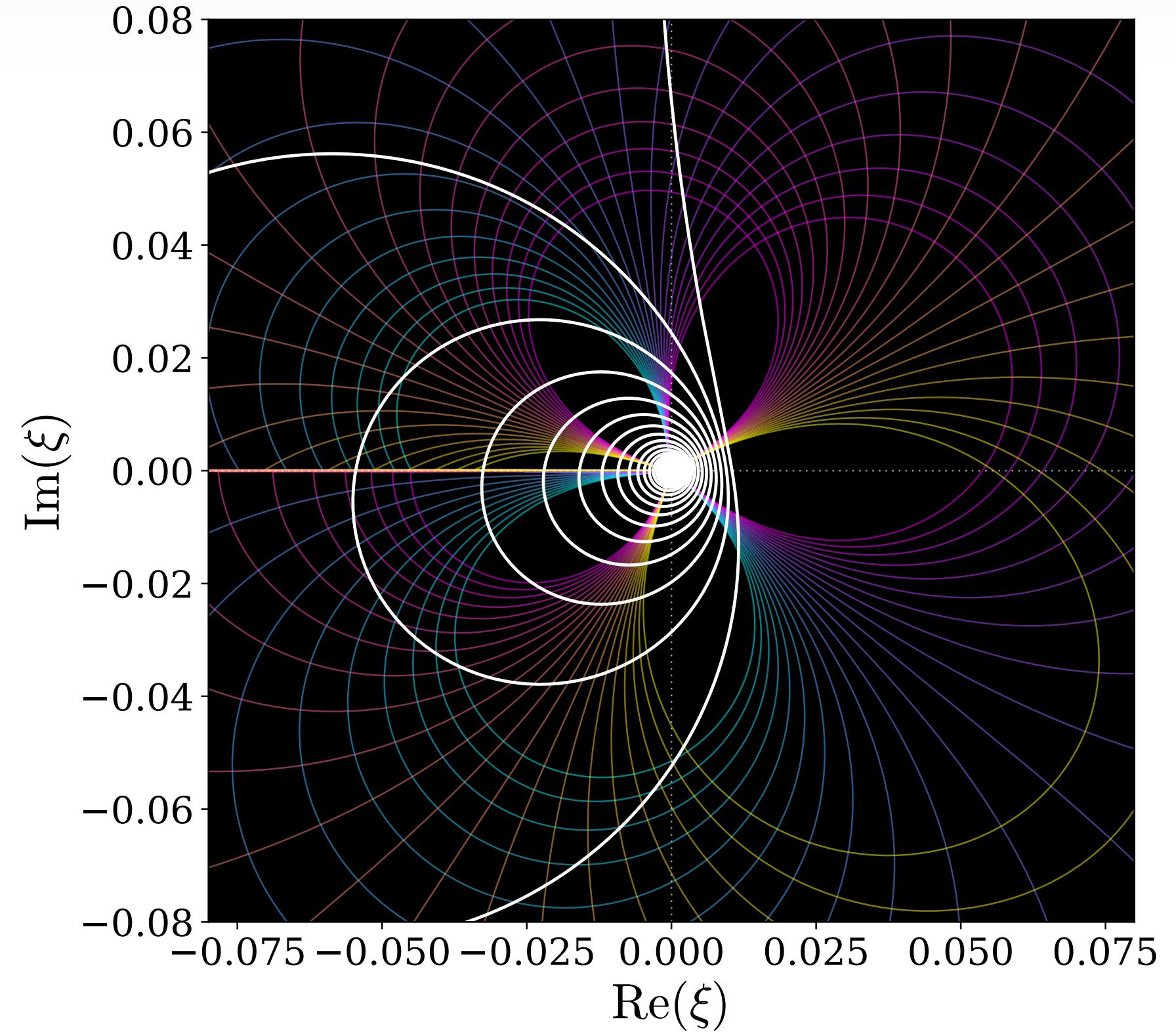
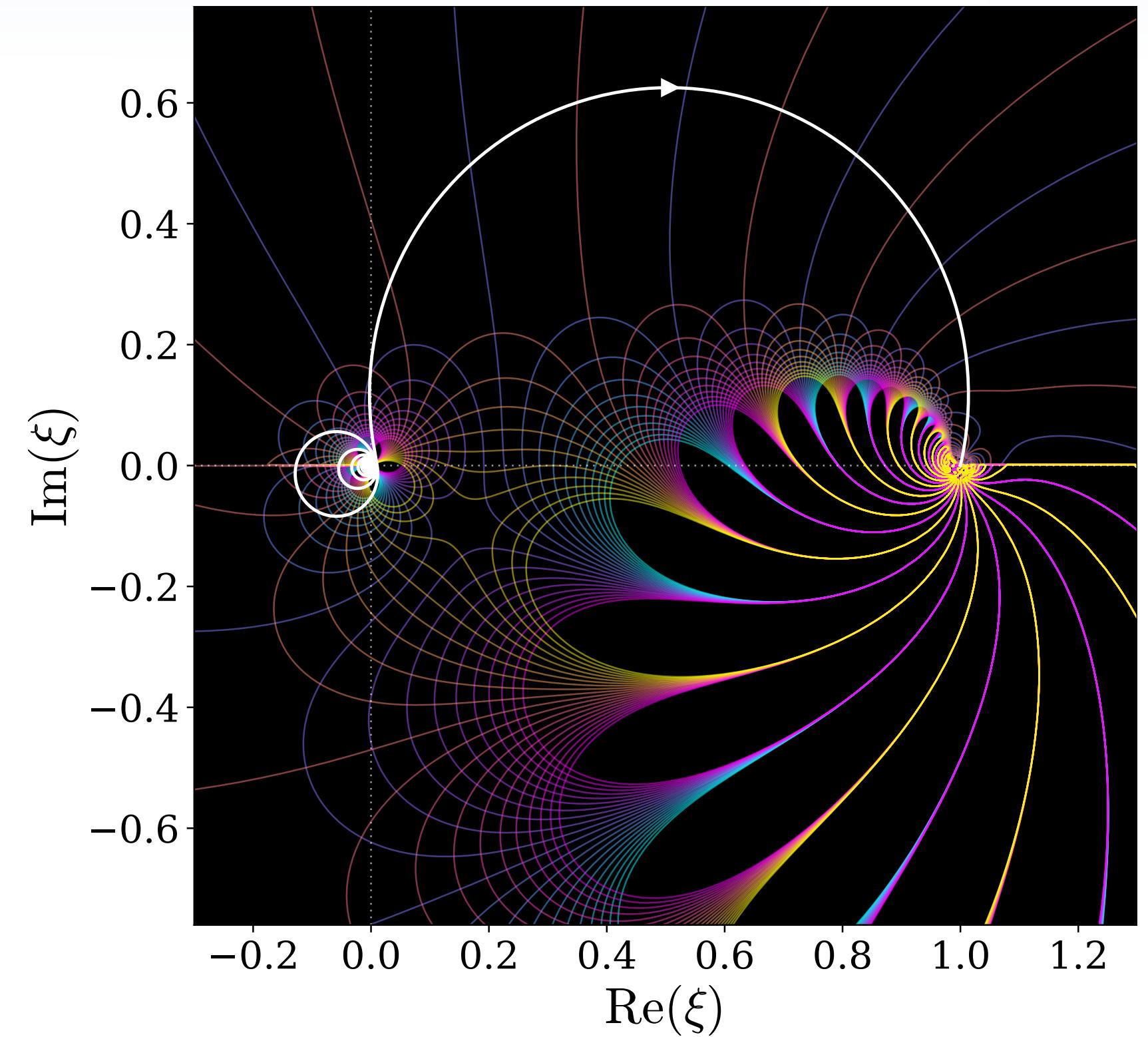


Teukolsky's Equations

Next lecture

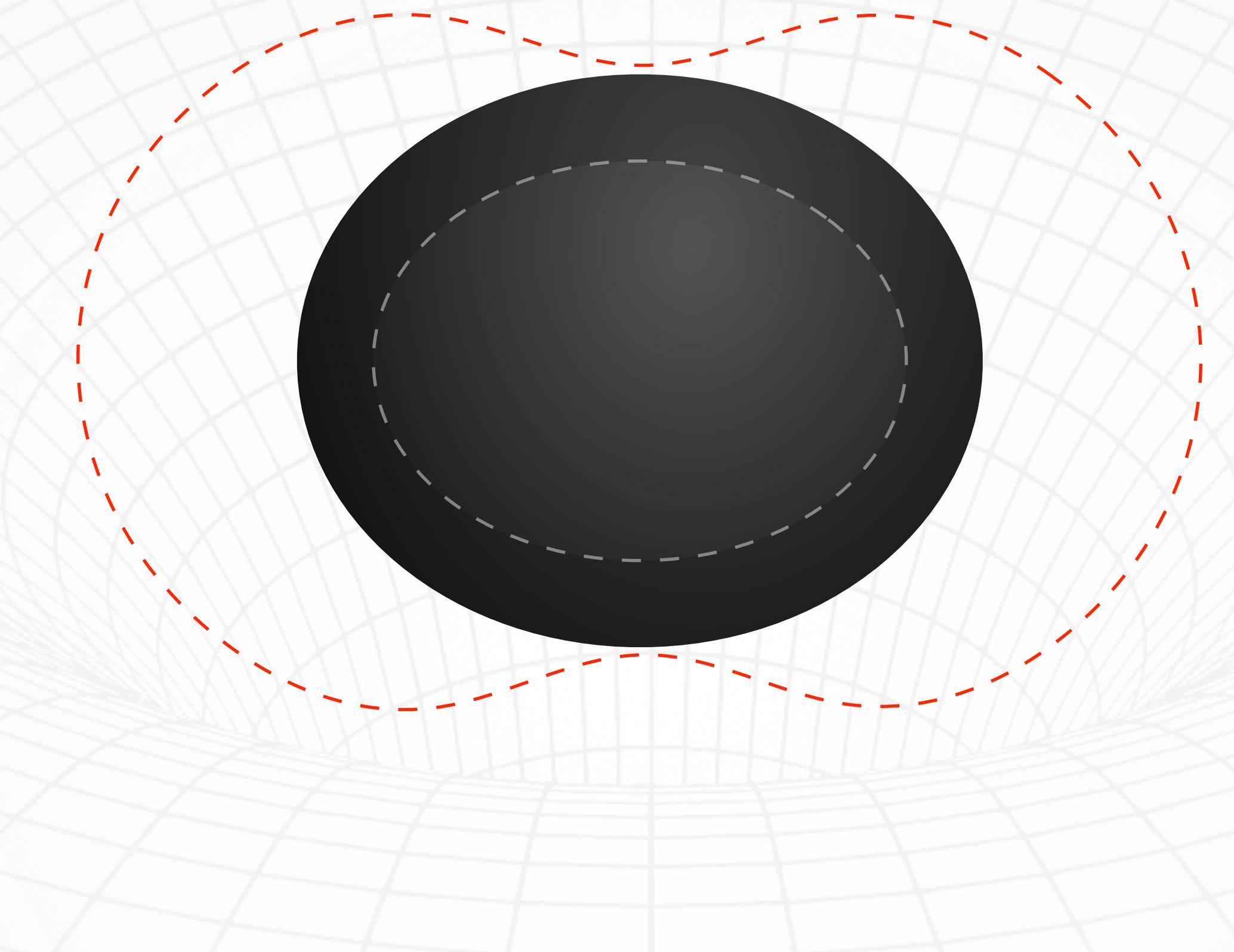
1.3 Black hole specific special functions

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Teukolsky's Equations

Next lecture



thanks!