1 Existence of a Concave Consumption Function

To show that (6) defines a sequence of continuously differentiable strictly increasing concave functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, ..., \mathrm{c}_{T-k}\}$, we start with a definition. We will say that a function $\mathrm{n}(z)$ is 'nice' if it satisfies

1. $\mathrm{n}(z)$ is well-defined iff $z > 0$
2. $\mathrm{n}(z)$ is strictly increasing
3. $\mathrm{n}(z)$ is strictly concave
4. $\mathrm{n}(z)$ is $\mathbf{C}^3$
5. $\mathrm{n}(z) < 0$
6. $\lim_{z \downarrow 0}\ \mathrm{n}(z) = -\infty$.

(Notice that an implication of niceness is that $\lim_{z \downarrow 0} \mathrm{n}'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = \mathrm{u}(m)$ and $\mathrm{u}(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $\mathfrak{v}_t(a)$ as

$$\mathfrak{v}_t(a) = \beta\, \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(\mathcal{R}_{t+1} a + \xi_{t+1}) \right]. \tag{1}$$

Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a \downarrow 0} \mathfrak{v}_t(a) = -\infty$ and $\lim_{a \downarrow 0} \mathfrak{v}_t'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchi (2003).)

Next define $\underline{\mathrm{v}}_t(m, c)$ as

$$\underline{\mathrm{v}}_t(m, c) = \mathrm{u}(c) + \mathfrak{v}_t(m - c) \tag{2}$$

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (6) can be written as

$$\mathrm{v}_t(m) = \max_{c}\ \underline{\mathrm{v}}_t(m, c). \tag{3}$$

$\underline{\mathrm{v}}_t$ is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c \downarrow 0} \underline{\mathrm{v}}_t(m, c) = \lim_{c \uparrow m} \underline{\mathrm{v}}_t(m, c) = -\infty$, $\frac{\partial^2 \underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c \downarrow 0} \frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c \uparrow m} \frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$. It follows that the $\mathrm{c}_t(m)$ defined by

$$\mathrm{c}_t(m) = \underset{0<c<m}{\arg\max}\ \underline{\mathrm{v}}_t(m, c) \tag{4}$$

exists and is unique, and (6) has an internal solution that satisfies

$$\mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(m - \mathrm{c}_t(m)). \tag{5}$$

Since both u and $\mathfrak{v}_t$ are strictly concave, both $\mathrm{c}_t(m)$ and $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t$ are three times continuously differentiable, using (5) we can conclude that $\mathrm{c}_t(m)$ is continuously differentiable and

$$\mathrm{c}_t'(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}. \tag{6}$$

Similarly we can easily show that $\mathrm{c}_t(m)$ is twice continuously differentiable (as is $\mathrm{a}_t(m)$) (See Appendix 2.) This implies that $\mathrm{v}_t(m)$ is nice, since $\mathrm{v}_t(m) = \mathrm{u}(\mathrm{c}_t(m)) + \mathfrak{v}_t(\mathrm{a}_t(m))$.

2 $\mathrm{c}_t(m)$ is Twice Continuously Differentiable

First we show that $\mathrm{c}_t(m)$ is $\mathbf{C}^1$. Define y as $y \equiv m + dm$. Since $\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))$ and $\frac{\mathrm{a}_t(y)-\mathrm{a}_t(m)}{dm} = 1 - \frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}$,

$$\frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)} = \left(\frac{\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y) - \mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)} \right) \frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim_{dm \to +0} \frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} < 0$ and $\lim_{dm \to +0} \frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ are satisfied. Then $\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ for sufficiently small dm. Hence we obtain a well-defined equation:

$$\frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm} = \frac{\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}{\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}.$$

This implies that the right-derivative, $\mathrm{c}_t'^{+}(m)$ is well-defined and

$$\mathrm{c}_t'^{+}(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}.$$

Similarly we can show that $\mathrm{c}_t'^{+}(m) = \mathrm{c}_t'^{-}(m)$, which means $\mathrm{c}_t'(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, $\mathrm{c}_t'(m)$ exists and is continuous. $\mathrm{c}_t'(m)$ is differentiable because $\mathfrak{v}_t''$ is $\mathbf{C}^1$, $\mathrm{c}_t(m)$ is $\mathbf{C}^1$ and $\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m)) < 0$. $\mathrm{c}_t''(m)$ is given by

$$\mathrm{c}_t''(m) = \frac{a_t'(m)\mathfrak{v}_t'''(a_t)\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right] - \mathfrak{v}_t''(a_t)\left[c_t'\mathrm{u}'''(c_t) + a_t'\mathfrak{v}_t'''(a_t)\right]}{\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right]^2}. \tag{7}$$

Since $\mathfrak{v}_t''(\mathrm{a}_t(m))$ is continuous, $\mathrm{c}_t''(m)$ is also continuous.

3 Proof that $\mathcal{T}$ Is a Contraction Mapping

We must show that our operator $\mathcal{T}$ satisfies all of Boyd's conditions.

Boyd's operator T maps from $\mathcal{C}_F(\mathcal{A},\mathcal{B})$ to $\mathcal{C}(\mathcal{A},\mathcal{B})$. A preliminary requirement is therefore that $\{\mathcal{T}\mathrm{z}\}$ be continuous for any $F-$bounded z, $\{\mathcal{T}\mathrm{z}\} \in \mathcal{C}(\mathbb{R}_{++},\mathbb{R})$. This is not difficult to show; see Hiraguchi (2003).

Consider condition (1). For this problem,

$$\{\mathcal{T}\mathrm{x}\}(m_t) \text{ is } \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{ \mathrm{u}(c_t) + \beta\, \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{x}\,(m_{t+1}) \right] \right\}$$
$$\{\mathcal{T}\mathrm{y}\}(m_t) \text{ is } \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{ \mathrm{u}(c_t) + \beta\, \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{y}\,(m_{t+1}) \right] \right\},$$

so $\mathrm{x}(\bullet) \leq \mathrm{y}(\bullet)$ implies $\{\mathcal{T}\mathrm{x}\}(m_t) \leq \{\mathcal{T}\mathrm{y}\}(m_t)$ by inspection.[1]

Condition (2) requires that $\{\mathcal{T}\mathbf{0}\} \in \mathcal{C}_{\mathcal{F}}\,(\mathcal{A}, \mathcal{B})$. By definition,

$$\{\mathcal{T}\mathbf{0}\}(m_t) = \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{ \left(\frac{c_t^{1-\rho}}{1-\rho} \right) + \beta 0 \right\}$$

the solution to which is patently $\mathrm{u}(\bar{\kappa} m_t)$. Thus, condition (2) will hold if $(\bar{\kappa} m_t)^{1-\rho}$ is $\mathcal{F}$-bounded. We use the bounding function

$$\mathcal{F}(m) = \eta + m^{1-\rho}, \tag{8}$$

for some real scalar $\eta > 0$ whose value will be determined in the course of the proof. Under this definition of $\mathcal{F}$, $\{\mathcal{T}\mathbf{0}\}(m_t) = \mathrm{u}(\bar{\kappa} m_t)$ is clearly $\mathcal{F}$-bounded.

Finally, we turn to condition (3), $\{\mathcal{T}(\mathrm{z} + \zeta\mathcal{F})\}(m_t) \leq \{\mathcal{T}\mathrm{z}\}(m_t) + \zeta\alpha\mathcal{F}(m_t)$. The proof will be more compact if we define č and ă as the consumption and assets functions[2] associated with $\mathcal{T}\mathrm{z}$ and ĉ and â as the functions associated with $\mathcal{T}(\mathrm{z} + \zeta\mathcal{F})$; using this notation, condition (3) can be rewritten

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z} + \zeta\mathcal{F})\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\breve{\mathrm{a}}) + \zeta\alpha\mathcal{F}.$$

Now note that if we force the $\smile$ consumer to consume the amount that is optimal for the $\wedge$ consumer, value for the $\smile$ consumer must decline (at least weakly). That is,

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\breve{\mathrm{a}}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$\begin{aligned} \mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z} + \zeta\mathcal{F})\}(\hat{\mathrm{a}}) &\leq \mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\alpha\mathcal{F} \\ \beta\{\mathsf{E}(\mathrm{z} + \zeta\mathcal{F})\}(\hat{\mathrm{a}}) &\leq \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\alpha\mathcal{F} \\ \beta\zeta\{\mathsf{E}\mathcal{F}\}(\hat{\mathrm{a}}) &\leq \zeta\alpha\mathcal{F} \\ \beta\{\mathsf{E}\mathcal{F}\}(\hat{\mathrm{a}}) &\leq \alpha\mathcal{F} \\ \beta\{\mathsf{E}\mathcal{F}\}(\hat{\mathrm{a}}) &< \mathcal{F}. \end{aligned}$$

where the last line follows because $0 < \alpha < 1$ by assumption.[3]

Using $\mathcal{F}(m) = \eta + m^{1-\rho}$ and defining $\hat{a}_t = \hat{\mathrm{a}}(m_t)$, this condition is

$$\beta\, \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}] - m_t^{1-\rho} < \eta(1 - \underbrace{\beta\, \mathbb{E}_t\, \Gamma_{t+1}^{1-\rho}}_{=\beth})$$

[1] For a fixed m_t, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

[2] Section 2.7 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

[3] The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

which by imposing PF-FVAC (equation (27), which says $\beth < 1$) can be rewritten as:

$$\eta > \frac{\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\xi_{t+1})^{1-\rho}\right]-m_t^{1-\rho}}{1-\beth}. \tag{9}$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing that the numerator of (9) is bounded from above:

$$\begin{aligned}
&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1})^{1-\rho}\right]-m_t^{1-\rho}\\
\le&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\mathsf{R}^{1-\rho}((1-\bar{\kappa})m_t)^{1-\rho}-m_t^{1-\rho}\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\wp\beta\mathsf{R}^{1-\rho}\left(\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)^{1-\rho}-1\right)\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\underbrace{\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}}_{<1\text{ by WRIC}}-1\right)
\end{aligned} \tag{10}$$

$$<(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\underline{\theta}/(1-\wp))^{1-\rho}\right]=\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}.$$

We can thus conclude that equation (9) will certainly hold for any:

$$\eta > \underline{\eta} = \frac{\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}}{1-\beth} \tag{11}$$

which is a positive finite number under our assumptions.

The proof that $\mathfrak{T}$ defines a contraction mapping under the conditions (37) and (33) is now complete.

3.1 $\mathfrak{T}$ and v

In defining our operator $\mathfrak{T}$ we made the restriction $\underline{\kappa}m_t \le c_t \le \bar{\kappa}m_t$. However, in the discussion of the consumption function bounds, we showed only (in (38)) that $\underline{\kappa}_t m_t \le \mathrm{c}_t(m_t) \le \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (6) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a slightly stronger (but still quite weak) condition. The place where these conditions have force is in the step at (10). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$\begin{aligned}
\wp\beta(\mathsf{R}(1-\bar{\kappa}_{T-1}))^{1-\rho} &< 1\\
(\wp\beta)^{1/(1-\rho)}(1-\bar{\kappa}_{T-1}) &> 1\\
(\wp\beta)^{1/(1-\rho)}(1-(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1}) &> 1
\end{aligned}$$

where we have used (36) for $\bar{\kappa}_{T-1}$ (and in the second step the reversal of the inequality occurs because we have assumed $\rho > 1$ so that we are exponentiating both sides by the negative number $1-\rho$). To see that this is a weak condition, note that for small values of $\wp$ this expression can be further simplified using $(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1} \approx 1-\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ so that it becomes

$$\begin{aligned}(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\text{Þ}_{\mathsf{R}} &> 1\\ (\wp\beta)\wp^{(1-\rho)/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1\\ \beta\wp^{1/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1.\end{aligned}$$

Calling the weak return patience factor $\text{Þ}_{\mathsf{R}}^{\wp} = \wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ and recalling that the WRIC was $\text{Þ}_{\mathsf{R}}^{\wp} < 1$, the expression on the LHS above is $\beta\text{Þ}_{\mathsf{R}}^{-\rho}$ times the WRPF. Since we usually assume β not far below 1 and parameter values such that $\text{Þ}_{\mathsf{R}} \approx 1$, this condition is clearly not very different from the WRIC.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique $\mathrm{v}(m)$. But since $\lim_{n\rightarrow\infty} \underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n\rightarrow\infty} \bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $\mathrm{v}(m)$ toward which these v_{T-n}'s are converging is the *same* $\mathrm{v}(m)$ that was the endpoint of the contraction defined by our operator $\mathcal{T}$. Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (6) converge, they converge to the same unique v defined by $\mathcal{T}$.[4]

References

Hiraguchi, Ryoji (2003): "On the Convergence of Consumption Rules," *Manuscript, Johns Hopkins University.*

[4] It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.