

¹For convenience, the equivalent (\equiv) mathematical statement of each condition is expressed nearby in parentheses.

respects. This notational convention is used in **category theory diagrams**,² to indicate that the diagram is not **commutative**.³

Negation of a condition is indicated by the reversal of the corresponding arrow. For example, negation of the **RIC**, $\overleftarrow{\text{RIC}} \equiv \mathbf{P} > R$, would be represented by moving the arrowhead from the bottom right to the top left of the line segment connecting \mathbf{P} and R .

If we were to start at R and then impose $\overleftarrow{\text{FWC}}$, that would reverse the arrow connecting R and Γ , but the Γ node would then have no exiting arrows so no further deductions could be made. However, if we *also* reversed **PF-GIC** (that is, if we imposed $\overleftarrow{\text{PF-GIC}}$), that would take us to the \mathbf{P} node, and we could deduce $R > \mathbf{P}$. However, we would have to stop traversing the diagram at this point, because the arrow exiting from the \mathbf{P} node points back to our starting point, which (if valid) would lead us to the conclusion that $R > R$. Thus, the reversal of the two earlier conditions (imposition of $\overleftarrow{\text{FWC}}$ and $\overleftarrow{\text{PF-GIC}}$) requires us also to reverse the final condition, giving us $\overleftarrow{\text{RIC}}$.⁴

Under these conventions, Figure 1 in the main text presents a modified version of the diagram extended to incorporate the **PF-FVAC** (reproduced here for convenient reference).

This diagram can be interpreted, for example, as saying that, starting at the \mathbf{P} node, it is possible to derive the **PF-FVAC**⁵ by imposing both the **PF-GIC** and the **FWC**; or by imposing **RIC** and $\overleftarrow{\text{FWC}}$. Or, starting at the Γ node, we can follow the imposition of the **FWC** (twice - reversing the arrow labeled $\overleftarrow{\text{FWC}}$) and then $\overleftarrow{\text{RIC}}$ to reach the conclusion that $\mathbf{P} < \Gamma$. Algebraically,

$$\begin{aligned} \text{FWC} : \quad & \Gamma < R \\ \overleftarrow{\text{RIC}} : \quad & R < \mathbf{P} \\ & \Gamma < \mathbf{P} \end{aligned} \tag{1}$$

which leads to the negation of both of the conditions leading into \mathbf{P} . $\overleftarrow{\text{PF-GIC}}$ is obtained directly as the last line in (1) and $\overleftarrow{\text{PF-FVAC}}$ follows if we start by multiplying the Return Patience Factor ($\text{RPF} = \mathbf{P}/R$) by the $\text{FWF} (= \Gamma/R)$ raised to the power $1/\rho - 1$, which is negative since we imposed $\rho > 1$. **FWC** implies $\text{FWF} < 1$ so when FWF is raised to a negative power the result is greater than one. Multiplying the RPF (which exceeds 1 because $\overleftarrow{\text{RIC}}$) by another number greater than one yields a product that must

²For a popular introduction to category theory, see Riehl (2017).

³But the rest of our notation does not necessarily abide by the other conventions of category theory diagrams.

⁴The corresponding algebra is

$$\begin{aligned} \overleftarrow{\text{FWC}} : \quad & R < \Gamma \\ \overleftarrow{\text{PF-GIC}} : \quad & \Gamma < \mathbf{P} \\ \Rightarrow \overleftarrow{\text{RIC}} : \quad & R < \mathbf{P}, \end{aligned}$$

⁵in the form $\mathbf{P} < (R/\Gamma)^{1/\rho} \Gamma$

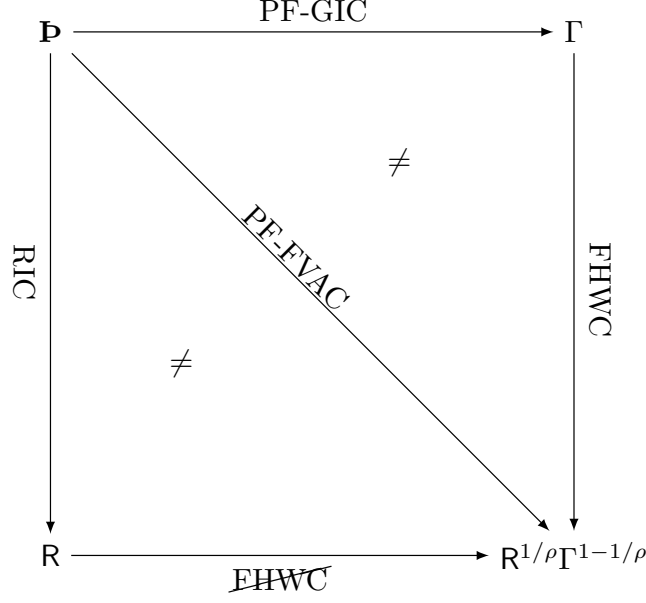


Figure 2 Relation of **PF-GIC**, **FHWC**, **RIC**, and **PF-FVAC**

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\mathbf{P} < R^{1/\rho}\Gamma^{1-1/\rho}$, which is an alternative way of writing the **PF-FVAC**, (28)

be greater than one:

$$\begin{aligned}
 1 &< \overbrace{\left(\frac{(R\beta)^{1/\rho}}{R}\right)}^{>1 \text{ from } \text{RIC}} \overbrace{\left(\frac{\Gamma}{R}\right)^{1/\rho-1}}^{>1 \text{ from FHWC}} \\
 1 &< \left(\frac{(R\beta)^{1/\rho}}{(R/\Gamma)^{1/\rho}R\Gamma/R}\right) \\
 R^{1/\rho}\Gamma^{1-1/\rho} &= (R/\Gamma)^{1/\rho}\Gamma < \mathbf{P}
 \end{aligned}$$

which is one way of writing **PF-FVAC**.

The complexity of this algebraic calculation illustrates the usefulness of the diagram, in which one merely needs to follow arrows to reach the same result.

After the warmup of constructing these conditions for the perfect foresight case, we can represent the relationships between all the conditions in both the perfect foresight case and the case with uncertainty as shown in Figure 3 in the paper (reproduced here).

References

RIEHL, EMILY (2017): *Category theory in context*. Courier Dover Publications.

