$$
\begin{tikzcd}
((\mathsf{R}\beta)^{1/\rho} \equiv)\ \mathbf{Þ} \arrow[rr, "\text{PF-GIC}(\equiv \mathbf{Þ} < \Gamma)"] \arrow[dr, "(\mathbf{Þ} < \mathsf{R} \equiv)\ \text{RIC}"'] \arrow[rr, phantom, shift right=6ex, "\neq"] & & \Gamma \arrow[dl, "\text{FHWC}\ \ (\equiv \Gamma < \mathsf{R})"] \\
& \mathsf{R} &
\end{tikzcd}
$$

Figure 1 Inequality Conditions for Perfect Foresight Model
(Start at a node and follow arrows)

1 Relational Diagrams for the Inequality Conditions

This appendix explains the paper's 'inequalities' diagrams (2,3).[1]

1.1 The Unconstrained Perfect Foresight Model

A simple illustration of our method is presented in Figure 1, whose three nodes represent values of the absolute patience factor $\mathbf{Þ}$, the permanent-income growth factor Γ, and the riskfree interest factor R. The arrows represent imposition of the labeled inequality condition (like, the uppermost arrow, pointing from $\mathbf{Þ}$ to Γ, reflects imposition of the PF-GIC condition).[2] Annotations (signified by parenthetical expressions containing $\equiv$) have no content: They are there to make the diagram readable for someone who may not immediately remember terms and definitions from the main text. (Such a reader might also to be reminded that R, β, and Γ are all in $\mathbb{R}_+$ and $\rho > 1$).

Navigation of the diagram is simple: Start at any node, and deduce a chain of inequalities by following any arrow that exits that node, and any arrows that exit from successive nodes. Traversal must stop when it reaches a node with no exiting arrows. So, for example, we can start at the $\mathbf{Þ}$ node and impose the PF-GIC and then the FHWC, and conclude that imposition of these conditions allows us to conclude that $\mathbf{Þ} < \mathsf{R}$.

Negation of a condition is indicated by the reversal of the corresponding arrow. So, for example, the negation of the RIC, $\cancel{\text{RIC}} \equiv \mathbf{Þ} > \mathsf{R}$, would be represented by moving the arrowhead from the bottom right to the top left of the line segment connecting $\mathbf{Þ}$ and R.

If we were to start at R and then impose $\cancel{\text{FHWC}}$, that would reverse the arrow connecting R and Γ, but the Γ node would then have no exiting arrows so no further

[1] Unless otherwise noted, the diagrams abide by the conventions that are used for constructing diagrams in category theory. In particular, the inequalities in the upper and lower triangular parts of the diagram indicate that this is not a commutative diagram.

[2] For convenience, the equivalent ($\equiv$) mathematical statement of each condition is expressed nearby in parentheses.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\textbf{Þ} \arrow[r, "\text{PF-GIC}"] \arrow[d, "\text{RIC}"'] \arrow[dr, "\text{PF-FVAC}"] & \Gamma \arrow[d, "\text{FHWC}"] \arrow[dl, phantom, "\neq"] \\
\mathsf{R} \arrow[r, "\cancel{\text{FHWC}}"'] \arrow[ur, phantom, "\neq"] & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}$$

Figure 2 Relation of PF-GIC, FHWC, RIC, and PF-FVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\textbf{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, which is an alternative way of writing the PF-FVAC, (**??**)

deductions could be made. However, if we *also* reversed PF-GIC (that is, if we imposed ~~PF-GIC~~), that would take us to the $\textbf{Þ}$ node, and we could deduce $\mathsf{R} > \textbf{Þ}$. However, we would have to stop traversing the diagram at this point, because the arrow exiting from the $\textbf{Þ}$ node points back to our starting point, which (if valid) would lead us to the conclusion that $\mathsf{R} > \mathsf{R}$. Thus, the reversal of the two earlier conditions (imposition of ~~FHWC~~ and ~~PF-GIC~~) requires us also to reverse the final condition, giving us ~~RIC~~. The corresponding algebra is

$$\begin{aligned}
\cancel{\text{FHWC}}: &\quad \mathsf{R} < \Gamma \\
\cancel{\text{PF-GIC}}: &\quad \Gamma < \textbf{Þ} \\
\Rightarrow \cancel{\text{RIC}}: &\quad \mathsf{R} < \textbf{Þ},
\end{aligned}$$

Under these conventions, the main text presents a version of the diagram extended to incorporate the PF-FVAC reproduced in Figure 2).[3]

This diagram can be interpreted, for example, as saying that, starting at the $\textbf{Þ}$ node, it is possible to derive the PF-FVAC[4] by imposing both the PF-GIC and the FHWC; or by imposing RIC and ~~FHWC~~. Or, starting at the Γ node, we can follow the imposition of the FHWC (twice - reversing the arrow labeled ~~FHWC~~) and then ~~RIC~~ to reach the

[3]For readers familiar with the commutative diagrams, it should be noted that despite the similar appearance, this diagram is not exactly commutative.

[4]in the form $\textbf{Þ} < (\mathsf{R}/\Gamma)^{1/\rho}\Gamma$

conclusion that $\text{Þ} < \Gamma$. Algebraically,

$$\begin{aligned} \text{FHWC}: \quad & \Gamma < \mathsf{R} \\ \cancel{\text{RIC}}: \quad & \mathsf{R} < \text{Þ} \\ & \Gamma < \text{Þ} \end{aligned} \tag{1}$$

which leads to the negation of both of the conditions leading into Þ. ~~PF-GIC~~ is obtained directly as the last line in (1) and ~~PF-FVAC~~ follows if we start by multipling the Return Patience Factor (RPF=$\text{Þ}/\mathsf{R}$) by the FHWF($=\Gamma/\mathsf{R}$) raised to the power $1/\rho - 1$, which is negative since we imposed $\rho > 1$. FHWC implies FHWF < 1 so when FHWF is raised to a negative power the result is greater than one. Multiplying the RPF (which exceeds 1 because ~~RIC~~) by another number greater than one yields a product that must be greater than one:

$$\begin{aligned} 1 &< \overbrace{\left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)}^{>1 \text{ from } \cancel{\text{RIC}}} \overbrace{(\Gamma/\mathsf{R})^{1/\rho-1}}^{>1 \text{ from FHWC}} \\ 1 &< \left(\frac{(\mathsf{R}\beta)^{1/\rho}}{(\mathsf{R}/\Gamma)^{1/\rho}\mathsf{R}\Gamma/\mathsf{R}}\right) \\ \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho} = (\mathsf{R}/\Gamma)^{1/\rho}\Gamma &< \text{Þ} \end{aligned}$$

which is one way of writing ~~PF-FVAC~~.

The complexity of this algebraic calculation illustrates the usefulness of the diagram, in which one merely needs to follow arrows to reach the same result.

After the warmup of constructing these conditions for the perfect foresight case, we can represent the relationships between all the conditions in both the perfect foresight case and the case with uncertainty as shown in Figure 3 in the paper (reproduced below).

$$
\begin{tikzcd}
 & & \underline{\Gamma} & \\
\wp^{1/\rho}\textbf{Þ} \arrow[r, "\text{because } \wp < 1"] \arrow[ddr, "\text{WRIC}"'] & \textbf{Þ} \arrow[ur, "\text{GIC}"] \arrow[rr, "\text{PF-GIC}"] \arrow[dr, "\text{FVAC}"] \arrow[dd, "\text{RIC}"] \arrow[ddr, "\text{PF-FVAC}"'] & & \Gamma \arrow[ul, "\text{because } \underline{\psi} > 1 \text{ and } \underline{\Gamma} \equiv \underline{\psi}\Gamma"'] \arrow[ddl, "\text{FHWC}"] \\
 & & \mathsf{R}^{1/\rho}\underline{\underline{\Gamma}}^{1-1/\rho} \arrow[d] & \\
 & \mathsf{R} & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho} \arrow[l, "\text{FHWC}"] &
\end{tikzcd}
$$

Figure 3 Relation of All Inequality Conditions

For Mateo and Xudong:

I performed all the Google searches I could think of to see if there was an established body of practice for constructing diagrams like the ones we have constructed, but did not find any examples that were particularly close.

But the idea is so useful and intuitive that I feel there must be SOME body of practice, that I should be citing (and whose conventions I should be, but may not be, following).

At first I thought that what we want to do could be done using the conventions of what I guess is called "category theory diagrams" (though that term does not seem to be widely used, I could not find any other term that seemed to encompass the set of practices that might be relevant). But now I think our diagrams do NOT abide by those conventions, which is why I have labeled them as “Relational” diagrams.

My understanding of category theory diagrams is that the "nodes" are sets (or some generalization of sets). I had been thinking of the nodes in diagrams like the simple one in 1 as quantities (numbers). To see the relation to category theory, I have worked out an interpretation in which you can think of the nodes as designating sets.

My interpretation of our diagrams, however, is one in which the content of a node depends on how you got to it. As best I can understand it, this is NOT the way category theory diagrams work.

For our diagrams, at the node where you start, you should consider the object in question as containing the set of all potential values of that object. That is, say in the case of **Þ**, all of the combinations of R and β and ρ which satisfy our intrinsic assumptions about those quantities (like $\rho > 1$).

So then the interpretation of PF-GIC is that it is an operation (“morphism?”) that, for every object in Þ, identifies the subset of values of the Γ set such that $\text{Þ} < \Gamma$ is satisfied. This is a one-to-one mapping: For each real Þ, exactly one (contiguous) set of values of Γ is identified. Or, if you start at the Γ node, then the items in that node are all possible values of $\Gamma \in (0, \infty)$, and imposition of the FHWC identifies, for each Γ_i, the set of values of R for which $\Gamma < \mathsf{R}$.

In this example, whether one thinks of the objects in Γ as values (if you start at Γ) or sets of values indexed by Þ (e.g., for Þ_i, the content of the node Γ reached by imposing PF-GIC is the SET of values satisfying $\Gamma_i > \text{Þ}_i$. This is different from what I have been able to understand about category theory diagrams, in which I believe that the nodes are supposed to contain sets whose content is not "operated upon" but simply mapped by prior operators.

In any case, the diagram is not commutative (if I understand commutativity in this context) in the sense that the set of values of R identified by the diagram depends on the route taken to get there. If we start (and end, because it has no exiting arrows) at the R node, its contents should be interpreted as $\mathsf{R} \in \mathbb{R}_+$. If we start at Γ and impose FHWC, then the interpretation of the R node is, for each Γ_i, the set of values of R that exceed that Γ_i. If we start at Þ and impose RIC, when we arrive at R it is viewed as containing the set of values that, for each feasible value of Þ_i, is greater than that Þ_i. And if we start at Þ and traverse the diagram by first imposing the PF-GIC to get to the Γ node, and then imposing the FHWC, the contents of R are doubly indexed. My understanding is that, for the diagram to be commutative, we would need (starting, say, at Þ) for the operations RIC and PF-GIC $\circ$ FHWC to be equivalent in their effects on R. The inequality sign in the center of the diagram is meant to signal the noncomutativity of the paths (a notational convention I found articulated on stackexchange).

Generalizing the point to the next diagram, I have put inequality signs in the upper and lower triangles to signal that the diagram is not commutative. That is, the inequality operators are intended to convey that if you reach the bottommost node by traversing the outside of the diagram (in either direction), that is NOT equivalent to imposing PF-FVACdirectly.

Let me emphasize that what we are doing is logically and mathematically sensible under these definitions. What I'm not sure about is whether

1. The diagram would be misunderstood by a category theorist who would assume that the structure and operations would mean something different than what we want to mean;

2. There is another way to do the diagram that would be equally transparent to non-category-theorists but would satisfy the conventions of category theory.