

Theoretical Foundations of Buffer Stock Saving

October 4, 2020

Christopher D. Carroll¹

Abstract

This paper builds theoretical foundations for rigorous and intuitive understanding of ‘buffer stock’ saving models, pairing each theoretical result with a quantitative exploration. After describing conditions under which the consumption function converges, the paper shows that a ‘target’ buffer stock exists only under conditions strictly stronger than those that guarantee convergence of the consumption and value functions. It also shows that the average growth rate of consumption equals the average growth rate of permanent income (in a small open economy populated by buffer stock savers). Together, the (provided) numerical tools and (proven) analytical results constitute a comprehensive toolkit for understanding buffer stock models.

Keywords Precautionary saving, buffer stock saving, marginal propensity
 to consume, permanent income hypothesis

JEL codes D81, D91, E21

Archive: <http://www.econ2.jhu.edu/people/ccarroll/BufferStockTheory.zip>
PDF: <http://llorracc.github.io/BufferStockTheory/BufferStockTheory.pdf>
Slides: <http://llorracc.github.io/BufferStockTheory/BufferStockTheory-Slides.pdf>
Web: <http://llorracc.github.io/BufferStockTheory/BufferStockTheory/>
Appendix: <http://llorracc.github.io/BufferStockTheory/BufferStockTheory#Appendices>
GitHub: <http://github.com/llorracc/BufferStockTheory>
(In GitHub repo, see /Code for tools for solving and simulating the model)

[CLICK HERE](#) for an interactive Jupyter Notebook that uses the [Econ-ARK/HARK](#) toolkit to produce all of the paper’s figures (warning: the notebook may take several minutes to launch). Information about citing the toolkit can be found at [Acknowledging Econ-ARK](#).

¹Contact: ccarroll1@jhu.edu, Department of Economics, 590 Wyman Hall, Johns Hopkins University, Baltimore, MD 21218, <http://econ.jhu.edu/people/ccarroll>, and National Bureau of Economic Research.

1 Introduction

In the presence of empirically realistic transitory and permanent shocks to income *à la* Friedman (1957), only one other ingredient is required to define a testable model of optimal consumption: A description of preferences. Modelers usually assume geometric discounting of a constant relative risk aversion (CRRA) utility function, because, starting with Zeldes (1989), a large literature has shown that models of this kind have quantitative predictions that can match microeconomic evidence reasonably well.

A companion theoretical literature has shown that standard numerical solution methods provide good approximations to limiting “true” mathematical solutions – but only for models more complex than the simple case with just shocks and utility. The extra complexity has been required because standard contraction mapping theorems (beginning with Bellman (1957) and including those following Stokey et. al. (1989)) cannot be applied when the utility function is unbounded (like CRRA - see [section 2.1](#)).¹

This paper’s first technical contribution is to articulate the (surprisingly loose) conditions under which the simple problem (without convenient shortcuts like a consumption floor or liquidity constraints) defines a contraction mapping with a nondegenerate consumption function (the main requirement is a ‘[Finite Value of Autarky](#)’ condition). Another contribution is to specify the conditions under which the resulting consumption function implies there is a ‘target’ wealth-to-permanent-income ratio (so the model exhibits ‘buffer stock’ saving behavior.) The key requirement for existence of a target is that the model’s parameters satisfy a “[Growth Impatience Condition](#)” (equation (34)) that relates preferences and uncertainty to the predictable growth rate of income.

Even without a formal proof, target saving of this kind has been intuitively understood to underlie central numerical results from the heterogeneous agent macroeconomics literature; for example, the logic of target saving is central to the explanation by Krueger, Mitman, and Perri (2016) of the fact that, during the Great Recession, middle-class consumers cut their consumption more than the poor or the rich. The theoretical logic articulated below explains this finding: Learning that the future has become more uncertain does not change the urgent imperatives of the poor (their high $u'(c)$) because they have little room to maneuver. Increased labor income uncertainty does not change the behavior of the rich because the increase in uncertainty does not threaten their consumption much. Only people in the middle have both the motivation and the wiggle-room to reduce their discretionary spending.

Conveniently, elements required for the convergence proof turn out to provide analytical foundations for many other results that have become familiar from the numerical

¹It is unclear whether newer methods such as those of Matkowski and Nowak (2011) could overcome this problem, or how difficult it would be to do so; but in any case this particular problem does not seem to have been tackled by those methods or any others.

(Carroll, Kaufman, Kazil, Palmer, and White (2018)); for reference to the toolkit itself see [Acknowledging Econ-ARK](#). Thanks to James Feigenbaum, Joseph Kaboski, Miles Kimball, Qingyin Ma, Misuzu Otsuka, Damiano Sandri, John Stachurski, Adam Szeidl, Metin Uyanik, Weifeng Wu, Xudong Zheng, and Jiaxiong Yao for comments on earlier versions of this paper, John Boyd for help in applying his weighted contraction mapping theorem, Ryoji Hiraguchi for extraordinary mathematical insight that improved the paper greatly, David Zervos for early guidance to the literature, and participants in a seminar at Johns Hopkins University and a presentation at the 2009 meetings of the Society of Economic Dynamics for their insights.

literature. All theoretical conclusions are paired with numerically computed illustrations (using an open-source toolkit available from the [Econ-ARK](#) project). All of the insights of this paper are instantiated in the toolkit, which algorithmically flags parametric choices under which a problem fails to define a contraction mapping, under which a target level of wealth does not exist, or under which the solution is otherwise degenerate.

Thus, the theoretical foundations provided here are valuable both because they provide intuition about the determinants of saving targets, and because they make it easier to develop reliable numerical solution methods (by providing tight restrictions that valid solutions must satisfy).

The paper proceeds in three parts.

The first part articulates the [conditions required](#) for the problem to define a unique nondegenerate limiting consumption function, and discusses the relation of the paper’s model to models previously considered in the literature. The required conditions are interestingly parallel to those required for the [liquidity constrained perfect foresight model](#); that parallel is explored and explained. Next, the paper derives some limiting properties of the consumption function as cash approaches infinity and as it approaches its lower bound, and the theorem is proven explaining when the problem defines a contraction mapping. Finally, a related class of commonly-used models (exemplified by Deaton (1991)) is shown to constitute a particular limit of this paper’s more general model.

The [next section](#) examines five key properties of the model. First, as [cash approaches infinity](#) the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as [cash approaches zero](#) the expected growth rate of consumption approaches infinity, and the MPC approaches a simple analytical limit. Third, if the consumer is ‘growth impatient,’ a [unique target cash-to-permanent-income ratio](#) will exist. Fourth, at the target cash ratio, the [expected growth rate of consumption](#) is slightly less than the expected growth rate of permanent noncapital income. Finally, the expected growth rate of consumption is [declining in the level of cash](#). The first four propositions are proven under general assumptions about parameter values; the last is shown to hold if there are no transitory shocks, but may fail in extreme cases if there are both transitory and permanent shocks.

Szeidl (2012) has shown that such an economy will be characterized by stable invariant distributions for the consumption ratio, the wealth ratio, and other variables.² Using Szeidl’s result, the final section discusses conditions under which, even with a fixed aggregate interest rate that differs from the time preference rate, an economy populated by buffer stock consumers converges to a balanced growth equilibrium in which the growth rate of consumption tends toward the (exogenous) growth rate of permanent income.

²Szeidl’s proof supplants the analysis in an earlier draft of this paper, which conjectured that such a result held and provided supportive simulation evidence.

2 The Problem

2.1 Setup

The consumer solves an optimization problem from period t until the end of life at T defined by the objective

$$\max \mathbb{E}_t \left[\sum_{n=0}^{T-t} \beta^n u(\mathbf{c}_{t+n}) \right] \quad (1)$$

where

$$u(\bullet) = \bullet^{1-\rho} / (1-\rho) \quad (2)$$

is a constant relative risk aversion utility function with $\rho > 1$.^{3,4} The consumer's initial condition is defined by market resources \mathbf{m}_t (which Deaton (1991) called 'cash-on-hand') and permanent noncapital income \mathbf{p}_t .

In the usual treatment, a dynamic budget constraint (DBC) simultaneously incorporates all of the elements that determine next period's \mathbf{m} given this period's choices; but for the detailed analysis here, it will be useful to disarticulate the steps so that individual ingredients can be separately examined:

$$\begin{aligned} \mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\ \mathbf{b}_{t+1} &= \mathbf{a}_t \mathbf{R} \\ \mathbf{p}_{t+1} &= \mathbf{p}_t \underbrace{\Gamma \psi_{t+1}}_{\equiv \Gamma_{t+1}} \\ \mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{p}_{t+1} \xi_{t+1}, \end{aligned} \quad (3)$$

where \mathbf{a}_t indicates the consumer's assets at the end of period t , which grow by a fixed interest factor $\mathbf{R} = (1 + r)$ between periods,⁵ so that \mathbf{b}_{t+1} is the consumer's financial ('bank') balances before next period's consumption choice;⁶ \mathbf{m}_{t+1} ('market resources' or 'money') is the sum of financial wealth \mathbf{b}_{t+1} and noncapital income $\mathbf{p}_{t+1} \xi_{t+1}$ (permanent noncapital income \mathbf{p}_{t+1} multiplied by a mean-one iid transitory income shock factor ξ_{t+1} ; future transitory shocks are assumed to satisfy $\mathbb{E}_t[\xi_{t+n}] = 1 \ \forall n \geq 1$). Permanent noncapital income in period $t+1$ is equal to its previous value, multiplied by a growth factor Γ , modified by a mean-one iid shock ψ_{t+1} , $\mathbb{E}_t[\psi_{t+n}] = 1 \ \forall n \geq 1$ satisfying

³The main results also hold for logarithmic utility which is the limit as $\rho \rightarrow 1$ but incorporating the logarithmic special case in the proofs is cumbersome and therefore omitted.

⁴We will define the infinite horizon solution as the limit of the finite horizon problem as the horizon $T - t$ approaches infinity.

⁵See Ma, Stachurski, and Toda (2018) for interesting new work that considers the case where capital returns are stochastic and liquidity constraints exist. Benhabib, Bisin, and Zhu (2015) examines implications of capital income risk for the distribution of wealth.

⁶Allowing a stochastic interest factor is straightforward but adds little insight. The effects are more interesting for analysis of the invariant distribution (Szeidl (2012)).

$\psi \in [\underline{\psi}, \bar{\psi}]$ for $0 < \underline{\psi} \leq 1 \leq \bar{\psi} < \infty$ where $\underline{\psi} = \bar{\psi} = 1$ is the degenerate case with no permanent shocks.^{7,8}

References

- ABOWD, JOHN M., AND DAVID CARD (1989): “On the Covariance Structure of Earnings and Hours Changes,” *Econometrica*, 57, 411–445.
- ALVAREZ, FERNANDO, AND NANCY L STOKEY (1998): “Dynamic programming with homogeneous functions,” *Journal of economic theory*, 82(1), 167–189.
- ATTANASIO, ORAZIO, AND GUGLIELMO WEBER (1995): “Is Consumption Growth Consistent with Intertemporal Optimization? Evidence from the Consumer Expenditure Survey,” *Journal of Political Economy*, 103(6), 1121–57.
- BELLMAN, RICHARD (1957): *Dynamic Programming*. Princeton University Press, Princeton, NJ, USA, 1 edn.
- BENHABIB, JESS, ALBERTO BISIN, AND SHENGHAO ZHU (2015): “The wealth distribution in Bewley economies with capital income risk,” *Journal of Economic Theory*, 159, 489–515, Available at <https://www.nber.org/papers/w20157.pdf>.
- BEWLEY, TRUMAN (1977): “The Permanent Income Hypothesis: A Theoretical Formulation,” *Journal of Economic Theory*, 16, 252–292.
- BLANCHARD, OLIVIER J. (1985): “Debt, Deficits, and Finite Horizons,” *Journal of Political Economy*, 93(2), 223–247.
- BLUNDELL, RICHARD, HAMISH LOW, AND IAN PRESTON (2008): “Decomposing Changes in Income Risk Using Consumption Data,” *Manuscript, University College London*.
- BOYD, JOHN H. (1990): “Recursive Utility and the Ramsey Problem,” *Journal of Economic Theory*, 50(2), 326–345.
- CARROLL, CHRISTOPHER D., KARSTEN CHIPENIUK, KIICHI TOKUOKA, AND WEIFENG WU (2020): “The Method of Moderation,” *Manuscript, Johns Hopkins University*.

⁷It is useful to emphasize that permanent noncapital income as defined here differs from what Deaton (1992) calls permanent income (which is often adopted in the macro literature). Deaton defines permanent income as the amount that a perfect foresight consumer could spend while leaving total (human and nonhuman) wealth constant. Relatedly, we refer to \mathbf{m}_t as ‘cash-on-hand’ or ‘market resources’ rather than as wealth to avoid any confusion for readers accustomed to thinking of the discounted value of future noncapital income as a part of wealth. The ‘market resources’ terminology is motivated by the model’s assumption that human wealth cannot be capitalized, an implication of anti-slavery laws.

⁸Hereafter for brevity we occasionally drop time subscripts, e.g. $\mathbb{E}[\psi^{-\rho}]$ signifies $\mathbb{E}_t[\psi_{t+1}^{-\rho}]$.

- CARROLL, CHRISTOPHER D., ALEXANDER M. KAUFMAN, JACQUELINE L. KAZIL, NATHAN M. PALMER, AND MATTHEW N. WHITE (2018): “The Econ-ARK and HARK: Open Source Tools for Computational Economics,” in *Proceedings of the 17th Python in Science Conference*, ed. by Fatih Akici, David Lippa, Dillon Niederhut, and M Pacer, pp. 25 – 30. doi: [10.5281/zenodo.1001067](https://doi.org/10.5281/zenodo.1001067).
- CARROLL, CHRISTOPHER D., AND MILES S. KIMBALL (1996): “On the Concavity of the Consumption Function,” *Econometrica*, 64(4), 981–992, <http://econ.jhu.edu/people/ccarroll/concavity.pdf>.
- CARROLL, CHRISTOPHER D., AND ANDREW A. SAMWICK (1997): “The Nature of Precautionary Wealth,” *Journal of Monetary Economics*, 40(1), 41–71.
- CARROLL, CHRISTOPHER D., JIRI SLACALEK, AND KIICHI TOKUOKA (2015): “Buffer-stock saving in a Krusell–Smith world,” *Economics Letters*, 132, 97–100, At <http://econ.jhu.edu/people/ccarroll/papers/BSinKS.pdf>.
- CARROLL, CHRISTOPHER D., JIRI SLACALEK, KIICHI TOKUOKA, AND MATTHEW N. WHITE (2017): “The Distribution of Wealth and the Marginal Propensity to Consume,” *Quantitative Economics*, 8, 977–1020, At <http://econ.jhu.edu/people/ccarroll/papers/cstwMPC>.
- CLARIDA, RICHARD H. (1987): “Consumption, Liquidity Constraints, and Asset Accumulation in the Face of Random Fluctuations in Income,” *International Economic Review*, XXVIII, 339–351.
- DEATON, ANGUS S. (1991): “Saving and Liquidity Constraints,” *Econometrica*, 59, 1221–1248, <http://www.jstor.org/stable/2938366>.
- (1992): *Understanding Consumption*. Oxford University Press, New York.
- DURÁN, JORGE (2003): “Discounting long run average growth in stochastic dynamic programs,” *Economic Theory*, 22(2), 395–413.
- FRIEDMAN, MILTON A. (1957): *A Theory of the Consumption Function*. Princeton University Press.
- JAPPELLI, TULLIO, AND LUIGI PISTAFERRI (2000): “Intertemporal Choice and Consumption Mobility,” *Econometric Society World Congress 2000 Contributed Paper Number 0118*.
- KRUEGER, DIRK, KURT MITMAN, AND FABRIZIO PERRI (2016): “Macroeconomics and Household Heterogeneity,” *Handbook of Macroeconomics*, 2, 843–921.
- LI, HUIYU, AND JOHN STACHURSKI (2014): “Solving the income fluctuation problem with unbounded rewards,” *Journal of Economic Dynamics and Control*, 45, 353–365.

- MA, QINGYIN, JOHN STACHURSKI, AND ALEXIS AKIRA TODA (2018): “The Income Fluctuation Problem with Capital Income Risk: Optimality and Stability,” *arXiv preprint arXiv:1812.01320*.
- (2020): “The Income Fluctuation Problem and the Evolution of Wealth,” *Journal of Economic Theory*, 187, 105003.
- MACURDY, THOMAS (1982): “The Use of Time Series Processes to Model the Error Structure of Earnings in a Longitudinal Data Analysis,” *Journal of Econometrics*, 18(1), 83–114.
- MARTINS-DA ROCHA, V FILIPE, AND YIANNIS VAILAKIS (2010): “Existence and uniqueness of a fixed point for local contractions,” *Econometrica*, 78(3), 1127–1141.
- MATKOWSKI, JANUSZ, AND ANDRZEJ S. NOWAK (2011): “On Discounted Dynamic Programming With Unbounded Returns,” *Economic Theory*, 46, 455–474.
- MUTH, JOHN F. (1960): “Optimal Properties of Exponentially Weighted Forecasts,” *Journal of the American Statistical Association*, 55(290), 299–306.
- RABAULT, GUILLAUME (2002): “When do borrowing constraints bind? Some new results on the income fluctuation problem,” *Journal of Economic Dynamics and Control*, 26(2), 217–245.
- RAMSEY, FRANK (1928): “A Mathematical Theory of Saving,” *Economic Journal*, 38(152), 543–559.
- RINCÓN-ZAPATERO, JUAN PABLO, AND CARLOS RODRÍGUEZ-PALMERO (2003): “Existence and uniqueness of solutions to the Bellman equation in the unbounded case,” *Econometrica*, 71(5), 1519–1555.
- SCHECHTMAN, JACK, AND VERA ESCUDERO (1977): “Some results on ‘An Income Fluctuation Problem’,” *Journal of Economic Theory*, 16, 151–166.
- SCHEINKMAN, JOSÉ, AND LAURENCE WEISS (1986): “Borrowing Constraints and Aggregate Economic Activity,” *Econometrica*, 54(1), 23–46.
- STOKEY, NANCY L., ROBERT E. LUCAS, AND EDWARD C. PRESCOTT (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press.
- STORESLETTEN, KJETIL, CHRIS I. TELMER, AND AMIR YARON (2004): “Consumption and Risk Sharing Over the Life Cycle,” *Journal of Monetary Economics*, 51(3), 609–633.
- SZEIDL, ADAM (2012): “Stable Invariant Distribution in Buffer-Stock Saving and Stochastic Growth Models,” *Manuscript, Central European University*.

- TOCHE, PATRICK (2005): “A Tractable Model of Precautionary Saving in Continuous Time,” *Economics Letters*, 87(2), 267–272, <http://ideas.repec.org/a/eee/ecolet/v87y2005i2p267-272.html>.
- YAO, JIAXIONG (2012): “The Theoretical Foundations of Buffer Stock Saving: A Note,” *Manuscript, Johns Hopkins University*.
- ZELDES, STEPHEN P. (1989): “Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence,” *Quarterly Journal of Economics*, 104(2), 275–298.