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PERTURBATIONS OF A ROTATING BLACK HOLE. II. DYNAMICAL STABILITY OF THE KERR METRIC*

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ABSTRACT

If unstable, a rotating black hole would spontaneously radiate gravitational waves and evolve dynamically to some new (unknown) final state. This paper tests the dynamical stability of rotating holes by numerical integration of the separable perturbation equations for the Kerr metric. No instabilities are found in any of the dozen or so lowest angular modes tested, for any value of specific angular momentum $0 \leq a < M$. Even in the limit $a \rightarrow M$, the hole appears to be stable. These stability results add credibility to the use of the Kerr metric in detailed astrophysical models. A numerical technique for preserving accuracy on an asymptotically small solution to an ordinary differential equation in the presence of an asymptotically large one is described.
Subject headings: black holes — gravitation — relativity — rotation

I. INTRODUCTION

The Schwarzschild (nonrotating) black hole has been shown by Vishveshwara (1970) to be dynamically stable against small perturbations; and more recent mathematical advances by Zerilli (1970) allow the proof of Schwarzschild stability to be put forth in about two sentences (see § II below). By contrast, almost nothing has been known about the stability of rotating black holes, described by the Kerr metric (of which Schwarzschild is a special case). The question of Kerr stability is interesting in principle, of course, and it is also vitally important astrophysically: black holes formed from a collapsed star are likely to be highly rotating (Bardeen 1970; Thorne 1973). If a Kerr metric were unstable, it could not be the endpoint of a dynamical collapse, and the whole question of black-hole uniqueness (cf. Carter 1973) and inevitability would have to be reexamined in quite a new light.

What do we mean by stability? At time $t = 0$ (constant time hypersurfaces are well defined because the metric is stationary) we imagine a Kerr black hole with a small gravitational perturbation, infinitesimal in amplitude. Physically, this perturbation may have been caused by a particle flying by or falling in; we take the perturbation as arbitrarily given, however, and do not consider its precise coupling to material sources. Since the perturbation is small, its time evolution is accurately treated by linearized perturbation equations, specifically the separable perturbation equations of the accompanying paper (Teukolsky 1973, hereafter cited as Paper I). If the perturbation radiates away down the hole and to infinity, remaining small and well behaved all the while, and finally goes to zero, then the linearized equations will have described the whole process accurately, and we can conclude that the black hole is stable against small perturbations.

On the other hand, the linearized equations might predict that the perturbation grows in time without bound, and that the black hole never returns to its quiescent state. In this case, the black hole is unstable. The linearized equations fail when the

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perturbations become large, so we cannot determine what the hole is evolving to, but we can be certain that it does not return to its original configuration: the linearized equations *are* valid in a neighborhood of that configuration.

What would be possible final states of an unstable Kerr black hole (if any were unstable)? First, since Schwarzschild is stable, it is not difficult to show that all Kerr holes in some neighborhood of Schwarzschild ("slowly rotating" holes) are also stable (Press 1972). Thus, an unstable Kerr hole might radiate away mass and angular momentum in a burst of gravitational radiation, until it settles down to become a Kerr hole somewhere in the stable region.

Second, the hole might become highly dynamical for a finite period of time, and finally settle down to a new non-Kerr stationary, axisymmetric configuration with a horizon and with no naked singularities (a new type of hole). Carter (1971) has proved that such configurations must occur in two-parameter families (like the Kerr family), and that they must be disjoint from the Kerr family. No such disjoint families are presently known; it is generally suspected that none exist.

A third possibility would be that a sequence of nonaxisymmetric holes bifurcates from the Kerr sequence at some finite, specific angular momentum, and that some or all Kerr holes with greater specific angular momentum are unstable against migrating dynamically to the new sequence. This picture would be the analog of the situation for classical fluid ellipsoids (see Chandrasekhar 1969) where—when there is any dissipation—stability passes from the axisymmetric Maclaurin to the nonaxisymmetric Jacobi sequence at their point of bifurcation. The similarities between fluid ellipsoids and black holes, and the possibility of similar instabilities, have been emphasized by Smarr (1973). By a theorem of Hawking (1972), any nonaxisymmetric sequence must be dynamic in its own right, but this does not mean that it cannot be stable: a hole on the new sequence will have a definite trajectory of time evolution, and this time evolution can be stable against small perturbations.

A fourth possibility would be the least pleasant. The hypothesized unstable black hole might evolve to a naked singularity, visible or asymptotically visible from asymptotically flat infinity. One hopes that naked singularities will at some future time be ruled out by something more general than a case-by-case analysis (Penrose's "cosmic censorship" hypothesis), but at present they are not.

We should note that the stability problem is also astrophysically important in connection with possible sources of gravitational waves. For a small mass m falling into a larger black hole of mass M, perturbation calculations (e.g., Davis *et al.* 1971, 1972) show that the rate of conversion of infalling mass to gravitational waves is never greater than

$$\frac{dm}{dt} \sim \left(\frac{m}{M}\right)^2$$

(units with $c = G = 1$). Thus, in the hole's characteristic time M the efficiency of mass conversion to a burst of waves is of order $m/M \ll 1$. In realistic astrophysical situations the accretion of matter into a black hole will tend to increase its specific angular momentum (Bardeen 1970). If instability sets in at some point, there will be a last particle of mass m whose capture pushes the hole into the unstable region. To return to a stable configuration, the hole must emit a burst of energy $\sim m$. Thus the particle—and all subsequent matter that accretes—is converted to gravitational waves with efficiency ~ 1. On the other hand, if the hole is not able to return to a stable configuration, then the initial burst of radiation emitted is even greater, perhaps even energy $\sim M$.

Having raised these striking—if speculative—possibilities, we turn to the more mundane task of disproving them: We have tested the dynamic stability of the Kerr metric by searching for an onset of instability as we "spin up" the hole from the

Schwarzschild configuration, known to be stable. Specifically, we have looked at the dozen or so lowest angular eigenvalues for vibrational modes of the hole, and have verified numerically that the corresponding vibrational frequencies do not cross the real axis of the complex frequency plane into the upper (unstable) half-plane, for any specific angular momenta a in the black-hole range $0 \leq a < M$. In other words, we find no configurations (not even the limit $a \to M$) which are marginally unstable at any real frequency, for any mode number $l \leq 3$ (for all $|m| \leq l$; l and m are spin-weighted spheroidal harmonic indices). We have also "spot-checked" a few higher modes, and find that they seem to go smoothly to an asymptotic limit, and show no tendency toward marginal stability.

We do not in this paper consider the possibility of an onset of instability through a zero-frequency mode (cf. Chandrasekhar and Friedman 1973). This, we claim, involves no loss of generality: A later paper in this series will show explicitly that there are no admissible zero-frequency solutions to the separable perturbation equation (or see Teukolsky 1972); and recent results by Wald (1973) show that there are no admissible solutions to the *full* set of perturbation equations, except those which are manifest in the separable equation.

Strictly speaking, this paper describes an unsuccessful search for instabilities rather than an actual proof of stability. However, we suspect that the data presented here are in fact sufficient to support a rigorous mathematical proof: Since the underlying equation is linear and suitably analytic, it is almost certain that instabilities must necessarily set in by passage of some mode from the stable lower half of the frequency plane, through a real-frequency mode, into the unstable upper half-plane. Also, although our numerical approach can sample only a discrete number of possible frequencies and a's, it is probably possible to derive analytic bounds on the "smoothness" of what is computed. Finally, our numerical results are also very smooth as a function of mode number, and a matching of the numerical results to asymptotic solutions of large mode number is probably possible at very moderate values of l.

Work which will strengthen the mathematical footing in this way is important; it may in fact lead to a purely analytic proof of stability. However, we feel that in the results here presented the essential evidence for stability is already clear.

In § II of this paper, we describe the formalism which relates the decoupled, separable perturbation equations for the Kerr metric to the stability problem. Section III and its related Appendices treat the actual numerical solution of the equations, and present the raw results. In integrating the separated radial equation we have used a technique which does not seem to have been described previously, a means of preserving accuracy on an asymptotically small solution in the presence of an asymptotically large one; this technique may be of some general use in unrelated problems. For the benefit of other workers on Kerr perturbation problems, we also describe our integration of the angular equations in some detail, and tabulate some of the eigenvalues. Section IV interprets the stability results and gives our conclusions.

II. USE OF THE DECOUPLED, SEPARATED WAVE EQUATION

The dynamical behavior of the Kerr black hole under small perturbations is described by a homogeneous wave equation in a single decoupled variable

$$\mathfrak{L}\psi(t, r, \theta, \varphi) = 0\,. \tag{2.1}$$

Here t, r, θ, φ are Boyer-Lindquist (1967) coordinates for the Kerr metric

$$\begin{aligned} ds^2 = {} & (1 - 2Mr/\Sigma)dt^2 + [4Mar\sin^2(\theta)/\Sigma]dtd\varphi \\ & - (\Sigma/\Delta)dr^2 - \Sigma d\theta^2 - \sin^2(\theta)[r^2 + a^2 + 2Ma^2r\sin^2(\theta)/\Sigma]d\varphi^2\,, \end{aligned} \tag{2.2}$$

where $\Delta \equiv r^2 - 2Mr + a^2$, $\Sigma \equiv r^2 + a^2 \cos^2 \theta$, M is the mass of the hole, a is its specific angular momentum, $0 \leq a \leq M$. The hyperbolic, second-order, linear differential operator $\mathfrak{L}$, as derived in Paper I, is

$$\mathfrak{L} = \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta\right] \frac{\partial^2}{\partial t^2} + \frac{4Mar}{\Delta} \frac{\partial^2}{\partial t \partial \varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta}\right] \frac{\partial^2}{\partial \varphi^2}$$
$$- \Delta^{-s} \frac{\partial}{\partial r}\left(\Delta^{s+1} \frac{\partial}{\partial r}\right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial}{\partial \theta}\right) - 2s\left[\frac{a(r - M)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta}\right] \frac{\partial}{\partial \varphi}$$
$$- 2s\left[\frac{M(r^2 - a^2)}{\Delta} - r - ia \cos \theta\right] \frac{\partial}{\partial t} + (s^2 \cot^2 \theta - s) \,. \tag{2.3}$$

Since we are interested in gravitational perturbations, we will always take $s = \pm 2$. The precise definition of ψ in terms of components of the Riemann tensor is given in Paper I and in Teukolsky (1972). For the stability problem, one need only know that ψ grows without bound if and only if physically measurable quantities (e.g., the energy fluxes of gravitational waves at infinity) grow without bound; this point is discussed below.

We now consider an initial-value problem where ψ and $\psi_{,t}$ are specified on an initial hypersurface $t = 0$, and where $\boxed{\mathfrak{L}\psi = 0}$ determines the subsequent evolution for $t > 0$. The function $\psi(t, r, \theta, \varphi)$ is assumed to satisfy boundary conditions of physical acceptability at radial infinity and on the horizon (see Paper I). A reasonable sufficient condition to ensure this is to take ψ and $\psi_{,t}$ nonzero only in a finite range of r outside the horizon, at $t = 0$. By Fourier analysis in the complex plane we have

$$\psi(t, r, \theta, \varphi) = (2\pi)^{-1/2} \int_{-\infty + i\tau_0}^{+\infty + i\tau_0} \psi_\omega e^{-i\omega t} d\omega \,, \qquad t > 0 \,, \tag{2.4}$$

where

$$\psi_\omega = \psi_\omega(r, \theta, \varphi) = (2\pi)^{-1/2} \int_0^\infty \psi e^{i\omega t} dt \tag{2.5}$$

and where τ_0 is a positive real number such that $\exp(\tau_0 t)$ is an upper bound on the growth of ψ at large times, i.e., faster than the fastest instability. (We will see below that such a bound exists.)

We now want to investigate the possibility of deforming the contour of integration in equation (2.4) into the lower half-plane, by letting τ_0 decrease and become negative. (Strictly speaking, convergence at infinity demands that the contour be left attached to the real axis at Re $\omega = \pm\infty$, but it can be deformed to Im $\omega = \tau_0 < 0$ in any finite range $|\text{Re}\, \omega| < B \to \infty$.) If ψ_ω, viewed as a function of ω, contains no poles (we will assume that branch cuts are ruled out by the form of $\mathfrak{L}$) in the region above the contour, then equation (2.4) remains a valid "reconstruction" of the complete field ψ. If, however, the contour deformation crosses a pole, then the pole's residue must be included, and we obtain

$$\boxed{\begin{aligned}\psi(t, r, \theta, \varphi) = (2\pi)^{-1/2} &\int_{-\infty + i\tau_0}^{+\infty + i\tau_0} \psi_\omega(r, \theta, \varphi) e^{-i\omega t} d\omega \\ &+ \sum_j F_j(r, \theta, \varphi) \exp(-i\omega_j t) \,,\end{aligned}} \tag{2.6}$$

where the sum is over all poles above the contour, with frequencies ω_j and with residues F_j.

The next point is that the solution $\psi = F_j(r, \theta, \varphi) \exp(-i\omega_j t)$ associated with any single pole must *by itself* satisfy $\mathfrak{L}\psi = 0$ with physically correct boundary conditions on the horizon and at infinity. Why? Because at late times $t \to \infty$ and at any fixed r, the contribution of the contour integral vanishes exponentially compared to the sum over the poles, while the total summed solution, by construction, satisfies the boundary condition at all times. Since the poles are a discrete set, their sum can satisfy the boundary condition only if each does separately.

Each ω_j represents a discrete frequency mode with the correct boundary conditions. At late times, any initial perturbation is asymptotically a superposition of these discrete modes; all other perturbations die away (i.e., radiate away to infinity or the horizon) faster than any exponential. (There is no contradiction with Price's 1972 power-law "tails"; his system was inhomogeneous with sources which became asymptotically static.) The discrete modes are what one calls the "vibrations" of the black hole, and have been seen numerically in previous work (Press 1971). Since the equation $\mathfrak{L}\psi = 0$ is separable, and its angular eigenfunctions are complete, we can write

$$F_j(r, \theta, \varphi) = \sum_{l,m} S^m{}_l(\theta) e^{im\varphi} R_{\omega lm}(r) , \tag{2.7}$$

where $S^m{}_l$ is a regular function which satisfies the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + E - s^2\right)S = 0 , \tag{2.8}$$

for eigenvalues $E^m{}_l$ (a "spin-weighted spheroidal harmonic"), and $R_{\omega lm}$ satisfies the separated radial equation

$$\Delta\frac{d^2R}{dr^2} + 2(s+1)(r-M)\frac{dR}{dr} + \left\{\frac{K^2 - 2is(r-M)K}{\Delta} + 4ir\omega s - \lambda\right\}R = 0 . \tag{2.9}$$

Here $K \equiv (r^2 + a^2)\omega - am$ and $\lambda \equiv E - 2am\omega + a^2\omega^2 - s(s+1)$. The boundary conditions on $R_{\omega lm}$ at the horizon ($r \to r_+$) and at infinity ($r \to \infty$) are, respectively,

$$R \sim \begin{cases} \Delta^{-s}e^{-ikr*} , & r \to r_+ , \\ e^{i\omega r*}/r^{(2s+1)} , & r \to \infty , \end{cases} \tag{2.10}$$

where $k \equiv \omega - ma/(2Mr_+)$, and r_* is defined by

$$dr_*/dr = (r^2 + a^2)/\Delta \tag{2.11}$$

(see Paper I for derivation and details).

Equations (2.8), (2.9), and (2.10) determine a nonlinear eigenvalue problem for the frequencies ω_j of vibration, nonlinear because ω is tied rather intimately into the equations as a parameter—for example, $E^m{}_l = E^m{}_l(a\omega)$. The stability problem is now very straightforward: if there are no solutions with ω_j in the upper half complex plane (for all angular modes l and m), then the Fourier reconstruction of equation (2.4) is valid with $\tau_0 \leq 0$ and there are no solutions which become unbounded in time; the hole is stable. If there *are* eigenfrequencies ω_j in the upper half-plane, then each corresponds to a perturbation which is well behaved for all $r \geq r_+$ at time $t = 0$, but which grows exponentially in time; these are instabilities.

How can one determine whether there are solutions corresponding to instabilities? The direct way, in principle, is to examine the entire upper half complex ω-plane by "shooting": for each ω, start a solution $R_{\omega lm}$ with the correct asymptotic form on the horizon

$$R = \Delta^{-s} e^{-ikr*}\,, \qquad r \to r_+\,; \tag{2.12}$$

integrate this solution outward with the radial equation (2.9) to large values of r, and resolve it into the two asymptotic solutions

$$R = Z_{\rm in} e^{-i\omega r*}/r + Z_{\rm out} e^{i\omega r*}/r^{(2s+1)}\,, \qquad r \to \infty\,. \tag{2.13}$$

The solutions we seek are zeros of $Z_{\rm in}$, or equivalently poles of $Z_{\rm out}/Z_{\rm in}$, viewed as a function of ω in the complex plane. Note that by looking at $Z_{\rm out}/Z_{\rm in}$ we cannot miss a zero of $Z_{\rm in}$ by $Z_{\rm out}$ going to zero at the same value of ω, for then R would be zero everywhere since it satisfies a linear second-order equation.

This paper reports a shooting search for poles of $Z_{\rm out}/Z_{\rm in}$ where we have restricted ω to the real axis, and tested a large number of values of a in the range $0 \leq a < M$. There are good reasons for this restriction: Since the Schwarzschild case $a = 0$ is known to be stable, there are no poles in the upper half-plane for this value. Since the linear differential operator $\mathfrak{L}$ depends continuously (in fact analytically) on the parameter a, and since the boundary conditions (2.10) are also analytic in a, it is virtually certain (but strictly speaking, not yet rigorously proved!) that the eigenvalues ω_j vary continuously with a. Thus, a pole can migrate to the upper half-plane only by crossing the real axis, and the restriction of our search to the real axis involves no loss of generality.

Going beyond the scope of this paper, it is worth noting that data on the real axis could rule out eigenvalues in the upper half-plane even if the continuity assumption failed. It is not difficult to show that $Z_{\rm in} = Z_{\rm in}(\omega)$ is regular as $\omega \to \infty$ in the upper half-plane; a proof that $Z_{\rm in}(\omega)$ is suitably meromorphic in the finite upper half-plane would rigorously justify a "phase shift analysis" on the real-ω axis: the change in phase of $Z_{\rm in}$ over the interval $-\infty < \omega < +\infty$ would directly measure the number of eigenvalues in the upper half-plane. This approach is discussed briefly in Appendix A. In that appendix we also show that $Z_{\rm in} \to 1$ for large $|\omega|$; this places a bound on the growth of the fastest instability, and justifies the choice of a constant τ_0 in equation (2.4). We postpone further discussion of the analyticity properties of the Kerr perturbation equations to a future paper.

Two loose ends can be tied up here. First, for the equation with spin-weight $s = -2$ (which for numerical reasons will turn out to be the actual case integrated, see § III), the ingoing piece of the solution $Z_{\rm in} \exp(-i\omega r_*)/r$ is asymptotically small compared to the outgoing piece $Z_{\rm out} \exp(i\omega r_*)r^3$. This is a direct consequence of the peeling theorems (see Newman and Penrose 1962, hereafter cited as NP): the Newman-Penrose field ψ_4 is "tailored" for outgoing waves, and suppresses ingoing waves by four powers of r. How do we know that the magnitude $Z_{\rm in}$ of this suppressed "ghost" of an incoming wave accurately reflects the physical magnitude of the wave (say, as an ingoing energy flux at infinity), so that a zero of $Z_{\rm in}$ is *necessary and sufficient* for the correct physical boundary condition, as claimed? The answer is given in Appendix B, where we give the relation between ingoing and outgoing wave energy flux, and $Z_{\rm in}$ and $Z_{\rm out}$ as defined above. It turns out that the factor relating $Z_{\rm in}$ to ingoing flux is rather complicated and cannot be exhibited in closed form for $a \neq 0$; but we prove that it is positive definite, without zeros or poles. A subsequent paper will give numerical results for this factor and discuss applications to the problem of superradiant scattering (Misner 1972; Press and Teukolsky 1972) of gravitational waves.

Second, we can give a very easy proof that a Schwarzschild black hole is stable (modeled on the work of Vishveshwara 1970 and Zerilli 1970). For the Schwarzschild case one can pose the analogous eigenvalue problem to that determined by equations (2.8), (2.9), and (2.10). Here, however, one can use the Regge-Wheeler (1957) and Zerilli (1970) radial equations instead of the more complicated equation (2.9) of the Kerr background. The Regge-Wheeler and Zerilli equations are both of the form

$$\psi'' = (V - \omega^2)\psi\,, \tag{2.14}$$

where a prime denotes d/dr_* and V is a real, positive function with no ω dependence. Thus the eigenvalue problem for ω^2 is linear, and self-adjoint. By self-adjointness, ω^2 must be real, so any instability must lie on the positive imaginary axis, $\omega = i\sigma$ say. Then the radial equations take the form

$$\psi'' = (\text{positive-definite function})\psi\,, \tag{2.15}$$

which manifestly has no solution that is regular at $r_* = \pm\infty$, QED. Marginal instabilities, with ω^2 real and positive, are equally easy to rule out, using the fact that the Wronskian of a solution with its complex conjugate is conserved.

III. NUMERICAL SOLUTION OF THE EQUATIONS

a) Angular Eigenfunctions and Eigenvalues

The angular equation (2.8) can be written as an eigenvalue equation involving the sum of two operators

$$(\mathfrak{H}_0 + \mathfrak{H}_1)S = -ES\,, \tag{3.1}$$

where

$$\mathfrak{H}_0 \equiv \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \left(\frac{m^2 + s^2 + 2ms\cos\theta}{\sin^2\theta}\right),$$

$$\mathfrak{H}_1 \equiv a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta\,. \tag{3.2}$$

The operator $\mathfrak{H}_0$ has no dependence on a or ω; the equation

$$\mathfrak{H}_0 S = -ES \tag{3.3}$$

has as its solutions the well-known spin-weighted spherical harmonics (Newman and Penrose 1966; Goldberg *et al.* 1967)

$$S(\theta) = {}_sY^m{}_l(\theta)\,, \qquad l = |s|, |s| + 1, \ldots\,,$$
$$E = l(l+1)\,, \qquad -l \le m \le +l\,. \tag{3.4}$$

We can view $\mathfrak{H}_1$ as a (noninfinitesimal) perturbation operator which takes us from the spherical ($a\omega = 0$) to the spheroidal ($a\omega \neq 0$) case. We will first consider the limiting case of $a\omega \ll 1$ and derive explicit perturbation solutions for ${}_sS^m{}_l(\theta, a\omega)$ and ${}_sE^m{}_l(a\omega)$; a numerical calculation then generalizes these solutions to arbitrary values of $a\omega$. In the present series of papers, we are primarily interested in the eigenvalues, which couple into the radial equation, and only secondarily interested in the details of the angular functions. Without loss of generality, we can restrict attention to cases with $s < 0$, $a\omega > 0$, since equation (3.2) admits the following symmetries:

$$
\begin{aligned}
{}_{-s}S^m{}_l(\theta, a\omega) &= {}_sS^m{}_l(\pi - \theta, a\omega)\,, \\
{}_{-s}E^m{}_l(a\omega) &= {}_sE^m{}_l(a\omega)\,, \qquad (3.5)\\
{}_sS^m{}_l(\theta, -a\omega) &= {}_sS^{-m}{}_l(\pi - \theta, a\omega)\,, \\
{}_sE^m{}_l(-a\omega) &= {}_sE^{-m}{}_l(a\omega)\,. \qquad (3.6)
\end{aligned}
$$

For small $a\omega$, ordinary perturbation theory gives the results

Typo: This should multiplied by a minus sign to the effect of swapping the primes in the denominator.

$$
\begin{aligned}
{}_sE^m{}_l &= l(l+1) - \langle slm|\mathfrak{H}_1|slm\rangle + \cdots\,, \\
{}_sS^m{}_l &= {}_sY^m{}_l + \sum_{l'\neq l} \frac{\langle sl'm|\mathfrak{H}_1|slm\rangle}{l(l+1) - l'(l'+1)}\, {}_sY^m{}_{l'} + \ldots\,. \qquad (3.7)
\end{aligned}
$$

Here

$$\langle sl'm|\mathfrak{H}_1|slm\rangle \equiv \int d\Omega\, {}_sY^m{}_{l'}{}^*\, {}_sY^m{}_l\, \mathfrak{H}_1\,. \qquad (3.8)$$

The spin-weighted spherical harmonics are related to the rotation matrix elements of quantum mechanics (see Campbell and Morgan 1971), so standard formulae are available for the integrated product of three such functions. The ones we need are

$$\langle sl'm|\cos^2\theta|slm\rangle = \tfrac{1}{3}\delta_{ll'} + \frac{2}{3}\left(\frac{2l+1}{2l'+1}\right)^{1/2} \langle l2m0|l'm\rangle\langle l2-s0|l'-s\rangle \qquad (3.9\text{a})$$

and

$$\langle sl'm|\cos\theta|slm\rangle = \left(\frac{2l+1}{2l'+1}\right)^{1/2} \langle l1m0|l'm\rangle\langle l1-s0|l'-s\rangle\,, \qquad (3.9\text{b})$$

where $\langle j_1 j_2 m_1 m_2|JM\rangle$ is a Clebsch-Gordan coefficient. Equations (3.7), (3.8), (3.9) yield an explicit construction of the perturbation eigenfunctions and eigenvalues. For example,

$$
\begin{aligned}
{}_sE^m{}_l(a\omega) &= l(l+1) - 2a\omega\frac{s^2 m}{l(l+1)} + O[(a\omega)^2]\,, && s \neq 0\,, \\
&= l(l+1) + 2a^2\omega^2\left[\frac{m^2 + l(l+1) - 1}{(2l-1)(2l+3)}\right] + O[(a\omega)^4]\,, && s = 0\,. \qquad (3.10)
\end{aligned}
$$

For the case when $a\omega$ is not small, the perturbation method generalizes to a continuation method (cf. Wasserstrom 1972), giving differential equations in the independent variable $a\omega$ which can be integrated numerically to arbitrarily large values. If we choose the spherical functions as a representation, so that

$${}_sS^m{}_l(\theta, a\omega) = \sum_{l'} {}_sA^m{}_{ll'}(a\omega)\, {}_sY^m{}_{l'}(\theta)\,, \qquad (3.11)$$

then the continuation equations take the form

$$\frac{dA_{ll'}}{d(a\omega)} = \sum_{\alpha,\beta,\gamma\neq l} \frac{A_{\gamma\alpha}A_{l\beta}}{{}_sE^m{}_l - {}_sE^m{}_\gamma} \langle\alpha, \beta\rangle A_{\gamma l'}\,, \qquad (3.12\text{a})$$

$$\frac{dE^m{}_l}{d(a\omega)} = \sum_{\alpha,\beta} A_{l\alpha}A_{l\beta}\langle\alpha, \beta\rangle\,, \qquad (3.12\text{b})$$

where

$$\langle \alpha, \beta \rangle \equiv \langle s\alpha m | d\mathfrak{H}_1/d(a\omega) | s\beta m \rangle$$
$$= \int d\Omega \; {}_sY^m{}_\alpha \; {}_sY^m{}_\beta \, (2a\omega \cos^2\theta - 2s\cos\theta)\,, \tag{3.13}$$

which is evaluated using equations (3.9a) and (3.9b). Our numerical calculations integrate equations (3.12a) and (3.12b), truncated at α, β, $\gamma = 20$ from $a\omega = 0$ to $a\omega = 3$, by fourth-order Runge-Kutta integration. A step size of $\Delta(a\omega) = 0.25$ is sufficient for five-place accuracy. The results are displayed in Appendix C as polynomial fits in the parameter $a\omega$ for the ranges $s = \pm 2$, $l \leq 6$, and $(a\omega) \lessgtr 3$.

The continuation method described here works well for "mass producing" eigenvalues and eigenfunctions. Standard shooting methods are also useful if only a few functional values are desired. Finite-difference methods should be used only with care, since the form of the functions is poorly suited to a uniform grid (see Keller 1968 for details on the standard methods).

b) Radial Equation

(Teukolsky+1973 ApJ)

Except for the simpler case $\omega = 0$, the radial equation (2.9) has regular singular points at the two roots of Δ, and an irregular singular point at infinity; thus, it is not soluble in terms of hypergeometric or other standard functions, nor do its solutions admit a known integral representation. Numerical integration of the equation appears to be the only direct line of attack.

There are actually two radial equations, corresponding to the spin-weights $s = \pm 2$. The two equations contain the same information, so we need integrate only one. Which one? The choice is largely dictated by the boundary condition on the horizon. There, the regular and irregular radial solutions are respectively

$$R \sim \exp(-ikr_*)\Delta^{-s} \sim \exp\left\{-ikr_* - s\left[\frac{2(M^2 - a^2)^{1/2}}{r_+{}^2 + a^2}\right] r_*\right\} \quad \text{(regular)}\,, \tag{3.14a}$$

$$R \sim \exp(ikr_*) \quad \text{(irregular)}\,. \tag{3.14b}$$

For $s = +2$ the regular solution is exponentially larger than the irregular solution near the horizon (large negative r_*). This is numerically very unpleasant: if we integrate the physically correct, regular solution out from the horizon, the integration will be unstable against bringing in an exponentially growing piece of the irregular solution. Likewise, if we integrate *in* to the horizon and attempt to impose boundary conditions there, we are faced with finding a zero of the exponentially small irregular solution, which is lost in numerical truncation.

The case $s = -2$ is pleasantly the reverse: integrating outward, any contamination of the irregular solution vanishes exponentially; or, integrating inward, the irregular solution is large, and can be zeroed definitively. On this basis, we choose to integrate the case $s = -2$ exclusively; we always integrate outward from the horizon through many e-folds of equation (3.14a); thus, we need not start the solution on the horizon very carefully—we have adopted the conservative procedure of always starting two "random" linearly independent solutions and verifying that they yield identical final results.

There is a price exacted at infinity for this convenience at the horizon, however. For large r, the solution is as given in equation (2.13). To find instabilities, we seek zeros of Z_{in} and for $s = -2$, Z_{in} is the coefficient of a function which decreases faster by four powers of r than does the dominant term multiplying Z_{out}. (As before, integrating *in* from infinity is no better—the solution is then numerically unstable against contamination with the other solution.) Fortunately, the problem here is only a power law, not an exponential. We know of two ways to circumvent it:

First, it is not too difficult to integrate with sufficiently high precision to maintain significance in the small solution. This "brute force" method is not without pitfalls: If one resolves Z_{in} and Z_{out} on the basis of function and derivative at one large radius $r = R_0$, then the outgoing (large, Z_{out}) solution at R_0 must be known analytically to much higher accuracy than just its leading term in equation (2.9)—it must be known to the accuracy of the small solution associated with Z_{in}. This is not easy: a WKB solution is not sufficiently accurate, for example; and even adding further terms to an asymptotic series does not always work, since the asymptotic series is not convergent at fixed R_0. A satisfactory way to resolve the Z_{in} and Z_{out} components is to fit (e.g., least-squares fit) a numerical solution to the functional form of equation (2.13) over a range in r_* of at least π/ω, so that all relative phases of the oscillatory parts contribute.

A second method is the one which we have actually used for most of the results here presented. The method is based on a differential transformation which takes a new independent variable

$$\chi \equiv \mathfrak{P}(r)\frac{dR}{dr} + \mathfrak{Q}(r)R\,. \tag{3.15}$$

The new variable χ satisfies a second-order, linear ordinary differential equation of its own, which is the equation actually integrated. The functions $\mathfrak{P}$ and $\mathfrak{Q}$ are chosen to make the two independent solutions at $r \to \infty$ have the same asymptotic behavior. This method, including a scheme for choosing $\mathfrak{P}$ and $\mathfrak{Q}$, is described in detail in Appendix D; the method does not appear to be in the literature, and it may have some utility in other unrelated applications where the two solutions of an ordinary differential equation differ by a power law.

The use of this second method also makes the problem of resolving Z_{in} and Z_{out} much easier: since the two solutions are of comparable size, accurate results are obtained by using WKB solutions for large r, and comparing to function and derivative of the numerically integrated variable at a single large R_0. Further description is in Appendix D.

In the limit of zero frequency, $\omega \to 0$, the radial equation is soluble analytically in terms of hypergeometric functions. Matching the zero-frequency solution to asymptotic small-ω solutions at infinity gives a limiting expression for $Z_{\text{out}}/Z_{\text{in}}$,

$$Z_{\text{out}}/Z_{\text{in}} = \frac{16\omega^4}{(l-1)l(l+1)(l+2)}[1 + O(a\omega)]\,, \qquad a\omega \ll 1\,. \tag{3.16}$$

(The pole at $\omega = 0$ does not correspond to an instability, but rather to the sole zero in the factor relating Z_{in} to physical gravitational wave flux at infinity [see Appendix B].)

c) *Numerical Results*

It is convenient to plot our results in terms of a real function with the same physical poles as $Z_{\text{out}}/Z_{\text{in}}$,

$$Z(\omega, a, l, m) \equiv \left|\frac{(l-1)l(l+1)(l+2)}{16\omega^4}\frac{Z_{\text{out}}}{Z_{\text{in}}}\right|\,. \tag{3.17}$$

By equation (3.16), $Z \to 1$ as $\omega \to 0$. Complex conjugating the radial equation (2.9), and reversing the sign of m and ω, gives the symmetry

$$Z(\omega, a, l, -m) = Z(-\omega, a, l, m)\,, \tag{3.18}$$

so we can restrict ourselves to nonnegative m without loss of generality. When $m = 0$, we can further take $\omega \geq 0$.

Figures 1–3 plot Z as a function of ω and a for all modes with $l = 2$ and $l = 3$ and for a sample mode of $l = 4$. (Recall that $l = 0, 1$ modes do not exist, since $l \geq |s| = 2$.) Calculations for several higher modes have also been made, with similar results. The following features of the numerical results are worth noting: (i) for low ω, Z is a very nearly linear function of ω, with a value of order unity; as $|\omega|$ becomes larger, the potential barrier between the horizon and infinity is surmounted, and the ingoing wave on the horizon matches to a nearly pure ingoing wave at infinity—Z drops to zero exponentially. This falloff is derived analytically in Appendix A. (ii) As one might expect intuitively, the effect of varying a is greatest for modes with $l = m$, $\omega > 0$ (roughly, wave packets in direct, equatorial orbits), and small for $m = 0$ and for $m = l$ with $\omega < 0$. (iii) Z is everywhere a very smooth function of a; not only is there no instability apparent, but it is difficult to find *any* particular value of a with distinguished behavior. (iv) There are no obvious qualitative differences between the $l = 2$ modes, the $l = 3$ modes, and higher modes; i.e., there is no particularly distinguished l. (v) For $a \sim M$ and $\omega > 0$, the transition to an exponentially falling function

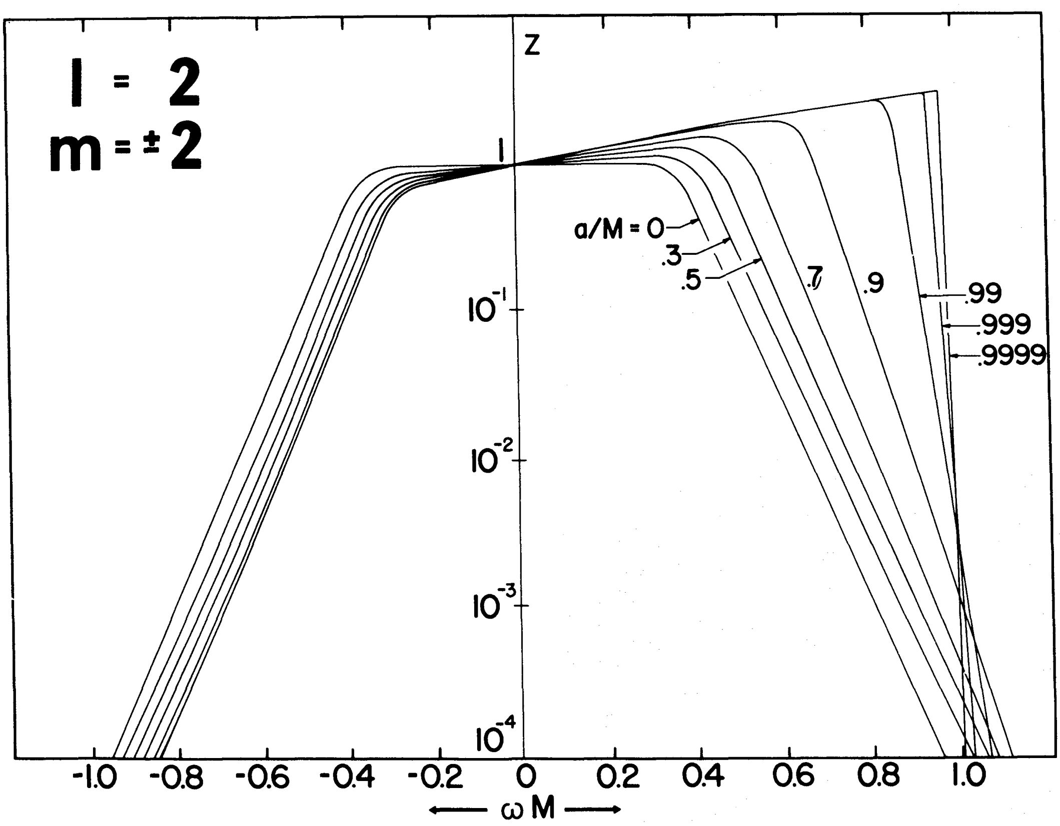

FIG. 1*a*

FIGS. 1*a*, *b*, *c*.—Asymptotic behavior at radial infinity of a real-frequency perturbation which is regular on the event horizon. Except for normalizing factors (see text for details), Z is the ratio of outgoing-wave to ingoing-wave solution far from the black hole. An instability is a perturbation which can support an outgoing wave with *no* ingoing wave required; hence an onset of instability (as a/M increases from zero, the stable Schwarzschild hole) would correspond to a pole in Z. These figures show that there are no such poles for any $l = 2$ mode. The figures are drawn for m positive. The curves for corresponding negative m are obtained by reflection through $\omega M = 0$. There are no $l = 0$ or $l = 1$ gravitational modes in the formalism used in this paper.

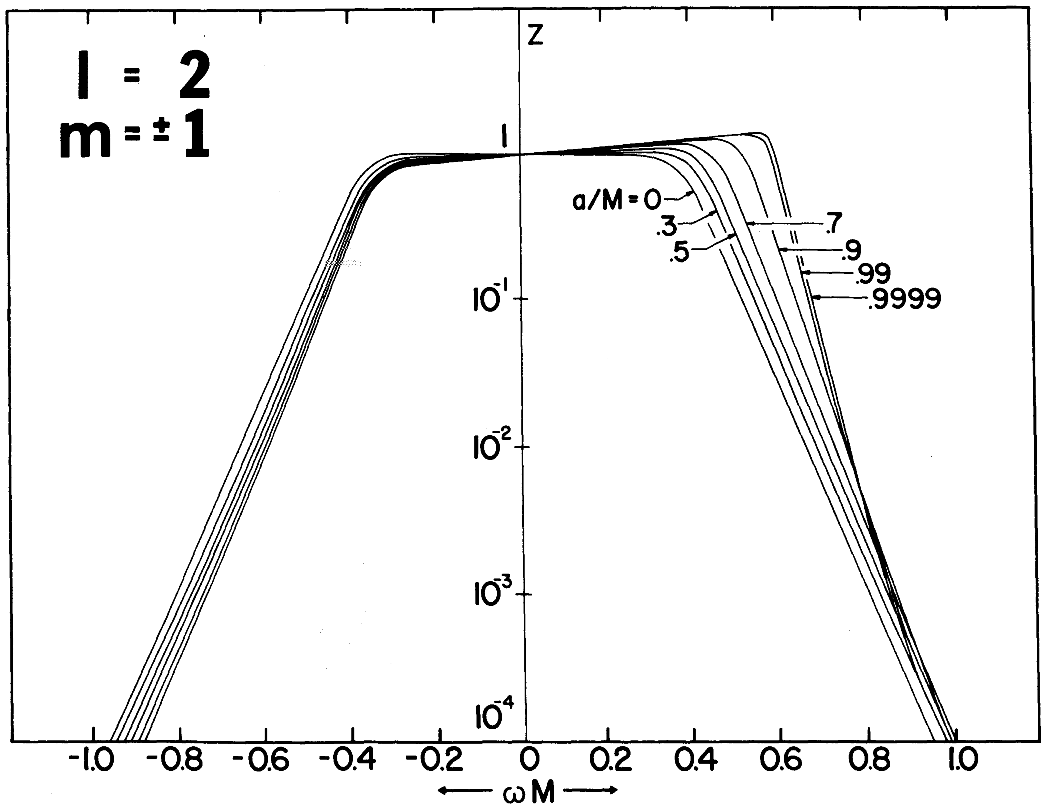
l = 2
m = ±1
Z
1
a/M = 0
.3
.5
.7
.9
.99
.9999
10^-1
10^-2
10^-3
10^-4
-1.0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
ωM

FIG. 1*b*

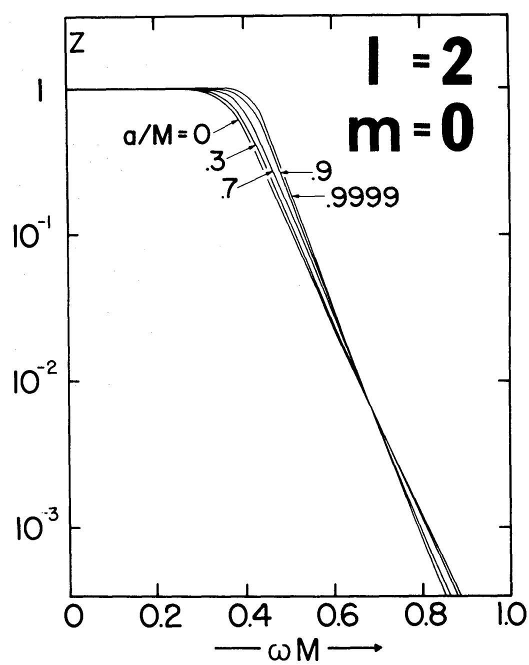
Z
l = 2
m = 0
1
a/M = 0
.3
.7
.9
.9999
10^-1
10^-2
10^-3
0
0.2
0.4
0.6
0.8
1.0
ωM

FIG. 1*c*

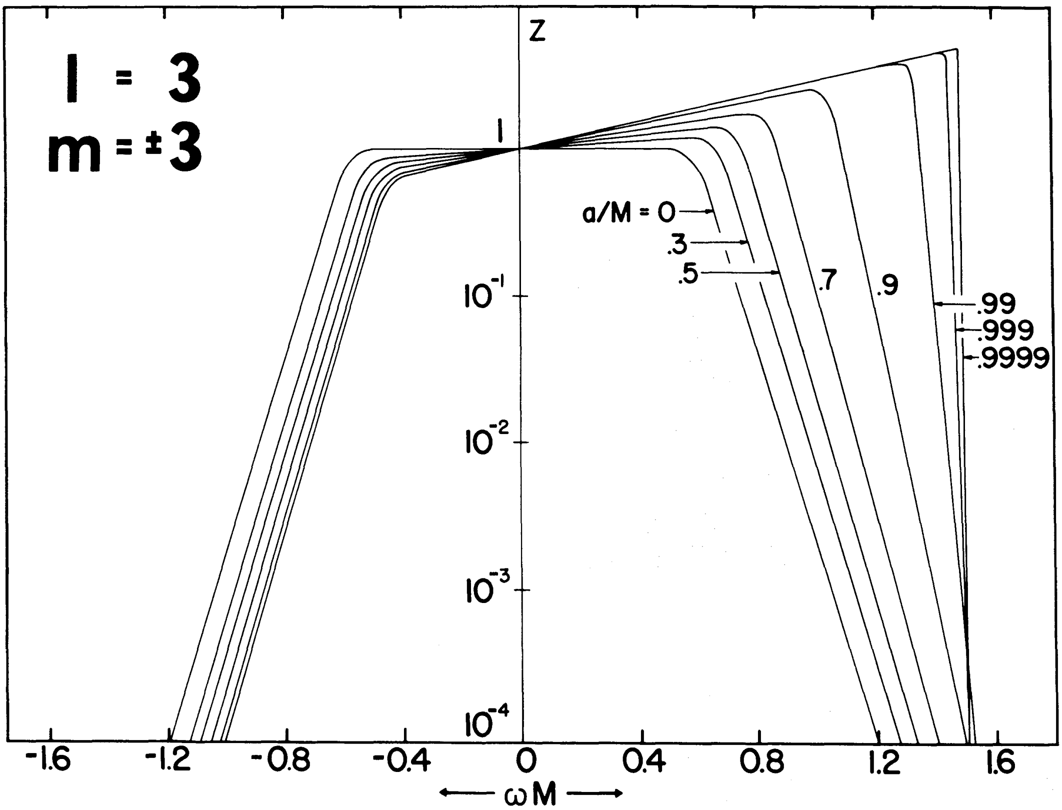

FIG. 2*a*

FIGS. 2*a*, *b*, *c*, *d*.—Asymptotic behavior at radial infinity for $l = 3$ modes. As for the $l = 2$ modes, the function Z has no poles corresponding to an onset of instability through a real frequency mode. It is almost certain that any instability must set in via a real frequency (see text); furthermore, the smoothness of the Z function in a/M argues directly that there is no sudden appearance of a pole in the upper-half complex plane. In this figure and fig. 1, note that there is not even a tendency toward marginal instability in the "unphysical" limit $a/M = 1$.

occurs more or less at m times the rotational frequency of the horizon $\omega \approx a/(2Mr_+)$. This phenomenon is related to the superradiance effect and will be discussed in a later paper in this series. (vi) Z is everywhere finite in the limiting case $a \rightarrow M$, but a sharp corner appears in the double limit $a \rightarrow M$, $\omega \rightarrow \frac{1}{2}m/M$; this behavior has previously been noted for the scalar wave equation in the Kerr background (Press and Teukolsky 1972; Starobinsky 1972).

IV. DISCUSSION AND CONCLUSIONS

In the data of figures 1 and 2, there is no onset of instability via a real frequency mode for any angular mode with $l \leq 3$, at any value of a, $0 \leq a/M \leq 0.9999$. The same conclusion holds for a number of higher angular modes that we have spot-checked, e.g., $l = 4$, $m = \pm 4$ in figure 3. We cannot, of course, rule out the possibility of an instability in some mode of high l that we have not examined. However, the lack of any qualitative differences between low discrete values of l, and the fact that even these moderately low values seem to tend to a smooth asymptotic behavior as l increases, seem to make the possibility unlikely.

As we have noted in § II, it is almost certain that an instability must set in through a real frequency mode; a rigorous proof would slightly extend known analyticity

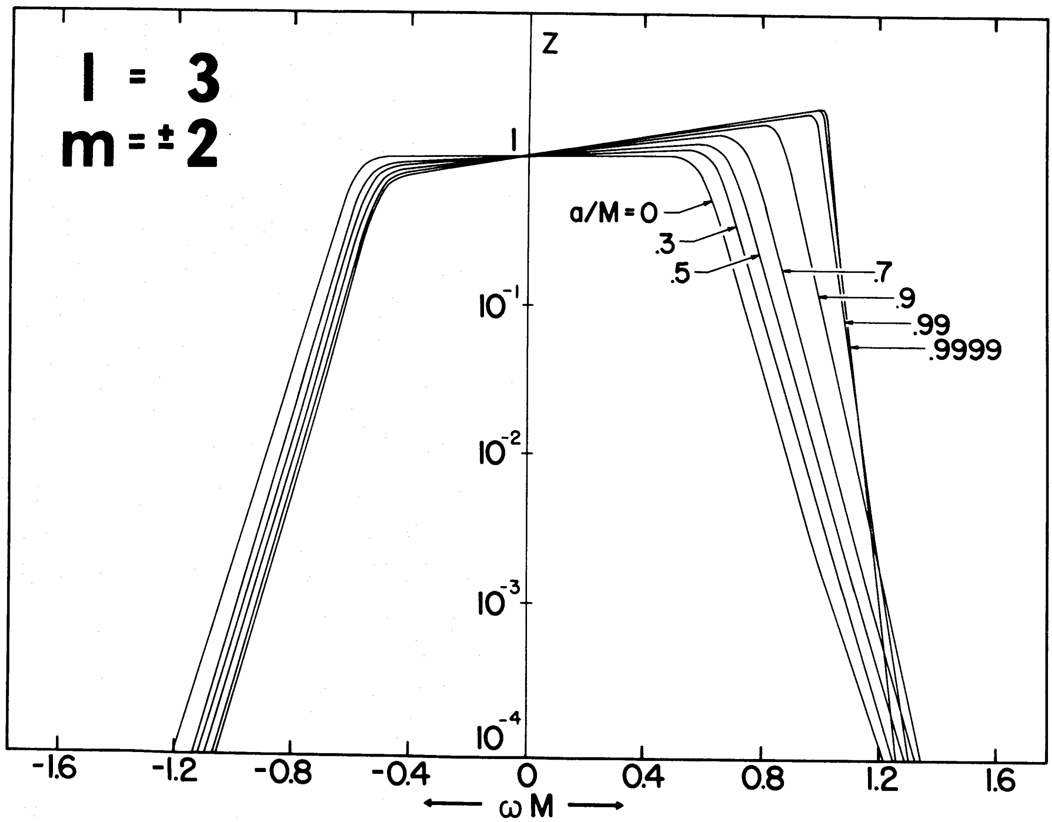
l = 3
m = ±2
Z
1
a/M = 0
.3
.5
.7
.9
.99
.9999
10^{-1}
10^{-2}
10^{-3}
10^{-4}
-1.6
-1.2
-0.8
-0.4
0
0.4
0.8
1.2
1.6
ωM

FIG. 2*b*

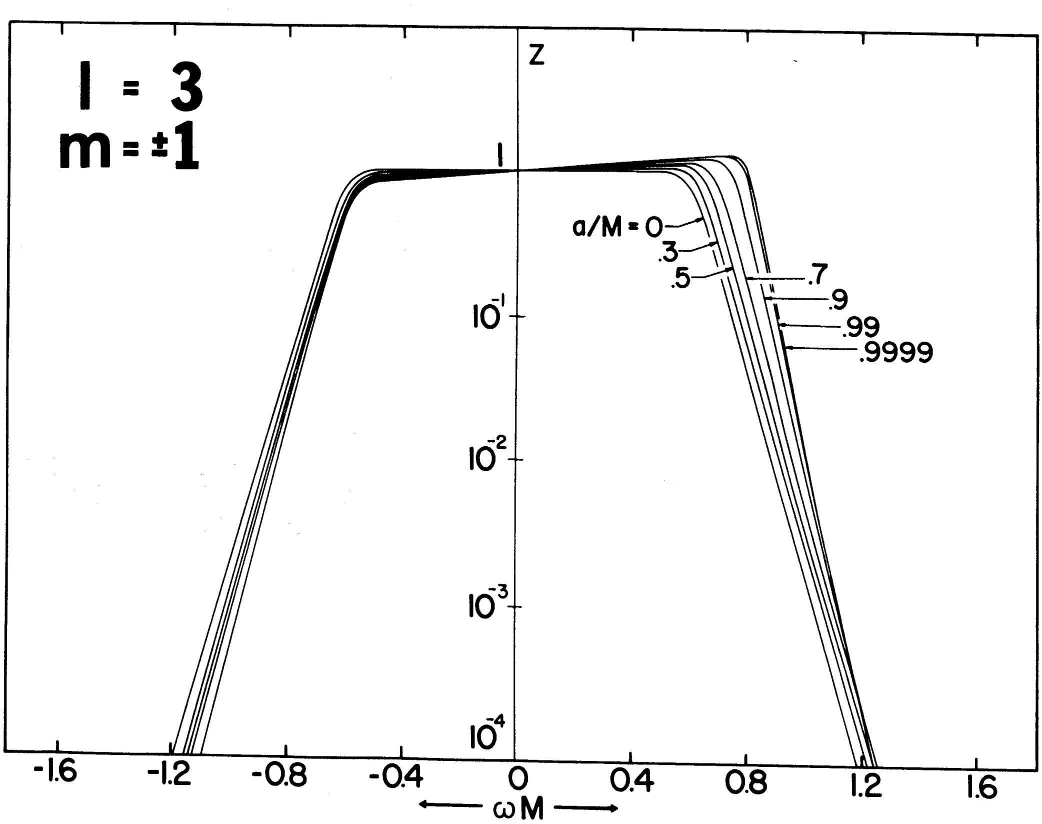
l = 3
m = ±1
Z
1
a/M = 0
.3
.5
.7
.9
.99
.9999
10^{-1}
10^{-2}
10^{-3}
10^{-4}
-1.6
-1.2
-0.8
-0.4
0
0.4
0.8
1.2
1.6
ωM

FIG. 2*c*

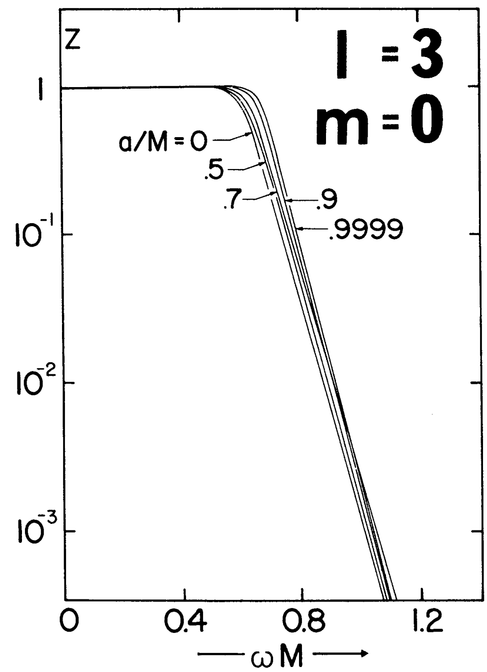

FIG. 2*d*

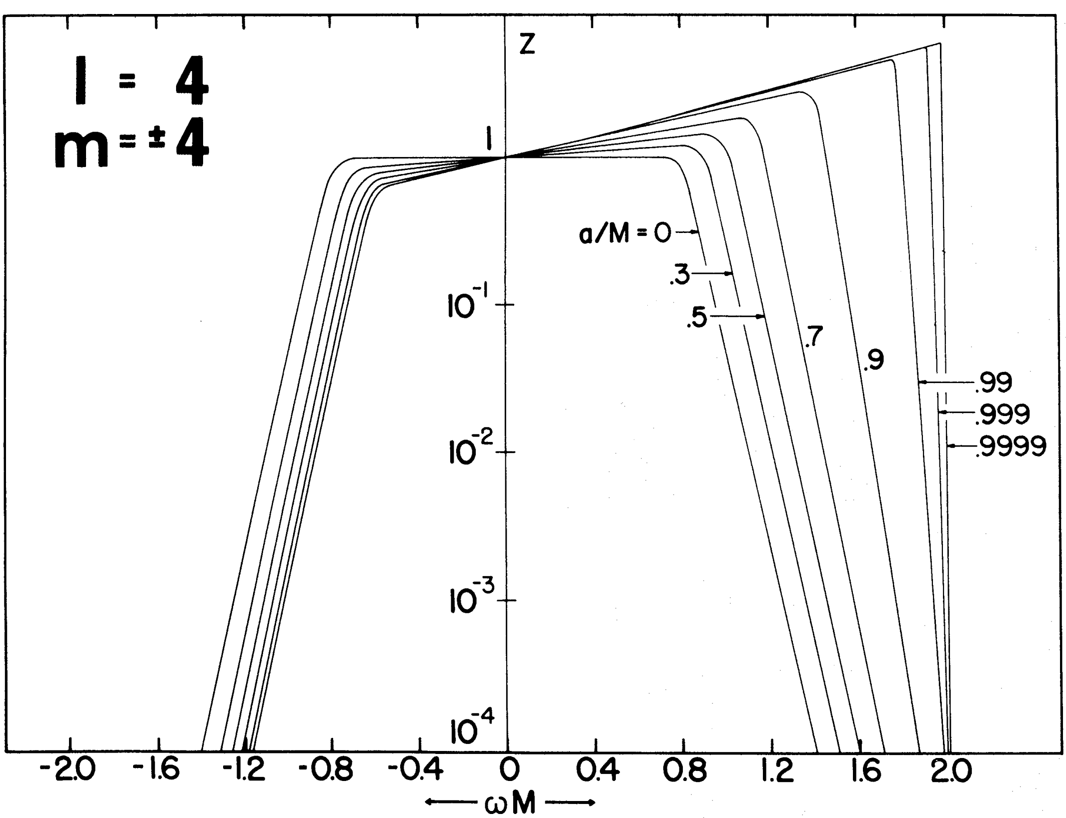

FIG. 3.—A sample mode with l = 4, which is also stable. Several higher modes have also been spot-checked, and no tendencies toward instability have been found.

theorems so as to include our radial equation (2.9). In the present case, moreover, the smoothness in a of the families of curves shown in figures 1–3 argues directly for this analyticity, and against the sudden appearance of a pole in the upper half-plane for some finite value of a: in general such a pole would be expected to change the functional form of $|Z_{\text{out}}/Z_{\text{in}}|$ on the real-ω axis, as well as introducing a new phase shift of 2π in arg $(Z_{\text{out}}/Z_{\text{in}})$.

Comparing the behavior of the data for $a/M = 0.99$, 0.999, and 0.9999, one sees no tendency toward instability even in the limit $a/M \rightarrow 1$, a value which Hawking (1973) has suggested as a possible candidate for a marginal instability. However, even this extrapolation is not necessary for astrophysical applications, since the natural upper limit to the spin-up of astrophysical black holes, $a/M \approx 0.9982$ (Thorne 1973), lies within the range of our calculations.

Our conclusion, that the Kerr metric is apparently dynamically stable, adds credibility to present and future detailed astrophysical models which include rotating black holes, e.g., models for variable X-ray sources in binary systems (Pringle and Rees 1973; Shakura and Sunyayev 1973; Novikov and Thorne 1973), and for infrared and radio emission at the galactic center (Lynden-Bell and Rees 1971); the conclusion also adds a bit of evidence to the conjecture (Carter 1973) that Kerr holes are the unique final states for collapsed isolated bodies; finally, the apparent stability also justifies the further use of the separable perturbation equations on the Kerr background for investigations of the dynamical behavior of rotating black holes in astrophysical contexts, e.g., tidal interactions and gravitational and electromagnetic wave processes, which subsequent papers in this series will investigate.

We have benefited greatly from discussions with J. M. Bardeen, B. Carter, S. W. Hawking, H. B. Keller, L. Smarr, and K. S. Thorne. We thank G. Fox and B. Zimmerman for assistance with some of the numerical work. One of us (W. H. P.) acknowledges the hospitality of the Institute of Theoretical Astronomy, Cambridge, England, where some early stages of this calculation were formulated, and thanks the Fannie and John Hertz Foundation for support.

APPENDIX A

ASYMPTOTIC FORMS OF Z_{out} AND Z_{in} FOR $\omega \rightarrow \infty$

Let $\psi = \Delta^{s/2}(r^2 + a^2)^{1/2}R$. Then equation (2.9) becomes

$$\psi_{,r*r*} + V\psi = 0\,, \tag{A1}$$

where

$$V = \frac{K^2 - 2is(r - M)K + \Delta(4ir\omega s - \lambda)}{(r^2 + a^2)^2} - G^2 - G_{,r*}\,,$$

$$G = \frac{s(r - M)}{r^2 + a^2} + \frac{r\Delta}{(r^2 + a^2)^2}\,. \tag{A2}$$

The conditions (2.12) and (2.13) become

$$\begin{aligned} &\psi \rightarrow \Delta^{-s/2}e^{-ikr*}\,, && r \rightarrow r_+\,, \\ &\psi \rightarrow Z_{\text{in}}r^s e^{-i\omega r*} + Z_{\text{out}}r^{-s}e^{-i\omega r*}\,, && r \rightarrow \infty\,. \end{aligned} \tag{A3}$$

Let $\psi_-{}^L$ be a solution of equation (A1) satisfying $\psi_-{}^L \rightarrow \Delta^{-s/2}e^{-ikr*}$ "on the left," at $r \to r_+$. Let $\psi_\pm{}^R$ be solutions satisfying $\psi_\pm{}^R \to r^{\mp s} e^{\pm i\omega r*}$ "on the right," at $r \to \infty$. Since equation (A1) has only two linearly independent solutions, we have

$$\psi_-{}^L = Z_{\text{in}}\psi_-{}^R + Z_{\text{out}}\psi_+{}^R \,. \tag{A4}$$

Take the Wronskian of equation (A4) with $\psi_+{}^R$. Then

$$\begin{aligned} Z_{\text{in}} &= W(\psi_-{}^L, \psi_+{}^R)/W(\psi_-{}^R, \psi_+{}^R) \\ &= W(\psi_-{}^L, \psi_+{}^R)/2i\omega \,. \end{aligned} \tag{A5}$$

Similarly

$$Z_{\text{out}} = -W(\psi_-{}^L, \psi_-{}^R)/2i\omega \,. \tag{A6}$$

Now suppose $\phi_\pm$ satisfies

$$\phi_{\pm, r*r*} + (V + U_\pm)\phi_\pm = 0 \,, \tag{A7}$$

$$\begin{aligned} &\phi_\pm \to \Delta^{\pm s/2} e^{\pm ikr*} \,, \qquad r \to r_+ \,, \\ &\phi_\pm \to r^{\mp s} e^{\pm i\omega r*} \,, \qquad r \to \infty \,. \end{aligned} \tag{A8}$$

(We shall later choose ϕ to be "close" to ψ for $\omega \to \infty$, i.e., $|U| \ll |V|$ for $\omega \to \infty$.) Multiply equation (A1) by $\phi_\pm$, equation (A7) by $\psi_-{}^L$, and subtract. Integrate from $r_* = -\infty$ to $r_* = \infty$ and use $\phi_\pm \to \psi_\pm{}^R$ at $r_* = \infty$ and $\psi_-{}^L \to \phi_-$ at $r_* = -\infty$. Thus find

$$Z_{\text{in}} = \frac{1}{2i\omega}\left[W(\phi_-, \phi_+)_{-\infty} - \int_{-\infty}^{\infty} U_+\phi_+\psi_-{}^L dr_*\right] , \tag{A9}$$

$$Z_{\text{out}} = \frac{1}{2i\omega}\int_{-\infty}^{\infty} U_-\phi_-\psi_-{}^L dr_* \,. \tag{A10}$$

For $\omega \to \infty$, a good approximation for $\psi_\pm$ is the WKB solution $\psi_\pm \sim \exp(\pm i \int V^{1/2} dr_*)$. It can be shown by methods analogous to those of Erdélyi *et al.* (1955) that for $\omega \to \infty$,

$$\lambda \sim \begin{cases} \alpha a\omega + O(1) \,, & \text{Re}\,\omega \neq 0 \,, \\ a^2\omega^2 + \beta a\omega + O(1) \,, & \text{Re}\,\omega = 0 \,, \end{cases} \tag{A11}$$

where α and β depend on l, m, and s, and α is real. Let us consider first Re $\omega \neq 0$. Then

$$V^{1/2} \sim H + O(\omega^{-1}) \,, \tag{A12}$$

where

$$H = \omega - \frac{am}{r^2 + a^2} - \frac{is(r - M)}{r^2 + a^2} + \frac{\Delta}{(r^2 + a^2)^2}\left(2irs - \frac{\alpha a}{2}\right) . \tag{A13}$$

Choose

$$\begin{aligned} \phi_\pm &= \exp(\pm i \textstyle\int H dr_*) \\ &= \Delta^{\pm s/2}(r^2 + a^2)^{\mp s} \exp\left[\pm i \displaystyle\int \left(\omega - \frac{am}{r^2 + a^2}\right) dr_*\right] \exp\left[\pm \frac{i\alpha}{4} \tan^{-1}\left(\frac{2a}{r}\right)\right] . \end{aligned} \tag{A14}$$

Then

$$U_\pm = H^2 \mp iH_{,r*} - V \sim O(1) \quad \text{as} \quad \omega \to \infty \,. \tag{A15}$$

Substituting ϕ_- for $\psi_-{}^L$ in equation (A9), we find

$$\begin{aligned} Z_{\text{in}} &\sim \frac{1}{2i\omega}\left[2iH(r_+) - \int_{-\infty}^{\infty} U_+\phi_+\phi_- dr_*\right] \\ &= \frac{1}{2i\omega}\left[2i\left(\omega - \frac{am}{r_+{}^2 + a^2} - \frac{is(r_+ - M)}{r_+{}^2 + a^2}\right) - \int_{-\infty}^{\infty} U_+ dr_*\right] \\ &\to 1 \quad \text{as} \quad \omega \to \infty \,. \end{aligned}$$

Thus the wave is almost perfectly transmitted for high frequencies. For Re $\omega = 0$, a similar argument can be given using a slightly different H satisfying the relation (A12). Again we find $Z_{\text{in}} \to 1$ as $\omega \to \infty$. This result further supports the assertion that instabilities must necessarily set in via a zero of Z_{in} crossing the real-ω axis to the upper half-plane: the possibility of a zero entering "from infinity" is excluded.

It is also interesting to show analytically that the exponential falloff in $Z_{\text{out}}/Z_{\text{in}}$, which is unmistakable in the numerical results for moderate ω, persists in the limit $\omega \to \pm\infty$ (ω real). This can be proved by replacing $\psi_-{}^L$ by ϕ_- in equation (A10):

$$Z_{\text{out}} \sim \frac{1}{2i\omega}\int_{-\infty}^{\infty} U_-\phi_-{}^2 dr_* = \frac{1}{2i\omega}\int_{r_+}^{\infty} U_-\phi_-{}^2 \frac{dr_*}{dr}\, dr \,. \tag{A16}$$

The factor $e^{-2i\omega r*}$ in $\phi_-{}^2$ allows a treatment of this expression by the saddle-point method. The saddle points occur at $dr_*/dr = 0$, i.e., at $r^2 + a^2 = 0$. Using the explicit form of U_-, we see that these are also poles of U_-. Moreover, as $a \to 0$ the two saddle points at $r = \pm ia$ coalesce. All these complications can be taken care of by Van der Waerden's (1951) treatment of the saddle-point method. The result of this analysis is that the dominant behavior of the integral is still determined by the value of $e^{-2i\omega r*}$ at a saddle point; the complications give rise to different power-law dependences on ω when $a = 0$ or $a \gg 0$, with a transition region described by an incomplete gamma function. For ω positive, $r = -ia$ is the dominant saddle point, while for ω negative $r = +ia$ dominates. Taking care to define the branches of the logarithms in $r_*(r)$ consistently, we find

$$Z_{\text{out}} \sim e^{-2|\omega|B} \times (\text{weak dependence on } \omega)$$

where

$$B = 2\pi + a + \frac{\tan^{-1}[(1 - a^2)^{1/2}/a]}{(1 - a^2)^{1/2}} - \frac{\pi}{2} \to \begin{cases} 2\pi & a \to 0 \\ 2\pi + \left(2 - \dfrac{\pi}{2}\right), & a \to 1 \,. \end{cases}$$

APPENDIX B

RELATION BETWEEN ψ_0 AND ψ_4 FOR INGOING WAVES

We wish to find the relation between $\psi_4{}^B$ and $\psi_0{}^B$ at infinity by using the asymptotic form of the NP equations. The notation in this Appendix is explained in Paper I, and we shall use the null tetrad equation (4.4) of that paper.

Since $\psi_4{}^B$ and $\psi_0{}^B$ are invariant under gauge transformations and infinitesimal tetrad rotations, we may make use of any convenient choice of gauge and tetrad. By infinitesimal transformations of types (i), (ii), and (iii), respectively (cf. Appendix A, Paper I), set $\lambda^B = \kappa^B = \epsilon^B = 0$. Set $D^B = 0$ by a gauge transformation. (Each of these transformations does not disturb the result of the previous ones.) For this choice of tetrad the necessary NP equations are:

NP 4.2a: $$D\rho^B = 2\rho\rho^B\,, \tag{B1}$$

NP 4.2b: $$D\sigma^B = (\rho + \rho^*)\sigma^B + \psi_0{}^B\,, \tag{B2}$$

NP 4.2d: $$D\alpha^B = \rho^B\alpha + \rho\alpha^B + \beta\sigma^{*B} + \rho\pi^B + \rho^B\pi\,, \tag{B3}$$

NP 4.2e: $$D\beta^B = (\alpha + \pi)\sigma^B + \rho^*\beta^B + \rho^{*B}\beta^B + \psi_1{}^B\,, \tag{B4}$$

NP 4.2g: $$-\delta^*\pi^B - \delta^{*B}\pi = \sigma^{*B}\mu + 2\pi\pi^B + (\alpha - \beta^*)^B\pi + (\alpha - \beta^*)\pi^B\,, \tag{B5}$$

NP 4.4: $$\delta^{*B}D - D\delta^{*B} = (\alpha + \beta^* - \pi)^B D - \sigma^{*B}\delta - \rho\delta^{*B} - \rho^B\delta^*\,, \tag{B6}$$

NP 4.5: $$(D - 4\rho)\psi_1{}^B = (\delta^* + \pi - 4\alpha)\psi_0{}^B\,, \tag{B7}$$

NP 4.5: $$(D - 3\rho)\psi_2{}^B = (\delta^* + 2\pi - 2\alpha)\psi_1{}^B + 3\rho^B\psi_2\,, \tag{B8}$$

NP 4.5: $$(D - 2\rho)\psi_3{}^B = (\delta^* + 3\pi)\psi_2{}^B + (\delta^* + 3\pi)^B\psi_2\,, \tag{B9}$$

NP 4.5: $$(D - \rho)\psi_4{}^B = (\delta^* + 4\pi + 2\alpha)\psi_3{}^B\,. \tag{B10}$$

For ingoing waves at infinity, $\psi_0{}^B \sim S_0(\theta)e^{-i\omega(t+r^*)}e^{im\varphi}/r$. Let all the perturbation quantities in equations (B1)–(B10) be asymptotically

$$(\text{Quantity})^B \sim \rho^n S(\theta)e^{-i\omega(t+r^*)}e^{im\varphi}/r\,,$$

where n is an integer to be determined along with the functions $S(\theta)$. Since the equations contain complex conjugate quantities, when we equate coefficients of $e^{-i\omega(t+r^*)}e^{im\varphi}$ we will get equations involving $S(\theta, \omega, m)$ and $S^*(\theta, -\omega, -m) \equiv S^\dagger(\theta, \omega, m)$. Because $\rho \sim 1/r$, we have

$$D(\text{Quantity})^B \sim -2i\omega(\text{Quantity})^B\,.$$

Thus, from equations (B1), (B2), (B7), (B4), and (B8) respectively, we find

$$\rho^B \sim 0\,, \qquad \sigma^B \sim \rho^0\,, \qquad \psi_1{}^B \sim \rho\,, \qquad \beta^B \sim \rho\,, \qquad \psi_2{}^B \sim \rho^2\,.$$

Let equation (B6) operate on ρ, and use the fact that, for this tetrad, $D\rho = \rho^2$, $\delta\rho = \rho\tau$, and $\delta^*\rho = -\rho\pi$. The result is

$$(-D + 3\rho)\delta^{*B}\rho = (\alpha + \beta^* - \pi)^B\rho^2 - \sigma^{*B}\rho\tau + \rho^B\rho\pi\,. \tag{B11}$$

Similarly, let equation (B6) operate on π, and use $D\pi = 2\rho\pi$, $\delta\pi = 2\pi(\tau + \beta)$, and $\delta^*\pi = -2\pi(\pi + \beta^*)$. Then find

$$2\pi\delta^{*B}\rho + (-D + 3\rho)\delta^{*B}\pi = (\alpha + \beta^* - \pi)^B 2\rho\pi - 2\sigma^{*B}(\tau\pi + \pi\beta) + 2\rho^B(\pi^2 + \pi\beta^*)\,. \tag{B12}$$

Equations (B3), (B5), (B11), and (B12) give

$$\alpha^B \sim \rho\,, \qquad \pi^B \sim \rho^0\,, \qquad \delta^{*B}\rho \sim \rho^2\,, \qquad \delta^{*B}\pi \sim \rho^3\,.$$

Since $\psi_2 = M\rho^3$, $\delta^{*B}\psi_2 \sim \rho^4$ and so this latter term can be neglected in equation (B9) in comparison with $\pi^B\psi_2$. Equations (B2), (B5), (B7)–(B10) therefore determine the angular function S_0 in terms of S_4:

$$(\partial_\theta + m \operatorname{cosec} \theta - a\omega \sin \theta + 2 \cot \theta)S_0 = 2^{3/2} i\omega S_1 , \tag{B13}$$

$$(\partial_\theta + m \operatorname{cosec} \theta - a\omega \sin \theta + \cot \theta)S_1 = 2^{3/2} i\omega S_2 , \tag{B14}$$

$$(\partial_\theta + m \operatorname{cosec} \theta - a\omega \sin \theta)S_2 = 2^{3/2} i\omega S_3 + 2^{1/2} 3 M S_\pi , \tag{B15}$$

$$(\partial_\theta + m \operatorname{cosec} \theta - a\omega \sin \theta - \cot \theta)S_3 = 2^{3/2} i\omega S_4 , \tag{B16}$$

$$(\partial_\theta + m \operatorname{cosec} \theta - a\omega \sin \theta - \cot \theta)S_\pi = -S_0^\dagger/(2^{3/2} i\omega) . \tag{B17}$$

The equations for the $S^\dagger$ can be found by complex conjugating equations (B13)–(B17) and letting $\omega \to -\omega$, $m \to -m$.

Given an angular dependence of ψ_4, some particular $S_4(\theta)$, these 10 equations (eqs. [B13]–[B17] and the corresponding equations for $S^\dagger$) explicitly determine a nonsingular angular function for ψ_0, namely $S_0(\theta)$, as follows: It is not difficult to check that there are five regular solutions to the homogeneous [i.e., $S_4(\theta) = 0$] set at $\theta = 0$, and five more at $\theta = \pi$. Finding the linear combination of these homogeneous solutions which regularizes the inhomogeneous set at both $\theta = 0$ and π is equivalent to a well-posed set of 10 linear equations in 10 unknowns, and always has a unique solution.

When $a = 0$, these equations can be solved explicitly with the appropriate spin-weighted spherical harmonics for each S, since the differential operators reduce to "edth" operators (Goldberg *et al.* 1967). If $S_4(\theta)e^{im\varphi} = {}_{-2}Y_{lm}(\theta, \varphi)$, then we find $S_0(\theta)e^{im\varphi} = A\,{}_2Y_{lm}(\theta, \varphi)$, where

$$A = \frac{64\omega^4}{l(l+1)(l-1)(l+2) + 12iM\omega} . \tag{B18}$$

Note that, if $\psi_4{}^B \sim \rho^4 e^{im\varphi} e^{-i\omega t} S_4(\theta)(Z_{\text{out}} r^3 e^{i\omega r*} + Z_{\text{in}} e^{-i\omega r*}/r)$ and

$$\psi_0{}^B \sim e^{im\varphi} e^{-i\omega t} S_0(\theta) Z_{\text{in}} e^{-i\omega r*}/r ,$$

then from equations (5.12) and (5.13) of Paper I, the ratio of outgoing to ingoing energy flux at infinity is

$$\frac{dE^{(\text{out})}/dt}{dE^{(\text{in})}/dt} = \frac{16|Z_{\text{out}}|^2 \int |S_4|^2 d(\cos\theta)}{|Z_{\text{in}}|^2 \int |S_0|^2 d(\cos\theta)} . \tag{B19}$$

APPENDIX C

ANGULAR EIGENVALUES FOR $s = \pm 2$

Table 1 gives the coefficients for polynomial approximations of the angular eigenvalues ${}_{\pm 2}E^m{}_l(a\omega)$ for $l \leq 6$ and all values of m. The polynomial approximation is accurate to about five decimal places for $0 \leq a\omega \leq 3$; for $a\omega > 3$ (but not $a\omega \gg 3$) the coefficients can be used as an extrapolating polynomial. Negative values of $a\omega$ can be converted to positive values by the symmetry relation (3.6).

Note that the polynomial approximations given here are not the same as power-series expansions around $a\omega = 0$. For example, there is not perfect symmetry between corresponding coefficients for m and $-m$; this is because the best interpolating polynomial for the range $0 \leq a\omega \leq 3$ is not identical to the best extrapolating polynomial for $-3 \leq a\omega \leq 0$.

All values in table 1 were computed by the continuation method described in § III*a*.

TABLE 1

POLYNOMIAL APPROXIMATIONS FOR THE ANGULAR EIGENVALUES ${}_{\pm 2}E^{m}{}_{\ell}(a\omega)$ WITH $\ell \leq 6$ (SEE TEXT FOR DETAILS)

ℓ=2 E= 6. +						
m= 2	-2.66704	-.733301	-.063112	.0071519	.0000556	-.00006648
m= 1	-1.33324	-.577448	-.044922	-.0109315	.0038306	-.00034314
m= 0	0.00088	-.528430	.008472	-.0209644	.0025263	-.00000657
m=-1	1.33250	-.572316	.038860	-.0012912	-.0045311	.00056402
m=-2	2.66684	-.736281	.059979	.0027459	-.0022491	.00012955
ℓ=3 E= 12. +						
m= 3	-1.99983	-.501002	-.047865	-.0055100	.0017950	-.00012886
m= 2	-1.33281	-.410252	.007111	-.0134938	.0005287	.00012146
m= 1	-0.66698	-.349910	.006907	.0060328	-.0044674	.00046121
m= 0	-0.00078	-.329262	-.007370	.0126218	-.0020442	-.00005615
m=-1	0.66749	-.356215	-.003271	-.0054137	.0042397	-.00051531
m=-2	1.33319	-.406766	-.002443	-.0087279	.0020020	-.00019510
m=-3	2.00012	-.500668	.051347	-.0046268	-.0000742	.00000116
ℓ=4 E= 20. +						
m= 4	-1.59991	-.359008	-.033376	-.0040907	.0002589	.00003745
m= 3	-1.20023	-.358292	.001576	.0013031	-.0019482	.00018990
m= 2	-0.80017	-.359597	.015870	.0013607	-.0004809	-.00006020
m= 1	-0.39976	-.362199	.015394	-.0003395	.0008162	-.00013044
m= 0	-0.00009	-.360510	-.001067	.0032991	-.0004615	.00006777
m=-1	0.40003	-.361041	-.012565	.0014466	.0002530	-.00005840
m=-2	0.79994	-.360094	-.017876	.0010567	-.0000151	.00007306
m=-3	1.19992	-.359176	-.005328	-.0005678	.0001105	-.00003007
m=-4	1.60003	-.358728	.034592	-.0038102	.0001703	-.00001149
ℓ=5 E= 30. +						
m= 5	-1.33335	-.270527	-.023430	-.0022118	-.0002452	.00004840
m= 4	-1.06678	-.314780	-.000553	.0005883	-.0003756	-.00001574
m= 3	-0.79994	-.350509	.012217	.0000083	.0002382	-.00006680
m= 2	-0.53331	-.375132	.013208	.0007410	.0000150	.00000002
m= 1	-0.26668	-.389812	.007902	.0010792	-.0000942	.00001032
m= 0	0.00000	-.394892	.000033	.0008844	.0000053	-.00000827
m=-1	0.26665	-.389792	-.008281	.0011075	-.0000355	.00000915
m=-2	0.53336	-.375119	-.012738	.0007261	.0001401	-.00000736
m=-3	0.79996	-.349946	-.011916	.0010161	-.0001063	.00002469
m=-4	1.06665	-.315278	-.000740	-.0000760	-.0000756	-.00000416
m=-5	1.33334	-.270700	.023249	-.0025948	.0001484	-.00000914
ℓ=6 E= 42. +						
m= 6	-1.14288	-.212695	-.016325	-.0014776	-.0002005	.00002073
m= 5	-0.95237	-.276181	-.001386	-.0004355	.0001442	-.00004094
m= 4	-0.76188	-.328010	.006747	.0002154	.0001270	-.00002272
m= 3	-0.57143	-.368133	.009365	.0006675	-.0000201	.00000378
m= 2	-0.38095	-.396898	.008394	.0006334	-.0000398	.00000310
m= 1	-0.19048	-.414164	.004852	.0005355	-.0000276	-.00000057
m= 0	-0.00000	-.419891	-.000045	.0005375	-.0000169	.00000097
m=-1	0.19047	-.414155	-.004861	.0005496	.0000232	.00000030
m=-2	0.38095	-.396910	-.008393	.0006096	.0000424	.00000017
m=-3	0.57143	-.368142	-.009387	.0006528	.0000085	.00000278
m=-4	0.76189	-.327805	-.006629	.0005684	-.0000883	.00000988
m=-5	0.95238	-.276093	.001534	-.0002084	-.0000593	-.00000041
m=-6	1.14286	-.212837	.016084	-.0017368	.0001037	-.00000605
	$\times (a\omega)$	$\times (a\omega)^2$	$\times (a\omega)^3$	$\times (a\omega)^4$	$\times (a\omega)^5$	$\times (a\omega)^6$

APPENDIX D

A METHOD FOR INTEGRATING EQUATION (2.9) AND SIMILAR EQUATIONS

We are given a linear, second-order equation, say

$$\varphi'' = A\varphi' + B\varphi \tag{D1}$$

(prime denotes d/dr), such that two asymptotic solutions as $r \to \infty$ are of the form

$$\begin{aligned} \varphi_1 &\sim \exp(i\omega r_*)[1 + O(1/r)]\,, \\ \varphi_2 &\sim [\exp(-i\omega r_*)/r^n][1 + O(1/r)]\,, \qquad n > 0\,. \end{aligned} \tag{D2}$$

Here $r_*(r)$ can be any function such that $dr_*/dr \sim O(1)$ at $r \to \infty$. The r^0 behavior of φ_1 at $r \to \infty$ involves no loss of generality, since a change of variable is allowed.

The problem is to integrate equation (D1) from some initial conditions at $r = r_0$ out to $r \to \infty$, without loss of significance to the small solution φ_2 in the presence of some large solution φ_1. Our method is as follows:

Let $S_1(r)$ be a function of r with the two properties, (i) $S_1(r) \neq 0$ for $r \geq r_0$, and (ii) for $r \to \infty$, S_1 and φ_1 agree asymptotically to order r^{-n}, i.e.,

$$\varphi_1 - S_1 = O(1/r^{n-1})\,, \qquad r \to \infty\,. \tag{D3}$$

Now define a new variable χ by

$$\chi \equiv (\varphi/S_1)'\,. \tag{D4}$$

The two asymptotic solutions for χ at infinity are

$$\begin{aligned} \chi_1 &= (\varphi_1/S_1)' = [1 + O(1/r^{n-1})]' = O(1/r^n)\,, \\ \chi_2 &= (\varphi_2/S_1)' = \{\exp(-2i\omega r_*)r^{-n}[1 + O(1/r)]\}' \\ &= -i\omega(dr_*/dr)r^{-n}[1 + O(1/r)]\,, \end{aligned} \tag{D5}$$

which manifestly decrease at the same asymptotic rate. Thus, a differential equation for χ can be integrated without loss of significance to either of the two solutions. Such an equation is obtainable by taking two successive derivatives of equation (D4) and using (D1) at each stage. The result is

$$\chi'' = \mathfrak{A}\chi' + \mathfrak{B}\chi \tag{D6}$$

where, if we define the intermediate quantities

$$\alpha \equiv A - \frac{2S_1'}{S_1}\,, \qquad \beta \equiv B + A\frac{S_1'}{S_1} - \frac{S_1''}{S_1}\,, \tag{D7}$$

$\mathfrak{A}$ and $\mathfrak{B}$ are given by

$$\mathfrak{A} \equiv \alpha + \beta'/\beta\,, \qquad \mathfrak{B} \equiv \alpha' + \beta - \alpha\beta'/\beta\,. \tag{D8}$$

At any radius r, one can recover φ and φ' from χ and χ' by algebraic relations,

$$\varphi = \frac{S_1}{\beta}(\chi' - \alpha\chi), \qquad \varphi' = \frac{S_1'}{\beta}\chi' + \left(S_1 - \frac{\alpha}{\beta}S_1'\right)\chi. \tag{D9}$$

Formally, the reverse relations are

$$\chi = \frac{1}{S_1}\varphi' - \frac{S_1'}{S_1^2}\varphi, \qquad \chi' = \frac{\alpha}{S_1}\varphi' + \left(\frac{\beta}{S_1} - \frac{\alpha S_1'}{S_1^2}\right)\varphi; \tag{D10}$$

but this transformation is ill-conditioned for large r; this is because the information in φ is very "delicately" encoded in the asymptotically small solution φ_2 while χ is not delicate.

The solution of a differential equation for χ instead of φ can be taken either as an analytic method (in which case $\mathfrak{A}$ and $\mathfrak{B}$ are worked out analytically once and for all) or as a numerical method, where the numerical evaluation of $\mathfrak{A}$ and $\mathfrak{B}$ is built into the actual integrating program and done at each integration step. In this latter case (which is the way we have actually proceeded) special care is needed in the numerical evaluation of β, since the defining formula (D7) contains three terms whose sum is very nearly zero, for large r. This difficulty is easily handled without loss of significance, however, since β is simply an algebraic function of r, and is not subject to the additional truncation error of a differencing scheme as φ would be.

It is generally not difficult to find a function S_1 on which to base the method. For example, one can take the form

$$S_1 = \exp\left(i\omega r_* + \frac{C_1}{r} + \frac{C_2}{r^2} + \frac{C_3}{r^3} + \cdots\right) \tag{D11}$$

and solve analytically for the first few C_i's by substitution into (D1). By construction S_1 is never zero, so property (i) above is satisfied.

Finally, the χ variable is well suited for directly resolving the components of the two independent solutions at large r; its two WKB solutions for $r \to \infty$ are

$$\chi = \exp\left[-\int_r^\infty dr(\tfrac{1}{2}\mathfrak{A} \pm iV^{1/2})\right], \tag{D12}$$

where

$$V \equiv -\mathfrak{B} + \tfrac{1}{2}\mathfrak{A}' - \tfrac{1}{4}\mathfrak{A}^2. \tag{D13}$$

For definiteness, here are the formulae which specifically apply the above method to equation (2.9) for $s = -2$ (a minor variation on the method is the factor r^3 left in φ_1):

Equation (D1):

$$\begin{aligned} A &= 2(r - M)/\Delta, \\ B &= -\left[\frac{K^2 + 4i(r - M)K - (8ir\omega + \lambda)\Delta}{\Delta^2}\right], \\ \varphi &\equiv R. \end{aligned} \tag{D14}$$

Equation (D11):

$$S_1 = \exp\left(i\omega r_* + 3\ln\frac{r}{M} + \frac{C_1}{r} + \frac{C_2}{r^2}\right), \tag{D15}$$

where

$$C_1 = -(\lambda + 2am\omega)/(2i\omega),$$
$$C_2 = (6a^2\omega^2 + 4iam\omega^2 M - 6am\omega + 6i\omega M - \lambda)/(4\omega^2). \tag{D16}$$

If the fundamental solutions for φ, equation (D2), are normalized by

$$\varphi_1 = r^3 \exp(i\omega r_*)[1 + O(1/r)],$$
$$\varphi_2 = r^{-1}\exp(-i\omega r_*)[1 + O(1/r)], \tag{D17}$$

then the corresponding solutions for χ are (eq. [D5])

$$\chi_1 = -3C_3 r^{-4}[1 + O(1/r)],$$
$$\chi_2 = -2i\omega r^{-4}\exp(-2i\omega r_*)[1 + O(1/r)], \tag{D18}$$

where

$$C_3 = -\{12\omega + 4ia^2\omega^2(m^2 + \lambda + 2am\omega + 3) + 8iam\omega(1 - 4\omega^2 M^2 - 3i\omega M) - i(\lambda + 2am\omega)[\lambda + 2am\omega - 12i\omega M - 2]\}/(24\omega^3). \tag{D19}$$

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