Theoretical Foundations of Buffer Stock Saving

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Abstract

This paper builds theoretical foundations for rigorous and intuitive understanding of 'buffer stock' saving models, pairing each theoretical result with a quantitative exploration. After describing conditions under which the consumption function exists, the paper shows that a 'target' buffer stock exists only under conditions strictly stronger than those that guarantee convergence of the consumption and value functions. Furthermore, the average growth rate of consumption equals the average growth rate of permanent income (in a small open economy populated by buffer stock savers). Together, the (provided) numerical tools and (proven) analytical results constitute a comprehensive toolkit for understanding buffer stock models.

Keywords Precautionary saving, buffer stock saving, marginal propensity to consume, permanent income hypothesis

JEL codes D81, D91, E21

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All figures and numerical results can be automatically reproduced using the Econ-ARK/HARK toolkit, which can be cited per our references (Carroll, Kaufman, Kazil, Palmer, and White (2018)); for reference to the toolkit itself see Acknowleding Econ-ARK. Thanks to the Consumer Financial Protection Bureau for funding the original creation of the Econ-ARK toolkit; and to the Sloan Foundation for funding Econ-ARK's extensive further development that brought it to the point where it could be used for this project. The toolkit can be cited with its digital object identifier,

1 Introduction

In the presence of empirically realistic transitory and permanent shocks to income *a la* Friedman (1957), only one further ingredient is required to construct a testable model of optimal consumption: A description of preferences. Modelers usually assume geometric discounting of a constant relative risk aversion utility function, because, starting with Zeldes (1989) and Deaton (1991), a large literature has constructed numerical solutions whose quantitative predictions match microeconomic evidence reasonably well.

A companion theoretical literature has shown that numerical solution methods provide good approximations to limiting "true" mathematical solutions – but only for models more complex than the simple case with just shocks and utility. The extra complexity has been required because standard contraction mapping theorems (beginning with Bellman (1957) and including those building on Stokey et. al. (1989)) cannot be applied when the utility function is unbounded (like CRRA - see section 2.1).[1]

This paper's first technical contribution is to articulate the (loose) conditions under which the simple problem (without convenient shortcuts like a consumption floor or liquidity constraints) defines a contraction mapping with a nondegenerate consumption function. The interesting requirement is a 'Finite Value of Autarky' condition. The paper's second contribution is to specify the conditions under which the resulting consumption function implies existence of a 'target' wealth-to-permanent-income ratio (the model exhibits 'buffer stock' saving behavior). Buffer stock behavior arises when the model's parameters satisfy a "Growth Impatience Condition" (equation (27)) that relates preferences and uncertainty to predictable income growth.

Even without a formal proof, target saving of this kind has been intuitively understood to underlie central quantitative results from the heterogeneous agent macroeconomics literature; for example, the logic of target saving is central to the explanation by Krueger, Mitman, and Perri (2016) of the fact that, during the Great Recession, middle-class consumers cut their consumption more than the poor or the rich. The theoretical logic articulated below explains why: Learning that the future has become more uncertain does not change the urgent imperatives of the poor (their high $\mathrm{u}'(c)$ means they have little room to maneuver). And, increased labor income uncertainty does not change the behavior of the rich because it does not threaten their consumption much. Only people in the middle have both the motivation and the wiggle-room to reduce their discretionary spending.

Conveniently, elements required for the convergence proof also provide analytical foundations for many other results that have become familiar from the numerical litera-

[1] It is unclear whether newer methods such as those of Matkowski and Nowak (2011) could overcome this problem, or how difficult it would be to do so; but in any case this particular problem does not seem to have been tackled by those methods or any others.

10.5281/zenodo.1001067, as is done in the paper's own references as Carroll, Kaufman, Kazil, Palmer, and White (2018). Thanks to James Feigenbaum, Joseph Kaboski, Miles Kimball, Qingyin Ma, Misuzu Otsuka, Damiano Sandri, John Stachurski, Adam Szeidl, Metin Uyanik, Mateo Velásquez-Giraldo, Weifeng Wu, Xudong Zheng, and Jiaxiong Yao for comments on earlier versions of this paper, John Boyd for help in applying his weighted contraction mapping theorem, Ryoji Hiraguchi for extraordinary mathematical insight that improved the paper greatly, David Zervos for early guidance to the literature, and participants in a seminar at Johns Hopkins University and a presentation at the 2009 meetings of the Society of Economic Dynamics for their insights.

ture. In this paper, all theoretical conclusions are paired with numerically computed illustrations (using the open-source Econ-ARK toolkit). The insights of the paper are instantiated in the toolkit, whose buffer stock saving module algorithmically flags parametric choices under which a problem fails to define a contraction mapping; a target level of wealth does not exist; or the solution is otherwise surprising.

Such theoretical foundations are valuable both because they provide intuition about the determinants of saving targets, and because they make it easier to develop reliable numerical solution methods (by providing necessary and sufficient restrictions that solutions must satisfy).

The paper proceeds in three parts.

The first part articulates the conditions required for the problem to define a nondegenerate limiting consumption function, and explains how the paper's model is more general than those previously considered in the literature. The conditions required for convergence are interestingly parallel to those required for the liquidity constrained perfect foresight model; that parallel is explored and explained. Next, the paper derives some limiting properties of the consumption function as resources ('cash') approach infinity and as they approach their lower bound; then the theorem is proven explaining when the problem defines a contraction mapping. Finally, a related class of commonly-used models (exemplified by Deaton (1991)) is shown to constitute a particular limit of this paper's model.

The next section examines five key properties of the model. First, as cash approaches infinity the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as cash approaches zero the expected growth rate of consumption approaches infinity, and the MPC approaches a simple analytical limit. Third, if the consumer is 'growth impatient,' a unique target cash-to-permanent-income ratio will exist. Fourth, at the target cash ratio, the expected growth rate of consumption is slightly less than the expected growth rate of permanent (noncapital) income. Finally, the expected growth rate of consumption is declining in the level of cash. The first four propositions are proven under general assumptions about parameter values; the last is shown to hold if there are no transitory shocks, but may fail in extreme cases if there are both transitory and permanent shocks.

Szeidl (2012) has shown that such an economy will be characterized by stable invariant distributions for the consumption ratio, the wealth ratio, and other variables.[2] Using Szeidl's result, the final section discusses conditions under which, even with a fixed aggregate interest rate that differs from the time preference rate, an economy populated by buffer stock consumers converges to a balanced growth equilibrium in which the growth rate of consumption tends toward the (exogenous) growth rate of permanent income.

[2] Szeidl's proof supplants the analysis in an earlier draft of this paper, which conjectured the result and provided supportive simulation evidence.

2 The Problem

2.1 Setup

The consumer solves a standard optimization problem from the current period t until the end of life at T:

$$\max\ \mathbb{E}_t\left[\sum_{n=0}^{T-t}\beta^n \mathrm{u}(\mathbf{c}_{t+n})\right]$$

where

$$\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho) \tag{1}$$

is a constant relative risk aversion utility function with $\rho > 1$.[3,4] The consumer's initial condition is defined by market resources $\mathbf{m}_t$ (Deaton (1991)'s 'cash-on-hand') and permanent noncapital income $\mathbf{p}_t$.

In the usual treatment, a dynamic budget constraint (DBC) incorporates several different elements that determine next period's $\mathbf{m}$ (given this period's choices); but for the detailed analysis here, it will be useful to disarticulate the steps so that individual ingredients can be separately examined:

$$\begin{aligned}
\mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\
\mathbf{b}_{t+1} &= \mathbf{a}_t \mathsf{R} \\
\mathbf{p}_{t+1} &= \mathbf{p}_t \underbrace{\Gamma\psi_{t+1}}_{\equiv \Gamma_{t+1}} \\
\mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{p}_{t+1}\xi_{t+1},
\end{aligned} \tag{2}$$

where $\mathbf{a}_t$ indicates the consumer's assets at the end of period t, which grow by a fixed interest factor $\mathsf{R} = (1+\mathsf{r})$ between periods,[5] so that $\mathbf{b}_{t+1}$ is the consumer's financial ('bank') balances before next period's consumption choice;[6] $\mathbf{m}_{t+1}$ ('market resources' or 'money') is the sum of financial wealth $\mathbf{b}_{t+1}$ and noncapital income $\mathbf{p}_{t+1}\xi_{t+1}$ (permanent noncapital income $\mathbf{p}_{t+1}$ multiplied by a mean-one iid transitory income shock factor ξ_{t+1}; future transitory shocks are assumed to satisfy $\mathbb{E}_t[\xi_{t+n}] = 1\ \forall\ n \geq 1$). Permanent noncapital income in period $t+1$ is equal to its previous value, multiplied by a growth factor Γ, modified by a mean-one iid shock ψ_{t+1}, $\mathbb{E}_t[\psi_{t+n}] = 1\ \forall\ n \geq 1$ satisfying

[3] The main results also hold for logarithmic utility which is the limit as $\rho \to 1$ but incorporating the logarithmic special case in the proofs is cumbersome and therefore omitted.

[4] We will define the infinite horizon solution as the limit of the finite horizon problem as the horizon $T-t$ approaches infinity.

[5] See Ma, Stachurski, and Toda (2018) for interesting new work that considers the case where capital returns are stochastic and liquidity constraints exist. Benhabib, Bisin, and Zhu (2015) examines implications of capital income risk for the distribution of wealth.

[6] Allowing a stochastic interest factor is straightforward but adds little insight.

$\psi \in [\underline{\psi}, \bar{\psi}]$ for $0 < \underline{\psi} \leq 1 \leq \bar{\psi} < \infty$ (and $\underline{\psi} = \bar{\psi} = 1$ is the degenerate case with no permanent shocks).[7,8]

In future periods $t+n \ \forall \ n \geq 1$ there is a small probability $\wp$ that income will be zero (a 'zero-income event'),

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/(1-\wp) & \text{with probability } (1-\wp) \end{cases} \tag{3}$$

where θ_{t+n} is an iid mean-one random variable ($\mathbb{E}_t[\theta_{t+n}] = 1 \ \forall \ n > 0$) with a distribution satisfying $\theta \in [\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} \leq 1 \leq \bar{\theta} < \infty$ (degenerately $\underline{\theta} = \bar{\theta} = 1$).[9] Call the cumulative distribution functions $\mathcal{F}_\psi$ and $\mathcal{F}_\theta$ (where $\mathcal{F}_\xi$ is derived trivially from (3) and $\mathcal{F}_\theta$). Permanent income and cash start out strictly positive, $\{\mathbf{p}_t, \mathbf{m}_t\} \in (0, \infty)$, and as usual the consumer cannot die in debt, so

$$\mathbf{c}_T \leq \mathbf{m}_T. \tag{4}$$

The model looks more special than it is. In particular, the assumption of a positive probability of zero-income events may seem objectionable. However, it is easy to show that a model with a nonzero minimum value of ξ (motivated, for example, by the existence of unemployment insurance) can be redefined by capitalizing the present discounted value of minimum income into current market assets,[10] analytically transforming that model back into the model analyzed here. Also, the assumption of a positive point mass (as opposed to positive density) for the worst realization of the transitory shock is inessential, but simplifies the proofs and is a powerful aid to intuition.[11]

This model differs from Bewley's (1977) classic formulation in several ways. The CRRA utility function does not satisfy Bewley's assumption that $\mathrm{u}(0)$ is well defined, or that $\mathrm{u}'(0)$ is well defined and finite; indeed, neither the value function nor the marginal value function will be bounded. It differs from Schectman and Escudero (1977) in that they impose liquidity constraints and positive minimum income. It differs from both of these in that it permits permanent growth in income, and also permanent shocks to income, which a large empirical literature finds are quantitatively important in micro data[12] and which since Friedman (1957) have been understood to be far more consequential for household welfare than are transitory fluctuations. It differs from Deaton (1991) because liquidity constraints are absent; there are separate transitory and permanent shocks (*a la* Muth (1960)); and the transitory shocks here can occasionally cause income

[7]It is useful to emphasize that permanent noncapital income as defined here differs from what Deaton (1992) calls permanent income (which is often adopted in the macro literature). Deaton defines permanent income as the amount that a perfect foresight consumer could spend while leaving total (human and nonhuman) wealth constant. Relatedly, we refer to $\mathbf{m}_t$ as 'cash-on-hand' or 'market resources' rather than as wealth to avoid any confusion for readers accustomed to thinking of the discounted value of future noncapital income as a part of wealth. The 'market resources' terminology is motivated by the model's assumption that human wealth cannot be capitalized, an implication of anti-slavery laws.

[8]Hereafter for brevity we occasionally drop time subscripts, e.g. $\mathbb{E}[\psi^{-\rho}]$ signifies $\mathbb{E}_t[\psi_{t+1}^{-\rho}]$.

[9]See Rabault (2002) and Li and Stachurski (2014) for analyses of cases where the shock processes have unbounded support.

[10]So long as this PDV is a finite number and unemployment benefits are proportional to $\mathbf{p}_t$; see the discussion in section 2.10.

[11]No key results would change if the unemployment state were persistent but mean-reverting, instead of IID.

[12]MaCurdy (1982); Abowd and Card (1989); Carroll and Samwick (1997); Jappelli and Pistaferri (2000); Storesletten, Telmer, and Yaron (2004); Blundell, Low, and Preston (2008)

to reach zero.[13] Finally, it differs from models found in Stokey et. al. (1989) because neither liquidity constraints nor bounds on utility or marginal utility are imposed.[14,15]

The incorporation of permanent shocks rules out application of the tools of Matkowski and Nowak (2011), who followed and corrected an error in the fundamental work on the local contraction mapping method developed in Rincón-Zapatero and Rodríguez-Palmero (2003). Martins-da Rocha and Vailakis (2010) provides a correction to Rincón-Zapatero and Rodríguez-Palmero (2003), and provides conditions that are easier to verify, but again only addresses the deterministic case.

2.2 The Problem Can Be Rewritten in Ratio Form

We establish a bit more notation by reviewing the standard result that in problems of this class (CRRA utility, permanent shocks) the number of relevant state variables can be reduced from two ($\mathbf{m}$ and $\mathbf{p}$) to one ($m = \mathbf{m}/\mathbf{p}$). Generically defining nonbold variables as the boldface counterpart normalized by $\mathbf{p}_t$ (as with m), assume that value in the last period of life is $\mathrm{u}(\mathbf{m}_T)$, and consider the problem in the second-to-last period,

$$\begin{aligned}
\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) &= \max_{\mathbf{c}_{T-1}} \ \mathrm{u}(\mathbf{c}_{T-1}) + \beta\, \mathbb{E}_{T-1}[\mathrm{u}(\mathbf{m}_T)] \\
&= \max_{c_{T-1}} \ \mathrm{u}(\mathbf{p}_{T-1} c_{T-1}) + \beta\, \mathbb{E}_{T-1}[\mathrm{u}(\mathbf{p}_T m_T)] \\
&= \mathbf{p}_{T-1}^{1-\rho} \left\{ \max_{c_{T-1}} \ \mathrm{u}(c_{T-1}) + \beta\, \mathbb{E}_{T-1}[\mathrm{u}(\Gamma_T m_T)] \right\},
\end{aligned} \tag{5}$$

where the last line follows because for the CRRA utility function (1), $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$.

Now, in a one-time deviation from the notational convention established in the last paragraph, define nonbold 'normalized value' not as $\mathbf{v}_t/\mathbf{p}_t$ but as $\mathrm{v}_t = \mathbf{v}_t/\mathbf{p}_t^{1-\rho}$, because this allows us to exploit features of the related problem,

$$\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{\{\mathrm{c}\}_t^T} \ \mathrm{u}(\mathrm{c}_t) + \beta\, \mathbb{E}_t[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(m_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
b_{t+1} &= (\mathsf{R}/\Gamma_{t+1}) a_t \ = \ \mathcal{R}_{t+1} a_t \\
m_{t+1} &= b_{t+1} + \xi_{t+1},
\end{aligned} \tag{6}$$

where $\mathcal{R}_{t+1} \equiv (\mathsf{R}/\Gamma_{t+1})$ is a 'growth-normalized' return factor, and the new problem's first order condition is

$$c_t^{-\rho} = \mathsf{R}\beta\, \mathbb{E}_t[\Gamma_{t+1}^{-\rho} c_{t+1}^{-\rho}]. \tag{7}$$

[13] Below it will become clear that the Deaton model is a particular limit of this paper's model.

[14] Similar restrictions to those in the cited literature are made in the well known papers by Scheinkman and Weiss (1986) and Clarida (1987). See Toche (2005) for an elegant analysis of a related but simpler continuous-time model.

[15] Alvarez and Stokey (1998) relaxed the bounds on the return function, but they address only the deterministic case.

Since $\mathrm{v}_T(m_T) = \mathrm{u}(m_T)$, defining $\mathrm{v}_{T-1}(m_{T-1})$ from (6), we obtain

$$\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) = \mathbf{p}_{T-1}^{1-\rho}\mathrm{v}_{T-1}(\underbrace{\mathbf{m}_{T-1}/\mathbf{p}_{T-1}}_{=m_{T-1}}).$$

This logic induces to all earlier periods, so that if we solve the normalized one-state-variable problem (6), we will have solutions to the original problem for any $t < T$ from:

$$\begin{aligned}\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) &= \mathbf{p}_t^{1-\rho}\mathrm{v}_t(m_t),\\ \mathbf{c}_t(\mathbf{m}_t, \mathbf{p}_t) &= \mathbf{p}_t\mathrm{c}_t(m_t).\end{aligned}$$

2.3 Definition of a Nondegenerate Solution

We say that this problem has a nondegenerate solution if, as the number of remaining periods of life gets arbitrarily large, the backward-iterating solution defines a unique limiting consumption function $c(\bullet)$ that satisfies

$$0 < c(m) < \infty \tag{8}$$

for every $0 < m < \infty$. ('Degenerate' limits will be cases where the limiting consumption function is $\mathrm{c}(m) = 0$ or $\mathrm{c}(m) = \infty$.)

2.4 Perfect Foresight Benchmarks

The familiar analytical solution to the perfect foresight specialization of the model, obtained by setting $\wp = 0$ and $\underline{\theta} = \bar{\theta} = \underline{\psi} = \bar{\psi} = 1$, allows us to define some remaining notation and terminology.

2.4.1 Human Wealth

The dynamic budget constraint, strictly positive marginal utility, and the can't-die-in-debt condition (4) imply an exactly-holding intertemporal budget constraint (IBC):

$$\mathrm{PDV}_t(\mathbf{c}) = \overbrace{\mathbf{m}_t - \mathbf{p}_t}^{\mathbf{b}_t} + \overbrace{\mathrm{PDV}_t(\mathbf{p})}^{\mathbf{h}_t}, \tag{9}$$

where $\mathbf{b}$ is nonhuman wealth and $\mathbf{h}_t$ is 'human wealth,' and with a constant $\mathcal{R} \equiv \mathsf{R}/\Gamma$,

$$\mathbf{h}_t = \mathbf{p}_t + \mathcal{R}^{-1}\mathbf{p}_t + \mathcal{R}^{-2}\mathbf{p}_t + ... + \mathcal{R}^{t-T}\mathbf{p}_t \tag{10}$$

$$= \underbrace{\left(\frac{1 - \mathcal{R}^{-(T-t+1)}}{1 - \mathcal{R}^{-1}}\right)}_{\equiv h_t}\mathbf{p}_t. \tag{11}$$

This equation shows that in order for $h \equiv \lim_{n\to\infty} h_{T-n}$ to be finite, we must impose the Finite Human Wealth Condition ('FHWC'):

$$\underbrace{\Gamma/\mathsf{R}}_{\equiv\mathcal{R}^{-1}} < 1. \tag{12}$$

Intuitively, for human wealth to be finite, the growth rate of (noncapital) income must be smaller than the interest rate at which that income is being discounted.

2.4.2 Unconstrained Solution

The consumption Euler equation holds in every period; with $\mathrm{u}'(\mathbf{c}) = \mathbf{c}^{-\rho}$,

$$\mathbf{c}_{t+1}/\mathbf{c}_t = (\mathsf{R}\beta)^{1/\rho} \equiv \text{Þ} \tag{13}$$

where the Old English letter 'thorn' represents what we will call the 'absolute patience factor' $(\mathsf{R}\beta)^{1/\rho}$.[16] The sense in which Þ captures patience is that if the 'absolute impatience condition' (AIC) holds,

$$\text{Þ} < 1, \tag{14}$$

the consumer will choose to spend an amount too large to sustain indefinitely (the level of consumption must fall over time). We call such a consumer 'absolutely impatient' (the key condition in Bewley (1977)).

We next define a 'return patience factor' that relates absolute patience to the return factor:

$$\text{Þ}_{\mathsf{R}} \equiv \text{Þ}/\mathsf{R} \tag{15}$$

and since consumption is growing by Þ but discounted by R:

$$\begin{aligned} \mathrm{PDV}_t(\mathbf{c}) &= \left(1 + \text{Þ}_{\mathsf{R}} + \text{Þ}_{\mathsf{R}}^2 + \ldots + \text{Þ}_{\mathsf{R}}^{T-t}\right) \mathbf{c}_t \\ &= \left(\frac{1 - \text{Þ}_{\mathsf{R}}^{T-t+1}}{1 - \text{Þ}_{\mathsf{R}}}\right) \mathbf{c}_t \end{aligned} \tag{16}$$

from which the IBC (9) implies

$$\mathbf{c}_t = \overbrace{\left(\frac{1 - \text{Þ}_{\mathsf{R}}}{1 - \text{Þ}_{\mathsf{R}}^{T-t+1}}\right)}^{\equiv \underline{\kappa}_t} (\mathbf{b}_t + \mathbf{h}_t) \tag{17}$$

which defines a normalized finite-horizon perfect foresight consumption function

$$\bar{\mathrm{c}}_{T-n}(m_{T-n}) = (\overbrace{m_{T-n} - 1}^{\equiv b_{T-n}} + h_{T-n}) \underline{\kappa}_{T-n} \tag{18}$$

where $\underline{\kappa}_t$ is the marginal propensity to consume (MPC) because it answers the question 'if the consumer had an extra unit of wealth, how much more would be spent.' (The overbar on c reflects the fact that this will be an upper bound as we modify the problem to incorporate constraints and uncertainty; analogously, the underbar for κ indicates that it is a lower bound). Equation (17) makes plain that for the limiting MPC to be strictly positive as $n = T - t$ goes to infinity we must impose the Return Impatience

[16] Impatience conditions of one kind or another have figured in intertemporal optimization problems since such problems were first formalized in economics, most notably by Ramsey (1928). Discussion of these issues was prominent in the literature of the 1960s and 1970s, and no brief citations here could do justice to the literature on the topic, so I refrain from the attempt.

Condition (RIC):

$$\text{Þ}_{\mathsf{R}} < 1, \tag{19}$$

so that

$$0 < \underline{\kappa} \equiv 1 - \text{Þ}_{\mathsf{R}} = \lim_{n \to \infty} \underline{\kappa}_{T-n}. \tag{20}$$

The RIC thus imposes a second kind of 'impatience:' The consumer cannot be so pathologically patient as to wish, in the limit as the horizon approaches infinity, to spend nothing today out of an increase in current wealth; that is, the condition rules out the degenerate limiting solution $\bar{c}(m) = 0$. Because the return patience factor Þ_{R} is the absolute patience factor divided by the return, we call equation (19) the 'return impatience condition' or RIC; we will say that a consumer who satisfies the condition is 'return impatient.'

Given that the RIC holds, and defining limiting objects by the absence of a time subscript (e.g., $\bar{c}(m) = \lim_{n\uparrow\infty} \bar{c}_{T-n}(m)$), the limiting consumption function will be

$$\bar{c}(m) = (m + h - 1)\underline{\kappa}, \tag{21}$$

and we now see that in order to rule out the degenerate limiting solution $\bar{c}(m) = \infty$ we need h to be finite; that is, we must impose the finite human wealth condition (12).

A final useful point is that since the perfect foresight growth factor for consumption is Þ, $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$ yields an analytical expression for value:

$$\begin{aligned} v_t &= \mathrm{u}(c_t) + \beta\mathrm{u}(c_t\text{Þ}) + \beta^2\mathrm{u}(c_t\text{Þ}^2) + ... \\ &= \mathrm{u}(c_t)\left(1 + \beta\text{Þ}^{1-\rho} + (\beta\text{Þ}^{1-\rho})^2 + ...\right) \\ &= \mathrm{u}(c_t)\left(\frac{1 - (\beta\text{Þ}^{1-\rho})^{T-t+1}}{1 - \beta\text{Þ}^{1-\rho}}\right) \end{aligned} \tag{22}$$

which asymptotes to a finite number as $n = T - t$ approaches $+\infty$ if $\beta\text{Þ}^{1-\rho} < 1$;[17] with a bit of algebra,[18] this requirement can be shown to be equivalent to the RIC. Thus, the same conditions that guarantee a nondegenerate limiting consumption function also guarantee a nondegenerate limiting value function (which, interestingly, will *not* be true when we incorporate uncertainty).

2.4.3 Constrained Solution

If a liquidity constraint requiring $b \geq 0$ is ever to be relevant, it must be relevant at the lowest possible level of market resources, $m_t = 1$, which obtains for a consumer who enters period t with $b_t = 0$. The constraint is 'relevant' if it prevents the choice that

[17] This is related to a condition in Alvarez and Stokey (1998).

[18]

$$\begin{aligned} \beta((\mathsf{R}\beta)^{1/\rho})^{1-\rho} &< 1 \\ \beta(\mathsf{R}\beta)^{1/\rho}/\mathsf{R}\beta &< 1 \\ (\mathsf{R}\beta)^{1/\rho}/\mathsf{R} &< 1 \end{aligned}$$

.

would otherwise be optimal; at $m_t = 1$ the constraint is relevant if the marginal utility from spending all of today's resources $c_t = m_t = 1$, exceeds the marginal utility from doing the same thing next period, $c_{t+1} = 1$; that is, if such choices would violate the Euler equation (7):

$$1^{-\rho} > \mathsf{R}\beta(\Gamma)^{-\rho}1^{-\rho}. \tag{23}$$

By analogy to the return patience factor, we therefore define a 'perfect foresight growth patience factor' (PF-GPF) as

$$\text{Þ}_\Gamma = \text{Þ}/\Gamma, \tag{24}$$

and define a 'perfect foresight growth impatience condition' (PF-GIC)

$$\underline{\Gamma} = \Gamma\underline{\psi} \tag{25}$$

and a compensated Growth Patience Pactor (GPF):

$$\begin{aligned} \text{Þ}_{\underline{\Gamma}} &= \text{Þ}/\underline{\Gamma} \\ &= \mathbb{E}\left(\frac{\text{Þ}}{\Gamma\psi}\right) \end{aligned} \tag{26}$$

and a straightforward derivation ((41) below) yields the conclusion that

$$\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] = \text{Þ}_{\underline{\Gamma}},$$

which implies that if we wish to prevent m from heading to infinity (that is, if we want m to be expected to fall for some large enough value of m) we must impose a generalized version of the Perfect Foresight Growth Impatience Condition (**??**); we call the 'growth impatience condition' (GIC) the requirement that the Growth Patience Factor must be less than 1:[19]

$$\text{Þ}_{\underline{\Gamma}} < 1 \tag{27}$$

which is stronger than the perfect foresight version (**??**) because

$$\underline{\Gamma} < \Gamma \tag{28}$$

(Jensen's inequality implies that $\underline{\psi} < 1$ for nondegenerate ψ).

2.4.4 Autarky Value

Analogously to (22), a consumer who spent exactly their permanent income every period would have value determined by the product of the expectation of the (independent) future shocks to permanent income:

$$\begin{aligned} \mathbf{v}_t &= \mathbb{E}_t\left[\mathrm{u}(\mathbf{p}_t) + \beta\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}) + ... + \beta^{T-t}\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}...\Gamma_T)\right] \\ &= \mathrm{u}(\mathbf{p}_t)\left(1 + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}] + ... + \beta^{T-t}\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}]...\,\mathbb{E}_t[\Gamma_T^{1-\rho}]\right) \\ &= \mathrm{u}(\mathbf{p}_t)\left(\frac{1 - (\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}])^{T-t+1}}{1 - \beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}]}\right) \end{aligned}$$

[19] Equation (27) is a bit easier to satisfy than the similar condition imposed by Deaton (1991): $\left(\mathbb{E}[\psi^{-\rho}]\right)^{1/\rho}\text{Þ}_\Gamma < 1$ to guarantee that his problem defined a contraction mapping.

which invites the definition of a utility-compensated equivalent of the permanent shock,

$$\underline{\underline{\psi}} = (\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)}$$

which will satisfy $\underline{\underline{\psi}} < 1$ for $\rho > 1$ and nondegenerate ψ (and $\underline{\underline{\psi}} < \underline{\psi}$ for the reasonable (though not required) case of $\rho > 2$); defining

$$\underline{\underline{\Gamma}} = \Gamma\underline{\underline{\psi}} \tag{29}$$

we can see that $\mathbf{v}_t$ will be finite as T approaches ∞ if

$$\begin{aligned} \overbrace{\beta\underline{\underline{\Gamma}}^{1-\rho}}^{\equiv\underline{\underline{\beth}}} &< 1 \\ \beta &< \underline{\underline{\Gamma}}^{\rho-1} \end{aligned} \tag{30}$$

which we call the 'finite value of autarky' condition (FVAC) because it guarantees that value is finite for a consumer who always consumes their (now stochastic) permanent income. For nondegenerate ψ, this condition is stronger (harder to satisfy in the sense of requiring lower β) than the perfect foresight version (**??**) because $\underline{\underline{\Gamma}} < \Gamma$.[20]

2.5 The Baseline Numerical Solution

Figure 1 depicts the successive consumption rules that apply in the last period of life ($\mathrm{c}_T(m)$), the second-to-last period, and various earlier periods under the baseline parameter values listed in Table 2. (The 45 degree line is labelled as $\mathrm{c}_T(m) = m$ because in the last period of life it is optimal to spend all remaining resources.)

In the figure, the consumption rules appear to converge as the horizon recedes. Our next purpose is to show that this appearance is not deceptive; we call the hypothesized limiting infinite-horizon consumption rule

$$\mathrm{c}(m) \equiv \lim_{n\to\infty} \mathrm{c}_{T-n}(m). \tag{31}$$

2.6 Concave Consumption Function Characteristics

A precondition for the main proof is that the maximization problem (6) defines a sequence of continuously differentiable strictly increasing strictly concave[21] functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, ...\}$.[22] The straightforward but tedious proof is relegated to appendix B. For present purposes, the most important point is the following intuition: $\mathrm{c}_t(m) < m$ for

[20] To see this, rewrite (30) as

$$\begin{aligned} \beta\mathsf{R} &< \mathsf{R}\underline{\underline{\Gamma}}^{\rho-1} \\ (\beta\mathsf{R})^{1/\rho} &< \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \\ \text{Þ}_\Gamma &< (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \end{aligned}$$

where the last equation is the same as the PF-FVAC condition except that the RHS is multiplied by $\underline{\underline{\psi}}^{1-1/\rho}$ which is strictly less than 1.

[21] There is one obvious exception: $\mathrm{c}_T(m)$ is a linear (and so only weakly concave) function.

[22] Carroll and Kimball (1996) proved concavity but not the other desired properties.

Table 1 Microeconomic Model Calibration

Calibrated Parameters			
Description	Parameter	Value	Source
Permanent Income Growth Factor	Γ	1.03	PSID: Carroll (1992)
Interest Factor	R	1.04	Conventional
Time Preference Factor	β	0.96	Conventional
Coefficient of Relative Risk Aversion	ρ	2	Conventional
Probability of Zero Income	$\wp$	0.005	PSID: Carroll (1992)
Std Dev of Log Permanent Shock	σ_{ψ}	0.1	PSID: Carroll (1992)
Std Dev of Log Transitory Shock	σ_{θ}	0.1	PSID: Carroll (1992)

Table 2 Model Characteristics Calculated from Parameters

Description	Symbol and Formula			Approximate Calculated Value
Finite Human Wealth Measure	$\mathcal{R}^{-1}$	$\equiv$	Γ/R	0.990
PF Finite Value of Autarky Measure	$\beth$	$\equiv$	$\beta\Gamma^{1-\rho}$	0.932
Growth Compensated Permanent Shock	$\underline{\psi}$	$\equiv$	$(\mathbb{E}[\psi^{-1}])^{-1}$	0.990
Uncertainty-Adjusted Growth	$\underline{\Gamma}$	$\equiv$	$\Gamma\underline{\psi}$	1.020
Utility Compensated Permanent Shock	$\underline{\underline{\psi}}$	$\equiv$	$(\mathbb{E}_t[\psi^{1-\rho}])^{1/(1-\rho)}$	0.990
Utility Compensated Growth	$\underline{\underline{\Gamma}}$	$\equiv$	$\Gamma\underline{\underline{\psi}}$	1.020
Absolute Patience Factor	Þ	$\equiv$	$(\mathsf{R}\beta)^{1/\rho}$	0.999
Return Patience Factor	Þ_{R}	$\equiv$	$\text{Þ}/\mathsf{R}$	0.961
PF Growth Patience Factor	Þ_{Γ}	$\equiv$	$\text{Þ}/\Gamma$	0.970
Growth Patience Factor	$\text{Þ}_{\underline{\Gamma}}$	$\equiv$	$\text{Þ}/\underline{\Gamma}$	0.980
Finite Value of Autarky Measure	$\underline{\underline{\beth}}$	$\equiv$	$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho}$	0.941

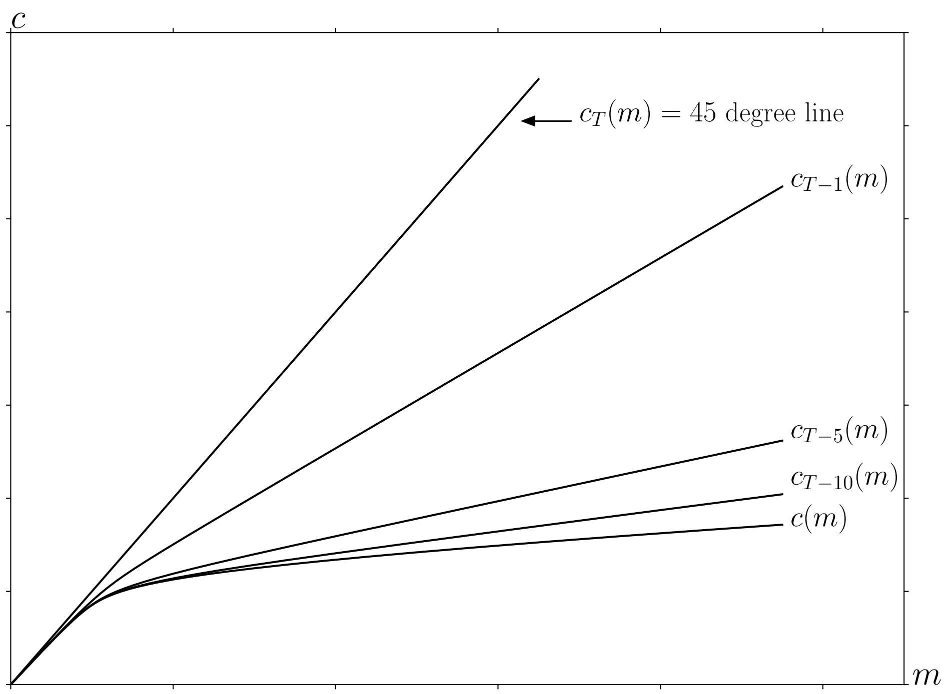


Figure 1 Convergence of the Consumption Rules

all periods $t < T$ because a consumer who spent all available resources would arrive in period $t+1$ with balances b_{t+1} of zero, and then might earn zero noncapital income over the remaining horizon (an unbroken series of zero-income events is unlikely but possible). In such a case, the budget constraint and the can't-die-in-debt condition mean that the consumer would be forced to spend zero, incurring negative infinite utility. To avoid this disaster, the consumer never spends everything. (This is an example of the 'natural borrowing constraint' induced by a precautionary motive, per Zeldes (1989)).[23]

2.7 Bounds for the Consumption Functions

The consumption functions depicted in Figure 1 appear to have limiting slopes as $m \downarrow 0$ and as $m \uparrow \infty$. This section confirms that impression and derives those slopes, which also turn out to be useful in the contraction mapping proof.[24,25]

Assume (as discussed above) that a continuously differentiable concave consumption function exists in period $t+1$, with an origin at $\mathrm{c}_{t+1}(0) = 0$, a minimal MPC $\underline{\kappa}_{t+1} > 0$, and maximal MPC $\bar{\kappa}_{t+1} \leq 1$. (If $t+1 = T$ these will be $\underline{\kappa}_T = \bar{\kappa}_T = 1$; for earlier periods they will exist by recursion from the following arguments.)

The MPC bound as wealth approaches infinity is easy to understand: In this case, under our imposed assumption that human wealth is finite, the proportion of consumption that will be financed out of human wealth approaches zero. The consequence is that the proportional difference between the solution to the model with uncertainty and the perfect foresight model shrinks to zero.

In the course of proving this, appendix F provides a useful recursive expression for the (inverse of the) limiting MPC:

$$\underline{\kappa}_t^{-1} = 1 + \text{Þ}_{\mathsf{R}} \underline{\kappa}_{t+1}^{-1}. \tag{32}$$

2.7.1 Weak RIC Conditions

There is a parallel expression for the limiting maximal MPC as $m \downarrow 0$: appendix equation (93) shows that, as $m_t \uparrow \infty$,

$$\bar{\kappa}_t^{-1} = 1 + \wp^{1/\rho} \text{Þ}_{\mathsf{R}} \bar{\kappa}_{t+1}^{-1}. \tag{33}$$

$\left\{\bar{\kappa}_{T-n}^{-1}\right\}_{n=0}^{\infty}$ is a decreasing convergent sequence if the 'weak return patience factor' $\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ satisfies:

$$0 \leq \wp^{1/\rho} \text{Þ}_{\mathsf{R}} < 1, \tag{34}$$

[23] It would perhaps be better to call it the 'utility-induced borrowing constraint' as it follows from the assumptions on the utility function (in particular, $\lim_{c \downarrow 0} \mathrm{u}(c) = -\infty$); for example, no such constraint arises if utility is of the (implausible) Constant Absolute Risk Aversion form.

[24] Benhabib, Bisin, and Zhu (2015) show that the consumption function becomes linear as wealth approaches infinity in a model with capital income risk and liquidity constraints; their results should generalize to the limits derived here if capital income risk were added to the model.

[25] Ma, Stachurski, and Toda (2020) establish the existence and uniqueness of a solution to a general income fluctuation problem with capital income risk in a Markovian setting and use such a model to study the tail behavior of wealth in the presence of risky returns to capital.

a condition that we dub the 'Weak Return Impatience Condition' (WRIC) because with $\wp < 1$ it will hold more easily (for a larger set of parameter values) than the RIC ($\text{\th}_{\mathsf{R}} < 1$).

The essence of the argument is that as wealth approaches zero, the overriding consideration that limits consumption is the (recursive) fear of the zero-income events. (That consideration is why the probability of the zero income event $\wp$ appears in the expression.)

We are now in position to observe that the optimal consumption function must satisfy

$$\underline{\kappa}_t m_t \ \leq \mathrm{c}_t(m_t) \leq \ \bar{\kappa}_t m_t \tag{35}$$

because consumption starts at zero and is continuously differentiable (as argued above), is strictly concave (Carroll and Kimball (1996)), and always exhibits a slope between $\underline{\kappa}_t$ and $\bar{\kappa}_t$ (the formal proof is provided in appendix D).

These limits are useful at least in the sense that they can be hard-wired into a solution algorithm for the model, which has the potential to make the solution more efficient (cf. Carroll, Chipeniuk, Tokuoka, and Wu (2020)). Alternatively, they can provide a useful check on the accuracy of a solution algorithm that does not impose them directly.

2.8 Conditions Under Which the Problem Defines a Contraction Mapping

To prove that the consumption rules converge, we need to show that the problem defines a contraction mapping. This cannot be proven using the standard theorems in, say, Stokey et. al. (1989), which require marginal utility to be bounded over the space of possible values of m, because the possibility (however unlikely) of an unbroken string of zero-income events for the remainder of life means that as m approaches zero c must approach zero (see the discussion in 2.6); thus, marginal utility is unbounded. Although a recent literature examines the existence and uniqueness of solutions to Bellman equations in the presence of 'unbounded returns' (see, e.g., Matkowski and Nowak (2011)), the techniques in that literature cannot be used to solve the problem here because the required conditions are violated by a problem that involves permanent shocks.[26]

Fortunately, Boyd (1990) provided a weighted contraction mapping theorem that Alvarez and Stokey (1998) showed could be used to address the homogeneous case (of which CRRA formulation is an example) in a deterministic framework; later, Durán (2003) showed how to extend the Boyd (1990) approach to the stochastic case.

Definition 1. *Consider any function $\bullet \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ where $\mathcal{C}(\mathcal{A}, \mathcal{B})$ is the space of continuous functions from $\mathcal{A}$ to $\mathcal{B}$. Suppose $F \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ with $\mathcal{B} \subseteq \mathbb{R}$ and $F > 0$. Then $\bullet$ is F-bounded if the F-norm of $\bullet$,*

$$\| \bullet \|_F = \sup_m \left[\frac{| \bullet (m)|}{F(m)} \right], \tag{36}$$

is finite.

[26]See Yao (2012) for a detailed discussion of the reasons the existing literature up through Matkowski and Nowak (2011) cannot handle the problem described here.

For $\mathcal{C}_{\digamma}(\mathcal{A},\mathcal{B})$ defined as the set of functions in $\mathcal{C}(\mathcal{A},\mathcal{B})$ that are $\digamma$-bounded; w, x, y, and z as examples of $\digamma$-bounded functions; and using $\mathbf{0}(m) = 0$ to indicate the function that returns zero for any argument, Boyd (1990) proves the following.

Boyd's Weighted Contraction Mapping Theorem. *Let* $\mathsf{T} : \mathcal{C}_{\digamma}(\mathcal{A},\mathcal{B}) \to \mathcal{C}(\mathcal{A},\mathcal{B})$ *such that*[27,28]

$$
\begin{aligned}
&(1)\mathsf{T} \textit{ is non-decreasing, i.e. } \mathrm{x} \leq \mathrm{y} \Rightarrow \{\mathsf{T}\mathrm{x}\} \leq \{\mathsf{T}\mathrm{y}\} \\
&(2)\{\mathsf{T}\mathbf{0}\} \in \mathcal{C}_{\digamma}(\mathcal{A},\mathcal{B}) \\
&(3) \textit{There exists some real } 0 < \alpha < 1, \textit{ such that} \\
&\quad \{\mathsf{T}(\mathrm{w} + \zeta\digamma)\} \leq \{\mathsf{T}\mathrm{w}\} + \zeta\alpha\digamma \quad \textit{holds for all real } \zeta > 0\,.
\end{aligned}
$$

Then T *defines a contraction with a unique fixed point.*

For our problem, take $\mathcal{A}$ as $\mathbb{R}_{>0}$ and $\mathcal{B}$ as $\mathbb{R}$, and define

$$\{\mathsf{E}\mathrm{z}\}(a_t) = \mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{z}(a_t\mathcal{R}_{t+1} + \xi_{t+1})\right].$$

Using this, we introduce the mapping $\mathfrak{T} : \mathcal{C}_{\digamma}(\mathcal{A},\mathcal{B}) \to \mathcal{C}(\mathcal{A},\mathcal{B})$,[29]

$$\{\mathfrak{T}\mathrm{z}\}(m_t) = \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \mathrm{u}(c_t) + \beta\left(\{\mathsf{E}\mathrm{z}\}(m_t - c_t)\right). \tag{37}$$

We can show that our operator $\mathfrak{T}$ satisfies the conditions that Boyd requires of his operator T, if we impose two restrictions on parameter values. The first is the WRIC necessary for convergence of the maximal MPC, equation (34) above. A more serious restriction is the utility-compensated Finite Value of Autarky condition, equation (30). (We discuss the interpretation of these restrictions in detail in section 2.10 below.) Imposing these restrictions, we are now in position to state the central theorem of the paper.

Theorem 1. $\mathfrak{T}$ *is a contraction mapping if the restrictions on parameter values* (34) *and* (30) *are true (that is, if the weak return impatience condition and the finite value of autarky condition hold).*

Intuitively, Boyd's theorem shows that if you can find a $\digamma$ that is everywhere finite but goes to infinity 'as fast or faster' than the function you are normalizing with $\digamma$, the normalized problem defines a contraction mapping. The intuition for the FVAC condition is just that, with an infinite horizon, with any initial amount of bank balances b_0, in the limit your value can always be made greater than you would get by consuming exactly your permanent income every period (say, by consuming $(\mathsf{r}/\mathsf{R})b_0 - \epsilon$ for some small $\epsilon > 0$).

The details of the proof are cumbersome, and are therefore relegated to appendix D. Given that the value function converges, appendix D.3 shows that the consumption functions converge.

[27] We will usually denote the function that results from the mapping as, e.g., $\{\mathsf{T}\mathrm{w}\}$.

[28] To non-theorists, this notation may be slightly confusing; the inequality relations in (1) and (3) are taken to mean 'for any specific element $\bullet$ in the domain of the functions in question' so that, e.g., $\mathrm{x} \leq \mathrm{y}$ is short for $\mathrm{x}(\bullet) \leq \mathrm{y}(\bullet)\ \forall \bullet \in \mathcal{A}$. In this notation, $\zeta\alpha\digamma$ in (3) is a *function* which can be applied to any argument $\bullet$ (because $\digamma$ is a function).

[29] Note that the existence of the maximum is assured by the continuity of $\{\mathsf{E}\mathrm{z}\}(a_t)$ (it is continuous because it is the sum of continuous $\digamma$-bounded functions z) and the compactness of $[\underline{\kappa} m_t, \bar{\kappa} m_t]$.

2.9 The Liquidity Constrained Solution as a Limit

This section shows that a related problem commonly considered in the literature (e.g., with a simpler income process, by Deaton (1991)), with a liquidity constraint and a positive minimum value of income, is the limit of the problem considered here as the probability $\wp$ of the zero-income event approaches zero.

The essence of the argument is easy to state. As noted above, the possibility of earning zero income over the remainder of the horizon prevents the consumer from ending the current period with zero assets because with some finite probability the consumer would be forced to consume zero, which would be infinitely painful.

But the *extent* to which the consumer feels the need to make this precautionary provision depends on the probability that it will turn out to matter. As $\wp \downarrow 0$, that probability becomes arbitrarily small, so the amount of precautionary saving approaches zero. But zero precautionary saving is the amount of saving that a liquidity constrained consumer with perfect foresight would choose.

Another way to understand this is just to think of the liquidity constraint as being imposed by specifying a component of the utility function that is zero whenever the consumer ends the period with (strictly) positive assets, but negative infinity if the consumer ended the period with (weakly) negative assets.

See appendix G for the formal proof justifying the foregoing intuitive discussion.

2.10 Discussion of Parametric Restrictions

The full relationship among all the conditions articulated above is represented in Figure 2. Though the diagram looks complex, it is merely a modified version of the earlier diagram with further (mostly intermediate) inequalities inserted. Again readers unfamiliar with such diagrams should Appendix) for a more detailed explanation.

2.10.1 The RIC

In the perfect foresight unconstrained problem (section 2.4.2), the RIC was required for existence of a nondegenerate solution. It is surprising, therefore, that in the presence of uncertainty, the RIC is neither necessary nor sufficient for a nondegenerate solution. We thus begin our discussion by asking what features the problem must exhibit (given the FVAC) if the RIC fails (that is, $\mathsf{R} < (\mathsf{R}\beta)^{1/\rho}$):

$$
\begin{aligned}
\mathsf{R} &< \overbrace{(\mathsf{R}\beta)^{1/\rho} \;<\; (\mathsf{R}(\Gamma\underline{\underline{\psi}})^{\rho-1})^{1/\rho}}^{\text{implied by FVAC}} \\
\mathsf{R} &< (\mathsf{R}/\Gamma)^{1/\rho}\Gamma\underline{\underline{\psi}}^{1-1/\rho} \\
\mathsf{R}/\Gamma &< (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \\
(\mathsf{R}/\Gamma)^{1-1/\rho} &< \underline{\underline{\psi}}^{1-1/\rho}
\end{aligned}
\tag{38}
$$

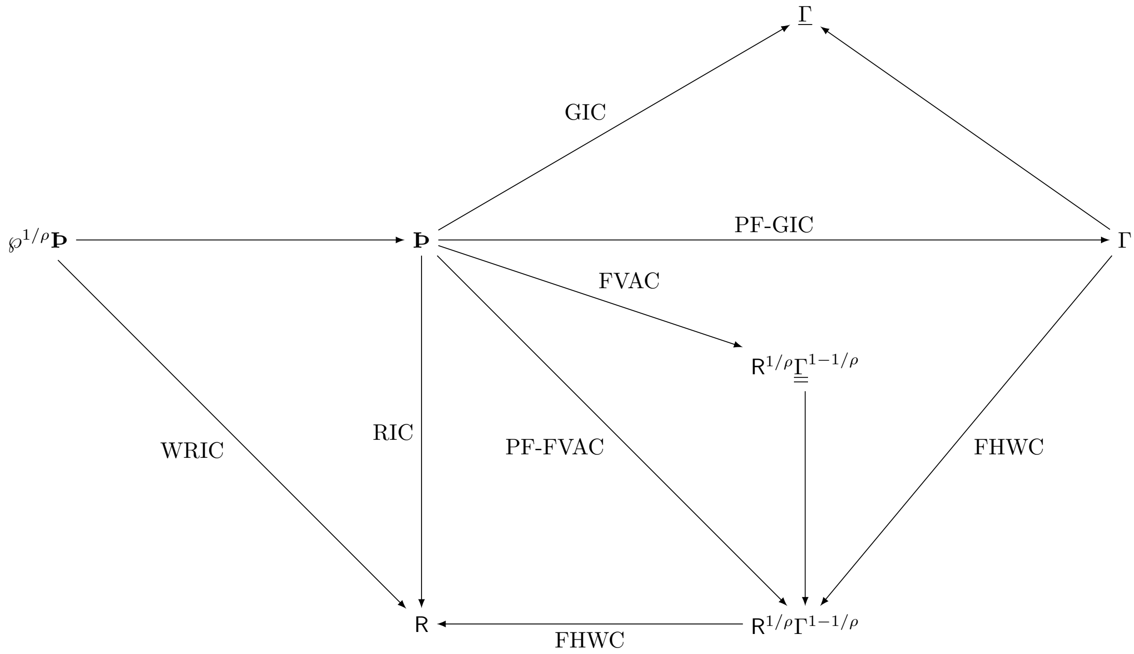


Figure 2 Relation of All Inequality Conditions

but since $\underline{\underline{\psi}} < 1$ and $0 < 1 - 1/\rho < 1$ (because we have assumed $\rho > 1$), equation (38) this can hold only if $\mathsf{R}/\Gamma < 1$; that is, given the FVAC, the RIC can fail only if human wealth is unbounded. Unbounded human wealth is permitted here, as in the perfect foresight liquidity constrained problem. But, from equation (32), an implication of ~~RIC~~ is that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Thus, interestingly, the presence of uncertainty both permits unlimited human wealth and at the same time prevents that unlimited wealth from resulting in infinite consumption. That is, in the presence of uncertainty, pathological patience (which in the perfect foresight model with finite wealth results in consumption of zero) plus infinite human wealth (which the perfect foresight model rules out because it leads to infinite consumption) combine here to yield a unique finite limiting MPC for any finite value of m. Note the close parallel to the conclusion in the perfect foresight liquidity constrained model in the {PF-GIC,~~RIC~~} case (for detailed analysis of this case see appendix A). There, too, the tension between infinite human wealth and pathological patience was resolved with a nondegenerate consumption function whose limiting MPC was zero.

2.10.2 The WRIC

The ‘weakness’ of the additional requirement for contraction, the weak RIC, can be seen by asking ‘under what circumstances would the FVAC hold but the WRIC fail?’ Algebraically, the requirement is

$$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho} < \ 1 \ < (\wp\beta)^{1/\rho}/\mathsf{R}^{1-1/\rho}. \tag{39}$$

If there were no conceivable parameter values that could satisfy both of these inequalities, the WRIC would have no force. And if we require $\mathsf{R} \geq 1$, the WRIC is indeed redundant because now $\beta < 1 < \mathsf{R}^{\rho-1}$, so that the RIC (and WRIC) must hold.

But neither theory nor evidence demands that we assume $\mathsf{R} \geq 1$. We can therefore approach the question of the WRIC's relevance by asking just how low R must be for the condition to be relevant. Suppose for illustration that $\rho = 2$, $\underline{\underline{\psi}}^{1-\rho} = 1.01$, $\Gamma^{1-\rho} = 1.01^{-1}$ and $\wp = 0.10$. In that case (39) reduces to

$$\beta < 1 < (0.1\beta/\mathsf{R})^{1/2}$$

but since $\beta < 1$ by assumption, the binding requirement is that

$$\mathsf{R} < \beta/10$$

so that for example if $\beta = 0.96$ we would need $\mathsf{R} < 0.096$ (that is, a perpetual riskfree rate of return of worse than -90 percent a year) in order for the WRIC to bind. The relevance of the WRIC is indeed "Weak."

Perhaps the best way of thinking about this is to note that the space of parameter values for which the WRIC is relevant shrinks out of existence as $\wp \to 0$, which section 2.9 showed was the precise limiting condition under which behavior becomes arbitrarily close to the liquidity constrained solution (in the absence of other risks). On the other hand, when $\wp = 1$, the consumer has no noncapital income (so that the FHWC holds) and with $\wp = 1$ the WRIC is identical to the RIC; but the RIC is the only condition required for a solution to exist for a perfect foresight consumer with no noncapital income. Thus the WRIC forms a sort of 'bridge' between the liquidity constrained and the unconstrained problems as $\wp$ moves from 0 to 1.

2.10.3 When the GIC Fails

If both the GIC and the RIC hold, the arguments above establish that the limiting consumption function asymptotes to the consumption function for the perfect foresight unconstrained function. The more interesting case is where the GIC fails. A solution that satisfies the combination FVAC and ~~GIC~~ is depicted in Figure 3. The consumption function is shown along with the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus that identifies the 'sustainable' level of spending at which m is expected to remain unchanged. The diagram suggests a fact that is confirmed by deeper analysis: Under the depicted configuration of parameter values (see the code for details), the consumption function never reaches the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus; indeed, when the RIC holds but the GIC does not, the consumption function's limiting slope $(1 - \mathbf{Þ}/\mathsf{R})$ is shallower than that of the sustainable consumption locus $(1 - \underline{\Gamma}/\mathsf{R})$,[30] so the gap between the two actually increases with m in the limit. Although a nondegenerate consumption function exists, a target level of m does not (or, rather, the target is $m = \infty$), because no matter how wealthy a consumer becomes, the consumer will always spend less than the amount that would keep m stable (in expectation).

Tables 3 and 4 present a summary of the connections between the various conditions in the presence and the absence of uncertainty.

[30] This is because $\mathbb{E}_t[m_{t+1}] = \mathbb{E}_t[\mathcal{R}_{t+1}(m_t - c_t)] + 1$; solve $m = (m - c)\mathcal{R}\underline{\psi}^{-1} + 1$ for c and differentiate.

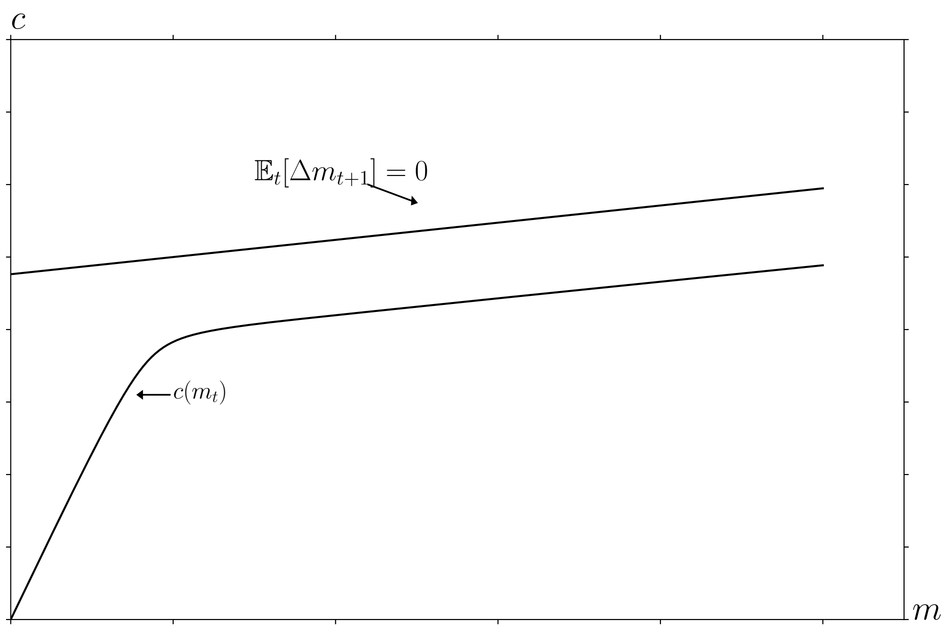


Figure 3 Example Solution when FVAC Holds but GIC Does Not

3 Analysis of the Converged Consumption Function

Figures 4 and 5a,b capture the main properties of the converged consumption rule when the RIC, GIC, and FHWC all hold.[31] Figure 4 shows the expected consumption growth factor $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ for a consumer behaving according to the converged consumption rule, while Figures 5a,b illustrate theoretical bounds for the consumption function and the marginal propensity to consume.

Five features of behavior are captured, or suggested, by the figures. First, as $m_t \uparrow \infty$ the expected consumption growth factor goes to $\mathbf{Þ}$, indicated by the lower bound in Figure 4, and the marginal propensity to consume approaches $\underline{\kappa} = (1 - \mathbf{Þ}_{\mathsf{R}})$ (Figure 5), the same as the perfect foresight MPC.[32] Second, as $m_t \downarrow 0$ the consumption growth factor approaches ∞ (Figure 4) and the MPC approaches $\bar{\kappa} = (1 - \wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}})$ (Figure 5). Third (Figure 4), there is a target cash-on-hand-to-income ratio $\check{m}$ such that if $m_t = \check{m}$ then $\mathbb{E}_t[m_{t+1}] = m_t$, and (as indicated by the arrows of motion on the $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ curve), the model's dynamics are 'stable' around the target in the sense that if $m_t < \check{m}$ then cash-on-hand will rise (in expectation), while if $m_t > \check{m}$, it will fall (in expectation). Fourth (Figure 4), at the target m, the expected rate of growth of consumption is slightly less than the expected growth rate of permanent noncapital income. The final proposition suggested by Figure 4 is that the expected consumption growth factor is declining in the level of the cash-on-hand ratio m_t. This turns out to be true in the absence of permanent shocks, but in extreme cases it can be false if permanent shocks are present.[33]

3.1 Limits as $m_t \uparrow \infty$

Define

$$\underline{\mathrm{c}}(m) = \underline{\kappa} m$$

which is the solution to an infinite-horizon problem with no noncapital income ($\xi_{t+n} = 0 \ \forall \ n \geq 1$); clearly $\underline{\mathrm{c}}(m) < \mathrm{c}(m)$, since allowing the possibility of future noncapital income cannot reduce current consumption.[34]

Assuming the FHWC holds, the infinite horizon perfect foresight solution (21) constitutes an upper bound on consumption in the presence of uncertainty, since Carroll and Kimball (1996) show that the introduction of uncertainty strictly decreases the level of consumption at any m.

[31] These figures reflect the converged rule corresponding to the parameter values indicated in Table 2.

[32] If the RIC fails, the limiting minimal MPC is 0; see appendix.

[33] Throughout the remaining analysis I make a final assumption that is not strictly justified by the foregoing. We have seen that the finite-horizon consumption functions $\mathrm{c}_{T-n}(m)$ are twice continuously differentiable and strictly concave, and that they converge to a continuous function $\mathrm{c}(m)$. It does not strictly follow that the limiting function $\mathrm{c}(m)$ is twice continuously differentiable, but I will assume that it is.

[34] We will assume the RIC holds here and subsequently so that $\underline{\kappa} > 0$; the situation is a bit more complex when the RIC does not hold. In that case the bound on consumption is given by the spending that would be undertaken by a consumer who faced binding liquidity constraints. Detailed analysis of this special case is not sufficiently interesting to warrant inclusion in the paper.

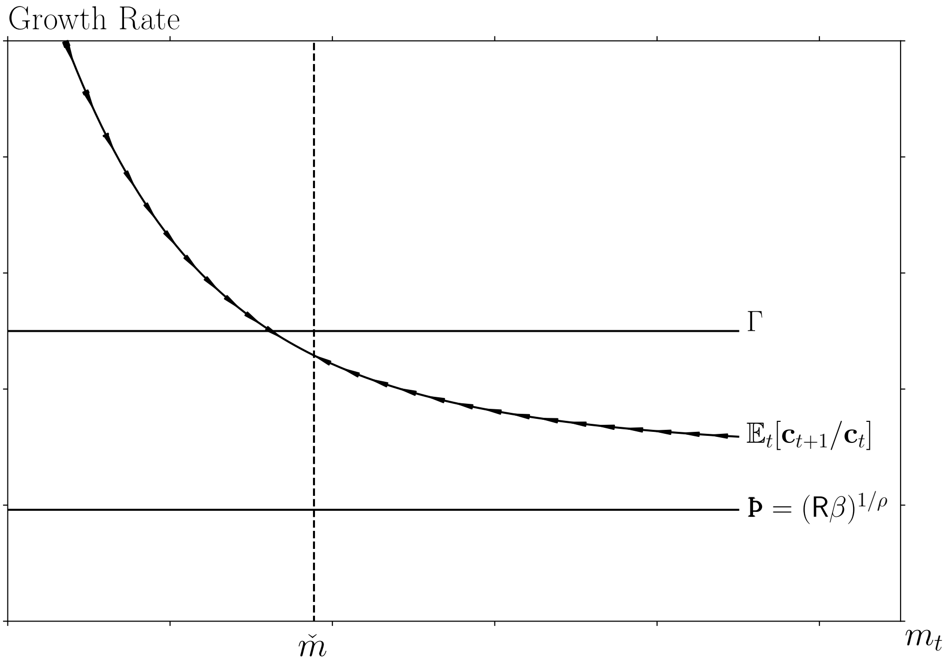


Figure 4 Target m, Expected Consumption Growth, and Permanent Income Growth

Thus, we can write

$$\begin{aligned} \underline{\mathrm{c}}(m) <& \mathrm{c}(m) < \bar{\mathrm{c}}(m) \\ 1 <& \mathrm{c}(m)/\underline{\mathrm{c}}(m) < \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m). \end{aligned} \tag{40}$$

But

$$\begin{aligned} \lim_{m\uparrow\infty} \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) &= \lim_{m\uparrow\infty} (m-1+h)/m \\ &= 1, \end{aligned}$$

so as $m \uparrow \infty, \mathrm{c}(m)/\underline{\mathrm{c}}(m) \to 1$, and the continuous differentiability and strict concavity of $\mathrm{c}(m)$ therefore implies

$$\lim_{m\uparrow\infty} \mathrm{c}'(m) = \underline{\mathrm{c}}'(m) = \bar{\mathrm{c}}'(m) = \underline{\kappa}$$

because any other fixed limit would eventually lead to a level of consumption either exceeding $\bar{\mathrm{c}}(m)$ or lower than $\underline{\mathrm{c}}(m)$.

Figure 5 confirms these limits visually. The top plot shows the converged consumption function along with its upper and lower bounds, while the lower plot shows the marginal propensity to consume.

Next we establish the limit of the expected consumption growth factor as $m_t \uparrow \infty$:

$$\lim_{m_t\uparrow\infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] = \lim_{m_t\uparrow\infty} \mathbb{E}_t[\Gamma_{t+1} c_{t+1}/c_t].$$

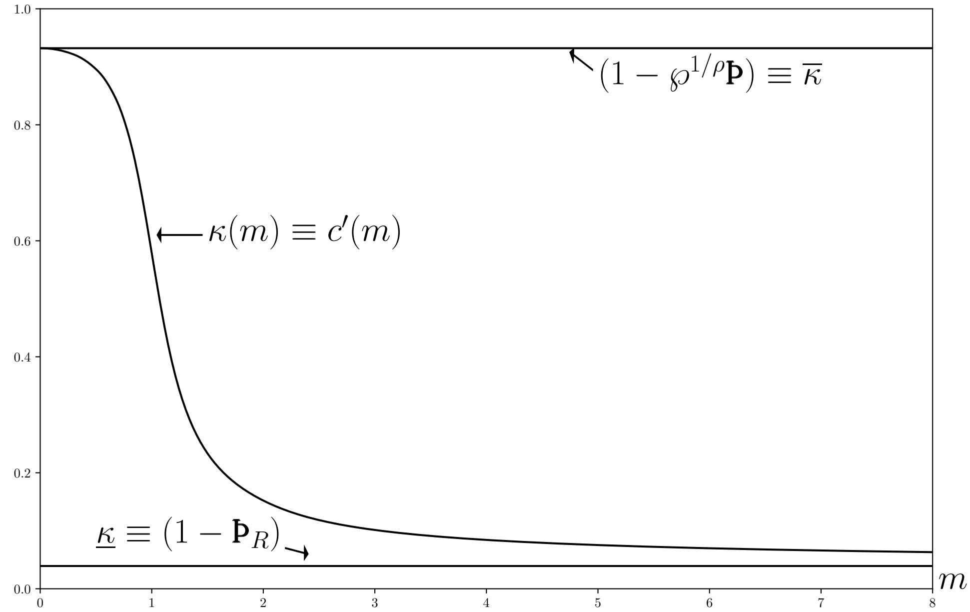


Figure 5 Limiting MPC's

But

$$\mathbb{E}_t[\Gamma_{t+1}\underline{c}_{t+1}/\bar{c}_t] \leq \mathbb{E}_t[\Gamma_{t+1}c_{t+1}/c_t] \leq \mathbb{E}_t[\Gamma_{t+1}\bar{c}_{t+1}/\underline{c}_t]$$

and

$$\lim_{m_t \uparrow \infty} \Gamma_{t+1}\underline{\mathrm{c}}(m_{t+1})/\bar{\mathrm{c}}(m_t) = \lim_{m_t \uparrow \infty} \Gamma_{t+1}\bar{\mathrm{c}}(m_{t+1})/\underline{\mathrm{c}}(m_t) = \lim_{m_t \uparrow \infty} \Gamma_{t+1}m_{t+1}/m_t,$$

while

$$\lim_{m_t \uparrow \infty} \Gamma_{t+1}m_{t+1}/m_t = \lim_{m_t \uparrow \infty} \left(\frac{\mathsf{R}\mathrm{a}(m_t) + \Gamma_{t+1}\xi_{t+1}}{m_t}\right) \tag{41}$$

$$= (\mathsf{R}\beta)^{1/\rho} = \boldsymbol{\text{Þ}} \tag{42}$$

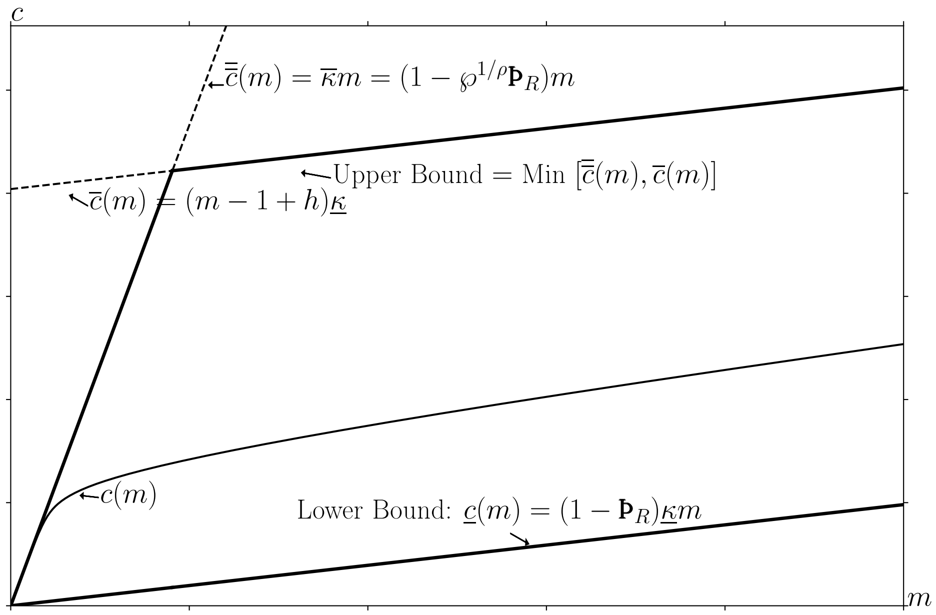
because $\lim_{m_t \uparrow \infty} \mathrm{a}'(m) = \boldsymbol{\text{Þ}}_{\mathsf{R}}$[35] and $\Gamma_{t+1}\xi_{t+1}/m_t \leq (\Gamma\bar{\psi}\bar{\theta}/(1-\wp))/m_t$ which goes to zero as m_t goes to infinity.

Hence we have

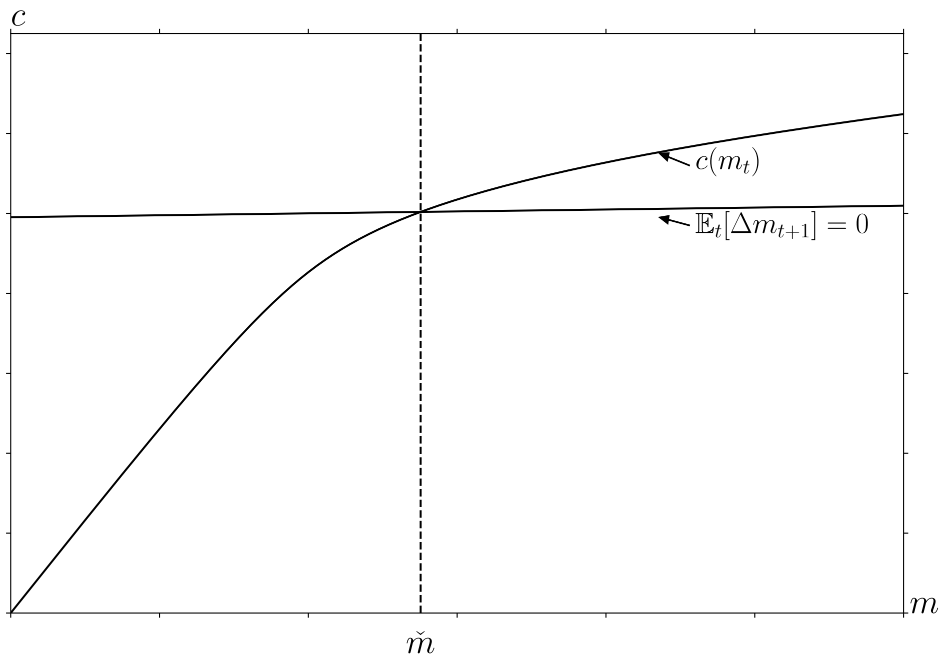
$$\boldsymbol{\text{Þ}} \leq \lim_{m_t \uparrow \infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] \leq \boldsymbol{\text{Þ}}$$

so as cash goes to infinity, consumption growth approaches its value $\boldsymbol{\text{Þ}}$ in the perfect foresight model.

[35]This is because $\lim_{m_t \uparrow \infty} \mathrm{a}(m_t)/m_t = 1 - \lim_{m_t \uparrow \infty} \mathrm{c}(m_t)/m_t = 1 - \lim_{m_t \uparrow \infty} \mathrm{c}'(m_t) = \boldsymbol{\text{Þ}}_{\mathsf{R}}$.



(a) Bounds



(b) Target m

Figure 6 The Consumption Function

This argument applies equally well to the problem of the restrained consumer, because as m approaches infinity the constraint becomes irrelevant (assuming the FHWC holds).

3.2 Limits as $m_t \downarrow 0$

Now consider the limits of behavior as m_t gets arbitrarily small.

Equation (33) shows that the limiting value of $\bar{\kappa}$ is

$$\bar{\kappa} = 1 - \mathsf{R}^{-1}(\wp \mathsf{R}\beta)^{1/\rho}.$$

Defining $\mathrm{e}(m) = \mathrm{c}(m)/m$ as before we have

$$\lim_{m \downarrow 0} \mathrm{e}(m) = (1 - \wp^{1/\rho} \text{Þ}_{\mathsf{R}}) = \bar{\kappa}.$$

Now using the continuous differentiability of the consumption function along with L'Hôpital's rule, we have

$$\lim_{m \downarrow 0} \mathrm{c}'(m) = \lim_{m \downarrow 0} \mathrm{e}(m) = \bar{\kappa}.$$

Figure 5 confirms that the numerical solution method obtains this limit for the MPC as m approaches zero.

For consumption growth, as $m \downarrow 0$ we have

$$\begin{aligned}
\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right) \Gamma_{t+1} \right] &> \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1} \mathrm{a}(m_t) + \xi_{t+1})}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \wp \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1} \mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&\quad + (1 - \wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1} \mathrm{a}(m_t) + \theta_{t+1}/(1 - \wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&> (1 - \wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\theta_{t+1}/(1 - \wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \infty
\end{aligned}$$

where the second-to-last line follows because $\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1} \mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.[36]

3.3 There Exists Exactly One Target Cash-on-Hand Ratio, which is Stable

We now prove the existence of a target cash-on-hand-to-income ratio $\check{m}$ towards which an agent's m expects to move. (The $\vee$ accent is meant to invoke the fact that this is the value that other m's 'point to.') We state the necessary conditions for the existence of $\check{m}$ and its properties in the following theorem.

[36] The same arguments establish $\lim_{m \downarrow 0} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] = \infty$ for the problem of the restrained consumer.

Theorem 2. *For the problem defined in section 2.1, if the GIC (27), and WRIC (34) hold, then there exists a unique cash-on-hand-to-income ratio $\check{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{43}$$

Moreover, $\check{m}$ is stable in the sense that

$$\begin{aligned} \forall m_t \in (0, \check{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\ \forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t. \end{aligned} \tag{44}$$

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

3.3.1 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$.

The consumption function exists because we have imposed the conditions (the WRIC and FVAC) that theorem 1 establishes are sufficient for its existence. (Indeed, Appendix C shows that $\mathrm{c}(m)$ is not just continuous, but twice continuously differentiable.)

Section 2.6 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + y_t$, even if y_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive under our foregoing assumptions. With $m_t > 0$, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

3.3.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

The logic in section 3.2 showing that $\lim_{m_t \downarrow 0} \mathbb{E}_t[c_{t+1}/c_t] = \infty$ transparently implies the same proposition for m_t: $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$ so the ratio is unbounded.

The limit as m_t goes to infinity is

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1}}{m_t}\right] \\ &= \mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})\text{\textbf{Þ}}_{\mathsf{R}}] \\ &= \mathbb{E}_t[\text{\textbf{Þ}}/\Gamma_{t+1}] \\ &< 1 \end{aligned} \tag{45}$$

where the last two lines are merely a restatement of the GIC (27).

The Intermediate Value Theorem tells us that if $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

3.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ *is monotonically decreasing.*

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned} \tag{46}$$

so that $\boldsymbol{\zeta}(\check{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1} - m_t)\right] \\
&= \bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1.
\end{aligned} \tag{47}$$

Note that the statement of theorem 2 did not require the RIC to hold. Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails ($\cancel{\text{RIC}}$).

If RIC holds. Equation 20 indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.7.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}
\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1 &< \bar{\mathcal{R}}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= \bar{\mathcal{R}}\text{Þ}_{\mathsf{R}} - 1 \\
&= \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}_{=\text{Þ}_{\underline{\Gamma}}} - 1
\end{aligned}$$

which is negative because the GIC says $\text{Þ}_{\underline{\Gamma}} < 1$.

If RIC fails. Under $\cancel{\text{RIC}}$, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) < \bar{\mathcal{R}}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (47) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < 1 \tag{48}$$

But the combination of the GIC holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}^{\text{Þ}_{\underline{\Gamma}}} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_{\mathsf{R}}},$$

and multiplying all three elements by $\mathsf{R}/\text{\textthorn}$ gives

$$\mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < \mathsf{R}/\text{\textthorn} < 1$$

which satisfies our requirement in (48).[37]

The foregoing arguments rely on the continuous differentiability of $\mathrm{c}(m)$, so the arguments do not directly go through for the restrained consumer's problem in which the existence of liquidity constraints can lead to discrete changes in the slope $\mathrm{c}'(m)$ at particular values of m. But we can use the fact that the restrained model is the limit of the baseline model as $\wp \downarrow 0$ to conclude that there is likely a unique target cash level even in the restrained model.

If consumers are sufficiently impatient, the limiting target level in the restrained model will be $\check{m} = \mathbb{E}_t[\xi_{t+1}] = 1$. That is, if a consumer starting with $m = 1$ will save nothing, $\mathrm{a}(1) = 0$, then the target level of m in the restrained model will be 1; if a consumer with $m = 1$ would choose to save something, then the target level of cash-on-hand will be greater than the expected level of income.

3.4 Expected Consumption Growth at Target m Is Less than Expected Permanent Income Growth

In Figure 4 the intersection of the target cash-on-hand ratio locus at $\check{m}$ with the expected consumption growth curve lies below the intersection with the horizontal line representing the growth rate of expected permanent income. This can be proven as follows.

Strict concavity of the consumption function implies that if $\mathbb{E}_t[m_{t+1}] = \check{m} = m_t$ then

$$\begin{aligned}
\mathbb{E}_t\left[\frac{\Gamma_{t+1}\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)}\right] &< \mathbb{E}_t\left[\left(\frac{\Gamma_{t+1}(\mathrm{c}(\check{m}) + \mathrm{c}'(\check{m})(m_{t+1} - \check{m}))}{\mathrm{c}(\check{m})}\right)\right] \\
&= \mathbb{E}_t\left[\Gamma_{t+1}\left(1 + \left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)(m_{t+1} - \check{m})\right)\right] \\
&= \Gamma + \left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)\mathbb{E}_t\left[\Gamma_{t+1}\left(m_{t+1} - \check{m}\right)\right] \\
&= \Gamma + \left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)\left[\mathbb{E}_t[\Gamma_{t+1}]\underbrace{\mathbb{E}_t[m_{t+1} - \check{m}]}_{=0} + \mathrm{cov}_t(\Gamma_{t+1}, m_{t+1})\right]
\end{aligned} \tag{49}$$

and since $m_{t+1} = (\mathsf{R}/\Gamma_{t+1})\mathrm{a}(\check{m}) + \xi_{t+1}$ and $\mathrm{a}(\check{m}) > 0$ it is clear that $\mathrm{cov}_t(\Gamma_{t+1}, m_{t+1}) < 0$ which implies that the entire term added to Γ in (49) is negative, as required.

[37] An interesting sidenote is that (48) is a close cousin to the FHWC: instead of $\mathsf{R}/\Gamma < 1$ we have $(\mathsf{R}/\Gamma)\underline{\psi} < 1$.

3.5 Expected Consumption Growth Is a Declining Function of m_t (or Is It?)

Figure 4 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$\boldsymbol{\Upsilon}(m_t) \equiv \Gamma_{t+1}\mathrm{c}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})/\mathrm{c}(m_t) = \mathbf{c}_{t+1}/\mathbf{c}_t$$

and the proposition in which we are interested is

$$(d/dm_t)\,\mathbb{E}_t[\underbrace{\boldsymbol{\Upsilon}(m_t)}_{\equiv \boldsymbol{\Upsilon}_{t+1}}] < 0$$

or differentiating through the expectations operator, what we want is

$$\mathbb{E}_t\left[\Gamma_{t+1}\left(\frac{\mathrm{c}'(m_{t+1})\mathcal{R}_{t+1}\mathrm{a}'(m_t)\mathrm{c}(m_t) - \mathrm{c}(m_{t+1})\mathrm{c}'(m_t)}{\mathrm{c}(m_t)^2}\right)\right] < 0. \tag{50}$$

Henceforth indicating appropriate arguments by the corresponding subscript (e.g. $\mathrm{c}'_{t+1} \equiv \mathrm{c}'(m_{t+1})$), since $\Gamma_{t+1}\mathcal{R}_{t+1} = \mathsf{R}$, the portion of the LHS of equation (50) in brackets can be manipulated to yield

$$\begin{aligned} \mathrm{c}_t\boldsymbol{\Upsilon}'_{t+1} &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\Gamma_{t+1}\mathrm{c}_{t+1}/\mathrm{c}_t \\ &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}. \end{aligned} \tag{51}$$

Now differentiate the Euler equation with respect to m_t:

$$\begin{aligned} 1 &= \mathsf{R}\beta\,\mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho}] \\ 0 &= \mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}\boldsymbol{\Upsilon}'_{t+1}] \\ &= \mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}]\,\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] + \mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}'_{t+1}) \\ \mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] &= -\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}'_{t+1})/\,\mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}] \end{aligned} \tag{52}$$

but since $\boldsymbol{\Upsilon}_{t+1} > 0$ we can see from (52) that (50) is equivalent to

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}'_{t+1}) > 0$$

which, using (51), will be true if

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}) > 0$$

which in turn will be true if both

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}) > 0$$

and

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}_{t+1}) < 0.$$

The latter proposition is obviously true under our assumption $\rho > 1$. The former will be true if

$$\mathrm{cov}_t\left((\Gamma\psi_{t+1}\mathrm{c}(m_{t+1}))^{-\rho-1}, \mathrm{c}'(m_{t+1})\right) > 0.$$

The two shocks cause two kinds of variation in m_{t+1}. Variations due to ξ_{t+1} satisfy the proposition, since a higher draw of ξ both reduces $c_{t+1}^{-\rho-1}$ and reduces the marginal propensity to consume. However, permanent shocks have conflicting effects. On the one hand, a higher draw of ψ_{t+1} will reduce m_{t+1}, thus increasing both $c_{t+1}^{-\rho-1}$ and c'_{t+1}. On the other hand, the $c_{t+1}^{-\rho-1}$ term is multiplied by $\Gamma\psi_{t+1}$, so the effect of a higher ψ_{t+1} could be to decrease the first term in the covariance, leading to a negative covariance with the second term. (Analogously, a lower permanent shock ψ_{t+1} can also lead a negative correlation.)

The software archive associated with this paper presents an example in which this perverse effect dominates. However, extreme assumptions were required (in particular, a very small probability of the zero-income shock) and the region in which $\boldsymbol{\Upsilon}'_{t+1} > 0$ was tiny. In practice, for plausible parametric choices, $\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] < 0$ should generally hold.

4 The Aggregate and Idiosyncratic Relationship Between Consumption Growth and Income Growth

This section examines the behavior of large collections of buffer-stock consumers with identical parameter values. Such a collection can be thought of as either a subset of the population within a single country (say, members of a given education or occupation group), or as the whole population in a small open economy.[38]

We have assumed infinite horizons. In practice, aggregative (macroeconomic) models of this kind usually incorporate mortality in some fashion. We omit mortality here because its incorporation in standard ways does not modify any of the derivations; but mortality can have important quantitative implications, which are discussed briefly in Section 4.3

Formally, we assume a continuum of *ex ante* identical households on the unit interval, with constant total mass normalized to one and indexed by $i \in [0, 1]$, all behaving according to the model specified above.[39]

Szeidl (2012) proves that such a population will be characterized by an invariant distribution of m that induces invariant distributions for c and a; designate these $\mathcal{F}^m$, $\mathcal{F}^a$, and $\mathcal{F}^c$.[40]

[38] We will continue to take the aggregate interest rate as exogenous and constant. It is also possible, and only slightly more difficult, to solve for the steady-state of a closed-economy version of the model where the interest rate is endogenous.

[39] One inconvenient aspect of the model as specified is that it does not exhibit a stationary distribution of idiosyncratic permanent noncapital income; the longer the economy lasts, the wider is the distribution. This problem can be remedied by assuming a constant probability of death, and replacing deceased households with newborns whose initial idiosyncratic permanent income matches the mean idiosyncratic permanent income of the population. For a fully worked-out general equilibrium version of such a model, see Carroll, Slacalek, and Tokuoka (2015).

[40] Szeidl's proof supplants simulation evidence of ergodicity that appeared in an earlier version of this paper.

4.1 Consumption and Income Growth at the Household Level

It is useful to define the operator $\mathbb{M}[\bullet]$ which yields the mean value of its argument in the population, as distinct from the expectations operator $\mathbb{E}[\bullet]$ which represents beliefs about the future.

An economist with a microeconomic dataset could calculate the average growth rate of idiosyncratic consumption, and would find

$$\begin{aligned}\mathbb{M}[\Delta \log \mathbf{c}_{t+1}] &= \mathbb{M}[\log c_{t+1}\mathbf{p}_{t+1} - \log c_t\mathbf{p}_t] \\ &= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t + \log c_{t+1} - \log c_t] \\ &= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t] + \mathbb{M}[\log c_{t+1} - \log c_t] \\ &= (\gamma - \sigma_\psi^2/2) + \mathbb{M}[\log c_{t+1} - \log c_t] \\ &= (\gamma - \sigma_\psi^2/2)\end{aligned}$$

where $\gamma = \log \Gamma$ and the last equality follows because the invariance of $\mathcal{F}^c$ (see Szeidl (2012)) means that $\mathbb{M}[\log c_{t+n}] = \mathbb{M}[\log c_t]$.[41]

Thus, in a population that has reached its invariant distribution, the growth rate of idiosyncratic log consumption matches the growth rate of idiosyncratic log permanent income.

4.2 Growth Rates of Aggregate Income and Consumption

Attanasio and Weber (1995) point out that concavity of the consumption function (or other nonlinearities) can imply that it is quantitatively important to distinguish between the growth rate of average consumption and the average growth rate of consumption.[42] We have just examined the average growth rate; we now examine the growth rate of the average.

Using boldface capital letters for aggregate variables, the growth factor for aggregate income is given by:

$$\begin{aligned}\mathbf{Y}_{t+1}/\mathbf{Y}_t &= \mathbb{M}[\xi_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t] / \mathbb{M}[\mathbf{p}_t\xi_t] \\ &= \Gamma\end{aligned}$$

because of the independence assumptions we have made about ξ and ψ.

The growth factor for aggregate assets is:

$$\begin{aligned}\left(\frac{\mathbf{A}_{t+1}}{\mathbf{A}_t}\right) &= \frac{\mathbb{M}[a_{t+1}\mathbf{p}_{t+1}]}{\mathbb{M}[a_t\mathbf{p}_t]} \\ &= \Gamma\left[\frac{\mathbb{M}[a_{t+1}\mathbf{p}_t\psi_{t+1}]}{\mathbb{M}[a_t\mathbf{p}_t]}\right] \\ &= \Gamma\left[\frac{\mathbb{M}[(a_t + (a_{t+1} - a_t))\mathbf{p}_t\psi_{t+1}]}{\mathbb{M}[a_t\mathbf{p}_t]}\right]\end{aligned}$$

[41] Papers in the simulation literature have observed an approximate equivalence between the average growth rates of idiosyncratic consumption and permanent income, but formal proof was not possible until Szeidl's proof of ergodicity.

[42] Since we assume number of the households are normalized to 1, aggregate and average variables are identical.

$$= \Gamma \left[1 + \frac{\mathbb{M}[(a_{t+1} - a_t)\mathbf{p}_t \psi_{t+1}]}{\mathbb{M}[a_t \mathbf{p}_t]} \right]$$

$$= \Gamma \left[1 + \frac{\mathbb{M}[a_{t+1} - a_t]\mathbb{M}[\mathbf{p}_t \psi_{t+1}] + \mathrm{cov}((a_{t+1} - a_t), \mathbf{p}_t \psi_{t+1})}{\mathbb{M}[a_t \mathbf{p}_t]} \right]$$

$$= \Gamma \left[1 + \frac{\mathrm{cov}(a_{t+1}, \mathbf{p}_t \psi_{t+1})}{\mathbb{M}[a_t \mathbf{p}_t]} \right]$$

where the second-to-last line follows from Szeidl (2012)'s proof the ergodicity of the distributions of normalized variables for this problem, which implies that $\mathbb{M}[a_{t+1} - a_t] = 0$.

Unfortunately, it is clear that the covariance term in the numerator, while generally small, will not in general be zero. This is because the realization of the permanent shock ψ_{t+1} has a nonlinear effect on a_{t+1}.

Matters are simpler if there are no permanent shocks; see Appendix E for a proof that in that case the growth rate of assets (and other variables) does eventually converge to the growth rate of aggregate permanent income.

One way of thinking about this problem is that it reflects the fact that, under our assumptions, the $\boldsymbol{p}$ variable does not have an ergodic distribution; the distribution of permanent income becomes forever wider and wider over time in this model.

4.3 Mortality and Redistribution

In practice most modelers incorporate a constant positive probability of death in their models, following Blanchard (1985). Carroll, Slacalek, Tokuoka, and White (2017) show that for probabilities of death that exceed a threshold that depends on the size of the permanent shocks, the distribution of permanent income has a finite variance. In such cases, numerical results confirm the intuition that the growth rate of aggregate assets ends up matching the growth rate of permanent income.

But the assumption of finite lifetimes requires us to specify what happens to the assets of the dying consumers.

4.3.1 Blanchard Lives

Blanchard (1985)'s solution is to include an annuitization scheme in which estates of the dying are redistributed to survivors in proportion to survivors' wealth, giving the recipients a higher effective rate of return. This treatment has several analytical advantages, the most notable of which is that the effect of mortality on the time preference factor is the exact inverse of its effect on the (effective) interest factor: If the probability of remaining alive is $\aleph$, then assuming that no utility accrues after death makes the effective discount factor $\hat{\beta} = \beta\aleph$. But the enhancement to the rate of return from the annuity scheme yields an effective interest rate of $\hat{\mathsf{R}}/\aleph$. In consequence, the effective patience factor in the new economy $\hat{\mathbf{Þ}}$ is unchanged from its value in the infinite horizon model:

$$\hat{\mathbf{Þ}} \equiv (\beta\aleph\mathsf{R}/\aleph)^{1/\rho} = (\mathsf{R}\beta)^{1/\rho} \equiv \mathbf{Þ}. \tag{53}$$

The only adjustments this requires to the analysis from prior parts of this paper are therefore to those elements that involve a role for R distinct from its contribution to $\mathbf{Þ}$. For example, the RIC ($\mathbf{Þ}/(\mathsf{R}/\aleph) < 1$) will be somewhat easier to satisfy because $\mathsf{R}/\aleph > \mathsf{R}$.

4.3.2 Modigliani Lives

Blanchard (1985)'s innovation was useful because the prevailing alternative, the Life Cycle model of Modigliani (1966), was unwieldy and difficult to use with the available computational technologies when Blanchard was writing. But the appeal of Blanchard's assumption is undermined by the empirical fact that in practice, little annuitization actually occurs (Pashchenko (2013), Brown, Kling, Mullainathan, and Wrobel (2008)).

Further, aside from the insight it yields, the Blanchard model's analytical solution is of little use today; all serious modeling now incorporates uncertainty, constraints, and other features that rule out analytical solutions anyway. Such frameworks can easily handle alternative assumptions about the disposition of assets at death.

The simplest such models are those that follow Modigliani in assuming there is no bequest motive; any wealth remaining at death occurs accidentally (this is not implausible, given the robust finding that for the great majority of households, bequests amount to less than 2 percent of lifetime earnings, Hendricks (2016)).

Some of that wealth will be absorbed by any estate tax that may exist; modelers have made a variety of assumptions about how any residue is distributed. Here, we consider the simplest choice, because it also represents something of a polar alternative to Blanchard: We assume that there is a 100 percent estate tax and that the reveneues from the estate tax are used to fund government expenditures that yield utility in a form that is separable from utility from nondurable consumption – say, for public goods (or, equivalently, the resources are thrown in the ocean). In that case, the estate-related wealth effectively simply vanishes from the economy.

This approach alters the conditions under which the economy has a fixed target wealth-to-income ratio. Effectively, the return on aggregate wealth is lower than the contingent-on-survival return on wealth at the individual level. The condition under which an aggregate target wealth-to-income ratio will exist then becomes an aggregate version of the Growth Impatience Condtion (GIC-Agg):

$$\aleph\mathbf{Þ}_{\Gamma} < 1. \tag{54}$$

Intuitively, the condition required to prohibit unbounded growth in the aggregate wealth-to-income ratio is the condition that prevents the wealth-to-income ratio of individual consumers from growing faster than the rate at which mortality diminishes their collective wealth.

Section 2.10.3 showed that the individual's problem can have a nondegenerate consumption rule for consumers who fail to satisfy the individual version of the GIC. The GIC-Agg therefore provides a bound on preferences which can accommodate a population in which individual consumers have no upper bound on target wealth, but the aggregate economy will nevertheless settle down to an equilibrium aggregate wealth-to-income ratio. Further analysis of these matters is beyond the scope of this paper, but the above-mentioned work of Carroll, Slacalek, Tokuoka, and White (2017) presents an example of the application of this point (and the associated toolkit reports the results not only of tests of the individual but also the aggregate versions of the GIC).

5 Conclusions

This paper provides theoretical foundations for many characteristics of buffer stock saving models that have heretofore been observed in numerical solutions but not proven. Perhaps the most important such proposition is the existence of a target cash-to-permanent-income ratio toward which actual resources will move. The intuition provided by the existence of such a target can be a powerful aid to understanding a host of numerical results.

Another contribution is integration of the paper's results with the open-source Econ-ARK toolkit, which is used to generate all of the quantitative results of the paper, and which integrally incorporates all of the analytical insights of the paper.

Table 3 Definitions and Comparisons of Conditions

Perfect Foresight Versions	Uncertainty Versions
Finite Human Wealth Condition (FHWC)	
$\Gamma/\mathsf{R} < 1$ The growth factor for permanent income Γ must be smaller than the discounting factor R for human wealth to be finite.	$\Gamma/\mathsf{R} < 1$ The model's risks are mean-preserving spreads, so the PDV of future income is unchanged by their introduction.
Absolute Impatience Condition (AIC)	
$\boldsymbol{\text{Þ}} < 1$ The unconstrained consumer is sufficiently impatient that the level of consumption will be declining over time: $c_{t+1} < c_t$	$\boldsymbol{\text{Þ}} < 1$ *If wealth is large enough,* the *expectation* of consumption next period will be smaller than this period's consumption: $\lim_{m_t \to \infty} \mathbb{E}_t[c_{t+1}] < c_t$
Return Impatience Conditions	
Return Impatience Condition (RIC)	Weak RIC (WRIC)
$\boldsymbol{\text{Þ}}/\mathsf{R} < 1$ The growth factor for consumption $\boldsymbol{\text{Þ}}$ must be smaller than the discounting factor R, so that the PDV of current and future consumption will be finite: $\mathrm{c}'(m) = 1 - \boldsymbol{\text{Þ}}/\mathsf{R} < 1$	$\wp^{1/\rho}\boldsymbol{\text{Þ}}/\mathsf{R} < 1$ If the probability of the zero-income event is $\wp = 1$ then income is always zero and the condition becomes identical to the RIC. Otherwise, weaker. $\mathrm{c}'(m) < 1 - \wp^{1/\rho}\boldsymbol{\text{Þ}}/\mathsf{R} < 1$
Growth Impatience Conditions	
PF-GIC	GIC
$\boldsymbol{\text{Þ}}/\Gamma < 1$ Guarantees that for an unconstrained consumer, the ratio of consumption to permanent income will fall over time. For a constrained consumer, guarantees the constraint will eventually be binding.	$\boldsymbol{\text{Þ}}\,\mathbb{E}[\psi^{-1}]/\Gamma < 1$ By Jensen's inequality, stronger than the PF-GIC. Ensures consumers will not expect to accumulate m unboundedly. $\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] = \boldsymbol{\text{Þ}}_{\underline{\Gamma}}$
Finite Value of Autarky Conditions	
PF-FVAC	FVAC
$\beta\Gamma^{1-\rho} < 1$ equivalently $\boldsymbol{\text{Þ}}/\Gamma < (\mathsf{R}/\Gamma)^{1/\rho}$ The discounted utility of constrained consumers who spend their permanent income each period should be finite.	$\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}] < 1$ By Jensen's inequality, stronger than the PF-FVAC because for $\rho > 1$ and nondegenerate ψ, $\mathbb{E}[\psi^{1-\rho}] > 1$.

Table 4 Sufficient Conditions for Nondegenerate[‡] Solution

Model	Conditions	Comments
PF Unconstrained	RIC, FHWC[°]	RIC $\Rightarrow \lvert \mathrm{v}(m)\rvert < \infty$; FHWC $\Rightarrow 0 < \lvert \mathrm{v}(m)\rvert$
		RIC prevents $\bar{\mathrm{c}}(m) = 0$
		FHWC prevents $\bar{\mathrm{c}}(m) = \infty$
PF Constrained	PF-GIC[*]	If RIC, $\lim_{m\to\infty} \mathring{c}(m) = \bar{c}(m), \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		If $\cancel{\text{RIC}}$, $\lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = 0$
Buffer Stock Model	FVAC, WRIC	FHWC $\Rightarrow \lim_{m\to\infty} \mathring{c}(m) = \bar{c}(m), \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		$\cancel{\text{FHWC}}$+RIC $\Rightarrow \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		$\cancel{\text{FHWC}}$+$\cancel{\text{RIC}}$ $\Rightarrow \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = 0$
		GIC guarantees finite target wealth ratio
		FVAC is stronger than PF-FVAC
		WRIC is weaker than RIC

[‡]For feasible m, the limiting consumption function defines the unique value of c satisfying $0 < c(m) < \infty$. [°]RIC, FHWC are necessary as well as sufficient. [*]Solution also exists for $\cancel{\text{PF-GIC}}$ and RIC, but is identical to the unconstrained model's solution for feasible $m \geq 1$.

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