1 Relational Diagrams for the Inequality Conditions

This appendix explains the paper's 'inequalities' diagrams (1,2).[1]

1.1 The Unconstrained Perfect Foresight Model

The basic idea is presented in Figure **??**, whose three nodes represent values of the absolute patience factor Þ, the permanent-income growth factor Γ, and the riskfree interest factor R. The arrows represent imposition of the labeled inequality condition (like, the uppermost arrow, pointing from Þ to Γ, reflects imposition of the PF-GIC).[2]

Traversal of the diagram is simple: Start at any node, and deduce a chain of inequalities by following any arrow that exits that node, and any arrows that exit from successive nodes, until reaching a point where no exit condition can be satisfied (that is, until the diagram defines no further inequalities).

Negation of a condition is also fairly simple. The negation of the PF-GIC is represented in the lower curved arrow pointing from Γ to Þ, labeled ~~PF-GIC~~. The only outgoing arrow from Þ points to the RIC, so the only further condition that the diagram maps is $\text{Þ} > \mathsf{R}$. But imposing this condition allows us to conclude that $\Gamma < \mathsf{R}$ because $\Gamma < \text{Þ}$ and $\text{Þ} < \mathsf{R}$. This illustrates the usefulness of the diagram: It can transparently show alternative ways to reach the same conclusion. (Generically, if you start at Γ and end up at R you know that the FHWC holds).

Notationally, rather than showing both arrows for every condition (PF-GIC and ~~PF-GIC~~, say), it is simpler is to define a convention that negation of a condition is indicated by the reversal of the corresponding arrow. So, for example, the negation of the RIC, ~~RIC~~ $\equiv \text{Þ} > \mathsf{R}$, would be represented by moving the arrowhead from the bottom right to the top left of the line segment connecting Þ and R.

Using this notation (and dropping the definitions) leads to the simpler diagram in part b of the diagram. So, for example, if we were to start at the Γ node and impose ~~PF-GIC~~ (reversing the arrow), we could traverse the diagram counterclockwise from Γ through Þ to R and, as before, impose the RIC, and reach the conclusion that $\Gamma > \mathsf{R}$.

So, if we were to start at R and then impose ~~FHWC~~, that would reverse the arrow connecting R and Γ. If we were then to impose ~~PF-GIC~~, we would follow the arrow to Þ. But

$$\begin{aligned} \cancel{\text{FHWC}}: &\quad \mathsf{R} < \Gamma \\ \cancel{\text{PF-GIC}}: &\quad \Gamma < \text{Þ} \\ \Rightarrow \cancel{\text{RIC}}: &\quad \mathsf{R} < \text{Þ}, \end{aligned}$$

so in this case reversing two arrows requires reversal of the third.

[1] Unless otherwise noted, the diagrams abide by the conventions that are used for constructing diagrams in category theory. In particular, the inequalities in the upper and lower triangular parts of the diagram indicate that this is not a commutative diagram.

[2] For convenience, the equivalent ($\equiv$) mathematical statement of each condition is expressed nearby in parentheses.

Under these conventions, the main text presents a version of the (simplified) diagram as extended to incorporate the PF-FVAC in Figure 1).[3]

$$\begin{tikzcd}
\mathbf{Þ} \arrow[rr, "\text{PF-GIC}"] \arrow[dd, "\text{RIC}"'] \arrow[ddrr, "\text{PF-FVAC}"'] & & \Gamma \arrow[dd, "\text{FHWC}"] \\
 & & \\
\mathsf{R} \arrow[rr, "\cancel{\text{FHWC}}"'] & & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}$$

Figure 1 Relation of PF-GIC, FHWC, RIC, and PF-FVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\mathbf{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, which is an alternative way of writing the PF-FVAC, (**??**)

This diagram can be interpreted, for example, as saying that, starting at the $\mathbf{Þ}$ node, it is possible to derive the PF-FVAC[4] by imposing both the PF-GIC and the FHWC; or by imposing RIC and ~~FHWC~~. Or, starting at the Γ node, we can follow the imposition of the FHWC (twice - reversing the arrow labeled ~~FHWC~~) and then ~~RIC~~ to reach the conclusion that $\mathbf{Þ} < \Gamma$. Algebraically,

$$\begin{aligned}
\text{FHWC}: \quad & \Gamma < \mathsf{R} \\
\cancel{\text{RIC}}: \quad & \mathsf{R} < \mathbf{Þ} \\
& \Gamma < \mathbf{Þ}
\end{aligned} \tag{1}$$

which leads to the negation of both of the conditions leading into $\mathbf{Þ}$. ~~PF-GIC~~ is obtained directly as the last line in (1) and ~~PF-FVAC~~ follows if we start by multipling the Return Patience Factor (RPF=$\mathbf{Þ}/\mathsf{R}$) by the FHWF(=Γ/R) raised to the power $1/\rho - 1$, which is negative since we imposed $\rho > 1$. FHWC implies FHWF < 1 so when FHWF is raised to a negative power the result is greater than one. Multiplying the RPF (which exceeds 1 because ~~RIC~~) by another number greater than one yields a product that must

[3]For readers familiar with the commutative diagrams, it should be noted that despite the similar appearance, this diagram is not exactly commutative.

[4]in the form $\mathbf{Þ} < (\mathsf{R}/\Gamma)^{1/\rho}\Gamma$

be greater than one:

$$1 < \overbrace{\left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)}^{>1 \text{ from } \cancel{\text{RIC}}} \overbrace{(\Gamma/\mathsf{R})^{1/\rho-1}}^{>1 \text{ from FHWC}}$$

$$1 < \left(\frac{(\mathsf{R}\beta)^{1/\rho}}{(\mathsf{R}/\Gamma)^{1/\rho}\mathsf{R}\Gamma/\mathsf{R}}\right)$$

$$\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho} = (\mathsf{R}/\Gamma)^{1/\rho}\Gamma < \textbf{Þ}$$

which is one way of writing ~~PF-FVAC~~.

The complexity of this algebraic calculation illustrates the usefulness of the diagram, in which one merely needs to follow arrows to reach the same result.

After the warmup of constructing these conditions for the perfect foresight case, we can represent the relationships between all the conditions in both the perfect foresight case and the case with uncertainty as shown in Figure 2 in the paper (reproduced below).

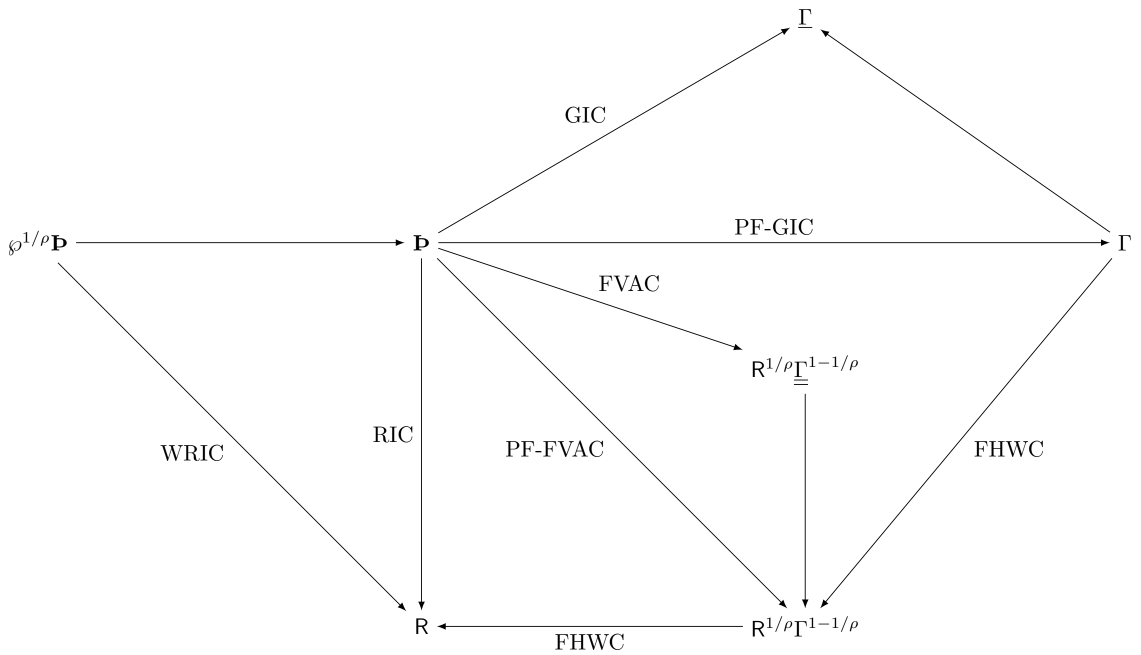

Figure 2 Relation of All Inequality Conditions