

# 1 The Limiting MPC's

For  $m_t > 0$  we can define  $e_t(m_t) = c_t(m_t)/m_t$  and  $a_t(m_t) = m_t - c_t(m_t)$  and the Euler equation (??) can be rewritten

$$\begin{aligned} e_t(m_t)^{-\rho} &= \beta R \mathbb{E}_t \left[ \left( e_{t+1}(m_{t+1}) \left( \frac{\overbrace{Ra_t(m_t) + \Gamma_{t+1}\xi_{t+1}}^{=m_{t+1}\Gamma_{t+1}}}{m_t} \right) \right)^{-\rho} \right] \\ &= (1 - \wp) \beta R m_t^\rho \mathbb{E}_t [(e_{t+1}(m_{t+1}) m_{t+1} \Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0] \\ &\quad + \wp \beta R^{1-\rho} \mathbb{E}_t \left[ \left( e_{t+1}(\mathcal{R}_{t+1} a_t(m_t)) \frac{m_t - c_t(m_t)}{m_t} \right)^{-\rho} \mid \xi_{t+1} = 0 \right]. \end{aligned}$$

Consider the first conditional expectation in (1), recalling that if  $\xi_{t+1} > 0$  then  $\xi_{t+1} \equiv \theta_{t+1}/(1 - \wp)$ . Since  $\lim_{m \downarrow 0} a_t(m) = 0$ ,  $\mathbb{E}_t[(e_{t+1}(m_{t+1}) m_{t+1} \Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0]$  is contained within bounds defined by  $(e_{t+1}(\underline{\theta}/(1 - \wp)) \Gamma \underline{\psi} \underline{\theta}/(1 - \wp))^{-\rho}$  and  $(e_{t+1}(\bar{\theta}/(1 - \wp)) \Gamma \bar{\psi} \bar{\theta}/(1 - \wp))^{-\rho}$  both of which are finite numbers, implying that the whole term multiplied by  $(1 - \wp)$  goes to zero as  $m_t^\rho$  goes to zero. As  $m_t \downarrow 0$  the expectation in the other term goes to  $\bar{\kappa}_{t+1}^{-\rho} (1 - \bar{\kappa}_t)^{-\rho}$ . (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting  $\bar{\kappa}_t$  satisfies  $\bar{\kappa}_t^{-\rho} = \beta \wp R^{1-\rho} \bar{\kappa}_{t+1}^{-\rho} (1 - \bar{\kappa}_t)^{-\rho}$ . Exponentiating by  $\rho$ , we can conclude that

$$\begin{aligned} \bar{\kappa}_t &= \wp^{-1/\rho} (\beta R)^{-1/\rho} R (1 - \bar{\kappa}_t) \bar{\kappa}_{t+1} \\ \underbrace{\wp^{1/\rho} R^{-1} (\beta R)^{1/\rho}}_{\equiv \wp^{1/\rho} \mathbf{P}_R} \bar{\kappa}_t &= (1 - \bar{\kappa}_t) \bar{\kappa}_{t+1} \end{aligned}$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$\begin{aligned} (\wp^{1/\rho} \mathbf{P}_R \bar{\kappa}_t)^{-1} &= (1 - \bar{\kappa}_t)^{-1} \bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1} (1 - \bar{\kappa}_t) &= \wp^{1/\rho} \mathbf{P}_R \bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1} &= 1 + \wp^{1/\rho} \mathbf{P}_R \bar{\kappa}_{t+1}^{-1}. \end{aligned}$$

As noted in the main text, we need the WRIC (??) for this to be a convergent sequence:

$$0 \leq \wp^{1/\rho} \mathbf{P}_R < 1, \quad (1)$$

Since  $\bar{\kappa}_T = 1$ , iterating (1) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n \rightarrow \infty} \bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1 - \wp^{1/\rho} \mathbf{P}_R \quad (2)$$

and we will therefore call  $\bar{\kappa}$  the ‘limiting maximal MPC.’

The minimal MPC's are obtained by considering the case where  $m_t \uparrow \infty$ . If the FHC holds, then as  $m_t \uparrow \infty$  the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving  $\xi_{t+1}$  in (1) can

be neglected, leading to a revised limiting Euler equation

$$(m_t e_t(m_t))^{-\rho} = \beta R \mathbb{E}_t [(e_{t+1}(a_t(m_t) \mathcal{R}_{t+1}) (R a_t(m_t)))^{-\rho}]$$

and we know from L'Hôpital's rule that  $\lim_{m_t \rightarrow \infty} e_t(m_t) = \underline{\kappa}_t$ , and  $\lim_{m_t \rightarrow \infty} e_{t+1}(a_t(m_t) \mathcal{R}_{t+1}) = \underline{\kappa}_{t+1}$  so a further limit of the Euler equation is

$$\begin{aligned} (m_t \kappa_t)^{-\rho} &= \beta R (\kappa_{t+1} R (1 - \underline{\kappa}_t) m_t)^{-\rho} \\ \underbrace{R^{-1} \mathbf{P}}_{\equiv \mathbf{P}_R = (1 - \underline{\kappa})} \underline{\kappa}_t &= (1 - \underline{\kappa}_t) \underline{\kappa}_{t+1} \end{aligned}$$

and the same sequence of derivations used above yields the conclusion that if the RIC  $0 \leq \mathbf{P}_R < 1$  holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1} \mathbf{P}_R \quad (3)$$

so that  $\{\underline{\kappa}_{T-n}^{-1}\}_{n=0}^{\infty}$  is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n \uparrow \infty} \underline{\kappa}_{T-n}^{-1} \quad (4)$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then  $\lim_{n \rightarrow \infty} \underline{\kappa}_{T-n}^{-1} = \infty$  and so the limiting MPC is  $\underline{\kappa} = 0$ .

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$c_t \underbrace{(1 + \mathbf{P}_R + \mathbf{P}_R^2 + \dots)}_{= 1 + \mathbf{P}_R(1 + \mathbf{P}_R \underline{\kappa}_{t+2}^{-1}) \dots} = c_t \underline{\kappa}_{T-n}^{-1}.$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t) \underline{\kappa}_t \quad (5)$$