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The analytic properties of the spin-weighted angular spheroidal functions introduced by Teukolsky are investigated by means of a series involving Jacobi polynomials. This approach facilitates the numerical determination of eigenvalues, particularly in the case of complex frequencies.

1. INTRODUCTION

Generalizations of angular spheroidal wavefunctions known as spin-weighted angular spheroidal functions were introduced by Teukolsky[1] in a paper dealing with perturbations of rotating black holes. These functions, which we shall denote by ${}_sS_{lm}(\gamma;x)$ are defined as the eigenfunctions of the differential equation

$$(1-x^2)\frac{d^2S}{dx^2}-2x\frac{dS}{dx}+\left(\gamma^2x^2-\frac{m^2+s^2}{1-x^2}-\frac{2msx}{1-x^2}\right.$$
$$\left.-2\gamma sx+{}_sE_l^m(\gamma)\right)S=0, \tag{1}$$

where ${}_sE_l^m(\gamma)$ is the eigenvalue. Clearly, when $s=0$, Eq. (1) reduces to the usual spheroidal wave equation. In a later paper Press and Teukolsky[2] used a perturbation technique for small real γ to obtain solutions of Eq. (1) in the form

$${}_sS_{lm}(\gamma;x)=\sum_{l'}{}_sA_{ll'}^m(\gamma)\,{}_sY^m{}_{l'}(x'). \tag{2}$$

For larger values of the parameter γ Press and Teukolsky used a continuation technique due to Wasserstrom.[3] This technique is extremely powerful for the purposes of numerical computation. However, these methods are not suitable for the elucidation of the analytic properties of Eq. (1).

In the present paper the analytic properties of the eigenfunctions of Eq. (1) are investigated by means of expansions of the form

$${}_sS_{lm}(\gamma;x)=\exp(\gamma x)\left(\frac{1-x}{2}\right)^{|m+s|/2}\left(\frac{1+x}{2}\right)^{|m-s|/2}$$
$$\times\sum_{r=0}^{\infty}{}_sA_{lm}^{(r)}(\gamma)P_r^{(|m+s|,\,|m-s|)}(x) \tag{3}$$

and

$${}_sS_{lm}(\gamma;x)=\exp(-\gamma x)\left(\frac{1-x}{2}\right)^{|m+s|/2}\left(\frac{1+x}{2}\right)^{|m-s|/2}$$
$$\times\sum_{r=0}^{\infty}{}_sB_{lm}^{(r)}(\gamma)P_r^{(|m+s|,\,|m-s|)}(x), \tag{4}$$

where $P_r^{(\alpha,\beta)}(x)$ is the Jacobi polynomial and the coefficients ${}_sA_{lm}^{(r)}(\gamma)$ and ${}_sB_{lm}^{(r)}(\gamma)$ satisfy separate three-term recurrence relations. Both of these recurrence relations give rise to a certain transcendental equation involving a continued fraction for the determination of the eigenvalues ${}_sE_l^m(\gamma)$.

2. TRANSFORMATIONS OF THE DIFFERENTIAL EQUATION

In order to investigate the eigenfunctions of the differential equation (1), we transform this equation by taking out the appropriate behavior of the eigenfunctions at the singular points of the differential equation, namely $x=-1$, $x=+1$, and $x=\infty$. We find that near $x=-1$ the dominant behavior is $(1+x)^{|m-s|/2}$, near $x=+1$ the dominant behavior is $(1-x)^{|m+s|/2}$, and at the point at infinity the behavior is either $\exp(\gamma x)$ or $\exp(-\gamma x)$. We therefore introduce the abbreviations

$$\alpha=|m+s|, \tag{5}$$
$$\beta=|m-s|, \tag{6}$$

and introduce new functions ${}_sU_{lm}(\gamma;x)$ and ${}_sV_{lm}(\gamma;x)$ by means of the equations

$${}_sS_{lm}(\gamma;x)=\exp(\gamma x)\left(\frac{1-x}{2}\right)^{\alpha/2}\left(\frac{1+x}{2}\right)^{\beta/2}{}_sU_{lm}(\gamma;x) \tag{7}$$

and

$${}_sS_{lm}(\gamma;x)=\exp(-\gamma x)\left(\frac{1-x}{2}\right)^{\alpha/2}\left(\frac{1+x}{2}\right)^{\beta/2}{}_sV_{lm}(\gamma;x). \tag{8}$$

Note that this implies

$${}_sV_{lm}(\gamma;x)=\exp(2\gamma x)\,{}_sU_{lm}(\gamma;x). \tag{9}$$

We readily find that ${}_sU_{lm}(\gamma;x)$ satisfies the differential equation

$$(1-x^2)\frac{d^2U}{dx^2}+[\beta-\alpha-(2+\alpha+\beta)x+2\gamma(1-x^2)]\frac{dU}{dx}$$
$$+\left[{}_sE_l^m(\gamma)+\gamma^2-\frac{\alpha+\beta}{2}\left(\frac{\alpha+\beta}{2}+1\right)+\gamma(\beta-\alpha)\right.$$
$$\left.-\gamma(\alpha+\beta+2+2s)x\right]U=0 \tag{10}$$

and that ${}_sV_{lm}(\gamma;x)$ satisfies the differential equation

$$(1-x^2)\frac{d^2V}{dx^2}+[\beta-\alpha-(2+\alpha+\beta)x-2\gamma(1-x^2)]\frac{dV}{dx}$$
$$+\left[{}_sE_l^m(\gamma)+\gamma^2-\frac{\alpha+\beta}{2}\left(\frac{\alpha+\beta}{2}+1\right)-\gamma(\beta-\alpha)\right.$$
$$\left.+\gamma(\alpha+\beta+2-2s)x\right]V=0. \tag{11}$$

3. EIGENFUNCTIONS AS SERIES OF JACOBI POLYNOMIALS

The differential equations (10) and (11) are closely related to the differential equation for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. These polynomials, defined by the

Rodrigues' formula

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\left(\frac{d}{dx}\right)^n [(1-x)^{\alpha+n}(1+x)^{\beta+n}], \tag{12}$$

satisfy the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}+[\beta-\alpha-(\alpha+\beta+2)x]\frac{dy}{dx}+n(n+\alpha+\beta+1)y=0 \tag{13}$$

to which both Eqs. (10) and (11) reduce when $\gamma=0$, provided we make the identification

$$_sE_l^m(0)=[n+(\alpha+\beta)/2][n+(\alpha+\beta)/2+1] \tag{14}$$

or

$$_sE_l^m(0)=l(l+1), \tag{15}$$

where

$$l=n+(\alpha+\beta)/2=n+\max(|m|,|s|). \tag{16}$$

Further, Jacobi polynomials also satisfy the recurrence relation

$$\begin{aligned}xP_n^{(\alpha,\beta)}(x)=&\frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n+1}^{(\alpha,\beta)}(x)\\&-\frac{(\alpha^2-\beta^2)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}P_n^{(\alpha,\beta)}(x)\\&+\frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-1}^{(\alpha,\beta)}(x)\end{aligned} \tag{17}$$

and the differentiation formula

$$\begin{aligned}&(1-x^2)\frac{d}{dx}P_n^{(\alpha,\beta)}(x)\\&=\frac{-2n(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}P_{n+1}^{(\alpha,\beta)}(x)\\&+\frac{(\alpha-\beta)2n(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}P_n^{(\alpha,\beta)}(x)\\&+\frac{2(n+\alpha)(n+\beta)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}P_{n-1}^{(\alpha,\beta)}(x).\end{aligned} \tag{18}$$

Consequently, if we expand $_sU_{lm}(\gamma;x)$ and $_sV_{lm}(\gamma;x)$ as infinite series of Jacobi polynomials,

$$_sU_{lm}(\gamma;x)=\sum_{r=0}^{\infty}{}_sA_{lm}^{(r)}(\gamma)P_r^{(\alpha,\beta)}(x) \tag{19}$$

and

$$_sV_{lm}(\gamma;x)=\sum_{r=0}^{\infty}{}_sB_{lm}^{(r)}(\gamma)P_r^{(\alpha,\beta)}(x), \tag{20}$$

we obtain three-term recurrence relations for the coefficients. Specifically, we find that

$$\begin{aligned}&\left[{}_sE_l^m(\gamma)+\gamma^2-\frac{\alpha+\beta}{2}\left(\frac{\alpha+\beta}{2}+1\right)+\frac{2(\alpha-\beta)s\gamma}{\alpha+\beta+2}\right]{}_sA_{lm}^{(0)}(\gamma)\\&+\frac{4\gamma(\alpha+1)(\beta+1)((\alpha+\beta)/2+1-s)}{(\alpha+\beta+2)(\alpha+\beta+3)}{}_sA_{lm}^{(1)}(\gamma)=0,\end{aligned} \tag{21}$$

and, for $r=1,2,\cdots,$

$$\begin{aligned}&\frac{4\gamma(r+\alpha+1)(r+\beta+1)(r+(\alpha+\beta)/2+1-s)}{(2r+\alpha+\beta+2)(2r+\alpha+\beta+3)}{}_sA_{lm}^{(r+1)}(\gamma)\\&+\left[{}_sE_l^m(\gamma)+\gamma^2-\left(r+\frac{\alpha+\beta}{2}\right)\left(r+\frac{\alpha+\beta}{2}+1\right)+\frac{2\gamma s(\alpha-\beta)(\alpha+\beta)}{(2r+\alpha+\beta)(2r+\alpha+\beta+2)}\right]{}_sA_{lm}^{(r)}(\gamma)\\&-\frac{4\gamma r(r+\alpha+\beta)(r+(\alpha+\beta)/2+s)}{(2r+\alpha+\beta-1)(2r+\alpha+\beta)}{}_sA_{lm}^{(r-1)}(\gamma)=0;\end{aligned} \tag{22}$$

similarly the equations for the $_sB_{lm}^{(r)}(\gamma)$ are

$$\begin{aligned}&\left[{}_sE_l^m(\gamma)+\gamma^2-\frac{\alpha+\beta}{2}\left(\frac{\alpha+\beta}{2}+1\right)+\frac{2(\alpha-\beta)s\gamma}{(\alpha+\beta+2)}\right]{}_sB_{lm}^{(0)}(\gamma)\\&-\frac{4\gamma(\alpha+1)(\beta+1)((\alpha+\beta)(2+1-s))}{(\alpha+\beta+2)(\alpha+\beta+3)}{}_sB_{lm}^{(1)}(\gamma)=0\end{aligned} \tag{23}$$

and, for $r=1,2,\cdots,$

$$\begin{aligned}&\frac{-4\gamma(r+\alpha+1)(r+\beta+1)(r+(\alpha+\beta)/2+1+s)}{(2r+\alpha+\beta+2)(2r+\alpha+\beta+3)}{}_sB_{lm}^{(r+1)}(\gamma)\\&+\left[{}_sE_l^m(\gamma)+\gamma^2-\left(r+\frac{\alpha+\beta}{2}\right)\left(r+\frac{\alpha+\beta}{2}+1\right)+\frac{2\gamma s(\alpha-\beta)(\alpha+\beta)}{(2r+\alpha+\beta)(2r+\alpha+\beta+2)}\right]{}_sB_{lm}^{(r)}(\gamma)\\&+\frac{4\gamma r(r+\alpha+\beta)(r+(\alpha+\beta)/2-s)}{(2r+\alpha+\beta-1)(2r+\alpha+\beta)}{}_sB_{lm}^{(r-1)}(\gamma).\end{aligned} \tag{24}$$

At first sight it would seem from either Eqs. (21) and (22) or Eqs. (23) and (24) that a well-behaved solution of the original differential equation (1) exists for any value of the constant $_sE_l^m(\gamma)$, since, given $_sA_{lm}^{(0)}(\gamma)$, all of the other $_sA_{lm}^{(r)}$ may be determined uniquely via Eqs. (21) and (22). However, unless $_sE_l^m(\gamma)$ is chosen appropriately, the $_sA_{lm}^{(r)}$ increase without bound and the series (19) does not converge. It may be shown directly from Eq. (22) that for sufficiently large r, either

$$_sA_{lm}^{(r)}(\gamma)\sim\frac{\mathrm{const}(-\gamma)^r}{\Gamma(r+(\alpha+\beta+3)/2-s)} \tag{25}$$

or

$$_sA_{lm}^{(r)}(\gamma)\sim\mathrm{const}(\gamma)^{-r}\Gamma\left(r+\frac{\alpha+\beta+1}{2}+s\right). \tag{26}$$

For the case of Eq. (26) the coefficients increase in absolute value without bound and the series (19) diverges for all values of x. In the case of Eq. (25), the series (19) converges uniformly in the finite complex x plane.[4] By an argument analogous to that given by Flammer[5] (cf. also Meixner and Schäfke,[6] Sec. 1–8), this convergence requires that $_sE_l^m(\gamma)$ satisfy a certain transcendental equation, which is most conveniently formulated in terms of continued fractions.

4. DETERMINATION OF THE EIGENVALUE

In order to obtain the equation which determines the eigenvalues $_sE_l^m(\gamma)$, we find it convenient to define quantities $_sN_r^{lm}(\gamma)$, and $_sK_r^{lm}(\gamma)$, and $_sL_r^{lm}(\gamma)$ by

$$ {}_sN_r^{lm}(\gamma) = \frac{2\gamma(r+\alpha)(r+\beta)(2r+\alpha+\beta-2s)}{(2r+\alpha+\beta)(2r+\alpha+\beta+1)} \frac{{}_sA_{lm}^{(r)}(\gamma)}{{}_sA_{lm}^{(r-1)}(\gamma)}, \tag{27} $$

$$ {}_sK_r^{lm}(\gamma) = \left(r+\frac{\alpha+\beta}{2}\right)\left(r+\frac{\alpha+\beta}{2}+1\right) - \gamma^2 - \frac{2\gamma s(\alpha-\beta)(\alpha+\beta)}{(2r+\alpha+\beta)(2r+\alpha+\beta+2)}, \tag{28} $$

$$ {}_sL_r^{lm}(\gamma) = \frac{2\gamma^2(r+\alpha)(r+\beta)(r+\alpha+\beta)(2r+\alpha+\beta+2s)(2r+\alpha+\beta-2s)}{(2r+\alpha+\beta)(2r+\alpha+\beta+3)(2r+\alpha+\beta-1)}, \tag{29} $$

where we have made the assumption that none of the ${}_sA_{lm}^{(r)}$ vanish.

Equation (21) may now be written as

$$ {}_sN_1^{lm}(\gamma) - {}_sK_0^{lm}(\gamma) + {}_sE_l^m(\gamma) = 0 \tag{30} $$

and (22) may be written as

$$ {}_sN_r^{lm}(\gamma) = {}_sL_r^{lm}(\gamma)/[{}_sE_l^m(\gamma) - {}_sK_r^{lm}(\gamma) + {}_sN_{r+1}^{lm}(\gamma)] \tag{31} $$

or as

$$ {}_sN_{r+1}^{lm}(\gamma) = {}_sK_r^{lm}(\gamma) - {}_sE_l^m(\gamma) + {}_sL_r^{lm}(\gamma)/{}_sN_r^{lm}(\gamma). \tag{32} $$

These expressions suggest that our treatment should be based upon continued fractions. Define quantities ${}_sN_r'^{lm}(\gamma)$ in terms of infinite continued fractions by

$$ {}_sN_r'^{lm}(\gamma) = \frac{{}_sL_r^{lm}(\gamma)}{{}_sE_l^m(\gamma) - {}_sK_r^{lm}(\gamma)} \;+\; \frac{{}_sL_{r+1}^{lm}(\gamma)}{{}_sE_l^m(\gamma) - {}_sK_{r+1}^{lm}(\gamma)} \;+\; \cdots . \tag{33} $$

From the analysis given by Perron,[7] this infinite continued fraction is convergent for all values of ${}_sE_l^m(\gamma)$. Consequently, from the definition (33) we have

$$ {}_sN_r'^{lm}(\gamma) = {}_sL_r^{lm}(\gamma)/[{}_sE_l^m(\gamma) - {}_sK_r^{lm}(\gamma) + {}_sN_{r+1}'^{lm}(\gamma)] \tag{34} $$

so that we can satisfy Eq. (31) by taking

$$ {}_sN_r^{lm}(\gamma) = {}_sN_r'^{lm}(\gamma), \tag{35} $$

with ${}_sN_r'^{lm}(\gamma)$ given by (33). But by iteration of Eq. (32) with the condition ${}_sA_{lm}^{(r)}(\gamma) = 0$ for $r < 0$ we also have

$$ {}_sN_{r+1}^{lm}(\gamma) = {}_sK_r^{lm}(\gamma) - {}_sE_l^m(\gamma) + \frac{{}_sL_r^{lm}(\gamma)}{{}_sK_{r-1}^{lm}(\gamma) - {}_sE_l^m(\gamma)} \;+\; \frac{{}_sL_{r-1}^{lm}(\gamma)}{{}_sK_{r-2}^{lm}(\gamma) - {}_sE_l^m(\gamma)} \;+\; \cdots \;+\; \frac{{}_sL_1^{lm}(\gamma)}{{}_sK_0^{lm}(\gamma) - {}_sE_l^m(\gamma)}. \tag{36} $$

Equating this continued fraction with the infinite convergent continued fraction ${}_sN_{r+1}'^{lm}(\gamma)$ obtained by putting $r = r+1$ in Eq. (33) gives rise to a transcendental equation for the determination of the eigenvalue ${}_sE_l^m(\gamma)$; specifically

$$ {}_sE_l^m(\gamma) = {}_sK_r^{lm}(\gamma) \;+\; \frac{{}_sL_r^{lm}(\gamma)}{{}_sK_{r-1}^{lm}(\gamma) - {}_sE_l^m(\gamma)} \;+\; \frac{{}_sL_{r-1}^{lm}(\gamma)}{{}_sK_{r-2}^{lm}(\gamma) - {}_sE_l^m(\gamma)} \;+\; \cdots \;+\; \frac{{}_sL_1^{lm}(\gamma)}{{}_sK_0^{lm}(\gamma) - {}_sE_l^m(\gamma)} + \frac{{}_sL_{r+1}^{lm}(\gamma)}{{}_sK_{r+1}^{lm}(\gamma) - {}_sE_l^m(\gamma)} \;+\; \frac{{}_sL_{r+2}^{lm}(\gamma)}{{}_sK_{r+2}^{lm}(\gamma) - {}_sE_l^m(\gamma)} \;+\; \cdots, \tag{37} $$

where r is less than or equal to the least value of l.

We wish to obtain a series expansion of the form

$$ {}_sE_l^m(\gamma) = \sum_{p=0}^{\infty} {}_sf_p^{lm}\gamma^p. \tag{38} $$

In light of Eqs. (15) and (16), it is convenient here to put $r = l - \max(|m|, |s|)$ in Eq. (37). The coefficients ${}_sf_p^{lm}$ are then obtained by substituting Eq. (38) into Eq. (37) and successively raising each denominator up to its associated numerator by binomial expansion and then equating coefficients. (Note that the smallest eigenvalue has $l = \max(|m|, |s|)$. If we let

$$ H(l) = \frac{[l^2 - (\alpha+\beta)^2/4][l^2 - s^2][l^2 - (\alpha-\beta)^2/4]}{2[l-1/2]l^3[l+1/2]}, \tag{39} $$

where

$$ \frac{\alpha+\beta}{2} = \max(|m|, |s|) \quad \text{and} \quad \frac{\alpha-\beta}{2} = \frac{ms}{\max(|m|, |s|)}, \tag{40} $$

then the first seven coefficients are given explicitly by

$$ {}_sf_0^{lm} = l(l+1), \tag{41a} $$

$$ {}_sf_1^{lm} = -2s^2m/l(l+1), \tag{41b} $$

$$ {}_sf_2^{lm} = H(l+1) - H(l) - 1, \tag{41c} $$

[8]

$$ {}_sf_3^{lm} = 2s^2m\left[\frac{H(l)}{(l-1)l^2(l+1)} - \frac{H(l+1)}{(l+2)(l+1)^2 l}\right], \tag{41d} $$

$$ \begin{aligned} {}_sf_4^{lm} = {} & 4s^4m^2\left[\frac{H(l+1)}{(l+2)^2(l+1)^4l^2} - \frac{H(l)}{(l-1)^2l^4(l+1)^2}\right] \\ & + \frac{1}{2}\left[\frac{H^2(l+1)}{(l+1)} + \frac{H(l+1)H(l)}{(l+1)l} - \frac{H^2(l)}{l}\right] \\ & + \frac{1}{4}\left[\frac{[l-1]H(l)H(l-1)}{l(l-1/2)} - \frac{[l+2]H(l+1)H(l+2)}{(l+1)(l+3/2)}\right], \end{aligned} \tag{41e} $$

$$ \begin{aligned} {}_sf_5^{lm} = {} & 8s^6m^3\left[\frac{H(l)}{(l-1)^3l^6(l+1)^3} - \frac{H(l+1)}{(l+2)^3(l+1)^6l^3}\right] \\ & + s^2m\left[\frac{3H^2(l)}{(l-1)l^3(l+1)} - \frac{[7l^2+7l+4]H(l)H(l+1)}{(l-1)l^3(l+1)^3(l+2)}\right. \\ & - \frac{3H^2(l+1)}{(l+2)(l+1)^3l} + \frac{1}{2}\left(\frac{[3l+7]H(l+1)H(l+2)}{(l+3)(l+3/2)(l+1)^3l}\right. \\ & \left.\left. - \frac{[3l-4]H(l)H(l-1)}{(l-2)(l-1/2)l^3(l+1)}\right)\right], \end{aligned} \tag{41f} $$

$$ \begin{aligned} {}_sf_6^{lm} = {} & 16s^8m^4\left[\frac{H(l+1)}{(l+2)^4(l+1)^8l^4} - \frac{H(l)}{(l-1)^4l^8(l+1)^4}\right] \\ & + 4s^4m^2\left[\frac{3H^2(l+1)}{(l+2)^2(l+1)^5l^2}\right. \\ & + \frac{[11l^4+22l^3+31l^2+20l+6]H(l)H(l+1)}{(l-1)^2l^5(l+1)^5(l+1)^2} \\ & - \frac{3H^2(l)}{(l-1)^2l^5(l+1)^2} + \frac{1}{2}\left(\frac{[3l^2-8l+6]H(l)H(l-1)}{(l-2)^2(l-1)(l-1/2)l^5(l+1)^2}\right. \end{aligned} $$

$$-\frac{[3l^2+14l+17]H(l+1)H(l+2)}{(l+3)^2(l+2)(l+3/2)(l+1)^5l^2}\Bigg)\Bigg]$$

$$+\frac{1}{4}\Bigg[\frac{2H^3(l+1)}{(l+1)^2}+\frac{[2l^2+4l+3]H^2(l)H(l+1)}{l^2(l+1)^2}$$

$$-\frac{[2l^2+1]H^2(l+1)H(l)}{(l+1)^2l^2}-\frac{2H^3(l)}{l^2}$$

$$+\frac{[l+2][3l^2+2l-3]H(l)H(l+1)H(l+2)}{4(l+3/2)^2(l+1)^2l}$$

$$-\frac{[l-1][3l^2+4l-2]H(l+1)H(l)H(l-1)}{4(l-1/2)^2l^2(l+1)}$$

$$+\frac{[l+2]H^2(l+2)H(l+1)}{4(l+3/2)^2(l+1)^2}-\frac{[l-1]^2H^2(l-1)H(l)}{4(l-1/2)^2l^2}$$

$$+\frac{[l-1][7l-3]H^2(l)H(l-1)}{4(l-1/2)^2l^2}$$

$$-\frac{[l+2][7l+10]H^2(l+1)H(l+2)}{4(l+3/2)^2(l+1)^2}$$

$$+\frac{[l+3]H(l+1)H(l+2)H(l+3)}{12(l+3/2)^2(l+1)}$$

$$-\frac{[l-2]H(l)H(l-1)H(l-2)}{12(l-1/2)^2l}\Bigg]. \tag{41g}$$

These coefficients may be used to give a first approximation to the eigenvalue. Successively closer approximations, together with accurate values of the ${}_sN_r^{lm}(\gamma)$ may be made by use of the method of Blanch[9] and Bouwkamp.[10] Finally the value of ${}_sA_{lm}^{(r)}(\gamma)/{}_sA_{lm}^{(0)}(\gamma)$ is given by

$$\frac{{}_sA_{lm}^{(r)}(\gamma)}{{}_sA_{lm}^{(0)}(\gamma)}=\frac{((\alpha+\beta)/2+1)_r((\alpha+\beta+1)/2+1)_r}{\gamma^r(\alpha+1)_r(\beta+1)_r((\alpha+\beta)/2-s+1)_r}\times\prod_{j=1}^{r}{}_sN_j^{lm}(\gamma). \tag{42}$$

The quantities ${}_sB_{lm}^{(r)}$ may be handled in a similar fashion. By defining the quantities ${}_sM_r^{lm}$, for integral $r\geqslant 1$ by

$${}_sM_r^{lm}(\gamma)=-\frac{\gamma(r+\alpha)(\alpha+\beta)[r+(\alpha+\beta)/2+s]}{[r+(\alpha+\beta)/2][r+(\alpha+\beta+1)/2]}\,\frac{{}_sB_{lm}^{(r)}(\gamma)}{{}_sB_{lm}^{(r-1)}(\gamma)}, \tag{43}$$

we find that ${}_sM_r^{lm}(\gamma)$ satisfies precisely the same equations as ${}_sN_r^{lm}(\gamma)$. Again the requirement that the series (20) should converge leads to precisely the same eigenvalue for the solution (20) as that obtained from the series (19). Consequently, ${}_sM_r^{lm}(\gamma)={}_sN_r^{lm}(\gamma)$.

From Eqs. (27) and (42) we find that

$$\frac{{}_sB_{lm}^{(r)}(\gamma)}{{}_sA_{lm}^{(r)}(\gamma)}=\frac{{}_sB_{lm}^{(0)}(\gamma)}{{}_sA_{lm}^{(0)}(\gamma)}(-1)^r\frac{((\alpha+\beta)/2-s+1)_r}{((\alpha+\beta)/2+s+1)_r}. \tag{44}$$

However, provided that $2s$ is a positive integer,

$$\frac{((\alpha+\beta)/2-s+1)_r}{((\alpha+\beta)/2+s+1)_r}=\frac{((\alpha+\beta)/2-s+1)_{2s}}{((\alpha+\beta)/2-s+r+1)_{2s}}. \tag{45}$$

Consequently,

$$\frac{{}_sB_{lm}^{(r)}(\gamma)}{{}_sA_{lm}^{(r)}(\gamma)}=(-1)^r\frac{{}_sB_{lm}^{(0)}(\gamma)}{{}_sA_{lm}^{(0)}(\gamma)}\,\frac{((\alpha+\beta)/2-s+1)_{2s}}{((\alpha+\beta)/2-s+r+1)_{2s}}, \tag{46}$$

and in a similar fashion we also have

$$\frac{{}_{-s}B_{lm}^{(r)}(\gamma)}{{}_sB_{lm}^{(r)}(\gamma)}=\frac{((\alpha+\beta)/2-s+r+1)_{2s}}{((\alpha+\beta)/2-s+1)_{2s}}\,\frac{{}_{-s}B_{lm}^{(0)}(\gamma)}{{}_sB_{lm}^{(0)}(\gamma)} \tag{47a}$$

and

$$\frac{{}_{-s}A_{lm}^{(r)}(\gamma)}{{}_sA_{lm}^{(r)}(\gamma)}=\frac{((\alpha+\beta)/2-s+1)_{2s}}{((\alpha+\beta)/2-s+r+1)_{2s}}\,\frac{{}_{-s}A_{lm}^{(0)}(\gamma)}{{}_sA_{lm}^{(0)}(\gamma)}. \tag{47b}$$

5. NORMALIZATION

So far we have been able to show that the convergence of either (19) or (20) leads to an equation for the determination of the eigenvalues ${}_sE_l^m(\gamma)$ and that, given a correctly determined eigenvalue, the two sets of coefficients ${}_sA_{lm}^{(r)}(\gamma)$ and ${}_sB_{lm}^{(r)}(\gamma)$ are determined for positive integral values of r in terms of ${}_sA_{lm}^{(0)}(\gamma)$ and ${}_sB_{lm}^{(0)}(\gamma)$. We now come to the problem of the normalization of the solution. Clearly, two conditions are required to determine the two unknowns, ${}_sA_{lm}^{(0)}(\gamma)$ and ${}_sB_{lm}^{(0)}(\gamma)$. One of these conditions may be obtained by substituting $x=1$ in Eqs (9), (19), and (20) to obtain

$${}_sB_{lm}^{(0)}(\gamma)\sum_{r=0}^{\infty}\frac{{}_sB_{lm}^{(r)}(\gamma)}{{}_sB_{lm}^{(0)}(\gamma)}(r+\alpha)(r+\alpha-1)$$

$$=\exp(2\gamma)\,{}_sA_{lm}^{(0)}(\gamma)\sum_{r=0}^{\infty}\frac{{}_sA_{lm}^{(r)}(\gamma)}{{}_sA_{lm}^{(0)}(\gamma)}(r+\alpha)(r+\alpha-1). \tag{48}$$

Provided neither of the infinite series vanishes, we obtain an equation for ${}_sB_{lm}^{(0)}/{}_sA_{lm}^{(0)}$. An equation for ${}_sA_{lm}^{(0)}(\gamma){}_sB_{lm}^{(0)}(\gamma)$ is provided by the normalization requirement

$$\int_{-1}^{1}[{}_sS_{lm}(\gamma;x)]^2\,dx=1 \tag{49}$$

since we can use (3) and (4) to write this equation as

$${}_sA_{lm}^{(0)}(\gamma)\,{}_sB_{lm}^{(0)}(\gamma)\int_{-1}^{1}\left(\frac{1-x}{2}\right)^{\alpha}\left(\frac{1+x}{2}\right)^{\beta}\left[\sum_{n=0}^{\infty}\frac{{}_sA_{lm}^{(n)}(\gamma)}{{}_sA_{lm}^{(0)}(\gamma)}P_n^{(\alpha,\beta)}(x)\right]$$

$$\times\left[\sum_{r=0}^{\infty}\frac{{}_sB_{lm}^{(r)}(\gamma)}{{}_sB_{lm}^{(0)}(\gamma)}P_r^{(\alpha,\beta)}(x)\right]dx=1, \tag{50}$$

which, in view of the uniform convergence of (19) and (20), becomes, on using Eq. (44),

$$\sum_{r=0}^{\infty}\frac{2(-1)^r\Gamma(r+\alpha+1)\Gamma(r+\beta+1)(1+(\alpha+\beta)/2-s)_r}{(2r+\alpha+\beta+1)r!\,\Gamma(r+\alpha+\beta+1)(1+(\alpha+\beta)/2+s)_r}$$

$$\times\left[\frac{{}_sA_{lm}^{(r)}(\gamma)}{{}_sA_{lm}^{(0)}(\gamma)}\right]^2=\frac{1}{{}_sA_{lm}^{(0)}(\gamma)\,{}_sB_{lm}^{(0)}(\gamma)}. \tag{51}$$

Consequently, we have equations for both ${}_sA_{lm}^{(0)}(\gamma)\,{}_sB_{lm}^{(0)}(\gamma)$ and ${}_sA_{lm}^{(0)}(\gamma)/{}_sB_{lm}^{(0)}(\gamma)$. The final determination of ${}_sA_{lm}^{(0)}(\gamma)$ and ${}_sB_{lm}^{(0)}(\gamma)$ is made by the requirement that the real part of ${}_sA_{lm}^{(0)}(\gamma)$ is to be positive.

6. REVERSING THE SIGN OF THE SPIN WEIGHT. RAISING AND LOWERING OPERATORS

Recently Teukolsky and Press[11] have given an explicit local transformation between quantities of opposite spin weight. In this section we rederive their results using our series (3) and (4). We begin by defining operators ${}_nT_m^-$ and ${}_nT_m^+$ by the equations

$${}_nT_m^-Q=-(1-x^2)^{1/2}\left(\frac{d}{dx}-\frac{m}{1-x^2}-\frac{nx}{1-x^2}\right)Q \tag{52}$$

and

$$ {}_nT_m^{+}Q = -(1-x^2)^{1/2}\left(\frac{d}{dx} + \frac{m}{1-x^2} + \frac{nx}{1-x^2}\right)Q. \tag{53} $$

Straightforward though tedious application of the identities (17) and (18) together with

$$ \frac{d}{dx}P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) $$

then gives for $s>0$ the results

$$ {}_{(1-s)}T_m^{-}\,{}_{(2-s)}T_m^{-}\cdots{}_sT_m^{-}\left[\left(\frac{1-x}{2}\right)^{\alpha/2}\left(\frac{1+x}{2}\right)^{\beta/2}P_r^{(\alpha,\beta)}\right] $$
$$ = (-1)^{[s+(\beta-\alpha)/2]}\left(\frac{\alpha+\beta}{2}-s+r+1\right)_{2s} $$
$$ \times\left(\frac{1-x}{2}\right)^{\beta/2}\left(\frac{1+x}{2}\right)^{\alpha/2}P_r^{(\beta,\alpha)} \tag{54} $$

and

$$ {}_{s-1}T_m^{+}\,{}_{s-2}T_m^{+}\cdots{}_{-s}T_m^{+}\left[\left(\frac{1-x}{2}\right)^{\beta/2}\left(\frac{1+x}{2}\right)^{\alpha/2}P_r^{(\beta,\alpha)}\right] $$
$$ = (-1)^{[s+(\alpha-\beta)/2]}\left(\frac{\alpha+\beta}{2}-s+r+1\right)_{2s} $$
$$ \times\left(\frac{1-x}{2}\right)^{\alpha/2}\left(\frac{1+x}{2}\right)^{\beta/2}P_r^{(\alpha,\beta)}. \tag{55} $$

Consequently, if we define the operator L_n by

$$ L_nQ = -(1-x^2)^{1/2}\left(\frac{d}{dx} - \frac{m}{1-x^2} + \gamma - \frac{nx}{1-x^2}\right)Q, \tag{56} $$

note that

$$ L_n(\exp(-\gamma x)Q) = \exp(-\gamma x)\,{}_nT_m^{-}Q \tag{57} $$

and use Eqs. (8), (20), (47a), and (54), we find that

$$ L_{-s+1}L_{-s+2}\cdots L_s\,{}_sS_{lm}(\gamma;x) $$
$$ = (-1)^{[s+(\beta-\alpha)/2]}\frac{{}_sB_{lm}^{(0)}(\gamma)}{{}_{-s}B_{lm}^{(0)}(\gamma)}\left(\frac{\alpha+\beta}{2}-s+1\right)_{2s}\,{}_{-s}S_{lm}(\gamma;x) \tag{58} $$

$$ = C_s\,{}_{-s}S_{lm}(\gamma;x), \tag{59} $$

say. But by a result of Teukolsky and Press[11] the constant C_s can be determined for any positive integral $2s$. The only cases of interest are $s=\frac{1}{2},1,2$ for which we have

$$ C_{1/2} = -(Q+\tfrac{1}{4})^{1/2}, \tag{60a} $$

$$ C_1 = (Q^2+4\gamma m-4\gamma^2)^{1/2}, \tag{60b} $$

and

$$ C_2 = \{(Q^2+4\gamma m-4\gamma^2)[(Q-2)^2+36\gamma m-36\gamma^2] + (2Q-1)(96\gamma^2-48\gamma m)-144\gamma^2\}^{1/2}, \tag{60c} $$

where

$$ Q = {}_sE_l^m(\gamma)+\gamma^2-2\gamma m. \tag{60d} $$

Thus

$$ {}_{-s}B_{lm}^{(0)}(\gamma) = (-1)^{[s+(\beta-\alpha)/2]}\frac{((\alpha+\beta)/2-s+1)_{2s}}{C_s}\,{}_sB_{lm}^{(0)}, \tag{61} $$

which enables us to determine ${}_{-s}B_{lm}^{(0)}(\gamma)$, once ${}_sB_{lm}^{(0)}(\gamma)$ is known. On using (47a) we also have

$$ {}_{-s}B_{lm}^{(r)}(\gamma) = (-1)^{[s+(\beta-\alpha)/2]}\frac{((\alpha+\beta)/2-s+r+1)_{2s}}{C_s}\,{}_sB_{lm}^{(r)}. \tag{62} $$

If we define $L_n^{\dagger}$ by

$$ L_n^{\dagger}Q = -(1-x^2)^{1/2}\left(\frac{d}{dx} + \frac{m}{1-x^2} - \gamma - \frac{nx}{1-x^2}\right)Q $$

and proceed in a similar fashion using Eqs. (7), (19), (47b), and (55), we obtain

$$ L_{-s+1}^{\dagger}L_{-s+2}^{\dagger}\cdots L_s^{\dagger}\,{}_{-s}S_{lm}(\gamma;x) $$
$$ = (-1)^{[s+(\alpha-\beta)/2]}\frac{{}_{-s}A_{lm}^{(0)}(\gamma)}{{}_sA_{lm}^{(0)}(\gamma)}\left(\frac{\alpha+\beta}{2}-s+1\right)_{2s}\,{}_sS_{lm}(\gamma;x). \tag{63} $$

In order to make the normalizations consistent, this latter expression must also be equal to

$$ C_s\,{}_sS_{lm}(\gamma;x). $$

Thus

$$ {}_{-s}A_{lm}^{(0)}(\gamma) = C_s\left(\frac{\alpha+\beta}{2}-s+1\right)_{2s}(-1)^{[s+(\alpha-\beta)/2]}\,{}_sA_{lm}^{(0)}(\gamma) \tag{64} $$

TABLE I. Typical eigenvalues for a range of $a\omega$.

	$s=2$	$l=4$	$m=5$
$a\omega$		Eigenvalue	
Real	Imag.	Real	Imag.
2.50	2.50	17.39773077	−5.66344689
2.50	2.00	17.01558312	−4.56332174
2.50	1.50	16.61410548	−3.45766590
2.50	1.00	16.28167956	−2.32605871
2.50	0.50	16.06600456	−1.16997224
2.50	0	15.99158250	0
2.00	2.50	18.57791007	−5.27323127
2.00	2.00	18.10954737	−4.13986680
2.00	1.50	17.69356737	−3.08480799
2.00	1.00	17.36799810	−2.05365496
2.00	0.50	17.16145520	−1.02705010
2.00	0	17.09077118	0
1.50	2.50	19.74556118	−4.69370520
1.50	2.00	19.15136680	−3.65113239
1.50	1.50	18.68072323	−2.69303775
1.50	1.00	18.33328939	−1.77884449
1.50	0.50	18.11917091	−0.88539869
1.50	0	18.04677552	0
1.00	2.50	20.76262438	−3.97052818
1.00	2.00	20.07601364	−3.08172634
1.00	1.50	19.54581684	−2.26033668
1.00	1.00	19.16479619	−1.48497071
1.00	0.50	18.93421146	−0.73644245
1.00	0	18.85693205	0
0.50	2.50	21.60196115	−3.16334833
0.50	2.00	20.84741563	−2.44354058
0.50	1.50	20.26302111	−1.78229220
0.50	1.00	19.84603024	−1.16507990
0.50	0.50	19.59561798	−0.57586109
0.50	0	19.51206790	0
0	2.50	22.26768762	−2.28909746
0	2.00	21.44873661	−1.74532034
0	1.50	20.81323599	−1.26012241
0	1.00	20.36086577	−0.81750859
0	0.50	20.09013160	−0.40216282

TABLE II. Typical expansion coefficients.

	$s=2$		$l=4$	$m=5$
$a\omega=1.5+i2.0$ Eigenvalue $=18.80496813-i4.41751318$				
	${}_sA_{lm}(r)\,(a\omega)$		${}_sB_{lm}(r)\,(a\omega)$	
r	Real	Imag.	Real	Imag.
0	0.912373602	−0.329432964	0.146796596	−6.114384929
1	−2.239413164	1.766013426	−1.755579943	−5.731174356
2	3.191155441	0.090524274	1.116787865	−2.649810011
3	−1.521163756	−1.668853332	0.948655825	−0.366449175
4	−0.124857053	1.194768320	0.267339123	0.137361234
5	0.426436151	−0.292867252	0.017802430	0.075593104
6	−0.180807050	−0.051354958	−0.010827490	0.014324403
7	0.023383874	0.054379069	−0.003767549	0.000134193
8	0.007732272	−0.014543782	−0.000474301	−0.000549990
9	−0.004040707	0.000744673	0.000024218	−0.000127115
10	0.000697596	0.000613857	0.000019049	−0.000009885
11	0.000011122	−0.000191956	0.000003030	0.000001384
12	−0.000030202	0.000020811	0.000000112	0.000000473
13	0.000006185	0.000001970	−0.000000042	0.000000052
14	−0.000000366	−0.000001007	−0.000000009	−0.000000000
15	−0.000000089	0.000000140	−0.000000001	−0.000000001
16	0.000000024	−0.000000002	0.000000000	−0.000000000
17	−0.000000002	−0.000000002	0.000000000	0.000000000
18	−0.000000000	0.000000000	0.000000000	0.000000000

and so

$$ {}_{-s}A^{(r)}_{lm}(\gamma)=C_s\,\frac{(-1)^{[s+(\alpha-\beta)/2]}}{((\alpha+\beta)/2-s+r+1)_{2s}}\,{}_sA^{(0)}_{lm}(\gamma). \qquad (65)$$

Note that (62) and (64) require that this normalization be equivalent to

$$ {}_sA^{(0)}_{lm}(\gamma)\,{}_sB^{(0)}_{lm}(\gamma)=(-1)^{\alpha-\beta}\,{}_{-s}A^{(0)}_{lm}(\gamma)\,{}_{-s}B^{(0)}_{lm}(\gamma) \qquad (66)$$

and, consequently,

$$ {}_sA^{(r)}_{lm}(\gamma)\,{}_sB^{(r)}_{lm}(\gamma)=(-1)^{\alpha-\beta}\,{}_{-s}A^{(r)}_{lm}\,{}_{-s}B^{(r)}_{lm}(\gamma). \qquad (67)$$

7. NUMERICAL COMPUTATION

The analytic techniques developed in this paper have been used by one of us (R. G. C.) to implement a computer program for the calculation of the eigenvalues and the expansion coefficients for a given frequency and for given values of s, l, and m. A feature of this program is that the eigenvalues are obtained to high accuracy (more than eight significant figures for both real and complex values of γ). Some typical eigenvalues and expansion coefficients are given in Tables I and II. The program has been checked both by comparison with published results for the eigenvalue for real frequencies and also by the use of the raising and lowering identities (66) and (67) in the case of complex frequencies.

8. CONCLUSION

We have shown how the analytic properties of spin-weighted angular spheroidal functions may be investigated by the use of infinite series of Jacobi polynomials.

*Part of the work was carried out whilst the author was at Monash University, Clayton, Victoria.

[1]S.A. Teukolsky, Astrophys J. 185, 635 (1973).

[2]W.H. Press and S.A. Teukolsky, Astrophys. J. 185, 649 (1973).

[3]E. Wasserstrom, J. Comp. Phys. 9, 53 (1972).

[4]Remark (3) following Theorem 9.2.1 in G. Szego, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol. XXIII (American Mathematical Society, Providence, R.I., 1939).

[5]Carson Flammer, *Spheroidal Wave Functions* (Stanford U.P., Stanford, California, 1957).

[6]J. Meixner and F.W. Schäfke, *Mathieusche Funktionen und Spharoidfunktionen* (Springer-Verlag, Berlin, 1954).

[7]O. Perron, *Die Lehre von den Kettenbruchen* (Teubner, Stuttgart, 1957), 3rd. ed., Vol. II.

[8]For $s=0$, Eq. (41d) gives the value $2[m^2-l(l+1)+\frac{1}{2}]/(2l-1)(2l+3)$ which agrees with the corresponding expression for $s=0$ given by Flammer in Ref. 5. The expression given by Press and Teukolsky [Eq. (3)—(10) of Ref. 2], however, differs from this and is wrong.

[9]G. Blanch, J. Math. and Phys. 25, 1 (1946).

[10]C.J. Bouwkamp, J. of Math. and Phys. 26, 79 (1947); also, Philips Res. Rep. 5, 87 (1950).

[11]S.A. Teukolsky and W.H. Press, Astrophys. J. 193, 443 (1974); see also A.A. Starobinskiĭ and S.M. Churilvo, Zh. Eksp. Teor. Fiz. 65, 3 (1973) [Sov. Phys. JETP 38, 1 (1974)].