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An Approach to Gravitational Radiation by a Method of Spin Coefficients*

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A new approach to general relativity by means of a tetrad or spinor formalism is presented. The essential feature of this approach is the consistent use of certain complex linear combinations of Ricci rotation coefficients which give, in effect, the spinor affine connection. It is applied to two problems in radiation theory; a concise proof of a theorem of Goldberg and Sachs and a description of the asymptotic behavior of the Riemann tensor and metric tensor, for outgoing gravitational radiation.

I. INTRODUCTION

In the study of gravitational radiation, two techniques have recently gained prominence; the tetrad calculus[1-6] and the spinor calculus.[7-9] In the present paper (Secs. II, III,[10] and IV) it is shown how the two techniques can be used to derive a very compact and useful set of equations, which are essentially linear combinations of the equations for the Riemann tensor expressed in terms of either Ricci rotation coefficients or the spinor affine connection. In Sec. V, we give a short proof of a theorem of Goldberg and Sachs[11] to the effect that if in empty space there exists a null geodesic congruence with vanishing shear, then the Riemann tensor of the space must be algebraically specilized (i.e., the Riemann tensor is not Petrov type I nondegenerate).

The last application of our formalism is to the asymptotic behavior of the Riemann tensor and metric tensor in empty space. In Sec. VI a coordinate system and tetrad are built around a hypersurface-orthogonal null-vector field. In Sec. VII, this special coordinate system and tetrad are used to prove essentially the following theorem. If a particular complex (tetrad) component of the Riemann tensor (complex because we are using a complex tetrad system) has an asymptotic behavior $O(r^{-5})$, the other four complex components are, respectively, $O(r^{-4})$, $O(r^{-3})$, $O(r^{-2})$, and $O(r^{-1})$. The last component represents the pure radiation field. Special cases of this theorem have been known for some time.[12,13] Our theorem is also a slight generalization of a similar result recently obtained by Bondi and Sachs.[6]

II. TETRAD CALCULUS

We deal with a four-dimensional Riemannian space with a signature -2. Into this space a tetrad system of vectors l_μ, m_μ, $\bar{m}_\mu$, n_μ is introduced, l_μ and n_μ being real null vectors and m_μ with its complex conjugate $\bar{m}_\mu$ being complex null vectors. The vector m_μ can be defined from a pair of real, orthogonal unit space-like vectors a_μ and b_μ by $m_\mu = (1/\sqrt{2})(a_\mu - ib_\mu)$. The orthogonality properties of the vectors are

$$\begin{gathered} l_\mu l^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = n_\mu n^\mu = 0, \\ l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1, \\ l_\mu m^\mu = l_\mu \bar{m}^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0. \end{gathered} \tag{2.1}$$

It is of great convenience to introduce the tetrad notation[14]

$$z_{\mathbf{m}\mu} = (l_\mu, n_\mu, m_\mu, \bar{m}_\mu), \qquad \mathbf{m} = 1, 2, 3, 4.$$

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1 A. Z. Petrov, Sci. Not. Kazan State Univ. **114**, 55 (1954).

2 F. A. E. Pirani, Acta Phys. Polon. **15**, 389 (1956); Phys. Rev. **105**, 1089 (1957); Bull. acad. polon. sci. **5**, 143 (1957a).

3 E. Newman, J. Math. Phys. **2**, 324 (1961).

4 J. Goldberg and R. Kerr, J. Math. Phys. **2**, 327 (1961).

5 R. Sachs, Infeld Volume and preprints.

6 H. Bondi and R. Sachs (private communication).

7 L. Witten, Phys. Rev. **113**, 357 (1959).

8 R. Penrose, Ann. Phys. (New York) **10**, 171 (1960).

9 J. Ehlers, Hamburg Lectures.

10 It is possible, if one has no familiarity with spinors to omit Sec. III, with but a small loss of continuity.

11 J. Goldberg and R. Sachs (to be published).

12 R. Sachs (to be published).

13 I. Robinson and A. Trautman, Phys. Rev. Letters **4**, 431 (1960).

14 Greek indices (values 1, 2, 3, 4) are tensor indices, bold face **a**, **b** ··· (values 1, 2, 3, 4) are tetrad indices, capital latin A, B ··· (values 0, 1) are spinor indices and small lightface latin a, b ··· (values 0, 1) are spinor "dyad" indices.

The tetrad indices can be raised or lowered by the flat-space metric $\eta_{\mathbf{mn}}$

$$\eta_{\mathbf{mn}} = \eta^{\mathbf{mn}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \qquad (2.2)$$

The following two relations are easily seen to be true;

$$g_{\mu\nu} = z_{\mathbf{m}\mu} z_{\mathbf{n}\nu} \eta^{\mathbf{mn}}$$
$$= l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu \qquad (2.3a)$$
$$\eta_{\mathbf{mn}} = z_{\mathbf{m}\mu} z_{\mathbf{n}\nu} g^{\mu\nu}. \qquad (2.3b)$$

Complex Ricci rotation coefficients are defined by

$$\gamma_{\mathbf{m}}{}^{\mathbf{np}} = z_{\mathbf{m}\mu;\nu} z^{\mathbf{n}\mu} z^{\mathbf{p}\nu}, \qquad (2.4)$$

with the symmetry

$$\gamma^{\mathbf{mnp}} = -\gamma^{\mathbf{nmp}}.$$

The tetrad components of the Riemann tensor defined by

$$R_{\mathbf{mnpq}} = R_{\alpha\beta\gamma\delta} z_{\mathbf{m}}^{\alpha} z_{\mathbf{n}}^{\beta} z_{\mathbf{p}}^{\gamma} z_{\mathbf{q}}^{\delta} \qquad (2.5)$$

can be expressed in terms of the rotation coefficients by[15]

$$R^{\mathbf{mnpq}} = \gamma^{\mathbf{mnp};\mathbf{q}} - \gamma^{\mathbf{mnq};\mathbf{p}} + \gamma_{\mathbf{l}}{}^{\mathbf{mq}} \gamma^{\mathbf{lnp}}$$
$$- \gamma_{\mathbf{l}}{}^{\mathbf{mp}} \gamma^{\mathbf{lnq}} + \gamma^{\mathbf{mnl}} (\gamma_{\mathbf{l}}{}^{\mathbf{pq}} - \gamma_{\mathbf{l}}{}^{\mathbf{qp}}), \qquad (2.6)$$

with

$$\gamma^{\mathbf{mnp};\mathbf{q}} \equiv \gamma^{\mathbf{mnp}}{}_{;\mu} z^{\mathbf{q}\mu}.$$

This can easily be derived from the Ricci identity

$$z_{\mathbf{m}\mu;[\alpha\beta]} = \tfrac{1}{2} z_{\mathbf{m}\nu} R^{\nu}{}_{\mu\alpha\beta} \qquad (2.7)$$

by repeated application of (2.4).

The relationship between the Riemann tensor, Weyl tensor, and Ricci tensor goes over in tetrad form unchanged[15]

$$R_{\mathbf{mnpq}} = C_{\mathbf{mnpq}} - \tfrac{1}{2}(\eta_{\mathbf{mp}} R_{\mathbf{nq}} - \eta_{\mathbf{mq}} R_{\mathbf{np}} + \eta_{\mathbf{nq}} R_{\mathbf{mp}}$$
$$- \eta_{\mathbf{np}} R_{\mathbf{mq}}) - (R/6)(\eta_{\mathbf{mq}} \eta_{\mathbf{np}} - \eta_{\mathbf{mp}} \eta_{\mathbf{nq}}). \qquad (2.8)$$

In the tetrad notation the Bianchi identities, $R_{\alpha\beta[\gamma\delta;\mu]} = 0$, take the form

$$R_{\mathbf{mn}[\mathbf{pq};\mathbf{r}]} = \gamma_{\mathbf{m}}{}^{\mathbf{l}}{}_{[\mathbf{r}} R_{\mathbf{pq}]\mathbf{ln}}$$
$$- \gamma_{\mathbf{n}}{}^{\mathbf{l}}{}_{[\mathbf{r}} R_{\mathbf{pq}]\mathbf{lm}} + 2R_{\mathbf{mnl}[\mathbf{p}} \gamma_{\mathbf{r}}{}^{\mathbf{l}}{}_{\mathbf{q}]}. \qquad (2.9)$$

Though they appear to be considerably more complicated, it will be seen in Sec. IV that with a new notation they take a simple and useful form in empty space.

In Eq. (2.6) we used the intrinsic derivative defined by $\varphi^{;\mathbf{m}} = \varphi_{;\mu} z^{\mathbf{m}\mu}$. It will be of great value to obtain the commutator of two intrinsic derivatives, $\varphi^{;\mathbf{m};\mathbf{n}} - \varphi^{;\mathbf{n};\mathbf{m}}$. We have

$$\varphi^{;\mathbf{m};\mathbf{n}} = (\varphi_{;\mu} z^{\mathbf{m}\mu})_{;\nu} z^{\mathbf{n}\nu} = \varphi_{,\mu\nu} z^{\mathbf{m}\mu} z^{\mathbf{n}\nu} + \varphi_{,\mu} z^{\mathbf{m}\mu}{}_{;\nu} z^{\mathbf{n}\nu}. \qquad (2.10)$$

By interchanging $\mathbf{m}$ and $\mathbf{n}$ in (2.10) and using $z^{\mathbf{m}\mu}{}_{;\nu} z^{\mathbf{n}\nu} = \gamma^{\mathbf{mpn}} z_{\mathbf{p}}^{\mu}$ obtained from (2.4), we see

$$\varphi^{;\mathbf{m};\mathbf{n}} - \varphi^{;\mathbf{n};\mathbf{m}} = \varphi^{;\mathbf{l}} [\gamma^{\mathbf{m}}{}_{\mathbf{l}}{}^{\mathbf{n}} - \gamma^{\mathbf{n}}{}_{\mathbf{l}}{}^{\mathbf{m}}]. \qquad (2.11)$$

In Sec. IV it will be advantageous to dispense with the semicolon notation for intrinsic derivatives and use the following;

$$D\varphi \equiv \varphi_{;\mu} l^\mu, \qquad \Delta\varphi \equiv \varphi_{;\mu} n^\mu$$
$$\delta\varphi \equiv \varphi_{;\mu} m^\mu, \qquad \bar{\delta}\varphi \equiv \varphi_{;\mu} \bar{m}^\mu. \qquad (2.12)$$

III. TWO-COMPONENT SPINOR CALCULUS

The connection between tensors and spinors[16] is achieved by means of a quantity $\sigma^\mu{}_{AB'}$, satisfying

$$g_{\mu\nu} \sigma^\mu{}_{AB'} \sigma^\nu{}_{CD'} = \epsilon_{AC} \epsilon_{B'D'}. \qquad (3.1)$$

For each value of μ, $\sigma_\mu{}^{AB'}$ is a (2×2) Hermitian matrix. The ϵ's are Levi-Civita symbols, that is, skew-symmetric expressions with $\epsilon_{01} = \epsilon_{0'1'} = \epsilon^{01} = \epsilon^{0'1'} = 1$, and they are used for lowering or raising spinor indices:

$$\xi^A = \epsilon^{AB} \xi_B, \qquad \xi_B = \xi^A \epsilon_{AB},$$
$$\eta^{A'} = \epsilon^{A'B'} \eta_{B'}, \qquad \eta_{B'} = \eta^{A'} \epsilon_{A'B'}. \qquad (3.2)$$

(Note the ordering of the indices.) The spinor equivalent of any tensor is a quantity having each tensor index replaced by a pair of spinor indices, one unprimed and one primed[17];

$$X^{\lambda\mu}{}_\nu \leftrightarrow X^{AB'CD'}{}_{EF'} = \sigma_\lambda{}^{AB'} \sigma_\mu{}^{CD'} X^{\lambda\mu}{}_\nu \sigma^\nu{}_{EF'}.$$

Inversely:

$$X^{\lambda\mu}{}_\nu = \sigma^\lambda{}_{AB'} \sigma^\mu{}_{CD'} X^{AB'CD'}{}_{EF'} \sigma_\nu{}^{EF'}.$$

Equation (3.1) tells us that $\epsilon_{AC}\epsilon_{B'D'}$ is the spinor equivalent of $g_{\mu\nu}$.

When taking the complex conjugate of a spinor, unprimed indices become primed, and primed indices become unprimed. For example, the complex conjugate[18] of $X^{AB'CD'}{}_{EF'}$ is $\bar{X}^{A'BC'D}{}_{E'F}$, whence

[15] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1960).

[16] See, for example, W. L. Bade and H. Jehle, Revs. Modern Phys. **25**, 714 (1953).

[17] We use primed rather than dotted indices for typographical reasons.

[18] Many authors omit the bar over the complex conjugate.

the condition for $X^{\lambda\mu}{}_{\nu}$ to be real is the Hermitian property

$$X^{AB'CD'}{}_{EF'} = \bar{X}^{B'AD'C}{}_{F'E}.$$

The covariant derivative $\xi_{A;\mu}$ of the spinor ξ_A is

$$\xi_{A;\mu} = \xi_{A,\mu} - \xi_B \Gamma^B{}_{A\mu}, \tag{3.3}$$

where $\Gamma^B{}_{A\mu}$ is the spinor affine connection. The corresponding quantity $\bar{\Gamma}^{B'}{}_{A'\mu}$ deals with the primed indices. The rules of covariant differentiation for spinor indices are exactly analogous to those for tensor indices. The choice of $\Gamma^B{}_{A\mu}$ is fixed by the requirement that the covariant derivatives of $\sigma^\mu{}_{AB'}, \epsilon_{AB}, \epsilon_{A'B'}$ shall all vanish.[16]

Observe that, by (3.1), the four expressions

$$\sigma^\mu{}_{00'}, \qquad \sigma^\mu{}_{11'}, \qquad \sigma^\mu{}_{01'}, \qquad \sigma^\mu{}_{10'} \tag{3.4}$$

satisfy the same orthogonality relations (2.1) as the four vectors l^μ, n^μ, m^μ, $\bar{m}^\mu$. We would like, therefore, to identify the expressions (3.4) as a convenient tetrad. However, this would not be strictly accurate and is a little misleading. The expressions (3.4) do not really denote vectors as they stand, as is exemplified by the fact that while covariant derivative of $\sigma^\mu{}_{AB'}$ is zero, this is not so for l^μ, m^μ, $\bar{m}^\mu$, n^μ [see Eq. (2.4)].

To get around this difficulty we introduce two basis spinors o^A, ι^A (a "dyad") normalized thus:

$$o_A \iota^A = \epsilon_{AB} o^A \iota^B = -\iota_A o^A = 1. \tag{3.5}$$

A dyad (in spin space) is analogous to a tetrad in vector space. We may put

$$l^\mu = \sigma^\mu{}_{AB'} o^A \bar{o}^{B'}, \qquad n^\mu = \sigma^\mu{}_{AB'} \iota^A \bar{\iota}^{B'},$$
$$m^\mu = \sigma^\mu{}_{AB'} o^A \bar{\iota}^{B'} \tag{3.6}$$

The covariant derivatives of these expressions will now involve the covariant derivatives of o^A and ι^B.

As with the tetrads, it is convenient to have a generic symbol for both o^A and ι^A. Define $\zeta_a{}^A$, $\bar{\zeta}_{a'}{}^{A'}$ by

$$\zeta_0{}^A = o^A, \qquad \zeta_1{}^A = \iota^A, \qquad \bar{\zeta}_{0'}{}^{A'} = \bar{o}^{A'},$$
$$\bar{\zeta}_{1'}{}^{A'} = \bar{\iota}^{A'}. \tag{3.7}$$

Then, for example, given a spinor $Y_{AB'C}$, we can define its *dyad components*

$$Y_{ab'c} = Y_{AB'C} \zeta_a{}^A \bar{\zeta}_{b'}{}^{B'} \zeta_c{}^C.$$

The lower-case indices behave the same way algebraically as ordinary (capital) spinor indices, but when covariant differentiation is applied, no term involving an affine connection appears for the lower case indices. Thus, the important formal difference between the lower-case and capital indices is simply the difference with respect to covariant differentiation.

Bearing this in mind, it is permissible to choose the components of $\zeta_a{}^A$ to be the Kronecker delta. The dyad components of any spinor will then, in fact, be identical with its spinor components. It is by no means essential to make this specialization but it will be convenient to do so here. The expression

$$\sigma^\mu{}_{ab'} = \sigma^\mu{}_{AB'} \zeta_a{}^A \bar{\zeta}_{b'}{}^{B'} \tag{3.8}$$

now gives us l^μ, n^μ, m^μ, $\bar{m}^\mu$ as ab' take, respectively, the values 00′, 11′, 01′, 10′, by (3.6). With this interpretation, the expressions (3.4) may indeed be though of as giving the required tetrad. However, it is essential to maintain the distinction between the dyad and spinor indices when covariant differentiation is involved.

The components of ζ_{aB} are now the same as those of ϵ_{AB}. Hence, $\zeta_{aA;\mu} = -\zeta_{aC}\Gamma^C{}_{A\mu} = \zeta_a^C \Gamma_{CA\mu} = \Gamma_{aA\mu}$ by (3.3). For the analog of the rotation coefficients[19] (2.4), we therefore have

$$\Gamma_{abcd'} = \zeta_{aA;\mu} \zeta_b{}^A \sigma^\mu{}_{cd'} \tag{3.9}$$

with the symmetry

$$\Gamma_{abcd'} = \Gamma_{bacd'}. \tag{3.10}$$

Writing

$$\varphi_{\cdot\cdot\cdot\cdot;\mu} \sigma^\mu{}_{ab'} \equiv \partial_{ab'} \varphi_{\cdot\cdot\cdot\cdot}, \tag{3.11}$$

the intrinsic derivatives (2.12) become

$$D \equiv \partial_{00'}, \quad \Delta \equiv \partial_{11'}, \quad \delta \equiv \partial_{01'}, \quad \bar{\delta} \equiv \partial_{10'}. \tag{3.12}$$

The commutation relations for these derivatives acting on scalars

$$\{\partial_{ab'}\, \partial_{cd'} - \partial_{cd'}\, \partial_{ab'}\}\varphi$$
$$\equiv \{\epsilon^{pq}(\Gamma_{pacd'}\, \partial_{qb'} - \Gamma_{pcab'}\, \partial_{qd'})$$
$$+ \epsilon^{r's'}(\bar{\Gamma}_{r'b'd'c}\, \partial_{as'} - \bar{\Gamma}_{r'd'b'a}\, \partial_{cs'})\}\varphi \tag{3.13}$$

can be obtained by direct calculation from (3.13) using (3.8) and (3.9). By a slight extension of this calculation, when the derivatives act on ζ_{aA} we obtain

$$\partial_{fe'}\Gamma_{acdb'} - \partial_{db'}\Gamma_{acfe'} = \epsilon^{pq}\{\Gamma_{apdb'}\Gamma_{qcfe'}$$
$$+ \Gamma_{acpb'}\Gamma_{qdfe'} - \Gamma_{apfe'}\Gamma_{qcdb'} - \Gamma_{acpe'}\Gamma_{qfdb'}\}$$
$$+ \epsilon^{r's'}\{\Gamma_{acdr'}\bar{\Gamma}_{s'b'e'f} - \Gamma_{acfr'}\Gamma_{s'e'b'd}\}$$
$$+ \Psi_{acdf}\epsilon_{e'b'} + \Lambda\epsilon_{e'b'}(\epsilon_{cd}\epsilon_{af} + \epsilon_{ab}\epsilon_{cf})$$
$$+ \Phi_{acb'e'}\epsilon_{fd}, \tag{3.14}$$

[19] The quantities (3.9) can be defined directly in terms of derivatives of the $\sigma_{\mu ab'}$, as follows:

$$\Gamma_{abcd'} = \tfrac{1}{2}\epsilon^{p'q'}\{\sigma_{cd'ap'bq'} - \sigma_{cd'bq'ap'} - \sigma_{ap'bq'cd'}\}$$

where

$$\sigma_{ab'cd'ef'} = \sigma^{[\mu}{}_{ab'}\sigma^{\nu]}{}_{cd'}\, \sigma_{\mu ef',\nu}$$

or

$$\Gamma_{abcd'} = \tfrac{1}{2}\epsilon^{p'q'}\sigma^\mu{}_{aq'}\sigma^\nu{}_{cd'}\sigma_{\mu bp';\nu}.$$

where the spinors Ψ_{ABCD}, $\Phi_{ABC'D'}$, and Λ correspond, respectively, to the Weyl tensor, trace-free part of the Ricci tensor, and scalar curvature. They have the symmetries:

$$\Psi_{ABCD} = \Psi_{(ABCD)},$$
$$\Phi_{ABC'D'} = \Phi_{(AB)(C'D')} = \bar{\Phi}_{C'D'AB} \qquad (3.15)$$

with

$$\Lambda = (1/24)R.$$

The spinor equivalent of the Riemann tensor $R_{\alpha\beta\gamma\delta}$ decomposes as follows[20]:

$$\begin{aligned} -R_{AE'BF'CG'DH'} &= \Psi_{ABCD}\epsilon_{E'F'}\epsilon_{G'H'} \\ &+ \epsilon_{AB}\epsilon_{CD}\bar{\Psi}_{E'F'G'H'} \\ &+ 2\Lambda(\epsilon_{AC}\epsilon_{BD}\epsilon_{E'F'}\epsilon_{G'H'} + \epsilon_{AB}\epsilon_{CD}\epsilon_{E'H'}\epsilon_{F'G'}) \\ &+ \epsilon_{AB}\Phi_{CDE'F'}\epsilon_{G'H'} + \epsilon_{CD}\Phi_{ABG'H'}\epsilon_{E'F'}. \end{aligned} \qquad (3.16)$$

The relations[8]

$$\partial_{(A}{}^{P'}\,\partial_{B)P'}\xi_C = -\Psi_{ABCD}\xi^D + \Lambda\xi_{(A}\epsilon_{B)C}, \qquad (3.17)$$
$$\partial_{C(P'}\,\partial^C{}_{Q')}\xi_A = \Phi_{ABP'Q'}\xi^B$$

have been used to obtain (3.14).

The Bianchi identities in spinor form are

$$\begin{aligned} \partial^D{}_{G'}\Psi_{ABCD} &= \partial_{(C}{}^{H'}\Phi_{AB)G'H'} \\ \partial^{AG'}\Phi_{ABG'H'} &= -3\,\partial_{BH'}\Lambda \end{aligned} \qquad (3.18)$$

from which we obtain

$$\begin{aligned} \partial^p{}_{d'}\Psi_{abcp} - \partial_{(c}{}^{t'}\Phi_{ab)d't'} &= \{3\Psi_{pr(ab}\Gamma_{c)}{}^{pr}{}_{d'} \\ &+ \Psi_{abcp}\Gamma^p{}_r{}^r{}_{d'}\} - 2\Gamma^p{}_{(ab}{}^{t'}\Phi_{c)pt'd'} \\ &- \{\bar{\Gamma}_{t'd'v'(a}\Phi_{bc)}{}^{t'v'} + \bar{\Gamma}_{t'}{}^{v'}{}_{v'(a}\Phi_{bc)}{}^{t'}{}_{d'}\} \end{aligned} \qquad (3.19)$$

and

$$\begin{aligned} 3\,\partial_{ab'}\Lambda &+ \partial^{pt'}\Phi_{apb't'} \\ &= \epsilon^{v'w'}\{\Phi_{ap}{}^{t'}{}_{w'}\bar{\Gamma}_{b't'v'}{}^p + \Phi_{apb'}{}^{t'}\bar{\Gamma}_{t'w'v'}{}^p\} \\ &- \{\Phi_{prb'}{}^{t'}\Gamma_a{}^{pr}{}_{t'} + \Phi_{apb'}{}^{t'}\Gamma^p{}_r{}^r{}_{t'}\}. \end{aligned} \qquad (3.20)$$

IV. THE SPIN COEFFICIENTS

In the present section we will show how the formalisms developed in Secs. II and III can be put into a relatively concise form, despite the fact that all summations will be written out explicitly.

Twelve complex functions (called spin coefficients) are defined in terms of either the rotation coefficients (2.4) or spinor affine connection (3.9).

$$\begin{aligned}
&\kappa = \gamma_{131} = l_{\mu;\nu}m^\mu l^\nu, \quad \pi = -\gamma_{241} = -n_{\mu;\nu}\bar{m}^\mu l^\nu, \\
&\epsilon = \tfrac{1}{2}(\gamma_{121} - \gamma_{341}) = \tfrac{1}{2}(l_{\mu;\nu}n^\mu l^\nu - m_{\mu;\nu}\bar{m}^\mu l^\nu), \\
&\rho = \gamma_{134} = l_{\mu;\nu}m^\mu\bar{m}^\nu, \quad \lambda = -\gamma_{244} = -n_{\mu;\nu}\bar{m}^\mu\bar{m}^\nu, \\
&\alpha = \tfrac{1}{2}(\gamma_{124} - \gamma_{344}) = \tfrac{1}{2}(l_{\mu;\nu}n^\mu\bar{m}^\nu - m_{\mu;\nu}\bar{m}^\mu\bar{m}^\nu), \\
&\sigma = \gamma_{133} = l_{\mu;\nu}m^\mu m^\nu, \quad \mu = -\gamma_{243} = -n_{\mu;\nu}\bar{m}^\mu m^\nu, \\
&\beta = \tfrac{1}{2}(\gamma_{123} - \gamma_{343}) = \tfrac{1}{2}(l_{\mu;\nu}n^\mu m^\nu - m_{\mu;\nu}\bar{m}^\mu m^\nu), \\
&\qquad\qquad \nu = -\gamma_{242} = -n_{\mu;\nu}\bar{m}^\mu n^\nu, \\
&\gamma = \tfrac{1}{2}(\gamma_{122} - \gamma_{342}) = \tfrac{1}{2}(l_{\mu;\nu}n^\mu n^\nu - m_{\mu;\nu}\bar{m}^\mu n^\nu), \\
&\qquad\qquad \tau = \gamma_{132} = l_{\mu;\nu}m^\mu n^\nu,
\end{aligned} \qquad (4.1a)$$

or

$$\Gamma_{abcd'} = \begin{array}{|c|c|c|c|} \hline cd' \backslash ab & 00 & 01 \text{ or } 10 & 11 \\ \hline 00' & \kappa & \epsilon & \pi \\ \hline 10' & \rho & \alpha & \lambda \\ \hline 01' & \sigma & \beta & \mu \\ \hline 11' & \tau & \gamma & \nu \\ \hline \end{array}\,. \qquad (4.1b)$$

It is seen that the spin coefficients appear more naturally when dealing with spinors than with tetrad vectors. This fact reappears when Eq. (2.6) is rewritten in terms of these new functions. The equations are rather unattractive until certain linear combinations are taken. These simpler equations are just the ones, (3.14), that arise naturally in the spinor calculus. Equation (3.14) or the appropriate linear combinations of Eq. (2.6) using (2.8), with the notation of (4.1), is

$$\begin{aligned} D\rho - \bar{\delta}\kappa &= (\rho^2 + \sigma\bar{\sigma}) + (\epsilon + \bar{\epsilon})\rho - \bar{\kappa}\tau \\ &\quad - \kappa(3\alpha + \bar{\beta} - \pi) + \Phi_{00} \end{aligned} \qquad (4.2a)$$

$$\begin{aligned} D\sigma - \delta\kappa &= (\rho + \bar{\rho})\sigma + (3\epsilon - \bar{\epsilon})\sigma \\ &\quad - (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa + \Psi_0 \end{aligned} \qquad (4.2b)$$

$$\begin{aligned} D\tau - \Delta\kappa &= (\tau + \bar{\pi})\rho + (\bar{\tau} + \pi)\sigma \\ &\quad + (\epsilon - \bar{\epsilon})\tau - (3\gamma + \bar{\gamma})\kappa + \Psi_1 + \Phi_{01} \end{aligned} \qquad (4.2c)$$

$$\begin{aligned} D\alpha - \bar{\delta}\epsilon &= (\rho + \bar{\epsilon} - 2\epsilon)\alpha + \beta\bar{\sigma} - \bar{\beta}\epsilon \\ &\quad - \kappa\lambda - \bar{\kappa}\gamma + (\epsilon + \rho)\pi + \Phi_{10} \end{aligned} \qquad (4.2d)$$

$$\begin{aligned} D\beta - \delta\epsilon &= (\alpha + \pi)\sigma + (\bar{\rho} - \bar{\epsilon})\beta \\ &\quad - (\mu + \gamma)\kappa - (\bar{\alpha} - \bar{\pi})\epsilon + \Psi_1 \end{aligned} \qquad (4.2e)$$

[20] These definitions of Ψ_{ABCD}, $\Phi_{ABC'D'}$ differ by a factor 2 from those given in reference 8. Also, the Riemann tensor used here is the negative of that used in reference 8.

$$\begin{aligned}D\gamma - \Delta\epsilon &= (\tau + \bar\pi)\alpha + (\bar\tau + \pi)\beta - (\epsilon + \bar\epsilon)\gamma \\ &\quad - (\gamma + \bar\gamma)\epsilon + \tau\pi - \nu\kappa + \Psi_2 - \Lambda + \Phi_{11}\end{aligned} \tag{4.2f}$$

$$\begin{aligned}D\lambda - \bar\delta\pi &= (\rho\lambda + \bar\sigma\mu) + \pi^2 + (\alpha - \bar\beta)\pi \\ &\quad - \nu\bar\kappa - (3\epsilon - \bar\epsilon)\lambda + \Phi_{20}\end{aligned} \tag{4.2g}$$

$$\begin{aligned}D\mu - \delta\pi &= (\bar\rho\mu + \sigma\lambda) + \pi\bar\pi - (\epsilon + \bar\epsilon)\mu \\ &\quad - \pi(\bar\alpha - \beta) - \nu\kappa + \Psi_2 + 2\Lambda\end{aligned} \tag{4.2h}$$

$$\begin{aligned}D\nu - \Delta\pi &= (\pi + \bar\tau)\mu + (\bar\pi + \tau)\lambda \\ &\quad + (\gamma - \bar\gamma)\pi - (3\epsilon + \bar\epsilon)\nu + \Psi_3 + \Phi_{21}\end{aligned} \tag{4.2i}$$

$$\begin{aligned}\Delta\lambda - \bar\delta\nu &= -(\mu + \bar\mu)\lambda - (3\gamma - \bar\gamma)\lambda \\ &\quad + (3\alpha + \bar\beta + \pi - \bar\tau)\nu - \Psi_4\end{aligned} \tag{4.2j}$$

$$\begin{aligned}\delta\rho - \bar\delta\sigma &= \rho(\bar\alpha + \beta) - \sigma(3\alpha - \bar\beta) \\ &\quad + (\rho - \bar\rho)\tau + (\mu - \bar\mu)\kappa - \Psi_1 + \Phi_{01}\end{aligned} \tag{4.2k}$$

$$\begin{aligned}\delta\alpha - \bar\delta\beta &= (\mu\rho - \lambda\sigma) + \alpha\bar\alpha + \beta\bar\beta - 2\alpha\beta \\ &\quad + \gamma(\rho - \bar\rho) + \epsilon(\mu - \bar\mu) - \Psi_2 + \Lambda + \Phi_{11}\end{aligned} \tag{4.2l}$$

$$\begin{aligned}\delta\lambda - \bar\delta\mu &= +(\rho - \bar\rho)\nu + (\mu - \bar\mu)\pi \\ &\quad + \mu(\alpha + \bar\beta) + \lambda(\bar\alpha - 3\beta) - \Psi_3 + \Phi_{21}\end{aligned} \tag{4.2m}$$

$$\begin{aligned}\delta\nu - \Delta\mu &= (\mu^2 + \lambda\bar\lambda) + (\gamma + \bar\gamma)\mu \\ &\quad - \bar\nu\pi + [\tau - 3\beta - \bar\alpha]\nu + \Phi_{22}\end{aligned} \tag{4.2n}$$

$$\begin{aligned}\delta\gamma - \Delta\beta &= (\tau - \bar\alpha - \beta)\gamma + \mu\tau - \sigma\nu \\ &\quad - \epsilon\bar\nu - \beta(\gamma - \bar\gamma - \mu) + \alpha\bar\lambda + \Phi_{12}\end{aligned} \tag{4.2o}$$

$$\begin{aligned}\delta\tau - \Delta\sigma &= (\mu\sigma + \bar\lambda\rho) + (\tau + \beta - \bar\alpha)\tau \\ &\quad - (3\gamma - \bar\gamma)\sigma - \kappa\bar\nu + \Phi_{02}\end{aligned} \tag{4.2p}$$

$$\begin{aligned}\Delta\rho - \bar\delta\tau &= -(\rho\bar\mu + \sigma\lambda) + (\bar\beta - \alpha - \bar\tau)\tau \\ &\quad + (\gamma + \bar\gamma)\rho + \nu\kappa - \Psi_2 - 2\Lambda\end{aligned} \tag{4.2q}$$

$$\begin{aligned}\Delta\alpha - \bar\delta\gamma &= (\rho + \epsilon)\nu - (\tau + \beta)\lambda \\ &\quad + (\bar\gamma - \bar\mu)\alpha + (\bar\beta - \bar\tau)\gamma - \Psi_3.\end{aligned} \tag{4.2r}$$

The notation for intrinsic derivatives (2.12) has been used. The quantities Ψ_0, Ψ_1, etc., Φ_{00}, etc., and Λ are, respectively, related to components of the Weyl tensor, Ricci tensor, and scalar curvature by the following;

$$\begin{aligned}\Psi_0 &= -C_{1313} = -C_{\alpha\beta\gamma\delta}l^\alpha m^\beta l^\gamma m^\delta = \Psi_{0000} \\ \Psi_1 &= -C_{1213} = -C_{\alpha\beta\gamma\delta}l^\alpha n^\beta l^\gamma m^\delta = \Psi_{0001} \\ \Psi_2 &= -\tfrac{1}{2}(C_{1212} + C_{1234}) = -\tfrac{1}{2}C_{\alpha\beta\gamma\delta} \\ &\quad \times (l^\alpha n^\beta l^\gamma n^\delta \mp l^\alpha n^\beta m^\gamma \bar m^\delta) = \Psi_{0011} \\ \Psi_3 &= C_{1224} = C_{\alpha\beta\gamma\delta}l^\alpha n^\beta n^\gamma \bar m^\delta = \Psi_{0111} \\ \Psi_4 &= -C_{2424} = -C_{\alpha\beta\gamma\delta}n^\alpha \bar m^\beta n^\gamma \bar m^\delta = \Psi_{1111}\end{aligned} \tag{4.3a}$$

$$\begin{aligned}\Phi_{00} &= -\tfrac{1}{2}R_{11} = \Phi_{000'0'} = \bar\Phi_{00}, \\ \Phi_{11} &= -\tfrac{1}{4}(R_{12} + R_{34}) = \Phi_{010'1'} \\ \Phi_{01} &= -\tfrac{1}{2}R_{13} = \Phi_{000'1'} = \bar\Phi_{10}, \\ \Phi_{12} &= -\tfrac{1}{2}R_{23} = \Phi_{011'1'} \\ \Phi_{10} &= -\tfrac{1}{2}R_{14} = \Phi_{010'0'} = \bar\Phi_{01}, \\ \Phi_{21} &= -\tfrac{1}{2}R_{24} = \Phi_{110'1'} \\ \Phi_{02} &= -\tfrac{1}{2}R_{33} = \Phi_{001'1'} = \bar\Phi_{20}, \\ \Phi_{22} &= -\tfrac{1}{2}R_{22} = \Phi_{111'1'} \\ \Phi_{20} &= -\tfrac{1}{2}R_{44} = \Phi_{110'0'}, \\ \Lambda &= R/24.\end{aligned} \tag{4.3b}$$

With the present notation the commutators (2.11) or (3.13) are

$$\begin{aligned}(\Delta D - D\Delta)\varphi &= [(\gamma + \bar\gamma)D + (\epsilon + \bar\epsilon)\Delta \\ &\quad - (\tau + \bar\pi)\bar\delta - (\bar\tau + \pi)\delta]\varphi \\ (\delta D - D\delta)\varphi &= [(\bar\alpha + \beta - \bar\pi)D + \kappa\Delta \\ &\quad - \sigma\bar\delta - (\bar\rho + \epsilon - \bar\epsilon)\,\delta]\varphi \\ (\delta\Delta - \Delta\delta)\varphi &= [-\bar\nu D + (\tau - \bar\alpha - \beta)\Delta \\ &\quad + \bar\lambda\bar\delta + (\mu - \gamma + \bar\gamma)\delta]\varphi \\ (\bar\delta\delta - \delta\bar\delta)\varphi &= [(\bar\mu - \mu)D + (\bar\rho - \rho)\Delta \\ &\quad - (\bar\alpha - \beta)\bar\delta - (\bar\beta - \alpha)\delta]\varphi.\end{aligned} \tag{4.4}$$

The Bianchi identities (2.9) or (3.18) when written out in general, are very long and unwieldy. However in empty-space, $R_{\alpha\beta} = \Phi_{ABC'D'} = 0$, they do have the simple form[21]

$$\begin{aligned}D\Psi_1 - \bar\delta\Psi_0 &= -3\kappa\Psi_2 + [2\epsilon + 4\rho]\Psi_1 \\ &\quad - [-\pi + 4\alpha]\Psi_0 \\ D\Psi_2 - \bar\delta\Psi_1 &= -2\kappa\Psi_3 + 3\rho\Psi_2 \\ &\quad - [-2\pi + 2\alpha]\Psi_1 - \lambda\Psi_0 \\ D\Psi_3 - \bar\delta\Psi_2 &= -\kappa\Psi_4 - [2\epsilon - 2\rho]\Psi_3 \\ &\quad + 3\pi\Psi_2 - 2\lambda\Psi_1 \\ D\Psi_4 - \bar\delta\Psi_3 &= -[4\epsilon - \rho]\Psi_4 \\ &\quad + [4\pi + 2\alpha]\Psi_3 - 3\lambda\Psi_2 \\ \Delta\Psi_0 - \delta\Psi_1 &= [4\gamma - \mu]\Psi_0 \\ &\quad - [4\tau + 2\beta]\Psi_1 + 3\sigma\Psi_2\end{aligned} \tag{4.5}$$

[21] For completeness, though it is never used in this paper, we give in the Appendix the formulas for the Bianchi identities in the presence of a Maxwell field as well as the Maxwell equations using the notation of this section.

$$\Delta\Psi_1 - \delta\Psi_2 = \nu\Psi_0 + [2\gamma - 2\mu]\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3$$

$$\Delta\Psi_2 - \delta\Psi_3 = 2\nu\Psi_1 - 3\mu\Psi_2 + [-2\tau + 2\beta]\Psi_3 + \sigma\Psi_4$$

$$\Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2 - [2\gamma + 4\mu]\Psi_3 + [-\tau + 4\beta]\Psi_4.$$

Before proceeding to applications of the formalism developed here, it is useful to examine briefly the geometric meaning of some of quantities (4.1) and (4.3) which will be used frequently in the remainder of the paper.

The spin coefficient κ is related to the first curvature of the congruence of which l_μ is the tangent vector by the equation

$$l_{\mu;\nu}l^\nu = -\kappa\bar{m}_\mu - \bar{\kappa}m_\mu + (\epsilon + \bar{\epsilon})l_\mu. \tag{4.6}$$

It is easily seen that if $\kappa = 0$, l_μ is tangent to a geodesic. By a change in scale $l_\mu \to \varphi l_\mu$, $\epsilon + \bar{\epsilon}$ can be made zero. In the case of a geodesic with the above choice of scaling for l_μ we have

$$\rho = \tfrac{1}{2}[-l^\mu{}_{;\mu} + i \operatorname{curl} l_\mu],$$

where

$$\operatorname{curl} l_\mu = (l_{[\mu;\nu]}l^{\mu;\nu})^{\frac{1}{2}} \tag{4.7}$$

and σ is the complex shear of l_μ satisfying[12,22]

$$\sigma\bar{\sigma} = \tfrac{1}{2}[l_{(\mu;\nu)}l^{\mu;\nu} - \tfrac{1}{2}(l^\mu{}_{;\mu})^2].$$

The quantity τ describes how the direction of l_μ changes as we move in the direction n_μ as follows from the equation

$$l_{\mu;\nu}n^\nu = -\tau\bar{m}_\mu - \bar{\tau}m_\mu + (\gamma + \bar{\gamma})l_\mu. \tag{4.8}$$

Again we can make $\gamma + \bar{\gamma}$ zero by the change $l_\mu \to \varphi l_\mu$.

The spin coefficients ν, μ, λ, π are analogous, respectively, to κ, $-\rho$, $-\sigma$, τ, the difference being that the congruence used is given by n_μ instead of l_μ.

If l_μ is taken tangent to a geodesic congruence and we wish to propogate the remainder of the tetrad system parallelly along this congruence, then

$$\kappa = \epsilon = \pi = 0. \tag{4.9}$$

If in addition to being tangent to geodesics, the l_μ are hypersurface orthogonal, that is, proportional to a gradiant field, we have

$$\rho = \bar{\rho}, \tag{4.10}$$

[22] I. Robinson, J. Math. Phys. **2**, 290 (1961).

if *equal* to a gradient field

$$\rho = \bar{\rho}, \qquad \tau = \bar{\alpha} + \beta. \tag{4.11}$$

One can understand the meaning of Ψ_0, Ψ_1, Ψ_2, Ψ_3, and Ψ_4 by the following:

Consider the five cases

(a) $\Psi_0 \neq 0$, others zero
(b) $\Psi_1 \neq 0$, others zero
(c) $\Psi_2 \neq 0$, others zero
(d) $\Psi_3 \neq 0$, others zero
(e) $\Psi_4 \neq 0$, others zero.

The Weyl tensor or the tetrad components of the Weyl tensor will have the following algebraic properties in each of the five cases;

(a) Petrov type N (or [4])[8] with propagation vector n_μ,
(b) Petrov type III (or [31]) with propagation vector n_μ,
(c) Petrov type D (or [22]) with propagation vector n_μ and l_μ,
(d) Petrov type III (or [31]) with propagation vector l_μ,
(e) Petrov type N (or [4]) with propagation vector l_μ.

By a propagation vector, we mean a repeated principal null vector.[8]

If in empty space the vector field l_μ satisfies the equation $l_{[\mu}R_{\alpha]\beta\gamma[\delta}l_{\nu]}l^\beta l^\gamma = 0$, then l_μ corresponds to one of the four principal null directions of the Riemann tensor and

$$\Psi_0 = 0. \tag{4.12}$$

If two or more of the principal null directions coincide and are represented by l_μ, they must satisfy $R_{\alpha\beta\gamma[\delta}l_{\mu]}l^\beta l^\gamma = 0$ or

$$\Psi_0 = \Psi_1 = 0. \tag{4.13}$$

(In this case, one refers to the Riemann tensor as being algebraically specialized.)

In the following section it will be shown that in empty space if the l_μ are tangents to a geodetic congruence whose shear σ vanishes, then (4.13) must be satisfied, and conversely.

V. GOLDBERG-SACHS THEOREM

In this section the conciseness attained by the use of spin coefficients will be illustrated by an example. Here and in the remainder of the paper it will be assumed that we are dealing with empty space, i.e.,

$$R_{\alpha\beta} = 0.$$

First we will prove that if the Riemann tensor is algebraically specialized, having $\Psi_0 = \Psi_1 = 0$, then $\sigma = \kappa = 0$. With these assumptions the pertinent Bianchi identities become

$$3\sigma\Psi_2 = 0,$$
$$-\delta\Psi_2 = -3\tau\Psi_2 + 2\sigma\Psi_3 \tag{5.1a}$$
$$\Delta\Psi_2 - \delta\Psi_3 = -3\mu\Psi_2 + (-2\tau + 2\beta)\Psi_3 + \sigma\Psi_4$$
$$-3\kappa\Psi_2 = 0$$
$$D\Psi_2 = -2\kappa\Psi_3 + 3\rho\Psi_2 \tag{5.1b}$$
$$D\Psi_3 - \bar{\delta}\Psi_2 = -\kappa\Psi_4 - (2\epsilon - 2\rho)\Psi_3 + 3\pi\Psi_2.$$

It is easily seen from this that unless the space is flat, $\sigma = 0$ by (5.1a) and $\kappa = 0$ by (5.1b).

The converse is more difficult to prove. We assume $\sigma = \kappa = 0$ and wish to prove $\Psi_0 = \Psi_1 = 0$. We can, by a transformation of the form $m_\mu \to e^{i\theta}m_\mu$ and by using a suitable scaling of l_μ, set $\epsilon = 0$.

The pertinent Eqs. (4.2) are then

$$D\rho = \rho^2 \tag{4.2a$'$}$$
$$0 = \Psi_0 \tag{4.2b$'$}$$
$$D\tau = (\tau + \bar{\pi})\rho + \Psi_1 \tag{4.2c$'$}$$
$$D\beta = \beta\bar{\rho} + \Psi_1 \tag{4.2e$'$}$$
$$\delta\rho = \rho(\bar{\alpha} + \beta) + (\rho - \bar{\rho})\tau - \Psi_1. \tag{4.2k$'$}$$

With the fact that $\Psi_0 = 0$, [Eq. (4.2b′)] the needed Bianchi identities and commutator are

$$\delta\Psi_1 = (4\tau + 2\beta)\Psi_1 \tag{5.2}$$
$$D\Psi_1 = 4\rho\Psi_1 \tag{5.3}$$
$$(D\delta - \delta D)\varphi = (\bar{\pi} - \bar{\alpha} - \beta)\, D\varphi + \bar{\rho}\, \delta\varphi. \tag{5.4}$$

There is yet a freedom in the choice of the vector n_μ, the freedom being that of the so-called "null rotations,"

$$\begin{aligned} l_\mu &\to l_\mu \\ m_\mu &\to m_\mu + a l_\mu \\ n_\mu &\to n_\mu + a\bar{m}_\mu + \bar{a}m_\mu + a\bar{a}l_\mu. \end{aligned} \tag{5.5}$$

This rotation of the tetrad does not change l_μ or disturb the relation $\epsilon = 0$. The complex function a can be chosen so that $\tau = 0$. [It is possible to do this only if $\rho \neq 0$. However from (4.2 k') it is easily seen that if $\rho = 0$, then $\Psi_1 = 0$, and our theorem is proved.]

Equations (5.2) and (5.3) are rewritten

$$\begin{aligned} \delta \ln \Psi_1 &= 2\beta \\ D \ln \Psi_1 &= 4\rho. \end{aligned} \tag{5.6}$$

Taking mixed derivatives and subtracting the two expressions, we have

$$\begin{aligned} (D\delta - \delta D) \ln \Psi_1 &= 2\, D\beta - 4\, \delta\rho \\ &= 2\beta\bar{\rho} - 4\rho(\bar{\alpha} + \beta) + 6\Psi_1 \end{aligned} \tag{5.7}$$

after using (4.2 e′) and (4.2 b′). The commutator (5.4) with $\varphi = \ln \Psi_1$, and using (5.6) is

$$(D\delta - \delta D) \ln \Psi_1 = 2\beta\bar{\rho} - 4\rho(\bar{\alpha} + \beta) + 4\rho\bar{\pi}. \tag{5.8}$$

Subtracting (5.8) from (5.7) we have

$$\Psi_1 = \tfrac{2}{3}\bar{\pi}\rho.$$

If this is compared with (4.2 c′), $\Psi_1 = -\bar{\pi}\rho$ we have, since ρ is assumed different from zero, $\Psi_1 = \bar{\pi} = 0$. This completes the proof.[23]

VI. SPECIAL COORDINATES

It is always possible, in a hyperbolic Riemannian manifold, to introduce a family of null hypersurfaces u = const, that is,

$$g^{\mu\nu}u_{,\mu}u_{,\nu} = 0. \tag{6.1}$$

The vectors $l^\mu = g^{\mu\nu}u_{,\nu}$ are tangent to the family of null geodesics lying in the hypersurfaces u = const, and satisfy

$$l^\mu{}_{;\nu}l^\nu = 0. \tag{6.2}$$

Robinson and Trautman[13] show that if one chooses as coordinates $u = x^1$ and an affine parameter[24] along the geodesics $r = x^2$, and two coordinates x^3, x^4 that label the geodesics on each surface u = constant, the metric takes the form $(i, j = 3, 4)$

$$g^{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & g^{22} & g^{23} & g^{24} \\ 0 & g^{23} & \multicolumn{2}{c}{\multirow{2}{*}{g^{ij}}} \\ 0 & g^{24} & & \end{bmatrix}. \tag{6.3}$$

(It is not always most convenient to use an affine parameter as x^2. Sachs uses a "luminosity" parameter, $\tilde{r} = 2/l^\mu{}_{;\mu}$ which makes the g^{12} different from unity. However, for our purposes an affine parameter seems simplest.)

With these coordinates the vector l_μ becomes

$$l_\mu = \delta^1_\mu, \qquad l^\mu = \delta^\mu_2. \tag{6.4}$$

[23] Though we have not seen all the details of the Goldberg-Sachs proof, we believe our proof to be essentially equivalent, but, due to the conciseness of our notation, much shorter.

[24] An affine parameter is a parameter along the geodesic, such that the equation for the geodesic takes the standard form. See, for example, E. M. Schrödinger, *Expanding Universes* (Cambridge University Press, New York, 1956).

To preserve $l_\mu n^\mu = 1$, and $l_\mu m^\mu = 0$, we have $(i = 3, 4)$

$$m^\mu = \omega\,\delta_2^\mu + \xi^i\,\delta_i^\mu$$
$$n^\mu = \delta_1^\mu + U\,\delta_2^\mu + X^i\,\delta_i^\mu. \tag{6.5}$$

The relation between the tetrad components (6.4) and (6.5) and the metric components (6.3) is

$$g^{22} = 2(U - \omega\bar{\omega}), \tag{6.6}$$
$$g^{2i} = X^i - (\xi^i\bar{\omega} + \bar{\xi}^i\omega),$$
$$g^{ij} = -(\xi^i\bar{\xi}^j + \bar{\xi}^i\xi^j),$$

$(i, j = 3, 4)$. This follows from (2.3a).

There is still complete freedom for the rotation of the tetrad vectors m^μ and n^μ leaving l^μ fixed. This freedom is eliminated by demanding m^μ and n^μ be parallelly propagated along l^μ. This requirement, in addition to the knowledge that l_μ is a gradient field, is stated in terms of the spin coefficients by (see Sec. IV)

$$\kappa = \pi = \epsilon = 0, \quad \rho = \bar{\rho}, \quad \tau = \bar{\alpha} + \beta. \tag{6.7}$$

With these simplifications the commutators (4.4) are

$$\begin{aligned}
(\Delta D - D\Delta)\varphi &= (\gamma + \bar{\gamma})\,D\varphi - \tau\,\bar{\delta}\varphi - \bar{\tau}\,\delta\varphi \\
(\delta D - D\delta)\varphi &= [\bar{\alpha} + \beta]\,D\varphi - \sigma\,\bar{\delta}\varphi - \rho\,\delta\varphi \\
(\delta\Delta - \Delta\delta)\varphi &= -\bar{\nu}\,D\varphi + \bar{\lambda}\,\bar{\delta}\varphi + [\mu + \bar{\gamma} - \gamma]\,\delta\varphi \\
(\bar{\delta}\delta - \delta\bar{\delta})\varphi &= (\mu - \bar{\mu})\,D\varphi - [\bar{\alpha} - \beta]\,\bar{\delta}\varphi - [\bar{\beta} - \alpha]\,\delta\varphi
\end{aligned} \tag{6.8}$$

with [using (6.5) and (2.12)]

$$D = \partial/\partial r, \qquad \delta = \omega\,\partial/\partial r + \xi^i\,\partial/\partial x^i$$
$$\Delta = U\,\partial/\partial r + \partial/\partial u + X^i\,\partial/\partial x^i. \tag{6.9}$$

In order to relate the tetrad components (or metric components) and the spin coefficients we replace φ by u, r, and x^i, respectively, in the four commutators. The result of this operation is $(i = 3, 4)$

$$D\xi^i = \rho\xi^i + \sigma\bar{\xi}^i \tag{6.10a}$$
$$D\omega = \rho\omega + \sigma\bar{\omega} - (\bar{\alpha} + \beta) \tag{6.10b}$$
$$DX^i = \tau\bar{\xi}^i + \bar{\tau}\xi^i \tag{6.10c}$$
$$DU = \tau\bar{\omega} + \bar{\tau}\omega - (\gamma + \bar{\gamma}) \tag{6.10d}$$
$$\delta X^i - \Delta\xi^i = (\mu + \bar{\gamma} - \gamma)\xi^i + \bar{\lambda}\bar{\xi}^i \tag{6.10e}$$
$$\delta\bar{\xi}^i - \bar{\delta}\xi^i = (\bar{\beta} - \alpha)\xi^i + (\bar{\alpha} - \beta)\bar{\xi}^i \tag{6.10f}$$
$$\delta\bar{\omega} - \bar{\delta}\omega = (\bar{\beta} - \alpha)\omega + (\bar{\alpha} - \beta)\bar{\omega} + (\mu - \bar{\mu}) \tag{6.10g}$$
$$\delta U - \Delta\omega = (\mu + \bar{\gamma} - \gamma)\omega + \bar{\lambda}\bar{\omega} - \bar{\nu}. \tag{6.10h}$$

We will refer to these as the metric equations.

To conclude this section we will write the Eqs. (4.2) and the Bianchi identities using the conditions (6.7).

$$D\rho = \rho^2 + \sigma\bar{\sigma} \tag{6.11a}$$
$$D\sigma = 2\rho\sigma + \Psi_0 \tag{6.11b}$$
$$D\tau = \tau\rho + \bar{\tau}\sigma + \Psi_1 \tag{6.11c}$$
$$D\alpha = \alpha\rho + \beta\bar{\sigma} \tag{6.11d}$$
$$D\beta = \beta\rho + \alpha\sigma + \Psi_1 \tag{6.11e}$$
$$D\gamma = \tau\alpha + \bar{\tau}\beta + \Psi_2 \tag{6.11f}$$
$$D\lambda = \lambda\rho + \mu\bar{\sigma} \tag{6.11g}$$
$$D\mu = \mu\rho + \lambda\sigma + \Psi_2 \tag{6.11h}$$
$$D\nu = \tau\lambda + \bar{\tau}\mu + \Psi_3 \tag{6.11i}$$
$$\Delta\lambda - \bar{\delta}\nu = 2\alpha\nu + (\bar{\gamma} - 3\gamma - \mu - \bar{\mu})\lambda - \Psi_4 \tag{6.11j}$$
$$\delta\rho - \bar{\delta}\sigma = (\beta + \bar{\alpha})\rho + (\bar{\beta} - 3\alpha)\sigma - \Psi_1 \tag{6.11k}$$
$$\delta\alpha - \bar{\delta}\beta = \mu\rho - \lambda\sigma - 2\alpha\beta + \alpha\bar{\alpha} + \beta\bar{\beta} - \Psi_2 \tag{6.11l}$$
$$\delta\lambda - \bar{\delta}\mu = (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 \tag{6.11m}$$
$$\delta\nu - \Delta\mu = \gamma\mu - 2\nu\beta + \bar{\gamma}\mu + \mu^2 + \lambda\bar{\lambda} \tag{6.11n}$$
$$\delta\gamma - \Delta\beta = \tau\mu - \sigma\nu + (\mu - \gamma + \bar{\gamma})\beta + \bar{\lambda}\alpha \tag{6.11o}$$
$$\delta\tau - \Delta\sigma = 2\tau\beta + (\bar{\gamma} + \mu - 3\gamma)\sigma + \bar{\lambda}\rho \tag{6.11p}$$
$$\Delta\rho - \bar{\delta}\tau = (\gamma + \bar{\gamma} - \bar{\mu})\rho - 2\alpha\tau - \lambda\sigma - \Psi_2 \tag{6.11q}$$
$$\Delta\alpha - \bar{\delta}\gamma = \rho\nu - \tau\lambda - \lambda\beta + (\bar{\gamma} - \gamma - \bar{\mu})\alpha - \Psi_3 \tag{6.11r}$$
$$D\Psi_1 - \bar{\delta}\Psi_0 = 4\rho\Psi_1 - 4\alpha\Psi_0 \tag{6.12a}$$
$$D\Psi_2 - \bar{\delta}\Psi_1 = 3\rho\Psi_2 - 2\alpha\Psi_1 - \lambda\Psi_0 \tag{6.12b}$$
$$D\Psi_3 - \bar{\delta}\Psi_2 = 2\rho\Psi_3 - 2\lambda\Psi_1 \tag{6.12c}$$
$$D\Psi_4 - \bar{\delta}\Psi_3 = \rho\Psi_4 + 2\alpha\Psi_3 - 3\lambda\Psi_2 \tag{6.12d}$$
$$\Delta\Psi_0 - \delta\Psi_1 = [4\gamma - \mu]\Psi_0 - [4\tau + 2\beta]\Psi_1 + 3\sigma\Psi_2 \tag{6.12e}$$
$$\Delta\Psi_1 - \delta\Psi_2 = \nu\Psi_0 + [2\gamma - 2\mu]\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 \tag{6.12f}$$
$$\Delta\Psi_2 - \delta\Psi_3 = 2\nu\Psi_1 - 3\mu\Psi_2 + [-2\tau + 2\beta]\Psi_3 + \sigma\Psi_4 \tag{6.12g}$$

$$\Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2 - [2\gamma + 4\mu]\Psi_3 + [-\tau + 4\beta]\Psi_4. \tag{6.12h}$$

VII. ASYMPTOTIC BEHAVIOR

We shall now investigate the asymptotic behavior of the Riemann tensor, spin coefficients and metric, for a general type of radiative empty space time. In order to do this, it is necessary to impose some condition of approach to flatness at infinity on the space time. This is usually done in terms of the metric tensor, but it is a little more satisfactory to impose restrictions on the Riemann tensor instead, as we shall do here.

The main condition that will be adopted here is[25]

$$\Psi_0 = O(r^{-5}) \tag{7.1}$$

but a condition

$$D\Psi_0 = O(r^{-6}) \tag{7.2}$$

on the r derivative of Ψ_0 will also be used. Furthermore, an assumption of "uniform smoothness" will be imposed, that as many as four or three derivatives with respect to x^3, x^4 do not spoil the above dependence:

$$d_i\Psi_0 = O(r^{-5}), \cdots, d_i\, d_j\, d_k\, d_l\Psi_0 = O(r^{-5}) \quad (i, j, k, l = 3, 4) \tag{7.3}$$
$$d_i\, D\Psi_0 = O(r^{-6}), \cdots, d_i\, d_j\, d_k\, D\Psi_0 = O(r^{-6})$$

where

$$d_i \equiv \partial/\partial x^i \qquad (i = 3, 4).$$

It will also be assumed that the hypersurfaces $u = \text{const}$ are not so chosen that they are "asymptotically cylindrical" or "asymptotically plane." The exact meaning of this condition will be explained later. It means, in effect, merely that certain very special choices of coordinate system are to be ruled out. From these assumptions[26] we shall prove:

$$\Psi_1 = O(r^{-4}), \qquad \Psi_2 = O(r^{-3})$$
$$\Psi_3 = O(r^{-2}), \qquad \Psi_4 = O(r^{-1}). \tag{7.4}$$

If, in addition, we were to assume that the Riemann tensor (in tetrad form) could be expanded[27] in negative powers of r, for large r, then (7.4) would tell us that the coefficient of $r^{-n}(n = 1, \cdots, 5)$ has the algebraic form of an empty-space Riemann tensor having the direction l^μ as a (5-n)-fold principal null direction. Bondi and Sachs[6] have obtained a similar result under somewhat different (and more restrictive) assumptions.

The choice of Ψ_0 as the quantity whose properties are specified in order to characterize the space-time is in accordance with a certain form of characteristic initial value problem. Given a suitable null hypersurface $u = \text{const}$, the function Ψ_0 on this hypersurface constitutes the main part (and sometimes all) of the initial data[28] for gravitation that is required for continuation. This matter will be discussed elsewhere,[29] but it is worth pointing out here that the determination of $\Psi_1, \cdots, \Psi_4$ from Ψ_0 seems to be a natural first step in this continuation problem. Other quantities such as σ or certain metric variables can be used equally as alternative initial data, but the choice of Ψ_0 seems simpler and is apparently the natural analog for gravitation, of certain characteristic initial data that are appropriate for other fields.

The exponent -5 in (7.1) is in agreement with what one would expect from the linear theory of a radiating quadripole. Also, (7.1) holds for a general null hypersurface in Schwarzschild's solution. If the hypersurfaces $u = \text{const}$ open out into the future (i.e., they are analogous to the future null cones given by constant advanced time) then the condition (7.1) would be expected to hold for an isolated system in the absence of incoming radiation, as is suggested by the linearized theory. In fact, even incoming radiation of sufficiently curtailed duration would not be expected to affect (7.1), (7.2), or (7.3).

We now proceed to prove (7.4) from our assumptions.[30] The proof of (7.4) for a particular null

[25] The meaning of the order symbols used here is that $f(r, u, x^i) = O[g(r)]$ means $|f(r, u, x^i)| < g(r)\, F(u, x^i)$ for some function F independent of r and for all large r, and $f(r, u, x^i) = o[g(r)]$ means

$$\lim_{r\to\infty} \frac{f(r, u, x^i)}{g(r)} = 0 \text{ for each } u, x^i.$$

[26] These assumptions, though stated in terms of a particular coordinate system appear to have a considerable amount of coordinate independence. For example, given a null geodesic with affine parameter r and tangent vector l_μ, if the r parameter of the original coordinate system can be so adjusted that

$$\bar{r} = r + o(r),\ \bar{l}_\mu = l_\mu + O(r^{-1}),$$

then (7.4) implies that $\widetilde{\Psi}_0 = O(r^{-5})$ also, where $\widetilde{\Psi}_0$ is the complex Riemann tensor component associated with $\bar{l}_\mu$. However, additional global assumptions appear to be necessary to ensure that r can always be so chosen.

[27] This may be a fairly strong restriction. It is, of course, stronger than just local analyticity in r since, for example, $r^{-n} \ln r$ cannot be expanded in negative powers of r.

[28] More properly, the quantity $D\Psi_0 - 5\rho\Psi_0$ may be the most significant one to specify on the hypersurface.

[29] R. Penrose (to be published).

[30] The necessity of (7.3) for the deduction of (7.4) and of ruling out the "asymptotically plane" case can be illustrated by considerations of certain plane waves. Plane waves can also be used to show that, for example, a local assumption merely of $\Psi_4 = O(r^{-1})$ or even $R_{\mu\nu\rho\sigma} = O(r^{-1})$ is quite inadequate for obtaining (7.4).

geodesic u, x^2, x^3 = const will depend on (7.1), (7.2), (7.3) only along this geodesic and its neighborhood within the hypersurface u = const. The order of procedure will be to obtain the r dependence of first ρ, σ and then the various x^i derivatives ($i = 3, 4$) of ρ, σ up to third order. Next, α, β, ξ^i, ω, Ψ_1, are similarly obtained, followed by their x^i derivatives. Then λ, μ, Ψ_2, are treated correspondingly, followed by Ψ_3, Ψ_4. The r dependence of the remaining quantities τ, γ, ν, X^i, U and hence g^{ij} may also be obtained if desired.

Writing

$$P = \begin{bmatrix} \rho & \sigma \\ \bar{\sigma} & \rho \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & \Psi_0 \\ \bar{\Psi}_0 & 0 \end{bmatrix}$$

(6.11a, b) become

$$DP = P^2 + Q. \tag{7.5}$$

This equation may be solved by

$$P = -(DY)Y^{-1}, \tag{7.6}$$

where

$$Y = \begin{bmatrix} y_1 & y_2 \\ \bar{y}_1 & \bar{y}_2 \end{bmatrix} \tag{7.7}$$

is a nonsingular solution (for given P) of

$$DY = -PY \tag{7.8}$$

and so satisfies

$$D^2 Y = -QY. \tag{7.9}$$

The asymptotic behavior of the solutions to (7.9) when $\int r\, |\Psi_0|\, dr = O(1)$, is[31]

$$DY = F + o(1) \tag{7.10}$$

$$Y = rF + o(r) \tag{7.11}$$

where F is a constant matrix. We can improve on this here since $Q = O(r^{-5})$. From (7.9) and (7.11) we get

$$D^2Y = -rQF + o(r^{-4}) = O(r^{-4}).$$

Hence, integrating[32] twice and comparing with (7.10), we get

$$DY = F + O(r^{-3}), \tag{7.12}$$

$$Y = rF + E + O(r^{-2}), \tag{7.13}$$

where E is another constant matrix. The solution (7.6) for P can now be used giving

$$P = -r^{-1}I + r^{-2}EF^{-1} + O(r^{-3}) \tag{7.14}$$

provided F is nonsingular. If F is singular, the asymptotic behavior of P is quite different. The case $|F| = 0$, $F \neq 0$, gives the "asymptotically cylindrical" case and P becomes asymptotically proportional to a singular matrix, with $\rho = -\frac{1}{2}r^{-1} + O(r^{-2})$. If $F = 0$, we get the "asymptotically plane" case and $P = O(r^{-3})$. [In each case, E must be such that there are two linearly independent columns among those of E, F. Otherwise it follows that in fact $|Y| = 0$ for all r so we do not get a solution of (7.5) for P.] These two exceptional cases are to be ruled out by assumption. For any given null geodesic u, x^2, x^3 = const the two types of exceptional behavior can always be avoided by an arbitrarily small change in the coordinate system which changes the relevant hypersurface u = const into a slightly different one through this null geodesic. Such exceptional solutions of the Eq. (7.5) do not occur for general choices of initial values for ρ and σ.

From (7.14) we get

$$\rho = -r^{-1} + O(r^{-2}), \qquad \sigma = O(r^{-2}). \tag{7.15}$$

In fact, a lot more can be obtained about the asymptotic behavior of ρ and σ, but (7.15) is all that will be needed here.

In order to proceed further, we shall require the following lemma:

Lemma. Let the complex ($n \times n$) matrix B and the complex column n vector b be given functions of r where

$$B = O(r^{-2}), \qquad b = O(r^{-2}). \tag{7.16}$$

Let the ($n \times n$) matrix A be independent of r and have no eigenvalue with a positive real part. Then all of the solution of

$$Dy = (Ar^{-1} + B)y + b \tag{7.17}$$

are bounded as $r \to \infty$, y being a complex column n vector function of r.

Proof. Put $r = e^l$, then (7.17) can be rewritten:

$$\frac{d}{dl}\begin{pmatrix} y \\ 1 \end{pmatrix} = \left\{ \left[\begin{array}{c|c} \begin{matrix} & & \\ & A & \\ & & \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & \cdots & 0 \end{matrix} & 0 \end{array}\right] + \left[\begin{array}{c|c} \begin{matrix} & & \\ & C & \\ & & \end{matrix} & c \\ \hline \begin{matrix} 0 & \cdots & 0 \end{matrix} & 0 \end{array}\right] \right\} \begin{pmatrix} y \\ 1 \end{pmatrix} \tag{7.18}$$

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[31] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill Book Publishers Inc., New York, 1955), p. 103.

[32] It is permissible to integrate order symbols formally but not to differentiate them.

where

$$C(l) = e^l B(e^l), \qquad c(l) = e^l b(e^l). \tag{7.19}$$

Now, the solutions ($z = \{\exp Al\}z_0$, $\zeta = \zeta_0$) of

$$\frac{d}{dl}\begin{pmatrix} z \\ \zeta \end{pmatrix} = \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 0 \end{array}\right)\begin{pmatrix} z \\ \zeta \end{pmatrix}$$

are all bounded. Also, by (7.16) and (7.19) it follows that the integrals with respect to l of the moduli of the elements of C and of c are bounded as $l \to \infty$. Hence by a theorem of N. Levinson,[33] the solutions of (7.18) are also all bounded [with $y = \{\exp Al\}y_0 + O(1)$]. This proves the lemma.[34]

Suppose, now, that B, b, and y are also functions of x^3, x^4. Then, differentiating (7.17) with respect to x^i we get

$$D(d_i y) = (Ar^{-1} + B)(d_i y) + \{(d_i B)y + d_i b\}, \qquad (i = 3, 4)$$

which is again of the form (7.17) provided that

$$d_i B = O(r^{-2}), \qquad d_i b = O(r^{-2}).$$

If this is the case, it follows from the lemma that $d_i y$ must also be bounded. Repeating this, we get the correspondiag results for higher derivatives.

Now consider the x^i derivatives of (6.11a, b) which can be put in the form

$$D\left\{r^2 d_i \begin{bmatrix} \rho \\ \sigma \\ \bar\sigma \end{bmatrix}\right\} = \begin{bmatrix} 2\rho + 2r^{-1} & \bar\sigma & \sigma \\ 2\sigma & 2\rho + 2r^{-1} & 0 \\ 2\bar\sigma & 0 & 2\rho + 2r^{-1} \end{bmatrix} \times r^2 d_i \begin{bmatrix} \rho \\ \sigma \\ \bar\sigma \end{bmatrix} + r^2 d_i \begin{bmatrix} 0 \\ \Psi_0 \\ \bar\Psi_0 \end{bmatrix}. \tag{7.20}$$

The lemma applies with $A = 0$, by (7.15), (7.3), whence

$$d_i\rho = O(r^{-2}), \quad d_i\sigma = O(r^{-2}) \quad (i = 3, 4). \tag{7.21}$$

The lemma applies again to the next two x^i derivatives of (7.20) successively, whence

$$d_i\, d_j\rho,\ d_i\, d_j\sigma,\ d_i\, d_j\, d_k\rho,\ d_i\, d_j\, d_k\sigma = O(r^{-2}). \tag{7.22}$$

Next, using (6.9), we can apply the lemma to (6.12a), (6.11d, e), (6.10a, b), and their complex conjugates, y being the column vector

$$\{r^4\Psi_1, r^4\bar\Psi_1, r\alpha, r\bar\alpha, r\beta, r\bar\beta, r\xi^3, r\bar\xi^3, r\xi^4, r\bar\xi^4, \omega, \bar\omega\}, \qquad b = 0$$

and

$$A = \left(\begin{array}{ccc|cc} 0 & \cdots & \cdots 0 & 0 & 0 \\ 0 & \cdots & \cdots 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots 0 & 0 & 0 \\ \hline & -a & & -1 & 0 \\ & & & 0 & -1 \end{array}\right),$$

$$a = \begin{bmatrix} 0\,0\,0\,1\,1\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,1\,0\,0\,0\,0 \end{bmatrix},$$

the elements of B being $O(r^{-2})$ expressions in r, ρ, σ, Ψ_0, $D\Psi_0$, and $d_i\Psi_0$. Hence

$$\Psi_1 = O(r^{-4}) \tag{7.23}$$

and

$$\alpha, \beta, \xi^3, \xi^4 = O(r^{-1}), \qquad \omega = 0(1). \tag{7.24}$$

Also, by (6.12a)

$$D\Psi_1 = O(r^{-5}). \tag{7.25}$$

Taking successive x^i derivatives and using (7.21), (7.22), and (7.3), the lemma also gives

$$d_i\Psi_1, \quad d_i\, d_j\Psi_1, \quad d_i\, d_j\, d_k\Psi_1 = O(r^{-4}),$$
$$d_i\alpha, \cdots, d_i\xi^4, \quad d_i\, d_j\alpha, \cdots, d_i\, d_j\, d_k\xi^4 = O(r^{-1}),$$
$$d_i\omega, \cdots, d_i\, d_j\, d_k\omega = O(1)$$

and so by (6.12a)

$$d_i\, D\Psi_1, \qquad d_i\, d_j\, D\Psi_1 = O(r^{-5}).$$

Next consider (6.12b) and (6.11g, h). The lemma applies again with y as the column vector $\{r^3\Psi_2, r\lambda, r\mu\}$, with $A = 0$ and B, b as certain $O(r^{-2})$ expressions in r, ρ, σ, Ψ_0, α, Ψ_1, ω, $D\Psi_1$, ξ^i, $d_i\Psi_1$. Thus

$$\Psi_2 = O(r^{-3}) \tag{7.28}$$

$$\lambda, \mu = O(r^{-1}). \tag{7.29}$$

Hence,

$$D\Psi_2 = O(r^{-4}),$$

[33] N. Levinson, Am. J. Math. **68**, 1 (1946).

[34] It may be seen from the proof to the lemma that conditions (7.16) are in fact, rather stronger than is necessary. They may be weakened to B, $b = O[f(r)]$ where $\int f\, dr = 0(1)$, $f > 0$. This enables condition (7.2) to be weakened to $D\,\Psi_0 = O(r^{-4}f(r))$ and (7.4) can still be obtained. Conditions (7.3) [and even (7.1)] can also be correspondingly weakened.

and continuing with the x^i derivatives we get

$$d_i\Psi_2, \quad d_id_j\Psi_2 = O(r^{-3});$$
$$d_i\lambda, \cdots, d_id_j\mu = O(r^{-1}); d_iD\Psi_2 = O(r^{-4}).$$

Similarly, the lemma applies to (6.12c) with $y = r^2\Psi_3$, then to its x^i derivative and then to (6.12d) with $y = r\Psi_4$ giving

$$\Psi_3 = O(r^{-2}) \tag{7.30}$$
$$d_i\Psi_3 = O(r^{-2}), \qquad D\Psi_3 = O(r^{-3})$$

and then

$$\Psi_4 = O(r^{-1}). \tag{7.31}$$

We may also continue the process and obtain

$$\tau = O(r^{-1}); \;\; \gamma, \nu, X^3, X^4 = O(1); \;\; U = O(r); \tag{7.32}$$

whence by (6.6)

$$g^{22} = O(r), \qquad g^{2i} = O(1), \qquad g^{ij} = O(r^{-2}), \qquad (i, j = 3, 4). \tag{7.33}$$

It is possible to obtain a great deal more information about the asymptotic behavior of all these quantities by examining the above procedure a little more closely and then substituting the expressions obtained back into the equations. Also, by specializing the coordinate system further many simplifications can be obtained. (We have, in fact, not even used $\tau = \bar{\alpha} + \beta$ here.) This, together with the integration of the remainder of the Eqs. (6.10) and (6.11), will be discussed elsewhere.

VIII. CONCLUSIONS

In the last section we showed that under certain fairly general assumptions of approach to flatness at infinity that are to be expected in radiative empty spaces, the Riemann tensor exhibits a characteristic asymptotic behavior. This is given by (7.1), (7.23), (7.28), (7.30), and; (7.31); namely

$$\Psi_n = O(r^{-5+n}) \qquad (n = 0, 1, \cdots, 4).$$

We may thus, in general, break the space up into five regions; namely, a near zone, where all terms are important; three transition zones, where Ψ_0, Ψ_1, Ψ_2 become negligible in turn; and finally the radiation zone, where only Ψ_4 remains important and the Riemann tensor is essentially null. The fourth zone is, of course, generally essentially type III and the third is essentially "algebraically special" or usually type II.[5,13] The second zone is essentially a region in which there are "goedesic rays" in the terminology of Sachs[12] and the first zone is of "general" type.[6] Thus, as we move backwards from infinity along a suitable null geodesic the principal null directions "peel off" one by one from the (outgoing) radial direction. This behavior was first observed by Sachs by considering the linear theory.[6] (It must be pointed out, however, that in many particular cases the actual positions of the principal null directions and the Petrov types encountered may not, in fact, agree with the above in detail since some of the Ψ's may be fortuitously small in some regions.)

The analogy between the above and the case of electrodynamics is striking. In the latter case there are three regions to consider, namely, the near zone where r^{-3} terms are important, the transition zone where r^{-2} terms are important and the radiation zone where the field goes essentially as r^{-1} and is null. The two electromagnetic principal null directions exhibit, in the general case, the same characteristic "peeling off" as in the gravitational case.

An interesting further question to consider will be the corresponding "peeling off" theorem for the Einstein-Maxwell theory. The relevant spin coefficient equations are given in an appendix. Another question of importance here is that of the extent to which the assumptions made here are coordinate independent.

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APPENDIX

In the spin coefficient notation Maxwell's equations take the form

$$\begin{aligned} D\Phi_1 - \bar{\delta}\Phi_0 &= (\pi - 2\alpha)\Phi_0 + 2\rho\Phi_1 - \kappa\Phi_2 \\ D\Phi_2 - \bar{\delta}\Phi_1 &= -\lambda\Phi_0 + 2\pi\Phi_1 + (\rho - 2\epsilon)\Phi_2 \\ \delta\Phi_1 - \Delta\Phi_0 &= (\mu - 2\gamma)\Phi_0 + 2\tau\Phi_1 - \sigma\Phi_2 \\ \delta\Phi_2 - \Delta\Phi_1 &= -\nu\Phi_0 + 2\mu\Phi_1 + (\tau - 2\beta)\Phi_2 \end{aligned} \tag{A1}$$

with

$$\Phi_0 = F_{\mu\nu}l^\mu m^\nu, \qquad \Phi_1 = \tfrac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \bar{m}^\mu m^\nu),$$
$$\Phi_2 = F_{\mu\nu}\bar{m}^\mu n^\nu.$$

If the Ricci tensor is proportional to the Maxwell stress tensor, so that

$$\Phi_{mn} = k\Phi_m\bar{\Phi}_n \qquad (m, n = 0, 1, 2) \tag{A2}$$

then the Bianchi identities become (choosing $k = 1$ for convenience)

$$
\begin{aligned}
&(\delta - \tau + 4\beta)\Psi_4 - (\Delta + 2\gamma + 4\mu)\Psi_3 + 3\nu\Psi_2 \\
&\qquad = \bar{\Phi}_1\,\Delta\Phi_2 - \bar{\Phi}_2\,\bar{\delta}\Phi_2 + 2(\bar{\Phi}_1\Phi_1\nu - \bar{\Phi}_2\Phi_1\lambda - \bar{\Phi}_1\Phi_2\gamma + \bar{\Phi}_2\Phi_2\alpha), \\
&(\delta - 2\tau + 2\beta)\Psi_3 + \sigma\Psi_4 - (\Delta + 3\mu)\Psi_2 + 2\nu\Psi_1 \\
&\qquad = \bar{\Phi}_1\,\delta\Phi_2 - \bar{\Phi}_2\,D\Phi_2 + 2(\bar{\Phi}_1\Phi_1\mu - \bar{\Phi}_2\Phi_1\pi - \bar{\Phi}_1\Phi_2\beta + \bar{\Phi}_2\Phi_2\epsilon), \\
&(\delta - 3\tau)\Psi_2 + 2\sigma\Psi_3 - (\Delta - 2\gamma + 2\mu)\Psi_1 + \nu\Psi_0 \\
&\qquad = \bar{\Phi}_1\,\Delta\Phi_0 - \bar{\Phi}_2\,\delta\Phi_0 + 2(\bar{\Phi}_1\Phi_0\gamma - \bar{\Phi}_2\Phi_0\alpha - \bar{\Phi}_1\Phi_1\tau + \bar{\Phi}_2\Phi_1\rho), \\
&(\delta - 4\tau - 2\beta)\Psi_1 + 3\sigma\Psi_2 - (\Delta - 4\gamma + \mu)\Psi_0 \\
&\qquad = \bar{\Phi}_1\,\delta\Phi_0 - \bar{\Phi}_2\,D\Phi_0 + 2(\bar{\Phi}_1\Phi_0\beta - \bar{\Phi}_2\Phi_0\epsilon - \bar{\Phi}_1\Phi_1\sigma + \bar{\Phi}_2\Phi_1\kappa), \\
&(D + 4\epsilon - \rho)\Psi_4 - (\bar{\delta} + 4\pi + 2\alpha)\Psi_3 + 3\lambda\Psi_2 \\
&\qquad = \bar{\Phi}_0\,\Delta\Phi_2 - \bar{\Phi}_1\,\bar{\delta}\Phi_2 + 2(\bar{\Phi}_0\Phi_1\nu - \bar{\Phi}_1\Phi_1\lambda - \bar{\Phi}_0\Phi_2\gamma + \bar{\Phi}_1\Phi_2\alpha), \\
&(D + 2\epsilon - 2\rho)\Psi_3 + \kappa\Psi_4 - (\bar{\delta} + 3\pi)\Psi_2 + 2\lambda\Psi_1 \\
&\qquad = \bar{\Phi}_0\,\delta\Phi_2 - \bar{\Phi}_1\,D\Phi_2 + 2(\bar{\Phi}_0\Phi_1\mu - \bar{\Phi}_1\Phi_1\pi - \bar{\Phi}_0\Phi_2\beta + \bar{\Phi}_1\Phi_2\epsilon), \\
&(D - 3\rho)\Psi_2 + 2\kappa\Psi_3 - (\bar{\delta} + 2\pi = 2\alpha)\Psi_1 + \lambda\Psi_0 \\
&\qquad = \bar{\Phi}_0\,\Delta\Phi_0 - \bar{\Phi}_1\,\delta\Phi_0 + 2(\bar{\Phi}_0\Phi_0\gamma - \bar{\Phi}_1\Phi_0\alpha - \bar{\Phi}_0\Phi_1\tau + \bar{\Phi}_1\Phi_1\rho), \\
&(D - 2\epsilon - 4\rho)\Psi_1 + 3\kappa\Psi_2 - (\bar{\delta} + \pi - 4\alpha)\Psi_0 \\
&\qquad = \bar{\Phi}_0\,\delta\Phi_0 - \bar{\Phi}_1\,D\Phi_0 + 2(\bar{\Phi}_0\Phi_0\beta - \bar{\Phi}_1\Phi_0\epsilon - \bar{\Phi}_0\Phi_1\sigma + \bar{\Phi}_1\Phi_1\kappa).
\end{aligned}
\tag{A3}
$$

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The Solution of a Transition Problem in a Superconducting Strip*

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The isothermal transition of a strip of superconducting material from the superconducting state to the normal state, under the influence of a supercritical external magnetic field, is studied on the basis of the London theory of superconductivity. The problem can be formulated as a free boundary problem with parabolic differential equations which is solved mainly by numerical methods; an analytical solution in the form of series expansions is given for the early part of the transition.

It is found that, in the beginning, the transition in the strip behaves almost like the transition in a half-space. However, the two differ quite drastically as the transition nears completion. It is pointed out that to predict the total transition time for the strip by extrapolating from the analytical solution for the half-space is incorrect. The results show that such an estimate for the transition time would be much too small.

I. INTRODUCTION

THE transition of a superconducting material from the superconducting state to the normal state has been considered in several recent papers. The problem has an obvious application in the use of superconductors in computer elements such as the cryotron. All of the work published thus far has been concerned with the case where the superconducting material fills an infinite half-space. For example, Ittner[1] and Duijvestijn[2] have considered the effects of latent heat and eddy currents on

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[1] W. B. Ittner, III, Phys. Rev. **111**, 1483 (1958).
[2] A. W. Duijvestijn, IBM J. Research Develop. **3**, 132 (1959).