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Radiation reaction in the Kerr gravitational field

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Abstract. The radiative Green functions for the massless scalar, electromagnetic and gravitational perturbations of the Kerr space–time are constructed using the Teukolsky formalism. The reaction force acting upon a test particle, which can emit radiation of any spin $s = 0, 1, 2$, is calculated and shown to account correctly for the energy and the angular momentum carried away by radiation to infinity and to the event horizon. The azimuthal component of the reaction force is found to remain finite for a particle at rest in the Boyer–Lindquist coordinates owing to non-zero angular momentum transfer to the rotating hole. This anomalous static force of radiation reaction emerges as the counteraction to Hawking's tidal friction.

1. Introduction

Despite the recent progress in treating the back action of gravitational radiation in the weak-field slow-motion approximation of general relativity (GR) (Misner *et al* 1973), the problem of radiation damping in GR still remains the subject of discussion (Ehlers 1980). The explicit proof of the equality between the energy carried away by gravitational radiation and the work produced by the radiation reaction force is so far known only for the weak-field limit. However, the powerful technique for solving the perturbation equations in the Kerr metric (Teukolsky 1973, Chrzanowski 1975, Chandrasekhar 1978) provides the possibility of studying this question for a strong background gravitational field also.

In the present paper the method of factorised Green functions initiated by Chrzanowski and Misner (1974) and Chrzanowski (1975) is developed further and it is shown in particular that the retarded factorised Green functions are real for any spin. Using this property we then construct the radiative Green functions in a form suitable for studying the energy and angular momentum balance between the particle and radiation field. The total conservation of these quantities is proved for all spins.

The explicit expressions for the reaction force due to scalar, electromagnetic or gravitational radiation obtained are then analysed in the slow-motion limit. It is found that the azimuthal component of the radiative force remains non-zero for a particle at rest with respect to the locally static frame when any physical radiation is absent. This force is due to the non-zero angular momentum transfer from the static external source of non-axisymmetric perturbations to the rotating black hole (Hawking and Hartle 1972). Thus the static radiation reaction in the Kerr metric plays the role of the counteraction to Hawking's tidal friction. It should be noted that earlier attempts to calculate the force acting on a static source using the retarded Green functions are not self-consistent, since in such an approach mass renormalisation is required.

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The existence of the anomalous static radiation reaction in the Kerr metric is also discussed briefly in the context of the equivalence principle.

2. The factorised Green functions and spin-weight reflection symmetry

In this section we develop further the concept of the factorised Green functions (Chrzanowski 1975) with particular attention to their symmetry properties. First we reformulate the theory in terms of Hermitian conjugate projectors (for a general discussion see Wald 1978). Then the symmetry relations under reflection of the spin weight are studied and it is proved that the retarded factorised Green functions are real for all spins.

The following notation for the Newman–Penrose (NP) quantities ${}_s\psi$ is used: ${}_0\psi=\psi$ is the scalar (real) massless field; ${}_1\psi=F_{\mu\nu}l^\mu m^\nu$ and ${}_{-1}\psi=\bar z^2F_{\mu\nu}\bar m^\mu n^\nu$ are the electromagnetic perturbations; ${}_2\psi=-C_{\mu\nu\lambda\tau}l^\mu m^\mu l^\lambda m^\tau$ and ${}_{-2}\psi=-\bar z^4C_{\mu\nu\lambda\tau}n^\mu\bar m^\nu n^\lambda\bar m^\tau$ are the tetrad projections of the perturbed Weil tensor ($z=r+\mathrm{i}a\cos\theta$, a is the parameter of rotation, $\mathbf{l}$, $\mathbf{n}$, $\mathbf{m}$ and $\bar{\mathbf{m}}$ form the Kinnersley tetrad, the Boyer–Lindquist coordinates are understood and the signature of the metric is $+---$). The units $G=c=1$ are used throughout, and a bar denotes complex conjugation.

The perturbations ${}_s\psi$ are subject to the Teukolsky equation

$$ {}_s\Box\,{}_s\psi=4\pi\sqrt{-g}\,{}_sT \tag{2.1}$$

where ${}_s\Box$ is the Teukolsky (1973) wave operator, which is real for $s=0$ (being the covariant D'Alembertian up to the factor $\sqrt{-g}$ and complex for $s\neq 0$. Let the tensor quantity ${}_s\mathbf{J}$ mean $(q/\mu)\,\mathrm{Tr}\,\mathbf{T}$ for $s=0$ ($\mathbf{T}$ is the energy–momentum tensor, q is the scalar charge and μ is the mass of a test particle), $e\mathbf{J}$ for $s=\pm1$ ($\mathbf{J}$ is the electromagnetic four-current) and $\mathbf{T}$ for $s=\pm2$. Then the source term in (2.1) can be written in the operator form

$$ {}_sT={}_s\boldsymbol{\tau}\cdot{}_s\mathbf{J} \tag{2.2}$$

with

$$\begin{aligned}
&{}_0\boldsymbol{\tau}=1\\
&{}_1\boldsymbol{\tau}=\bar z^{-2}z^{-1}(\mathscr{D}_0 z\,\mathbf{m}-\tfrac12\sqrt2\mathscr{L}_0^{+}\mathbf{l})\bar z^2\qquad {}_{-1}\boldsymbol{\tau}=\bar z^{-2}z^{-1}(\tfrac12\Delta\mathscr{D}_0\bar{\mathbf{m}}+\tfrac12\sqrt2\mathscr{L}_0 z\,\mathbf{n})\bar z^3\\
&{}_2\boldsymbol{\tau}=\bar z^{-4}z^{-1}[\tfrac12\sqrt2(\mathscr{L}_{-1}^{+}\bar z^4z^{-2}\mathscr{D}_0+\mathscr{D}_0\bar z^4z^{-2}\mathscr{L}_{-1}^{+})z^2(\mathbf{l}\otimes\mathbf{m}+\mathbf{m}\otimes\mathbf{l})\\
&\qquad-\mathscr{L}_{-1}^{+}\bar z^4\mathscr{L}_0^{+}z^{-1}\mathbf{l}\otimes\mathbf{l}-2\mathscr{D}_0\bar z^4\mathscr{D}_0 z\,\mathbf{n}\otimes\mathbf{m}]\\
&{}_{-2}\boldsymbol{\tau}=-\bar z^{-4}z^{-1}[\tfrac14\sqrt2\Delta(\mathscr{D}_{-1}^{+}\bar z^4z^{-2}\mathscr{L}_{-1}+\mathscr{L}_{-1}\bar z^4z^{-2}\mathscr{D}_{-1}^{+})\\
&\qquad\times\Sigma^2(\mathbf{n}\otimes\bar{\mathbf{m}}+\bar{\mathbf{m}}\otimes\mathbf{n})+\mathscr{L}_{-1}\bar z^4\mathscr{L}_0\bar z\Sigma\mathbf{n}\otimes\mathbf{n}\\
&\qquad+\tfrac12\Delta^2\mathscr{D}_{-1}^{+}\bar z^4\mathscr{D}_0^{+}\bar z^2z^{-1}\bar{\mathbf{m}}\otimes\bar{\mathbf{m}}]\\
&\qquad\Delta=r^2-2Mr+a^2\qquad \Sigma=z\bar z=r^2+a^2\cos^2\theta
\end{aligned} \tag{2.3}$$

where a dot means a contraction over tensor indices, the operators $\mathscr{L}_s$ and $\mathscr{D}_n$ are defined as

$$\begin{aligned}
&\mathscr{L}_s=\partial_\theta-(\mathrm{i}/\sin\theta)\,\partial_\varphi-\mathrm{i}a\sin\theta\,\partial_t+s\cot\theta\\
&\mathscr{D}_n=\partial_r+[(r^2+a^2)/\Delta]\,\partial_t+(a/\Delta)\,\partial_\varphi+2n[(r-M)/\Delta]
\end{aligned} \tag{2.4}$$

and the symbol $+$ corresponds to the change $\partial_t\to-\partial_t$, $\partial_\varphi\to-\partial_\varphi$.

The variables in (2.1) separate into terms of the spin-weighted spheroidal harmonics ${}_sZ_\Lambda(\theta,\varphi)={}_sS_\Lambda(\theta)\,\mathrm{e}^{im\varphi}$, $\{\Lambda\}=\{\omega, l, m\}$

$$ {}_s\psi=\sum_\Lambda {}_sR_\Lambda(r)\,{}_sZ_\Lambda(\theta,\varphi)\,\mathrm{e}^{-i\omega t} \tag{2.5}$$

where Σ_Λ denotes the integral over ω and the summation over $l\leqslant|s|$ and m, $|m|\leqslant 1$. When all quantities are expanded in harmonics $m\omega$, the symbol + will correspond to the operation $\{m,\omega\}\to\{-m,-\omega\}$, e.g. ${}_sZ^+_\Lambda={}_sZ_{l-m-\omega}$. For ${}_sZ_\Lambda$ the usual normalisation is assumed, and the phase is chosen so that

$$ {}_sZ_\Lambda=(-1)^lP_{-s}Z_\Lambda \qquad {}_sZ_\Lambda=(-1)^{s+m}{}_{-s}\bar{Z}^+_\Lambda \tag{2.6}$$

where P is the operator of space inversion, $Pf(\theta,\varphi)=f(\pi-\theta,\pi+\varphi)$ (in the following the cumulative label Λ will often be omitted for brevity).

With the definitions $\mathrm{d}r^*=[(r^2+a^2)/\Delta]\,\mathrm{d}r$, ${}_sR_\Lambda=\Delta^{-s/2}(\tau^2+a^2)^{-1/2}{}_su$, for the homogeneous case the equation for the radial functions becomes

$$ \mathrm{d}^2{}_su/\mathrm{d}r^{*2}+{}_sV_su=0 \tag{2.7}$$

where the effective potential satisfies ${}_sV={}_{-s}V$. Taking the asymptotic form of the potential

$$ V_s=\begin{cases}\omega(\omega+2is/r) & r^*\to\infty\\ k^2\kappa_s^2 \qquad r^*\to-\infty, & \kappa_s=1-\dfrac{is(r_+-M)}{2kMr_+}\end{cases} \tag{2.8}$$

where $k=\omega-m\omega_+$, $\omega_+=a/2Mr_+$ is the rotation velocity of the horizon and $r_+=M+(M^2-a^2)^{1/2}$, we parametrise the two independent radial solutions as

$$ \begin{aligned} {}_su^{\mathrm{in}}&=\alpha_s\begin{cases}|\omega|^{-1/2}(r^2\,\mathrm{e}^{-i\omega r^*}+\sigma_sr^{-s}\,\mathrm{e}^{i\omega r^*}) & r^*\to\infty\\ |k|^{-1/2}\tau_s\Delta^{-s/2}\,\mathrm{e}^{-ikr^*} & r^*\to-\infty\end{cases}\\ {}_su^{\mathrm{up}}&=\beta_s\begin{cases}|\omega|^{-1/2}r^{-s}\,\mathrm{e}^{i\omega r^*} & r^*\to\infty\\ |k|^{-1/2}\dfrac{\omega k}{|\omega k|}(\mu_s\Delta^{s/2}\,\mathrm{e}^{ikr^*}+\nu_s\Delta^{-s/2}\,\mathrm{e}^{-ikr^*}) & r^*\to-\infty.\end{cases}\end{aligned} \tag{2.9}$$

The notation 'in' and 'up' follows that of Chrzanowski (1975). His complex conjugate solutions ${}_su^{\mathrm{out}}={}_{-s}\bar{u}^{\mathrm{in}}$ and ${}_su^{\mathrm{down}}={}_{-s}\bar{u}^{\mathrm{up}}$ will also be used.

The complex coefficients α_s, β_s, σ_s, τ_s, μ_s and ν_s are subject to numerous relations, which can be divided into two groups. The first consists of the Wronskian relations. Equating the asymptotic values of $W({}_su^{\mathrm{in}},{}_su^{\mathrm{up}})$, $W({}_{-s}\bar{u}^{\mathrm{in}},{}_su^{\mathrm{up}})$ and $W({}_{-s}\bar{u}^{\mathrm{in}},{}_su^{\mathrm{in}})$ one gets

$$ \mu_s=\kappa_s^{-1}\tau_s^{-1} \qquad \nu_s=-\kappa_s^{-1}\bar{\sigma}_{-s}\bar{\tau}_{-s}^{-1} \tag{2.10}$$

and the 'unitarity' condition

$$ \sigma_s\bar{\sigma}_{-s}+\frac{\omega k}{|\omega k|}\kappa_s\tau_s\bar{\tau}_{-s}=1. \tag{2.11}$$

The other connections follow from the relations of Teukolski and Press (1974) for radial solutions with a different sign for s,

$$ {}_sU_sR_\Lambda={}_{-s}R_\Lambda, \tag{2.12}$$

when the operators of spin–weight inversion ${}_sU$ are given explicitly by

$$
\begin{aligned}
{}_{-1}U &= 2B^{-1}\mathscr{D}_0^2 & {}_1U &= \tfrac{1}{2}\Delta B^{-1}\mathscr{D}_0^{+2}\Delta \\
{}_{-2}U &= 4C^{-1}\mathscr{D}_0^4 & {}_2U &= \tfrac{1}{4}\Delta^2\bar{C}^{-1}\mathscr{D}_0^{+4}\Delta^2
\end{aligned}
\tag{2.13}
$$

with

$$B = (F^2 + 4m\gamma - 4\gamma^2)^{1/2}$$

$$C = \{B^2[(F-2)^2 + 36\gamma(m-\gamma)] + 48\gamma(2F-1)(2\gamma - m) - 144\gamma^2\}^{1/2} + 12\mathrm{i}M\omega$$

$$F = {}_sE + \gamma^2 - 2m\gamma \qquad \gamma = a\omega$$

and where ${}_sE$ is the angular eigenvalue. We impose the conditions (2.12) on the independent 'in' and 'up' solutions; the corresponding relations for 'out' and 'down' radial functions will be fixed by their definitions. Using the asymptotic relation valid as $r^* \to \infty$

$$\mathscr{D}_0^{2|s|}\Delta^{|s|}\,\mathrm{e}^{-ikr^*} = (-4)^{|s|}(2Mr_+k)^{2|s|}\kappa_{-|s|}(\kappa_{-1}\kappa_1)^{|s|-1}\Delta^{-|s|}\,\mathrm{e}^{-ikr^*} \tag{2.14}$$

the following relations can be obtained:

$$
\begin{aligned}
&\beta_{-s} = 2^s\omega^{2s}(\bar{Q}_{|s|})^{-s}\beta_s & &\alpha_{-s} = 2^{-3s}\omega^{-2s}(Q_{|s|})^s\alpha_s \\
&\sigma_{-s} = (2\omega)^{4s}|Q_{|s|}|^{-2s}\sigma_s & &\tau_{-|s|} = \left(\frac{\omega}{2Mr_+k}\right)^{2|s|}(\kappa_{-1}\kappa_1)^{1-|s|}\kappa_{-|s|}^{-1}\tau_{|s|} \qquad (s \neq 0)
\end{aligned}
\tag{2.15}
$$

where $Q_0 = 1$, $Q_1 = -B$ and $Q_2 = C^{1/2}$. The remaining free parameters in (2.9) will be chosen later to provide the desired form of the Green functions.

With these preliminaries completed we now turn to the construction of the factorised retarded Green functions. Denote by ${}_s\mathbf{P}$ the field perturbations ${}_0\mathbf{P} = \psi$, ${}_{\pm1}\mathbf{P} = \mathbf{A}$ (the electromagnetic four-potential) and ${}_{\pm2}\mathbf{P} = \mathbf{h}$ ($h_{\mu\nu}$ is the metric perturbation), positive and negative signs of s corresponding to the outgoing and ingoing gauges in the notation of Chrzanowski (1975). According to conjecture of the factorised Green functions any field perturbation ${}_s\mathbf{P}$ may be computed via the formula

$${}_s\mathbf{P}(x) = \int {}_s\mathbf{G}^{\mathrm{ret}}(x, x') \cdot {}_s\mathbf{J}(x')\sqrt{-g(x')}\,\mathrm{d}^4x' \tag{2.16}$$

where the retarded Green function has the factorised form

$$
\begin{aligned}
{}_s\mathbf{G}^{\mathrm{ret}}(x, x') = \sum_{\Lambda,p} \mathrm{i}\delta_s \frac{\omega}{|\omega|} &\big({}_s\boldsymbol{\pi}^{\mathrm{up}}_{\Lambda p}(x) \otimes {}_s\bar{\boldsymbol{\pi}}^{\mathrm{out}}_{\Lambda p}(x')\theta(r - r') \\
&+ {}_s\boldsymbol{\pi}^{\mathrm{in}}_{\Lambda p}(x) \otimes {}_s\bar{\boldsymbol{\pi}}^{\mathrm{down}}_{\Lambda p}(x')\theta(r' - r)\big)
\end{aligned}
\tag{2.17}
$$

where δ_s is some normalisation constant and $p = \pm 1$ is the helicity index. The crucial point of the theory is the determination of the expansion modes ${}_s\boldsymbol{\pi}_{\Lambda p}$. Their explicit form was found by Chrzanowski (1975) via integration by parts in (2.16) and subsequent comparison of the resulting ${}_s\psi$ with those obtained from the Teukolski equation. A simpler way consists in using the Hermitian conjugate projectors (a similar idea was suggested by Wald (1978)).

Define the scalar product of two tensor-valued functions $\boldsymbol{\varphi}(x)$ and $\boldsymbol{\psi}(x)$ of equal rank as

$$(\boldsymbol{\varphi}, \boldsymbol{\psi}) \equiv \int \bar{\boldsymbol{\varphi}}(x) \cdot \boldsymbol{\psi}(x)\sqrt{-g}\,\mathrm{d}^4x \tag{2.18}$$

where the dot means the contraction over the tensor indices. For every tensor operator M taking the n-index field $\boldsymbol{\psi}$ into the k-index field $\mathbf{M}\boldsymbol{\psi}$ the Hermitian conjugate M^* (labelled with an asterisk to avoid confusion with the + introduced previously) will be defined through the relation

$$(\boldsymbol{\varphi}, \mathbf{M}\boldsymbol{\psi}) = (\mathbf{M}^*\boldsymbol{\varphi}, \boldsymbol{\psi}). \tag{2.19}$$

$\mathbf{M}^*$ acts on the space of k-index tensors taking them into n-index ones. Clearly, the scalar operators of multiplication, such as z, will be transformed to their complex conjugate, e.g. $z^* = \bar{z}$. The usual rule $(AB)^* = B^*A^*$ also holds. The Hermitian conjugates to the operators $\mathscr{L}_s$ and $\mathscr{D}_n$ are

$$\mathscr{L}_s^* = -\Sigma^{-1}\mathscr{L}_{1-s}^+\Sigma \qquad \mathscr{D}_n^* = -\Sigma^{-1}\mathscr{D}_{-n}\Sigma \tag{2.20}$$

Now we introduce the mode projectors ${}_s\boldsymbol{\tau}_{\omega m}$ which are just the ${}_s\boldsymbol{\tau}$ given previously (2.3) with ∂_t replaced by $-\mathrm{i}\omega$ and ∂_φ by $\mathrm{i}m$. By the definition (2.3) the operators ${}_s\boldsymbol{\tau}_{\omega m}$ act in the space of s-rank tensors ${}_s\mathbf{J}$ taking them into scalars. So the Hermitian conjugates ${}_s\boldsymbol{\tau}^*_{\omega m}$ act on the space of scalar functions taking them into tensors of rank s. These operators are just what is needed to contract the modes ${}_s\boldsymbol{\pi}_{\Lambda p}$ out of the scalar expansion modes, the proposed expression being

$${}_s\boldsymbol{\pi}_{\Lambda p} = I_{p\,-s}\boldsymbol{\tau}^*_{\omega m\,-s}Z_{\Lambda\,s}R_\Lambda\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{2.21}$$

where $I_p = 1 + pP$ and $p = \pm 1$ is the helicity projector. Using equations (2.3) and the conjugation rules (2.20) one can verify explicitly that the proposed modes coincide up to numerical factors with the modes given by Chrzanowski (1975), the factor δ_s in (2.17) being given by $\delta_s = 2^{s-|s|}\lambda_{|s|}$, $\lambda_0 = 1$, $\lambda_1 = -1$, $\lambda_2 = -\frac{1}{2}$.

Once the field perturbations ${}_s\mathbf{P}$ are found, the reconstruction of the NP quantities ${}_s\psi = {}_s\mathbf{M}\cdot{}_s\mathbf{P}$ is greatly facilitated by use of the following representation for the projectors ${}_s\mathbf{M}$:

$$\begin{aligned}
{}_1\mathbf{M} &= \mathscr{D}_0^2({}_1\boldsymbol{\tau}^*)^{-1} = \tfrac{1}{2}\mathscr{L}_0^+\mathscr{L}_1^+({}_{-1}\boldsymbol{\tau}^*)^{-1}\\
{}_{-1}\mathbf{M} &= \tfrac{1}{2}\mathscr{L}_0\mathscr{L}_1({}_1\boldsymbol{\tau}^*)^{-1} = \tfrac{1}{4}\Delta\mathscr{D}^{+2}\Delta({}_{-1}\boldsymbol{\tau}^*)^{-1}\\
{}_2\mathbf{M} &= \tfrac{1}{4}\mathscr{D}_0^4({}_2\boldsymbol{\tau}^*)^{-1} = \tfrac{1}{2}\mathscr{L}_{-1}^+\mathscr{L}_0^+\mathscr{L}_1^+\mathscr{L}_2^+({}_{-2}\boldsymbol{\tau}^*)^{-1}\\
{}_{-2}\mathbf{M} &= \tfrac{1}{8}\mathscr{L}_{-1}\mathscr{L}_0\mathscr{L}_1\mathscr{L}_2({}_2\boldsymbol{\tau}^*)^{-1} = \tfrac{1}{16}\Delta^2\mathscr{D}_0^{+4}\Delta^2({}_2\boldsymbol{\tau}^*)^{-1}
\end{aligned} \tag{2.22}$$

where the inverse operators are defined by $({}_s\boldsymbol{\tau}^*)\cdot{}^{-1}({}_s\boldsymbol{\tau}^*) = 1$. Note that every NP scalar ${}_s\psi$ can be obtained through equations (2.22) in two different ways corresponding to the outgoing and ingoing gauges. The following relations appear to be important for establishing that both gauges actually give the same result:

$$\sum_p ({}_{-a}\boldsymbol{\pi}_{\Lambda p}\cdot{}_s\mathbf{J}) = (4\bar{Q}_{|s|}Q_{|s|}^{-1})^{-s}\sum_p ({}_s\boldsymbol{\pi}_{\Lambda p}\cdot{}_s\mathbf{J}) \tag{2.23}$$

$${}_{-s}\boldsymbol{\tau}^*_{\omega m\,-s}Z_\Lambda = (-1)^{l+m}P_{-s}\bar{\boldsymbol{\tau}}^{*+}_{\omega m\,-s}\bar{Z}^+_\Lambda. \tag{2.24}$$

Combining equations (2.16), (2.17), (2.21) and (2.22) we return to the expansion (2.5) with some numerical factor. For consistency the relation

$$\alpha_s\beta_s = 2^{3|s|+s-2}(Q_{|s|})^{-s}|Q_{|s|}|^{s-|s|} \tag{2.25}$$

has to be imposed. Now all the coefficients are determined except for an overall normalisation constant. For later convenience the following choice is suitable:

$$\alpha_s = 2^{3/2(|s|+s)-1}\omega^s(Q_{|s|})^{-(s+|s|)/2} \qquad \beta_s = 2^{3/2|s|-1/2s-1}\omega^{-s}(\bar{Q}_{|s|})^{(s-|s|)/2}. \tag{2.26}$$

With such a normalisation all the homogeneous radial solutions ('in', 'up', 'out' and 'down') satisfy

$$ {}_sR^{+}_{\Lambda} = (-1)^s {}_s\bar{R}_{\Lambda}. \tag{2.27}$$

Now we are able to prove that the retarded factorised Green functions so defined are real for all s. Indeed, using the relations (2.23) and (2.26) one can show that in terms of the quantities

$$ {}_s\mathbf{f}_{\Lambda} = {}_{-s}\boldsymbol{\tau}^{*}_{\omega m}\, {}_{-s}S_{\Lambda}\, {}_sR_{\Lambda} \tag{2.28}$$

the Green functions (2.17) may be presented as

$$ {}_s\mathbf{G}^{\mathrm{ret}}(x,x') = 4\,\mathrm{Re}\sum_{\Lambda p} \mathrm{i}\delta_s \frac{\omega}{|\omega|} \exp[\mathrm{i}m(\varphi-\varphi') - \mathrm{i}\omega(t-t')] $$
$$ \times ({}_s\mathbf{f}^{\mathrm{up}}_{\Lambda}(x) \otimes {}_s\mathbf{f}^{\mathrm{out}}_{\Lambda}(x')\theta(r-r') + {}_s\mathbf{f}^{\mathrm{in}}_{\Lambda}(x) \otimes {}_s\bar{\mathbf{f}}^{\mathrm{down}}_{\Lambda}(x')\theta(r'-r)). \tag{2.30}$$

3. Radiative Green functions

To calculate the radiation reaction force upon the point test particle the radiative Green functions are needed; these can be written, with account for the reality of the retarded ones, as

$$ {}_s\mathbf{G}^{\mathrm{rad}}(x,x') = \tfrac{1}{2}({}_s\mathbf{G}^{\mathrm{ret}}(x,x') - {}_s\mathbf{G}^{\mathrm{adv}}(x,x')) = \tfrac{1}{2}({}_s\mathbf{G}^{\mathrm{ret}}(x,x') - {}_s\mathbf{G}^{\mathrm{ret}}(x',x)). \tag{3.1}$$

Using the representation (2.17) one gets

$$ {}_s\mathbf{G}^{\mathrm{rad}}(x,x') = \sum_{\Lambda p} \frac{\mathrm{i}\delta_s\omega}{2|\omega|} (I_{p\,-s}\boldsymbol{\tau}^{*}_{\omega m}(x)\,{}_{-s}Z(\theta,\varphi)) \otimes (I_{p\,-s}\boldsymbol{\tau}^{*}_{\omega m}(x')\,{}_{-s}Z(\theta',\varphi'))\,\mathrm{e}^{-\mathrm{i}\omega(t-t')} $$
$$ \times[({}_sR^{\mathrm{up}}(r)\,{}_s\bar{R}^{\mathrm{out}}(z') + {}_sR^{\mathrm{down}}(r)\,{}_s\bar{R}^{\mathrm{in}}(r'))\theta(r-r') $$
$$ + ({}_sR^{\mathrm{in}}(r)\,{}_s\bar{R}^{\mathrm{down}}(r') + {}_sR^{\mathrm{out}}(r)\,{}_s\bar{R}^{\mathrm{up}}(r'))\theta(r'-r)]. \tag{3.2}$$

Define the 'denormalised' radial functions as ${}_sv^{\mathrm{up}} = \beta_s^{-1}\,{}_su^{\mathrm{up}}$ and ${}_sv^{\mathrm{in}} = \alpha_s^{-1}\,{}_su^{\mathrm{in}}$. According to the asymptotic formulae (2.9) they are connected by the relations

$$ \begin{aligned} {}_sv^{\mathrm{down}} &= {}_{-s}\bar{v}^{\mathrm{up}} = {}_sv^{\mathrm{in}} - \sigma_{ss}v^{\mathrm{up}} \\ {}_sv^{\mathrm{out}} &= {}_{-s}\bar{v}^{\mathrm{in}} = \kappa_s\tau_s\bar{\tau}_{-s}(\omega k/|\omega k|)\,({}_sv^{\mathrm{up}} + \bar{\sigma}_{-ss}v^{in}). \end{aligned} \tag{3.3}$$

From here, with account for the relation

$$ \kappa_{-s}\tau_{-s}\bar{\tau}_s = \kappa_s\tau_s\bar{\tau}_{-s} \tag{3.4}$$

following from the last of equations (2.15), we get the connection formulae for the spin-reflected radial functions of 'out' and 'down' types:

$$ {}_sU_s R^{\mathrm{out(down)}} = \bar{\alpha}_{-s}\bar{\beta}_{-s}(\alpha_s\beta_s)^{-1}\,{}_{-s}R^{\mathrm{out(down)}}. \tag{3.5}$$

For further transformations we write down with the aid of equations (3.3) the following quadratic combination:

$$
\begin{aligned}
{}_{-s}v^{\mathrm{up}}(r)_{-s}v^{\mathrm{in}}(r') + {}_s\bar{v}^{\mathrm{up}}(r)_s\bar{v}^{\mathrm{up}}(r)_s\bar{v}^{\mathrm{in}}(r') \\
&= (\omega k/|\omega k|)\,(\kappa_{-s}\tau_{-s}\bar{\tau}_s)^{-1}[({}_s\bar{v}^{\mathrm{in}}(r) - \bar{\sigma}_{s-s}v^{\mathrm{in}}(r))_{-s}v^{\mathrm{in}}(r') \\
&\quad + ({}_{-s}v^{\mathrm{in}}(r) - \sigma_{-ss}\bar{v}^{\mathrm{in}}(r))_s\bar{v}^{\mathrm{in}}(r')]. \qquad (3.6)
\end{aligned}
$$

On the other hand

$$
{}_s\bar{v}^{\mathrm{up}}(r)_{-s}v^{\mathrm{up}}(r') = (\kappa_{-s}\tau_{-s}\bar{\tau}_s)^{-2}({}_{-s}v^{\mathrm{in}}(r) - \sigma_{-ss}\bar{v}^{\mathrm{in}}(r)({}_s\bar{v}^{\mathrm{in}}(r') - \bar{\sigma}_{s-s}v^{\mathrm{in}}(r')). \qquad (3.7)
$$

Eliminating from equations (3.6) and (3.7) the terms proportional to ${}_s\bar{v}^{\mathrm{in}}(r)_s\bar{v}^{\mathrm{in}}(r')$ and ${}_{-s}v^{\mathrm{in}}(r)_{-s}v^{\mathrm{in}}(r')$ and using the 'unitarity' condition (2.11) we obtain

$$
{}_{-s}v^{\mathrm{up}}(r)_{-s}v^{\mathrm{in}}(r') + {}_s\bar{v}^{\mathrm{up}}(r)_s\bar{v}^{\mathrm{in}}(r') = {}_s\bar{v}^{\mathrm{in}}(r)_{-s}v^{\mathrm{in}}(r') + \kappa_{-s}\tau_{-s}\bar{\tau}_s(\omega k/|\omega k|)_s\bar{v}^{\mathrm{up}}(r)_{-s}v^{\mathrm{up}}(r'). \qquad (3.8)
$$

Note that the left-hand side of this relation is invariant under combined complex conjugation and spin–weight reflection $s \leftrightarrow -s$, while on the right-hand side the arguments r and r' will be interchanged. Consequently, the right-hand side of (3.8) has to be symmetric in r and r'. Taking this into account and using equations (2.26), (3.4) and (3.5), we transform the radiative Green function (3.2) to

$$
{}_s\mathbf{G}^{\mathrm{rad}}(x, x') = \sum_{\Lambda p} \frac{\mathrm{i}\delta_s\omega}{|\omega|} 2^{s-1}({}_s\boldsymbol{\pi}^{\mathrm{out}}_{\Lambda p}(x) \otimes {}_s\bar{\boldsymbol{\pi}}^{\mathrm{out}}_{\Lambda p}(x') + \frac{\omega k}{|\omega k|}\kappa_s\tau_s\bar{\tau}_{-s}\boldsymbol{\pi}^{\mathrm{down}}_{\Lambda p}(x) \otimes {}_s\bar{\boldsymbol{\pi}}^{\mathrm{down}}_{\Lambda p}(xy')). \qquad (3.9)
$$

Note that the radiative Green function does not have a factorised form in contrast to the case of flat space–time. The existence of the second term in (3.9) is due to the absorption of radiation at the event horizon of the Kerr metric. The possibility of a negative sign for the second term is related to the superradiance phenomenon.

4. Equations of motion and the conservation laws in radiative processes

Within the linearised theory adopted, all types of radiation may be considered independently. First we write the equations of motion for a test particle, interacting with the scalar field ${}_0\psi \equiv \psi$, the interaction constant being q. In accordance with the wave equations (2.1) we have

$$
\frac{\mathrm{D}}{\mathrm{d}\tau}[\mu + q\psi(z(\tau))]u_\mu = q\left.\frac{\partial\psi}{\partial x^\mu}\right|_{x=z(\tau)} \qquad (4.1)
$$

where $u_\mu = g_{\mu\nu}\,\mathrm{d}z^\nu/\mathrm{d}\tau$ is the covariant four-velocity and $\mathrm{D}/\mathrm{d}\tau = u^\lambda\nabla_\lambda$ is the covariant derivative along the world line of the particle $z^\mu(\tau)$. For the electric charge one can write

$$
\mu\frac{\mathrm{D}u_\mu}{\mathrm{d}\tau} = \mathrm{e}\left(\frac{\partial A_\nu}{\partial x^\mu}u^\nu - \frac{\mathrm{d}A_\mu}{\mathrm{d}\tau}\right). \qquad (4.2)
$$

The motion of the neutral particle taking correct account of its gravitational field can be put into a form similar to (4.1):

$$\mu\frac{\mathrm{D}}{\mathrm{d}\tau}(1-\tfrac{1}{2}h_{\lambda\nu}u^\lambda u^\nu)u_\mu=\frac{\mu}{2}\frac{\partial h_{\lambda\nu}}{\partial x^\mu}u^\lambda u^\nu-\frac{\mathrm{d}}{\mathrm{d}\tau}h_{\mu\nu}u^\nu. \tag{4.3}$$

In all three cases the left-hand sides of the equations of motion are just the covariant derivatives of the covariant canonical four-momentum p_μ with respect to the background metric. The right-hand sides of equations (4.1)–(4.3) describe the action of the proper fields on the particle, in all three cases being of the same structure. So after the renormalisation on mass, similar to that in the case of electrodynamics, we obtain equations of motion of the same form with only the radiative part of the proper fields present on the right-hand sides. Discarding the total derivative terms on the right-hand sides of equations (4.2) and (4.3), which do not contribute to the irreversible losses due to radiation, we can write the equations of motion taking account of the radiative reaction in the unified form

$$\mathrm{D}p_\mu/\mathrm{d}\tau=\mathrm{d}p_\mu/\mathrm{d}\tau-\tfrac{1}{2}p^\nu u^\lambda\,\partial g_{\lambda\nu}/\partial x^\mu={}_{|s|}f^{\mathrm{rad}}_\mu \tag{4.4}$$

where the radiative forces are given by

$${}_0f^{\mathrm{rad}}_\mu=q\,\partial\psi^{\mathrm{rad}}/\partial x^\mu \tag{4.5}$$

$${}_1f^{\mathrm{rad}}_\mu=eu^\nu\,\partial A^{\mathrm{rad}}_\nu/\partial x^\mu \tag{4.6}$$

$${}_2f^{\mathrm{rad}}_\mu=\tfrac{1}{2}\mu u^\nu u^\lambda\,\partial h^{\mathrm{rad}}_{\nu\lambda}/\partial x^\mu. \tag{4.7}$$

Since the Kerr metric $g_{\nu\lambda}$ does not depend on the coordinates t and φ, equations (4.4) for $\mu=i$, $\{i\}=\{t,\varphi\}=\{0,3\}$ are simplified to

$$\mathrm{d}p_i/\mathrm{d}\tau={}_{|s|}f^{\mathrm{rad}}_i. \tag{4.8}$$

In the absence of the radiative force these equations express the conservation of energy and angular momentum of a particle in the Kerr space–time. Consider now the energy and angular momentum radiative losses for a large proper time interval $(-T,T)$, $T\to\infty$ (for periodic motion it can be easily renormalised to unit time):

$${}_s\mathscr{E}_i=-(-1)^i\int_{-\infty}^{\infty}(\mathrm{d}p_i/\mathrm{d}\tau)\,\mathrm{d}\tau=-(-1)^i\int_{-\infty}^{\infty}{}_{|s|}f^{\mathrm{rad}}_i\,\mathrm{d}\tau\qquad i=0,3 \tag{4.9}$$

where the integral is taken along the world line of a particle. The radiative fields may be computed via the formula

$${}_s\mathbf{P}^{\mathrm{rad}}(x)=\int{}_s\mathbf{G}^{\mathrm{rad}}(x,x')\cdot{}_s\mathbf{J}(x')\sqrt{-g(x')}\,\mathrm{d}^4x', \tag{4.10}$$

the source terms for different s being

$$\begin{gathered}{}_s\mathbf{J}(x)=\int\delta^{(4)}(x,z(\tau))(-g)^{-1/2}\,\mathrm{d}\tau_{|s|}\mathbf{H}(\tau)\\ {}_0H=q\qquad {}_1H^\mu=eu^\mu\qquad {}_2H^{\mu\nu}=\mu u^\mu u^\nu\end{gathered} \tag{4.11}$$

where the δ function is normalised according to

$$\int\delta^{(4)}(x,x')\,\mathrm{d}^4x'=1. \tag{4.12}$$

Using the radiative Green functions (3.9) we obtain the following universal expression for the radiative losses valid in all three cases of scalar ($s = 0$), electromagnetic ($s = \pm 1$) or gravitational ($s = \pm 2$) radiation:

$$ {}_s\mathscr{E}_i^{\mathrm{rad}} = \sum_{\Lambda p} |\omega|(m/\omega)^{i/3} 2^{2s-|s|-1} \left(|({}_s\boldsymbol{\pi}_{\Lambda p}^{\mathrm{out}}, {}_s\mathbf{J})|^2 + \kappa_s \tau_s \bar{\tau}_{-s} \frac{\omega k}{|\omega k|} |({}_s\boldsymbol{\pi}_{\Lambda p}^{\mathrm{down}}, {}_s\mathbf{J})|^2 \right). \tag{4.13} $$

The scalar products in this formula are defined according to equation (2.18). With the aid of equations (2.23) and (2.24) one can prove that the right-hand side of (4.13) does not depend on the sign of s, i.e. both the ingoing and outgoing gauges give the same result. Note that in the original formulation by Chrzanowski (1975) different gauges were used for ingoing and outgoing radiation.

Two terms in equation (4.13) are easily identified with the energy and angular momentum losses carried away by the radiation fields to infinity and to the horizon respectively. Indeed, using the expressions for radiation fluxes in the NP formalism (Teukolsky and Press 1974) and applying the retarded Green function (2.17), for the desired quantities at infinity we get

$$ {}_s\mathscr{E}_i^{\infty} = \sum_{\Lambda p} 2^{|s|-1} |\omega|(m/\omega)^{i/3} |({}_{|s|}\boldsymbol{\pi}_{\Lambda p}^{\mathrm{out}}, {}_s\mathbf{J})|^2 \tag{4.14} $$

and at the horizon of the black hole

$$ {}_s\mathscr{E}_i^{+} = \sum_{\Lambda p} 2^{-3|s|-1} \kappa_{-|s|} \tau_{-|s|} \bar{\tau}_{|s|} \omega (m/\omega)^{i/3} (k/|k|) |({}_{-|s|}\boldsymbol{\pi}_{\Lambda p}^{\mathrm{down}}, {}_s\mathbf{J}). \tag{4.15} $$

Comparing equations (4.14) and (4.15) with (4.13) we have

$$ {}_s\mathscr{E}_i^{\mathrm{rad}} = {}_s\mathscr{E}_i^{\infty} + {}_s\mathscr{E}_i^{+}, \tag{4.15} $$

i.e. the radiation losses computed locally as a result of the radiation reaction are actually identical with the corresponding quantities associated with the radiation fields in two wave zones at infinity and at the horizon of a hole. The analogous theorem can be proved in the case of periodic motion for the quantities per unit time.

5. Slow-motion approximation

Suppose that the particle motion is slow enough for the frequency of radiation to be small compared with the inverse mass of a hole, $\omega M \ll 1$. Under this condition, $a\omega \ll 1$ and the angular functions become the spin-weighted spherical harmonics ${}_sY_{lm}$. The radial functions of 'in' type are also known (Starobinski and Churilov 1973), with our normalisation being

$$ \begin{aligned} {}_sR^{\mathrm{in}} &= \alpha_s \tau_s |k|^{-1/2} 2^{-2s} (M^2 - a^2)^{-s} (2Mr_+)^{-1/2} x^{-s+iQ} \\ &\quad \times (x+1)^{-s-iQ} F(-l-s, l-s+1; 1-s+2iQ; -x) \\ Q &= kMr_+/(M-r_+) \qquad x = (r-r_+)/2(r_+-M) \end{aligned} \tag{5.1} $$

where F is the hypergeometric function. The 'up' solution may be written in terms of the hypergeometric function of the argument $1/x$ as follows

$$
{}_sR^{\mathrm{up}} = |k|^{-1/2} \frac{\omega k}{|\omega k|} \frac{\beta_s}{\kappa_s \tau_s (2Mr_+)^{1/2}} \frac{(l-s)!}{(2l+1)!} \frac{\Gamma(l+1+2\mathrm{i}Q)}{\Gamma(-s+2\mathrm{i}Q)} x^{-l-1+\mathrm{i}Q}
$$
$$
\times (1+x)^{-s-\mathrm{i}Q} F(l-s+1, l+1-2\mathrm{i}Q; 2l+2; -1/x). \tag{5.2}
$$

Matching (5.1) and (5.2) with the confluent hypergeometric function solutions valid at large x leads to the following values for the barrier coefficients:

$$
\tau_s = |k/\omega|^{1/2} 2^{2s} (M^2-a^2)^s \frac{(2Mr_+)^{1/2}}{2(r_+-M)} (-2\mathrm{i}\nu)^{l-s+1} \frac{(l-s)!}{2l!} \frac{(l+s)!}{(2l+1)!} \frac{\Gamma(l+1+2\mathrm{i}Q)}{\Gamma(1-s+2i\mathrm{Q})}
$$
$$
\sigma_s = (-1)^{-l+s-1} (\mathrm{i}\nu)^{-2s} (M^2-a^2)^s \frac{(l+s)!}{(l-s)!} \qquad \nu = 2\omega(r_+-M) \tag{5.3}
$$

where Γ is the gamma function.

Using these expressions the radiative Green functions can be constructed explicitly and the radiation reaction forces can be found. The resulting formulae, however, are too cumbersome and we shall give them only for the large-distance limit $r \gg M$. In this case the radiative Green functions have the explicit form

$$
{}_s\mathbf{G}^{\mathrm{rad}}(x,x') = \sum_{\Lambda p} \frac{\mathrm{i}\delta_s 2^{-s}}{[(2l+1)!]^2} \bigg[2^{4s+2l+1} |\alpha_{-s}|^2 \omega^{2l+2s+1} [(l-s)!]^2 (rr')^{l-s}
$$
$$
+ \frac{|\beta_{-s}|^2}{4Mr_+k(\mathrm{i}\nu)^{2s}} \left(\frac{4(M^2-a^2)}{rr'}\right)^{l+s-1} \frac{1}{\kappa_{-s}} \left(\frac{2\mathrm{i}Q-s}{2\mathrm{i}Q+s}\right) \left|\frac{\Gamma(l+1+2\mathrm{i}Q)(l+s)!}{\Gamma(s+2\mathrm{i}Q)}\right|^2 \bigg]
$$
$$
\times (I_{p-s}\boldsymbol{\tau}^*_{\omega m}(x)_{-s}Z(\theta,\varphi)) \otimes (I_{p-s}\boldsymbol{\tau}^*_{\omega m}(x')_{-s}Z(\theta',\varphi'))\, \mathrm{e}^{-\mathrm{i}\omega(t-t')}. \tag{5.4}
$$

For small ω the l series in (5.4) are rapidly convergent, the leading term being $l=|s|$. The following formulae are obtained taking only the leading terms in both 'out' and 'down' contributions in (5.3) into account. For simplicity the case of circular motion with radius r and angular velocity Ω in the equatorial plane $\theta = \frac{1}{2}\pi$ is considered.

5.1. Scalar radiation

Under the conditions $\omega M \ll 1$, $r \gg M$ the non-zero components of the radiative force are

$$
{}_0f_\varphi^{\mathrm{rad}} = \tfrac{1}{3}q^2\left(r^2\Omega^3 + \frac{2M^3r_+}{r^4}(\Omega-\omega_+)\right) \tag{5.5}
$$

$$
{}_0f_t^{\mathrm{rad}} = -\Omega\, {}_0f_\varphi^{\mathrm{rad}}. \tag{5.6}
$$

The first term in (5.5) corresponds to radiation going to infinity and the second to radiation absorbed by a hole (scattered and amplified when $\Omega < \omega_+$). Note that the first term falls more rapidly with decreasing Ω, and for sufficiently small angular velocity of the particle the second term becomes dominant. In the case $\Omega = \omega_+$ the black hole does not absorb radiation.

5.2. Electromagnetic radiation

Under the same conditions the components of the radiative force are given by equations (5.5) and (5.6) with q^2 replaced by $2e^2$, so the same conclusions are valid. In addition, the radiative force acting upon a point magnetic dipole $\boldsymbol{\mu}$ at the same motion has been computed. At the locally static frame the components of the radiative force for sufficiently slow motion ('down' terms dominating) are

$$F_{\hat{r}} = -2g\mu_{\hat{r}}\mu_{\hat{\varphi}} \qquad F_{\hat{\theta}} = g\mu_{\hat{\theta}}\mu_{\hat{\varphi}} \qquad F_{\hat{\varphi}} = g(\mu_{\hat{\varphi}}^2 + 4\mu_{\hat{r}}^2)$$
$$g = \tfrac{4}{3}M^2 r_+(\omega_+ - \Omega)/r^7. \tag{5.7}$$

A rotating moment of the radiative force on a dipole also exists.

5.3. *Gravitational radiation*

The non-zero components of the radiative force are

$${}_2f_\varphi^{\mathrm{rad}} = \tfrac{32}{5}\mu^2\Omega^5 r^4 + \tfrac{16}{5}\mu^2\frac{M^5 r_+}{r^6}(\Omega - \omega_+)(1 + 3a^2/M^2) \tag{5.8}$$

$${}_2f_t^{\mathrm{rad}} = -\Omega\,{}_2f_\varphi^{\mathrm{rad}}. \tag{5.9}$$

Here, as previously, the first term corresponds to gravitational radiation going to infinity and the second to radiation absorbed by a hole. For geodesic motion the second term is $(M/r)^4$ smaller than the first (for sufficiently small a). In the ultrarelativistic case (geodesic synchrotron radiation) both terms become equal.

6. Static radiation reaction in the Kerr metric and tidal friction

The common feature of expressions (5.5) and (5.6) for the azimuthal component of radiation reaction forces derived in the case of a circularly moving particle is the non-zero limiting value when $\Omega \to 0$, i.e. for a particle at rest with respect to the locally static frame. In this limit neither real radiation is present; indeed, the t components of the reaction force (5.6) and (5.9) tend to zero when $\Omega \to 0$. For a particle at rest at the point (r, θ) the azimuthal components of the radiative force in scalar, electromagnetic and gravitational cases are (for $r \gg M$)

$${}_0f_\varphi^{\mathrm{rad}} = -\tfrac{1}{3}aq^2M^2\sin^2\theta/r^4 \tag{6.1}$$

$${}_1f_\varphi^{\mathrm{rad}} = -\tfrac{2}{3}ae^2M^2\sin^2\theta/r^4 \tag{6.2}$$

$${}_2f_\varphi^{\mathrm{rad}} = -\tfrac{8}{5}a\mu^2\frac{M^4\sin^2\theta}{r^6}\left(1 + \frac{3a^2}{4M^2}(5\sin^2\theta - 1)\right). \tag{6.3}$$

The sign of these expressions corresponds to acceleration of a particle in the direction of rotation of the black hole. Consequently, the rotation of the hole itself, by the global conservation of angular momentum, must slow down. This effect is just the well known tidal friction (Hawking and Hartle 1972). Indeed, one can easily recognise in equation (6.3) the familiar expression for the rate of angular momentum loss by the Kerr black hole in the static external field of a point-like source.

The question of back reaction upon the external source of non-axisymmetric perturbations of the Kerr space-time has been discussed previously (see, e.g., Chrzanowski 1976), but the nature of the back force as the radiation damping force was not established. Chrzanowski (1976) claimed that the force acting upon a source of perturbations in the tidal friction problem can be computed using the retarded potentials. However, in this case the Coulomb-like infinite terms are present and some regularisation procedure is needed to exclude them. It was actually the part of the angular momentum absorbed by a hole that was computed in the reference cited above. As we have seen here, the physical counteracting force to the Hawking tidal force is just the force of radiation reaction, the azimuthal component of which remains non-zero even in the absence of real radiation.

From the general point of view the existence of the radiation reaction force in situations when no radiation takes place might seem paradoxical; in a sense, the 'inverse' Born paradox (radiation present, but no reaction force). It should be noted, however, that in the present situation the static force of radiation reaction does not work and energy is conserved unlike in the Born case.

7. 'Violation' of the equivalence principle

One could consider the existence of the static force of radiation reaction in the Kerr metric, in addition to the classical renormalisation of mass of a test particle in the Schwarzschild metric (Frolov and Zel'nikov 1980), as a non-trivial example of effective 'violation' of the equivalence principle due to the non-local effects of interaction. Of course, this does not at all contradict the 'first principles' of general relativity: the radiative effects mean that the particle can no longer be considered as the test one. From the physical point of view it is interesting, however, to compare the magnitude of the 'anomalous' gravitational force (6.3) and the 'usual' gravitational force. For $\theta = \frac{1}{2}\pi$ the ratio of the radiative force (6.3) to the Newtonian force appears to be

$$\frac{{}_2f_{\text{rad}}}{f_{\text{Newton}}} = \frac{8}{5}\frac{a}{M}\frac{\mu}{M}\left(\frac{M}{r}\right)^5\left(1+\frac{3a^2}{M^2}\right). \tag{7.1}$$

Though this formula is strictly valid only for $r \gg M$, for a rough estimate one can put $r \sim M$ (corresponding more accurate expressions for the radiative force can be found using formulae (5.1)–(5.3)). So the maximal 'violation' of Newton's law is of order $\mu a/M^2$; the ratio μ/M has to be regarded as a small quantity within the linearised theory used.

For an elementary particle, e.g. a proton, the ratio of the anomalous force (6.2) to the Newtonian force is

$$\frac{{}_1f_{\text{rad}}}{f_{\text{Newton}}} = \frac{2}{3}\left(\frac{e}{\mu_{\text{p}}}\right)^2\frac{\mu_{\text{p}}}{M}\frac{a}{M}\left(\frac{M}{r}\right)^3. \tag{7.2}$$

Putting $a \sim M \sim r$ for the rough estimate, we shall see that it becomes of the order of unity for a black hole of mass $M \sim 10^{14}\mu_{\text{p}}$. The question of whether the effects considered would be significant at the quantum level, in particular, for the picture of the black hole evaporation should be considered.

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