$$
\begin{tikzcd}
((\mathsf{R}\beta)^{1/\rho} \equiv)\ \mathbf{Þ} \arrow[rr, "\text{PF-GIC}(\equiv \mathbf{Þ} < \Gamma)"] \arrow[dr, "(\mathbf{Þ} < \mathsf{R} \equiv)\ \text{RIC}"'] \arrow[drr, phantom, "\neq"] & & \Gamma \arrow[dl, "\text{FHWC}\ (\equiv \Gamma < \mathsf{R})"] \\
& \mathsf{R} &
\end{tikzcd}
$$

Figure 1 Inequality Conditions for Perfect Foresight Model
(Start at a node and follow arrows)

1 Relational Diagrams for the Inequality Conditions

This appendix explains in detail the paper's 'inequalities' diagrams (Figures 1,3).

1.1 The Unconstrained Perfect Foresight Model

A simple illustration is presented in Figure 1, whose three nodes represent values of the absolute patience factor $\mathbf{Þ}$, the permanent-income growth factor Γ, and the risk-free interest factor R. The arrows represent imposition of the labeled inequality condition (like, the uppermost arrow, pointing from $\mathbf{Þ}$ to Γ, reflects imposition of the PF-GIC condition (clicking PF-GIC should take you to its definition at https://econ-ark.github.io/BufferStockTheory#PFGIC; definitions of other conditions are also linked below).[1] Annotations inside parenthetical expressions containing $\equiv$ have no content: They are there to make the diagram readable for someone who may not immediately remember terms and definitions from the main text. (Such a reader might also want to be reminded that R, β, and Γ are all in $\mathbb{R}_{++}$, and that $\rho > 1$).

Navigation of the diagram is simple: Start at any node, and deduce a chain of inequalities by following any arrow that exits that node, and any arrows that exit from successive nodes. Traversal must stop upon arrival at a node with no exiting arrows. So, for example, we can start at the $\mathbf{Þ}$ node and impose the PF-GIC and then the FHWC, and see that imposition of these conditions allows us to conclude that $\mathbf{Þ} < \mathsf{R}$.

One could also impose $\mathbf{Þ} < \mathsf{R}$ directly (without imposing PF-GIC and FHWC) by following the downward-sloping diagonal arrow exiting $\mathbf{Þ}$. Although alternate routes from one node to another, all justify the same core conclusion ($\mathbf{Þ} < \mathsf{R}$, in this case), $\neq$ symbol in the center is meant to convey that these routes are not identical in other

[1]For convenience, the equivalent ($\equiv$) mathematical statement of each condition is expressed nearby in parentheses.

respects. This notational convention is used in category theory diagrams,[2] to indicate that the diagram is not commutative.[3]

Negation of a condition is indicated by the reversal of the corresponding arrow. For example, negation of the RIC, $\cancel{\text{RIC}} \equiv \mathbf{Þ} > \mathsf{R}$, would be represented by moving the arrowhead from the bottom right to the top left of the line segment connecting $\mathbf{Þ}$ and R.

If we were to start at R and then impose ~~FHWC~~, that would reverse the arrow connecting R and Γ, but the Γ node would then have no exiting arrows so no further deductions could be made. However, if we *also* reversed PF-GIC (that is, if we imposed ~~PF-GIC~~), that would take us to the $\mathbf{Þ}$ node, and we could deduce $\mathsf{R} > \mathbf{Þ}$. However, we would have to stop traversing the diagram at this point, because the arrow exiting from the $\mathbf{Þ}$ node points back to our starting point, which (if valid) would lead us to the conclusion that $\mathsf{R} > \mathsf{R}$. Thus, the reversal of the two earlier conditions (imposition of ~~FHWC~~ and ~~PF-GIC~~) requires us also to reverse the final condition, giving us ~~RIC~~.[4]

Under these conventions, Figure 1 in the main text presents a modified version of the diagram extended to incorporate the PF-FVAC (reproduced here for convenient reference).

This diagram can be interpreted, for example, as saying that, starting at the $\mathbf{Þ}$ node, it is possible to derive the PF-FVAC[5] by imposing both the PF-GIC and the FHWC; or by imposing RIC and ~~FHWC~~. Or, starting at the Γ node, we can follow the imposition of the FHWC (twice - reversing the arrow labeled ~~FHWC~~) and then ~~RIC~~ to reach the conclusion that $\mathbf{Þ} < \Gamma$. Algebraically,

$$\begin{aligned} \text{FHWC}: &\quad \Gamma < \mathsf{R} \\ \cancel{\text{RIC}}: &\quad \mathsf{R} < \mathbf{Þ} \\ &\quad \Gamma < \mathbf{Þ} \end{aligned} \tag{1}$$

which leads to the negation of both of the conditions leading into $\mathbf{Þ}$. ~~PF-GIC~~ is obtained directly as the last line in (1) and ~~PF-FVAC~~ follows if we start by multipling the Return Patience Factor (RPF=$\mathbf{Þ}/\mathsf{R}$) by the FHWF($=\Gamma/\mathsf{R}$) raised to the power $1/\rho - 1$, which is negative since we imposed $\rho > 1$. FHWC implies FHWF < 1 so when FHWF is raised to a negative power the result is greater than one. Multiplying the RPF (which exceeds 1 because ~~RIC~~) by another number greater than one yields a product that must

---

[2] For a popular introduction to category theory, see Riehl (2017).

[3] But the rest of our notation does not necessarily abide by the other conventions of category theory diagrams.

[4] The corresponding algebra is

$$\begin{aligned} \cancel{\text{FHWC}}: &\quad \mathsf{R} < \Gamma \\ \cancel{\text{PF-GIC}}: &\quad \Gamma < \mathbf{Þ} \\ \Rightarrow \cancel{\text{RIC}}: &\quad \mathsf{R} < \mathbf{Þ}, \end{aligned}$$

.

[5] in the form $\mathbf{Þ} < (\mathsf{R}/\Gamma)^{1/\rho}\Gamma$

$$
\begin{tikzcd}
\text{Þ} \arrow[r, "\text{PF-GIC}"] \arrow[d, "\text{RIC}"'] \arrow[dr, "\text{PF-FVAC}"] \arrow[dr, phantom, shift left=4, "\neq"] \arrow[dr, phantom, shift right=4, "\neq"] & \Gamma \arrow[d, "\text{FHWC}"] \\
\mathsf{R} \arrow[r, "\cancel{\text{FHWC}}"'] & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}
$$

Figure 2 Relation of PF-GIC, FHWC, RIC, and PF-FVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\text{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, which is an alternative way of writing the PF-FVAC, (28)

be greater than one:

$$
\begin{aligned}
1 &< \overbrace{\left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)}^{>1 \text{ from } \cancel{\text{RIC}}} \overbrace{(\Gamma/\mathsf{R})^{1/\rho-1}}^{>1 \quad \text{from FHWC}} \\
1 &< \left(\frac{(\mathsf{R}\beta)^{1/\rho}}{(\mathsf{R}/\Gamma)^{1/\rho}\mathsf{R}\Gamma/\mathsf{R}}\right) \\
\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho} = (\mathsf{R}/\Gamma)^{1/\rho}\Gamma &< \text{Þ}
\end{aligned}
$$

which is one way of writing ~~PF-FVAC~~.

The complexity of this algebraic calculation illustrates the usefulness of the diagram, in which one merely needs to follow arrows to reach the same result.

After the warmup of constructing these conditions for the perfect foresight case, we can represent the relationships between all the conditions in both the perfect foresight case and the case with uncertainty as shown in Figure 3 in the paper (reproduced here).

References

Riehl, Emily (2017): *Category theory in context*. Courier Dover Publications.

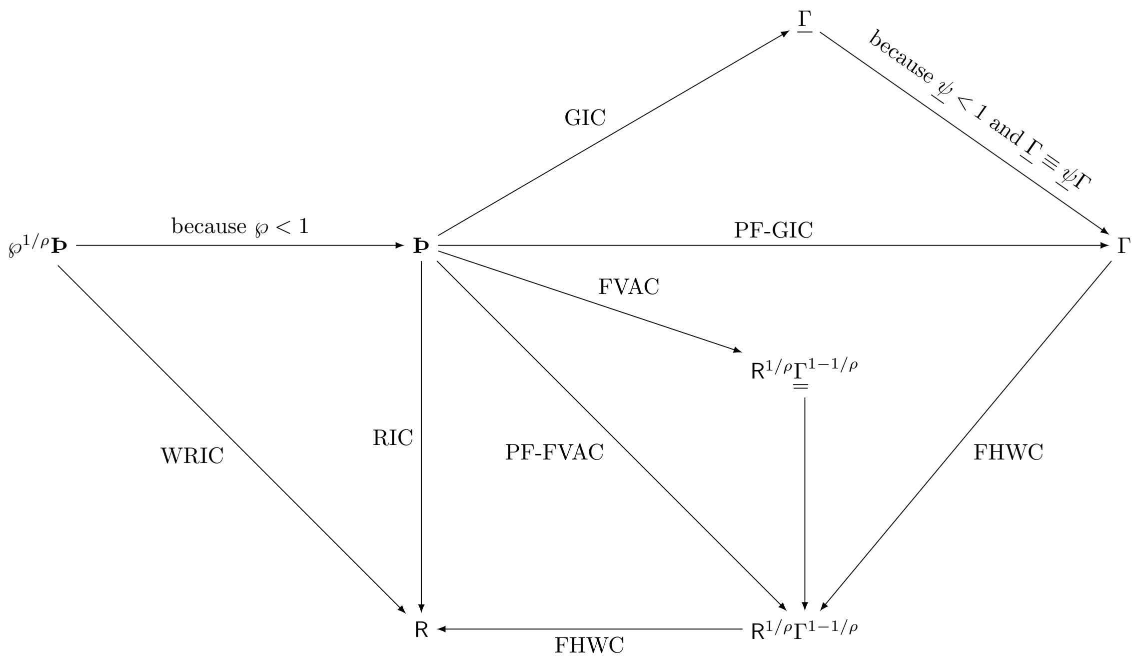

Figure 3 Relation of All Inequality Conditions