Theoretical Foundations of Buffer Stock Saving

September 28, 2020

Christopher D. Carroll[1]

Abstract

This paper builds theoretical foundations for rigorous and intuitive understanding of 'buffer stock' saving models, pairing each theoretical result with a quantitative exploration. After describing conditions under which the consumption function converges, the paper shows that a 'target' buffer stock exists only under conditions strictly stronger than those that guarantee convergence of the consumption and value functions. It also shows that the average growth rate of consumption equals the average growth rate of permanent income (in a small open economy populated by buffer stock savers). Together, the (provided) numerical tools and (proven) analytical results constitute a comprehensive toolkit for understanding buffer stock models.

Keywords Precautionary saving, buffer stock saving, marginal propensity to consume, permanent income hypothesis

JEL codes D81, D91, E21

Archive: `http://www.econ2.jhu.edu/people/ccarroll/BufferStockTheory.zip`
PDF: `http://llorracc.github.io/BufferStockTheory/BufferStockTheory.pdf`
Slides: `http://llorracc.github.io/BufferStockTheory/BufferStockTheory-Slides.pdf`
Web: `http://llorracc.github.io/BufferStockTheory/BufferStockTheory/`
Appendix: `http://llorracc.github.io/BufferStockTheory/BufferStockTheory#Appendices`
GitHub: `http://github.com/llorracc/BufferStockTheory`
(In GitHub repo, see `/Code` for tools for solving and simulating the model)

CLICK HERE for an interactive Jupyter Notebook that uses the Econ-ARK/HARK toolkit to produce all of the paper's figures (warning: the notebook may take several minutes to launch). Information about citing the toolkit can be found at Acknowleding Econ-ARK.

[1]Contact: `ccarroll@jhu.edu`, Department of Economics, 590 Wyman Hall, Johns Hopkins University, Baltimore, MD 21218, `http://econ.jhu.edu/people/ccarroll`, and National Bureau of Economic Research.

All numerical results herein were produced using the Econ-ARK/HARK toolkit, which can be cited per our references

1 Introduction

In the presence of empirically realistic transitory and permanent shocks to income *a la* Friedman (1957), only one other ingredient is required to define a testable model of optimal consumption: A description of preferences. Modelers usually assume geometric discounting of a constant relative risk aversion (CRRA) utility function, because, starting with Zeldes (1989), a large literature has shown that models of this kind have quantitative predictions that can match microeconomic evidence reasonably well.

A companion theoretical literature has shown that standard numerical solution methods provide good approximations to limiting "true" mathematical solutions – but only for models more complex than the simple case with just shocks and utility. The extra complexity has been required because standard contraction mapping theorems (beginning with Bellman (1957) and including those following Stokey et. al. (1989)) cannot be applied when the utility function is unbounded (like CRRA - see section 2.1).[1]

This paper's first technical contribution is to articulate the (surprisingly loose) conditions under which the simple problem (without convenient shortcuts like a consumption floor or liquidity constraints) defines a contraction mapping with a nondegenerate consumption function (the main requirement is a 'Finite Value of Autarky' condition). Another contribution is to specify the conditions under which the resulting consumption function implies there is a 'target' wealth-to-permanent-income ratio (so the model exhibits 'buffer stock' saving behavior.) The key requirement for existence of a target is that the model's parameters satisfy a "Growth Impatience Condition" (equation (**??**)) that relates preferences and uncertainty to the predictable growth rate of income.

Even without a formal proof, target saving of this kind has been intuitively understood to underlie central numerical results from the heterogeneous agent macroeconomics literature; for example, the logic of target saving is central to the explanation by Krueger, Mitman, and Perri (2016) of the fact that, during the Great Recession, middle-class consumers cut their consumption more than the poor or the rich. The theoretical logic articulated below explains this finding: Learning that the future has become more uncertain does not change the urgent imperatives of the poor (their high $\mathrm{u}'(c)$) because they have little room to maneuver. Increased labor income uncertainty does not change the behavior of the rich because the increase in uncertainty does not threaten their consumption much. Only people in the middle have both the motivation and the wiggle-room to reduce their discretionary spending.

Conveniently, elements required for the convergence proof turn out to provide analytical foundations for many other results that have become familiar from the numerical

[1] It is unclear whether newer methods such as those of Matkowski and Nowak (2011) could overcome this problem, or how difficult it would be to do so; but in any case this particular problem does not seem to have been tackled by those methods or any others.

(Carroll, Kaufman, Kazil, Palmer, and White (2018)); for reference to the toolkit itself see Acknowleding Econ-ARK. Thanks to James Feigenbaum, Joseph Kaboski, Miles Kimball, Qingyin Ma, Misuzu Otsuka, Damiano Sandri, John Stachurski, Adam Szeidl, Metin Uyanik, Weifeng Wu, Xudong Zheng, and Jiaxiong Yao for comments on earlier versions of this paper, John Boyd for help in applying his weighted contraction mapping theorem, Ryoji Hiraguchi for extraordinary mathematical insight that improved the paper greatly, David Zervos for early guidance to the literature, and participants in a seminar at Johns Hopkins University and a presentation at the 2009 meetings of the Society of Economic Dynamics for their insights.

literature. All theoretical conclusions are paired with numerically computed illustrations (using an open-source toolkit available from the Econ-ARK project). All of the insights of this paper are instantiated in the toolkit, which algorithmically flags parametric choices under which a problem fails to define a contraction mapping, under which a target level of wealth does not exist, or under which the solution is otherwise degenerate.

Thus, the theoretical foundations provided here are valuable both because they provide intuition about the determinants of saving targets, and because they make it easier to develop reliable numerical solution methods (by providing tight restrictions that valid solutions must satisfy).

The paper proceeds in three parts.

The first part articulates the conditions required for the problem to define a unique nondegenerate limiting consumption function, and discusses the relation of the paper's model to models previously considered in the literature. The required conditions are interestingly parallel to those required for the liquidity constrained perfect foresight model; that parallel is explored and explained. Next, the paper derives some limiting properties of the consumption function as cash approaches infinity and as it approaches its lower bound, and the theorem is proven explaining when the problem defines a contraction mapping. Finally, a related class of commonly-used models (exemplified by Deaton (1991)) is shown to constitute a particular limit of this paper's more general model.

The next section examines five key properties of the model. First, as cash approaches infinity the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as cash approaches zero the expected growth rate of consumption approaches infinity, and the MPC approaches a simple analytical limit. Third, if the consumer is 'growth impatient,' a unique target cash-to-permanent-income ratio will exist. Fourth, at the target cash ratio, the expected growth rate of consumption is slightly less than the expected growth rate of permanent noncapital income. Finally, the expected growth rate of consumption is declining in the level of cash. The first four propositions are proven under general assumptions about parameter values; the last is shown to hold if there are no transitory shocks, but may fail in extreme cases if there are both transitory and permanent shocks.

Szeidl (2012) has shown that such an economy will be characterized by stable invariant distributions for the consumption ratio, the wealth ratio, and other variables.[2] Using Szeidl's result, the final section discusses conditions under which, even with a fixed aggregate interest rate that differs from the time preference rate, an economy populated by buffer stock consumers converges to a balanced growth equilibrium in which the growth rate of consumption tends toward the (exogenous) growth rate of permanent income.

[2]Szeidl's proof supplants the analysis in an earlier draft of this paper, which conjectured that such a result held and provided supportive simulation evidence.

2 The Problem

2.1 Setup

The consumer solves an optimization problem from period t until the end of life at T defined by the objective

$$\max \ \mathbb{E}_t \left[\sum_{n=0}^{T-t} \beta^n \mathrm{u}(\mathbf{c}_{t+n}) \right] \tag{1}$$

where

$$\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho) \tag{2}$$

is a constant relative risk aversion utility function with $\rho > 1$.[3,4] The consumer's initial condition is defined by market resources $\mathbf{m}_t$ (which Deaton (1991) called 'cash-on-hand') and permanent noncapital income $\mathbf{p}_t$.

In the usual treatment, a dynamic budget constraint (DBC) simultaneously incorporates all of the elements that determine next period's $\mathbf{m}$ given this period's choices; but for the detailed analysis here, it will be useful to disarticulate the steps so that individual ingredients can be separately examined:

Appendices

A Perfect Foresight Liquidity Constrained Solution

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\grave{\mathrm{c}}(m)$ that arise under various parametric conditions. Results are summarized in table 1.

A.1 If PF-GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails ($\cancel{\text{PF-GIC}}$, $1 < \text{Þ}/\Gamma$). Under $\cancel{\text{PF-GIC}}$ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathsf{R}\beta)^{1/\rho}/\Gamma$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[5]

$$1 < (\mathsf{R}\beta)^{1/\rho}\Gamma^{-1} \tag{3}$$

$$1 < \mathsf{R}\beta\Gamma^{-\rho} \tag{4}$$

[3] The main results also hold for logarithmic utility which is the limit as $\rho \to 1$ but incorporating the logarithmic special case in the proofs is cumbersome and therefore omitted.

[4] We will define the infinite horizon solution as the limit of the finite horizon problem as the horizon $T-t$ approaches infinity.

[5] The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $\mathrm{u}'(c_{\#}) = \mathsf{R}\beta\mathrm{u}'(\Gamma)$ which is $c_{\#} = \Gamma/(\mathsf{R}\beta)^{1/\rho}$ and the consumption function will be defined by $\grave{\mathrm{c}}(m) = \min[m, c_{\#} + (m - c_{\#})\underline{\kappa}]$.

$$\mathrm{u}'(1) < \mathsf{R}\beta\mathrm{u}'(\Gamma). \tag{5}$$

Similar logic shows that under these circumstances the constraint will never bind at $m = 1$ for a constrained consumer with a finite horizon of n periods, so for $m \geq 1$ such a consumer's consumption function will be the same as for the unconstrained case examined in the main text.

If the RIC fails ($1 < \text{Þ}_{\mathsf{R}}$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$\grave{\mathrm{c}}_{T-n}(m) = 0(b_t + h). \tag{6}$$

(that is, consumption is zero for any level of human or nonhuman wealth).

If the RIC fails and the FHWC fails, human wealth limits to $h = \infty$ so the consumption function limits to either $\grave{\mathrm{c}}_{T-n}(m) = 0$ or $\grave{\mathrm{c}}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞.[6]

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying ~~PF-GIC~~ we must impose the RIC (and the FHWC can be shown to be a consequence of ~~PF-GIC~~ and RIC). In this case, the consumer's optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer would choose $c = m$ from equation (**??**):

$$m_{\#} = (m_{\#} - 1 + h)\underline{\kappa} \tag{7}$$

$$m_{\#}(1 - \underline{\kappa}) = (h - 1)\underline{\kappa} \tag{8}$$

$$m_{\#} = (h - 1)\left(\frac{\underline{\kappa}}{1 - \underline{\kappa}}\right) \tag{9}$$

which (under these assumptions) satisfies $0 < m_{\#} < 1$.[7] For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m; for such m, the constrained consumer is obliged to choose $\grave{\mathrm{c}}(m) = m$.[8] For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{\mathrm{c}}(m)$.

(Stachurski and Toda (2019) obtain a similar lower bound on consumption and use it to study the tail behavior of the wealth distribution.)

A.2 If PF-GIC Holds

Imposition of the PF-GIC reverses the inequality in (5), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t, but perpetually thereafter.

Now define $b^n_{\#}$ as the b_t such that an unconstrained consumer holding $b_t = b^n_{\#}$ would behave so as to arrive in period $t + n$ with $b_{t+n} = 0$ (with $b^0_{\#}$ trivially equal to 0); for

[6] The knife-edge case is where $\text{Þ} = \Gamma$, in which case the two quantites counterbalance and the limiting function is $\grave{\mathrm{c}}(m) = \min[m, 1]$.

[7] Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by ~~PF-GIC~~.

[8] As an illustration, consider a consumer for whom $\text{Þ} = 1$, $\mathsf{R} = 1.01$ and $\Gamma = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\Gamma < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

example, a consumer with $b_{t-1} = b^1_{\#}$ was on the 'cusp' of being constrained in period $t-1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the PF-GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m^0_{\#} = 1 + b^0_{\#} = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c^0_{\#} = 1$.

We can construct the entire 'prehistory' of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must have been growing according to Þ_Γ, so consumption n periods in the past must have been

$$c^n_{\#} = \text{Þ}_\Gamma^{-n} c_t = \text{Þ}_\Gamma^{-n}. \tag{10}$$

The PDV of consumption from $t-n$ until t can thus be computed as

$$\begin{aligned}
\mathbb{C}^t_{t-n} &= c_{t-n}(1 + \text{Þ}/\mathsf{R} + \ldots + (\text{Þ}/\mathsf{R})^n) \\
&= c^n_{\#}(1 + \text{Þ}_{\mathsf{R}} + \ldots + \text{Þ}^n_{\mathsf{R}}) \\
&= \text{Þ}_\Gamma^{-n}\left(\frac{1 - \text{Þ}^{n+1}_{\mathsf{R}}}{1 - \text{Þ}_{\mathsf{R}}}\right) \qquad (11) \\
&= \left(\frac{\text{Þ}_\Gamma^{-n} - \text{Þ}_{\mathsf{R}}}{1 - \text{Þ}_{\mathsf{R}}}\right) \qquad (12)
\end{aligned}$$

and note that the consumer's human wealth between $t-n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post-t income) is

$$h^n_{\#} = 1 + \ldots + \mathcal{R}^{-n} \tag{13}$$

while the intertemporal budget constraint says

$$\mathbb{C}^t_{t-n} = b^n_{\#} + h^n_{\#}$$

from which we can solve for the $b^n_{\#}$ such that the consumer with $b_{t-n} = b^n_{\#}$ would unconstrainedly plan (in period $t-n$) to arrive in period t with $b_t = 0$:

$$b^n_{\#} = \mathbb{C}^t_{t-n} - \overbrace{\left(\frac{1 - \mathcal{R}^{-(n+1)}}{1 - \mathcal{R}^{-1}}\right)}^{h^n_{\#}}. \tag{14}$$

Defining $m^n_{\#} = b^n_{\#} + 1$, consider the function $\grave{c}(m)$ defined by linearly connecting the points $\{m^n_{\#}, c^n_{\#}\}$ for integer values of $n \geq 0$ (and setting $\grave{c}(m) = m$ for $m < 1$). This function will return, for any value of m, the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes; for infinitesimal ϵ the MPC of a consumer with assets $m = m^n_{\#} - \epsilon$ is discretely higher than for a consumer with assets $m = m^n_{\#} + \epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (14) for the entire domain of positive real values of b, we need $b_{\#}^{n}$ to become arbitrarily large with n. That is, we need

$$\lim_{n\to\infty} b_{\#}^{n} = \infty. \tag{15}$$

A.2.1 If FHWC Holds

The FHWC requires $\mathcal{R}^{-1} < 1$, in which case the second term in (14) limits to a constant as $n \uparrow \infty$, and (15) reduces to a requirement that

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n} - (\text{Þ}_{\mathsf{R}}/\text{Þ}_{\Gamma})^{n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n} - \mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty.
\end{aligned}$$

Given the PF-GIC $\text{Þ}_{\Gamma}^{-1} > 1$, this will hold iff the RIC holds, $\text{Þ}_{\mathsf{R}} < 1$. But given that the FHWC $\mathsf{R} > \Gamma$ holds, the PF-GIC is stronger (harder to satisfy) than the RIC; thus, the FHWC and the PF-GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because the date at which the constraint binds becomes arbitrarily distant, so the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m\to\infty} \grave{\mathrm{c}}(m) - \bar{\mathrm{c}}(m) = 0. \tag{16}$$

A.2.2 If FHWC Fails

If the FHWC fails, matters are a bit more complex.

Given failure of FHWC, (15) requires

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\mathcal{R}^{-n}\text{Þ}_{\mathsf{R}} - \text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right) + \left(\frac{1-\mathcal{R}^{-(n+1)}}{\mathcal{R}^{-1}-1}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\mathsf{R}}}{\text{Þ}_{\mathsf{R}}-1} - \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right)\mathcal{R}^{-n} - \left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right) &= \infty
\end{aligned} \tag{17}$$

If RIC Holds. When the RIC holds, rearranging (17) gives

$$\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right) - \mathcal{R}^{-n}\left(\frac{\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}} + \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right) = \infty$$

and for this to be true we need

$$\begin{aligned}
\text{Þ}_{\Gamma}^{-1} &> \mathcal{R}^{-1} \\
\Gamma/\text{Þ} &> \Gamma/\mathsf{R} \\
1 &> \text{Þ}/\mathsf{R}
\end{aligned}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n\to\infty} \kappa_{\#}^{n} = \lim_{n\to\infty} \left(\frac{c_{\#}^{n}}{b_{\#}^{n}}\right) \tag{18}$$

which with a bit of algebra[9] can be shown to asymptote to the MPC in the perfect foresight model:[10]

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 1 - \text{Þ}_{\mathsf{R}}. \tag{20}$$

If RIC Fails. Consider now the $\cancel{\text{RIC}}$ case, $\text{Þ}_{\mathsf{R}} > 1$. We can rearrange (17)as

$$\lim_{n\to\infty} \left(\frac{\text{Þ}_{\mathsf{R}}(\mathcal{R}^{-1}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)} - \frac{\mathcal{R}^{-1}(\text{Þ}_{\mathsf{R}}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)}\right)\mathcal{R}^{-n} - \left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right) = \infty. \tag{21}$$

which makes clear that with $\cancel{\text{FHWC}} \Rightarrow \mathcal{R}^{-1} > 1$ and $\cancel{\text{RIC}} \Rightarrow \text{Þ}_{\mathsf{R}} > 1$ the numerators and denominators of both terms multiplying $\mathcal{R}^{-n}$ can be seen transparently to be positive. So, the terms multiplying $\mathcal{R}^{-n}$ in (17) will be positive if

$$\begin{aligned}
\text{Þ}_{\mathsf{R}}\mathcal{R}^{-1} - \text{Þ}_{\mathsf{R}} &> \mathcal{R}^{-1}\text{Þ}_{\mathsf{R}} - \mathcal{R}^{-1} \\
\mathcal{R}^{-1} &> \text{Þ}_{\mathsf{R}} \\
\Gamma &> \text{Þ}
\end{aligned}$$

which is merely the PF-GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving $\mathcal{R}^{-n}$ goes to $+\infty$ faster than the term involving $-\text{Þ}_{\Gamma}^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned}
\mathcal{R}^{-1} &> \text{Þ}_{\Gamma}^{-1} \\
\Gamma/\mathsf{R} &> \Gamma/\text{Þ} \\
\text{Þ}/\mathsf{R} &> 1
\end{aligned}$$

which merely confirms the starting assumption that the RIC fails.

What is happening here is that the $c_{\#}^{n}$ term is increasing backward in time at rate dominated in the limit by $\Gamma/\text{Þ}$ while the $b_{\#}$ term is increasing at a rate dominated by Γ/R term and

$$\Gamma/\mathsf{R} > \Gamma/\text{Þ} \tag{22}$$

because $\cancel{\text{RIC}} \Rightarrow \text{Þ} > \mathsf{R}$.

Consequently, while $\lim_{n\uparrow\infty} b_{\#}^{n} = \infty$, the limit of the *ratio* $c_{\#}^{n}/b_{\#}^{n}$ in (18) is zero. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the

[9] Calculate the limit of

$$\left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\Gamma}^{-n}/(1-\text{Þ}_{\mathsf{R}}) - (1-\mathcal{R}^{-1}\mathcal{R}^{-n})/(1-\mathcal{R}^{-1})}\right) = \left(\frac{1}{1/(1-\text{Þ}_{\mathsf{R}}) + \mathcal{R}^{-n}\mathcal{R}^{-1}/(1-\mathcal{R}^{-1})}\right) \tag{19}$$

[10] For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the Mathematica software archive.

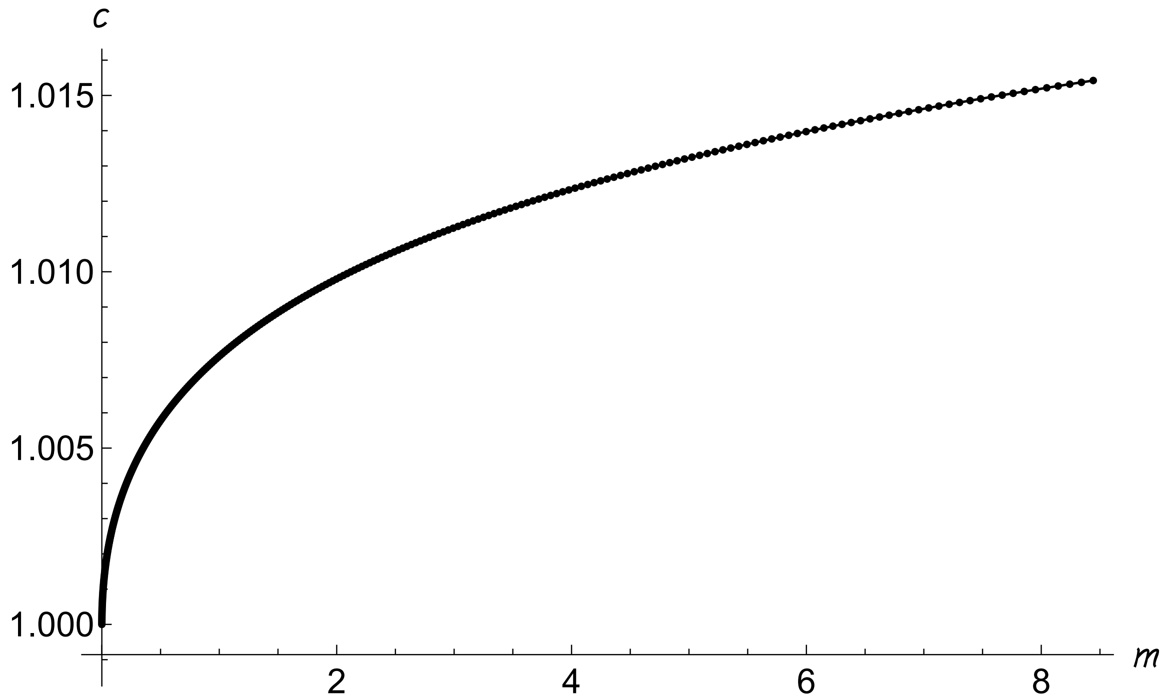

Figure 1 Nondegenerate Consumption Function with ~~FHWC~~ and ~~RIC~~

RIC fails. It remains true that ~~RIC~~ implies a limiting MPC of zero,

$$\lim_{m \to \infty} \grave{\boldsymbol{\kappa}}(m) = 0, \tag{23}$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\grave{\mathrm{c}}(m) = 0$. (Figure 1 presents an example for $\rho = 2$, $\mathsf{R} = 0.98$, $\beta = 1.00$, $\Gamma = 0.99$; note that the horizontal axis is bank balances $b = m{-}1$; the part of the consumption function below the depicted points is uninteresting – $c = m$ – so not worth plotting).

We can summarize as follows. Given that the PF-GIC holds, the interesting question is whether the FHWC holds. If so, the RIC automatically holds, and the solution limits into the solution to the unconstrained problem as $m \uparrow \infty$. But even if the FHWC fails, the problem has a well-defined and nondegenerate solution, whether or not the RIC holds.

Although these results were derived for the perfect foresight case, we know from work elsewhere in this paper and in other places that the perfect foresight case is an upper bound for the case with uncertainty. If the upper bound of the MPC in the perfect foresight case is zero, it is not possible for the upper bound in the model with uncertainty to be greater than zero, because for any $\kappa > 0$ the level of consumption in the model with uncertainty would eventually exceed the level of consumption in the absence of uncertainty.

Ma and Toda (2020) characterize the limits of MPC in a more general framework that allows for non-CRRA utility as well as capital and labor income risks in a Markovian setting, and find that in that much more general framework the limiting MPC is also zero.

Table 1 Taxonomy of Perfect Foresight Liquidity Constrained Model Outcomes

For constrained $\grave{c}$ and unconstrained $\bar{\mathrm{c}}$ consumption functions

Main Condition Subcondition	Math	Outcome, Comments or Results
~~PF-GIC~~	$1 < \text{Þ}/\Gamma$	Constraint never binds for $m \geq 1$
and RIC	$\text{Þ}/\mathsf{R} < 1$	FHWC holds ($\mathsf{R} > \Gamma$); $\grave{\mathrm{c}}(m) = \bar{\mathrm{c}}(m)$ for $m \geq 1$
and ~~RIC~~	$1 < \text{Þ}/\mathsf{R}$	$\grave{\mathrm{c}}(m)$ is degenerate: $\grave{\mathrm{c}}(m) = 0$
PF-GIC	$\text{Þ}/\Gamma < 1$	Constraint binds in finite time for any m
and RIC	$\text{Þ}/\mathsf{R} < 1$	FHWC may or may not hold
		$\lim_{m\uparrow\infty} \bar{\mathrm{c}}(m) - \grave{\mathrm{c}}(m) = 0$
		$\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
and ~~RIC~~	$1 < \text{Þ}/\mathsf{R}$	~~FHWC~~
		$\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~PF-GIC~~ and RIC both hold, while the third row indicates that when the PF-GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the PF-GIC holds, the constraint will bind in finite time.

References

Bellman, Richard (1957): *Dynamic Programming.* Princeton University Press, Princeton, NJ, USA, 1 edn.

Carroll, Christopher D., Alexander M. Kaufman, Jacqueline L. Kazil, Nathan M. Palmer, and Matthew N. White (2018): "The Econ-ARK and HARK: Open Source Tools for Computational Economics," in *Proceedings of the 17th Python in Science Conference*, ed. by Fatih Akici, David Lippa, Dillon Niederhut, and M Pacer, pp. 25 – 30. doi: 10.5281/zenodo.1001067.

Deaton, Angus S. (1991): "Saving and Liquidity Constraints," *Econometrica*, 59, 1221–1248, http://www.jstor.org/stable/2938366.

Friedman, Milton A. (1957): *A Theory of the Consumption Function.* Princeton University Press.

Krueger, Dirk, Kurt Mitman, and Fabrizio Perri (2016): "Macroeconomics and Household Heterogeneity," *Handbook of Macroeconomics*, 2, 843–921.

Ma, Qingyin, and Alexis Akira Toda (2020): "Asymptotic Marginal Propensity to Consume," Manuscript, Australian National University.

Matkowski, Janusz, and Andrzej S. Nowak (2011): "On Discounted Dynamic Programming With Unbounded Returns," *Economic Theory*, 46, 455–474.

Stachurski, John, and Alexis Akira Toda (2019): "An Impossibility Theorem for Wealth in Heterogeneous-agent Models with Limited Heterogeneity," *Journal of Economic Theory*, 182, 1–24.

Stokey, Nancy L., Robert E. Lucas, and Edward C. Prescott (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press.

Szeidl, Adam (2012): "Stable Invariant Distribution in Buffer-Stock Saving and Stochastic Growth Models," *Manuscript, Central European University.*

Zeldes, Stephen P. (1989): "Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence," *Quarterly Journal of Economics*, 104(2), 275–298.