1 The Steady-State Distribution of Permanent Income

Imagine an economy that begins functioning at date 1 with a population of size L_0 living at locations indexed by ℓ.[1] Immediately prior to date 1, all agents have a value of 'permanent income' $\wp_{0,\ell} = 1$.

Here we derive the dynamics of the distribution of $\wp_{t,\ell}$ across locations, and the limiting distribution as $t \to \infty$, when each period the agent at each location receives an idiosyncratic shock to $\wp = \log \wp$ until 'death,' at which point the agent is replaced by a 'newborn.'[2] To make the computational instantiation transparent (and debuggable), we are more explicit than usual in denoting sets.

1. The first thing that happens at date τ is that agents living at a randomly selected constant proportion $\mho$ of locations die and are replaced by newborns; $\{\mho_\tau\}$ is the set of such locations. Locations where death does not occur are in the set $\{\Lambda_\tau\}$ (death has been canceled - for now), and of course $\Lambda = 1 - \mho$ and $0 < \mho < 1$.

2. The initial permanent income of newborns is $\wp = 1$ (formally, $\wp_{\tau,\ell} = 1 \ \forall \ \ell \in \{\mho_\tau\}$)

3. Next, every location is subjected to a mean-one lognormally distributed shock

$$\psi \equiv \log \Psi \sim \mathcal{N}(\mu, \sigma_\psi^2) \tag{1}$$

where $\mu = -\sigma_\psi^2/2$. That is, for all $\wp \in \Lambda_\tau$,

$$\wp_\tau = \wp_{\tau-1} + \psi_\tau \tag{2}$$

while for $\ell \in \mho$

$$\wp_\tau = \psi_\tau \tag{3}$$

4. Under these assumptions, $\mathbb{E}_{\{\Lambda_\tau\}}[\wp_{\tau+1,\ell}] = \mathbb{E}_{\{\Lambda_\tau\}}[\wp_{\tau,\ell}\Psi_{\tau+1,\ell}] = \wp_{\tau,\ell}$ (where $\mathbb{E}_{\{[\text{set}]_\tau\}}$ denotes the expectation of the value of its argument at the *locations* occupied in period τ for the agents in the set, while $\mathbb{E}[\wp_{\tau,\ell}]$ denotes the expectation across all persons and locations)

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[1] These will be represented computationally by a simulated population of integer size L_0 in which each member of the population at any date τ occupies a slot at integer location ℓ).

[2] This is a discrete version of the 'perpetual youth' model of Blanchard (1985).

5. Because dying agents are selected at random, the expectation of $\wp_{\tau,\ell}$ for period-τ survivors must match the expectation of $\wp_\tau$ for the entire population:

$$\mathbb{E}_{\{\Lambda_\tau\}}[\wp_{\tau,\ell}] = \mathbb{E}[\wp_{\tau,\ell}] \tag{4}$$

Thus for an economy in which the average value of $\wp$ is 1 in period τ, the population-mean level of $\wp$ in period $\tau + 1$ will remain at 1:

$$\mathbb{E}[\wp_{\tau+1}] = \mho \overbrace{\mathbb{E}_{\{\Lambda_\tau\}}[\wp_{\tau,\ell}\Psi_{\tau+1,\ell}]}^{\text{survivors have } \wp = 1} + (1 - \mho) \overbrace{\mathbb{E}_{\{\mho_\tau\}}[1 \times \Psi_{\tau+1,\ell}]}^{\text{newborns have expected } \wp = 1}$$
$$= \mho + (1 - \mho)$$

At t we denote the locations occupied by persons who have been alive for exactly m periods as the set $\{\ell_t[m]\}$. Agents for whom $m = 1$ will have experienced only a single shock, the one at the beginning of the period; for $m > 0$ the agents at those locations (that is, for $\ell \in \{\ell_t\}[m]$),

$$\wp_{t,\ell} = \sum_{i=1}^{m} \psi_{i,\ell}. \tag{5}$$

For legibility defining $\{\wp_t[m]\} = \{\wp_{t,\{\ell_t[m]\}}\}$ as the values of $\wp$ across the set of agents who in period t have been alive for exactly m periods, the independence of the ψ's means that for every $\ell \in \{\ell_t[m]\}$,

$$\wp_{t,\ell}[m] \sim \mathcal{N}(m\mu, m\sigma_\psi^2) \tag{6}$$

because such agents have experienced m IID permanent shocks (recall that the variance of the sum of independent variables is equal to the sum of their variances).

So the PDF of an agent who has been alive for m periods is the PDF of $\mathcal{N}(m\mu, m\sigma_\psi^2)$,

$$\text{PDF}(\wp_t[m]) = \left(\sigma_\psi\sqrt{m2\pi}\right)^{-1} \exp\left(-\frac{(\wp - \mu m)^2}{2m\sigma_\psi^2}\right) \tag{7}$$

The set of all values of $\wp$ in period t will be the union of the $\wp$'s of persons in the sets of people who have been alive for each possible number of periods (1, for newborns):

$$\{\wp_t\} = \bigcup_{m=1}^{t} \{\wp_t[m]\} \tag{8}$$

but since each of the individual $\wp_{t,\ell}[m]$ objects is a set of identical gaussians, the set as a whole will be the corresponding gaussian with a weight equal to the size of its population. The cohort's population mass starts at size $(1 - \Lambda)$ and shrinks by a factor Λ each period:

$$k_m = (1 - \Lambda)\Lambda^{m-1} \tag{9}$$

and the total population mass after t periods is the sum of the masses of the living cohorts:

$$K_t = \sum_{m=1}^{t} k_m \tag{10}$$

$$= (1 - \Lambda) \sum_{m=1}^{t} \Lambda^{i-1} \tag{11}$$

$$= (1 - \Lambda) \left(\frac{1 - \Lambda^t}{1 - \Lambda} \right) \tag{12}$$

$$= 1 - \Lambda^t \tag{13}$$

and we use ϱ for the proportion of the population of the population at date t constituted by agents who have lived m periods:

$$\varrho_{t,m} = k_m / K_t \tag{14}$$

$$= \left(\frac{(1 - \Lambda)\Lambda^{m-1}}{1 - \Lambda^t} \right) \tag{15}$$

For example, in the first period of operation of the economy, $t = m = 1$ and $\varrho_{t,m} = 1$.

Thus $\wp_t$ is distributed as a weighted sum of gaussians:

$$\wp_t \sim \sum_{m=1}^{t} \varrho_{t,m} \mathcal{N}(m\mu, m\sigma_\psi^2) \tag{16}$$

We are finally in a position to derive the results we need.

The set of values of permanent income at all locations at date t, $\{\wp_t\}$, is a weighted collection of normally distributed random variables, with the weights corresponding to the population masses of the cohorts of persons who have lived m periods given by $\varrho_{t,m}$. The PDF of the entire distribution is the weighted sum of the PDFs of the constituent populations:

$$\text{PDF}(\wp_t) = \frac{\sum_{m=1}^{t} \Lambda^{m-1} \exp\left(-\frac{(\wp_t - m\mu)^2}{2m\sigma^2}\right)}{\sigma_\psi \sqrt{2\pi} \underbrace{\sum_{m=1}^{t} \Lambda^m}_{(1-\Lambda^t)/(1-\Lambda)}} \tag{17}$$

2 Simulation Algorithm

This describes the simulation algorithm.

2.1 A Useful Trick

We will repeatedly be using a trick that greatly reduces the number of agents necessary to get well-behaved results. If there is a normally distributed random variable ψ that will be drawn for a population of size $K \in \mathbb{N}^+$, we first construct the $K-$point equiprobable approximation to the normal distribution, where the probability mass of each element is $1/K$ and the value at location ℓ is $\psi[\ell]$.

The trick is that, rather than actually drawing a value of ψ independently for each ℓ, instead we just randomly reshuffle the order of the elements in $\boldsymbol{\psi}$ (the vector of values of ψ. In the limit as $k \uparrow \infty$ this approaches the asymptote defined by the true distribution. But in a finite population, we guarantee that the population-mean level and variance of the collection of draws of $\boldsymbol{\psi}$ matches the theoretical values of these objects:

$$\mu = (1/K) \sum_{\ell=1}^{K} \psi_\ell \tag{18}$$

$$\sigma_\psi^2 = (1/K) \sum_{\ell=1}^{K} (\psi_\ell - \mu)^2 \tag{19}$$

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Call a version of $\boldsymbol{\psi}$ that has been reshuffled at date τ $\tilde{\boldsymbol{\psi}}$, and define

$$\tilde{\boldsymbol{\psi}}_\Lambda = \text{reshuffled } \psi \text{ list of size of suvivors} \tag{20}$$

$$\tilde{\boldsymbol{\psi}}_\mho = \text{reshuffled } \psi \text{ list of size of newborns} \tag{21}$$

2.2 Monte Carlo Simulation Method

The first step is to determine how many agents to simulate.

We choose a value of $\mho$ so that $1/\mho$ is an integer. Our population size needs to be at least $\Xi = 3/\mho$ because this guarantees that there will be at least three newborns in the population, which allows us to use the trick above to construct the minimal sensible approximation to a normal distribution - with three points each of which is realized with probability one third.

To simulate a population that has existed for t periods, loop over the following steps t times:

1. Select a random set $\{\mho_\tau\}$, which corresponds to an integer number Ξ locations; these are the locations at which the agents will be replaced

2. For the locations in $\{\mho_\tau\}$, set the values of $\wp$ to $\tilde{\boldsymbol{\psi}}_\mho$

3. For the locations in $\{\Lambda_\tau\}$, add to the existing value of $\wp$ the list of values $\tilde{\boldsymbol{\psi}}_\mho$

   - Note that in a finite sample, the covariance between $\wp$ and $\tilde{\boldsymbol{\psi}}_\Lambda$ will not be zero, which means that the sample mean and variance will not be exactly equal to their expectations.

4. Adjust the distribution (see the algorithm for details) so that its mean and variance exactly match the theoretical expectation

3 Direct Construction of Approximation to Ergodic Distribution

With this formula, it is easy to (numerically) construct an approximation to the ergodic distribution of permanent income.

4 Discretized Transition Matrix

Esmaeil Zadeh Soudjani and Abate (2014) present an admirable analysis of discrete approximations of Markov on continuous state spaces in discrete time. Notation and setup here follow them.

The setup is identical to ?

4.1 The Continuous-State-Space Problem We Want to Discretize

Consider a discrete time Markov process $\mathcal{M}$ defined over a continuous state space $\mathcal{S}$ which is characterized by a pair $(\mathcal{S}, \mathfrak{T}_s)$. We denote by $(\mathcal{S}, \mathcal{B}(\mathcal{S}), \mathcal{P})$ the probability structure on $\mathcal{S}$, with $\mathcal{B}(\mathcal{S})$ being the associated $\sigma-$algebra, and $\mathcal{P}$ is a probability measure to be defined shortly. $\mathfrak{T}_s$ is a conditional stochastic kernel that assigns to each point $s \in \mathcal{S}$ a probability measure $\mathfrak{T}_s(\bullet|s)$ so that for any measurable set $A \in \mathcal{B}(\mathcal{S}), \mathcal{P}(s(1) \in A|s(0) = s) \equiv \int_A \mathfrak{T}_s(d\bar{s}|s)$ ($s(1)$ is at date 1 while $s(0)$ is at date 0). We assume that the stochastic kernel $\mathfrak{T}$ admits a density function t_s, namely $\mathfrak{T}_s(d\bar{s}|s) = \mathsf{t}_s(\bar{s}|s)d\bar{s}$.

$\mathfrak{T}$ for 'transition'

'stochastic kernel': valid pty description

$\mathfrak{T}$ maps from a particular point at τ to any set of points ('event') at $\tau + 1$.

The initial state of the Markov process $\mathcal{M}$ is random and distributed according to the density function $\pi_0 : \mathcal{S} \to \mathbb{R}^{\geq 0}$. The state distribution of $\mathcal{M}$ at time $\tau \in \mathbb{N} \doteq \{1, 2, 3, ...\}$ is characterized by a density function $\pi_\tau : \mathcal{S} \to \mathbb{R}^{\geq 0}$ such that for all $A \in \mathcal{B}(\mathcal{S})$,

Could be deterministic instead

For all measurable sets, there is a measurable probability of any specific sequence of events leading up to τ

$$\mathbb{P}(s(\tau) \in A) \equiv \int_A \pi_\tau(s)ds \tag{22}$$

where $\mathbb{P}$ indicates the probability associated to events over the product space $\mathcal{S}^{\tau+1}$ with elements $\boldsymbol{s} = [s(0), s(1), \ldots, s(\tau)]$, where $\boldsymbol{s}$ indicates a vector of individual values of s (the same notation applies below).

combine $(\tau + 1)$ of $\mathcal{S}$

The state density functions $\pi_\tau(\bullet)$ can be computed recursively:

prb of $s_{\tau+1}$: sum of ways to get there from all s_t

$$\pi_{\tau+1}(\bar{s}) = \int_{\mathcal{S}} \mathsf{t}_s(\bar{s}|s)\pi_\tau(s)ds \quad \forall \, \bar{s} \in \mathcal{S} \tag{23}$$

We make further assumptions that correspond to Assumptions 1 and 2 in ?, with the result that their Theorem 3 holds:

Theorem 3: *The functions $\pi_\tau(\bullet)$ can be approximated by piecewise*

constant functions $\hat{\pi}(\bullet)$, *defined as*

$$\hat{\pi}_\tau(s) = \sum_{k=1}^{n} \frac{q_\tau(k)}{\mathcal{L}(A_k)} \mathbb{I}_{A_k}(s) \quad \forall \, \tau \in \mathbb{N}^+ \tag{24}$$

where $\mathbb{I}_A(\bullet)$ is the indicator function of a set $A \subseteq \mathcal{S}$ (that is, $\mathbb{I}_A(s) = 1$ if $s \in A$; else $\mathbb{I}_A(s) = 0$).

We consider a finite Markov chain $\mathcal{M}_f$ with $n+1$ discrete abstract states over this space characterized by the transition probability matrix $P = [P_{ij}]$ where the probability of jumping from state i to state j $(P_{i,j})$ is computed as:

$$P_{i,j} = \left(\frac{1}{\mathcal{L}(A_i)} \right) \int_{A_i} \left(\int_{A_j} \mathfrak{t}_s(\bar{s}|s)d\bar{s} \right) ds \quad \forall j \in \mathbb{N}_n \tag{25}$$

$$P_{(n+1),i} = \delta_{(n+1),i}$$

for all $j \in \mathbb{N}_{(n+1)}$, and where $\delta_{(n+1),j}$ is the Kronecker delta function (the abstract state $(n+1)$ of $\mathcal{M}_f$ is absorbing) and $\mathcal{L}(\bullet)$ denotes the Lebesgue measure of a set.[3]

The initial distribution of $\mathcal{M}_f$ is the probability mass function (pmf) $\boldsymbol{q}_0 = [q_0(1), q_0(2), \ldots, q_0(n+1)]$ obtained from π_0 as the average value of q_0 in each of the i intervals: $q_0(i) = \int_{A_i} \pi_0(s)ds, \ \forall \, i \in \mathbb{N}_n$.[4] Under Esmaeil Zadeh Soudjani and Abate (2014)'s further assumptions, the pmf associated to the state distribution of $\mathcal{M}_f$ at time $\tau \in \mathbb{N} \doteq \{1, 2, 3, \ldots\}$ can be computed as $\boldsymbol{q}_\tau = \boldsymbol{q}_0 P^\tau$. $\boldsymbol{q}_\tau$ is used to construct an approximation function, denoted by $\hat{\pi}_\tau$, of the density function π_τ, where $\hat{\pi}_\tau$ is a piecewise constant approximation (with values that are the entries of the pmf $\boldsymbol{q}_\tau$ normalized by the Lebesgue measure of the associated partition set) of the original density function π_τ.

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[3]If there are any point masses in the set, they are captured as a case where $\underline{A} = \bar{A}$; that is, all point masses are allocated an "interval" of measure zero at the location of the point mass.

[4]If there is a point mass at A_i, $q_0(i) = \int_{A_i} \delta(i)\pi_{0,i} = \mathbb{P}_0(s) > 0$ where δ is the Dirac δ).

5 Permanent Income Shocks

We now adapt the foregoing to the case of calculating an approximation to the steady state distribution of permanent income. We specialize the space $\mathcal{S}$ to contain points on the real number line so that for all $s \in \mathcal{S}$, $s \in \mathbb{R}$.

Suppose we have partitioned the space into a set of $(n+1)$ successive intervals indexed by i such that $i \in \mathbb{N}_{(n+1)} \doteq \{1, 2, \ldots, n, (n+1)\}$, whose boundaries are given by $\{\underline{A}_i, \bar{A}_i\}$ where $\underline{A}_1$ is the lower bound of the space A and $\bar{A}_n$ is the upper bound, and the upper bound of A_i is the lower bound of A_{i+1} (that is, $\bar{A}_i = \underline{A}_{i+1}$). The set A_i therefore contains all $s \in [\underline{A}_i, \underline{A}_{i+1})$, and $A_{(n+1)}$ is the mass of points (if any) excluded by the bounding process. _{Leaves any point mass at $\bar{A}_i$ in $i+1$}

We now have the notation to describe our discrete approximation to the distribution of the level of permanent income $\wp$.

Per (25), we can calculate the transition probabilities from:

$$P_{i,j} = \left(\underbrace{\frac{1}{\mathcal{L}(A_i)}}_{A_i \text{ Pop mass}} \right) \overbrace{\int_{A_i}}^{s \in [\underline{A}_i, \bar{A}_i)} \left(\underbrace{\overbrace{\int_{A_j}}^{\bar{s} \in [\underline{A}_j, \bar{A}_j)} \mathfrak{t}_s(\bar{s}|s) d\bar{s}}_{\equiv \bar{q}(j|s)} \right) ds \quad \forall i \in \mathbb{N}_n \qquad (26)$$

where $\bar{q}(j|s)$ is the probability mass for being in interval j next period, given that you are at point s this period.

Defining $\mathcal{F}^{\bullet}$ as the cumulative distribution function of $\bullet$, now recall from (2) that for $s \in \Lambda_{\tau+1}$ (locations where the occupant survives),

$$\bar{s} = s + \psi \qquad (27)$$

so

$$\mathfrak{t}_s(\bar{s}|s, \bar{s} \in \Lambda_{\tau+1}) = d\mathcal{F}^{\psi}(\bar{s} - s). \qquad (28)$$

In words: To get from s to $\bar{s}$ requires a shock of size $\psi = \bar{s} - s$ whose probability density is measured by $d\mathcal{F}^{\psi}(\bar{s} - s)$. For a given s the probability mass for the set of draws of ψ that put you in A_j is therefore:

$$\begin{aligned} \bar{\hat{q}}(j|s, \bar{s} \in \Lambda_{\tau+1}) &= \int_{\underline{A}_j - s}^{\bar{A}_j - s} d\mathcal{F}^{\psi} \\ &= \mathcal{F}^{\psi}(\bar{A}_j - s) - \mathcal{F}^{\psi}(\underline{A}_j - s) \end{aligned} \qquad (29)$$

For locations whose occupant is replaced, the calculation is even simpler.

At those locations, permanent income is reset to its starting value of 0 and then a shock is drawn, so

$$\bar{s} = \psi \tag{30}$$

so

$$\bar{\hat{q}}(j|s, \bar{s} \notin \Lambda_{\tau+1}) = \int_{\underline{A}_j}^{\bar{A}_j} d\mathcal{F}^\psi \tag{31}$$

$$= \mathcal{F}^\psi(\bar{A}_j) - \mathcal{F}^\psi(\underline{A}_j)$$

so the unconditional transition process is a weighted mixture of the two distributions:

$$\bar{\hat{q}}(j|s) = \Lambda \bar{\hat{q}}(j|s, \bar{s} \in \Lambda_{\tau+1}) + (1 - \Lambda)\bar{\hat{q}}(j|s, \bar{s} \notin \Lambda_{\tau+1}). \tag{32}$$

The overall transition probability structure is therefore

$$\hat{P}_{i,j} = \left(\frac{1}{\mathcal{L}(A_i)}\right) \int_{\underline{A}_i}^{\bar{A}_i} \bar{q}(j|s) d\mathcal{F}^{\pi_\tau}(s) \tag{33}$$

$$= \left(\frac{1}{\mathcal{L}(A_i)}\right) \int_{\underline{A}_i}^{\bar{A}_i} \bar{\hat{q}}(j|s) d\mathcal{F}^{\hat{\pi}_\tau}(s). \tag{34}$$

The idea here is that we are replacing the respective set of shocks that would be sufficient, in the continuous state space, to move from each point in any partition i to some point in partition j (weighted by each shock's probability) by, in the discretized state space, a single shock with that same probability that moves the (now assumed to be identical) occupants of discrete bin i to discrete bin j.

5.1 Intervals

There are many possible ways of choosing the set of intervals A_i. A particularly transparent choice is the equiprobable case: The intervals are chosen so that in the steady state there should be an equal population mass in each interval. If we have the "true" CDF of the limiting distribution $\mathcal{F}^\wp$ then we can define:

Define a set of points from $\sharp_0$ to $\sharp_n$ on the $[0, 1]$ interval as the elements of the set $\sharp = \{0, 1/n, 2/n, \ldots, 1\}$.[5] Call the inverse of the $\wp$ distribution

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[5]These points define intervals that constitute a partition of the domain of $\mathcal{F}^\wp$.

$(\mathcal{F}^\wp)^{-1}$, and define the points $s_i = (\mathcal{F}^\wp)^{-1}(\natural_i)$. Then the conditional mean of $\wp$ in each of the intervals numbered 1 to n is:

$$\wp_i \equiv \mathbb{E}[\wp | \natural_{i-1}^{-1} \leq \wp < \natural_i^{-1}] \;\; = \;\; \int_{\underline{A}_i}^{\bar{A}_i} \wp \, d\mathcal{F}^\wp(\wp) \tag{35}$$

5.2 What Exactly Are We Approximating?

In the original, truly-continuous-state-space economy, the homotheticity of the problem allowed us to eliminate a state variable by dividing the problem by the level of permanent income. This required that the anticipated distribution of the shocks to (log) permanent income be identical no matter what the consumer's level of permanent income.

In our approximation, however, the size of the jump in s cannot be independent of the level of s. This is most obvious at the outermost bins of the distribution: It is not possible to jump to a level of permanent income that is lower than the lowest bin, or higher than the highest bin, while those two possibilities are equally likely for the point at the midpoint of the $\mathcal{F}$.

There are two possible interpretations of what we are doing:

- We are modeling a 'true' economy in which, in truth, the magnitude of the permanent shocks is independent of their level

- We are imagining that we have moved the population of the true economy to a new and different economy that follows the structure we have derived as an approximation to the true economy

In the former case, we think of our model consumers as actually living in the true economy. Their expectations, and the "true" evolution of the population, reflect the homotheticity/scaling assumptions we made earlier. In the latter case, consumers would need to keep track of their permanent income as a state variable, because the true characteristics of the shocks they would experience would depend on the level of their permanent income.

6 Markov Transition for a

Designating a next-period variable $\bullet$ by $\grave{\bullet}$, for a person who ends the current period with assets a, market resources at the beginning of the next period will be

$$\grave{m} = \grave{\mathcal{R}}a + \grave{\xi}. \tag{36}$$

We assume that we have found converged consumption and asset functions $\mathrm{a}(m) = m - \mathrm{c}(m)$.

Now we want to calculate the discrete Markov approximation to the true continuous evolution of the economy.

$$\grave{a} = \mathrm{a}(\grave{\mathcal{R}}a + \grave{\xi}) \tag{37}$$

The period-τ distribution of $\mathcal{M}_f$ is defined by a probability mass function (pmf) $\boldsymbol{q}_\tau = [q_\tau(1), q_\tau(2), \ldots, q_\tau(n+1)]$.[6] The transition matrix is

$$P_{i,j} = \left(\frac{1}{\mathcal{L}(\mu_i)}\right) \overbrace{\int_{\underline{\mu}_i}^{}}^{a\in[\underline{\mu}_i, \bar{\mu}_i)} \underbrace{\overbrace{\int_{\underline{\mu}_j}^{}}^{\grave{a}\in[\underline{\mu}_j, \bar{\mu}_j)} \mathsf{t}_a(\grave{a}|a)d\grave{a}}_{\equiv \grave{q}(j|a)} \, da \quad \forall i \in \mathbb{N}_n \tag{38}$$

where $\grave{q}(j|a)$ is the probability of being in interval j next period, given that you are at point a this period. The only structural difference with the case above is that the transition function t_a now depends on two shocks, $\grave{\psi}$ and $\grave{\xi}$, instead of just one ($\grave{\psi}$).

We now want to inspect $\grave{q}(j|a)$ more closely:

$$\grave{q}(j|a) = \int_{\underline{\mu}_j}^{\bar{\mu}_j} \mathsf{t}_a(\grave{a}|a)d\grave{a} \tag{39}$$

In words, we integrate over the combinations of realizations of $\grave{\psi}$ and $\grave{\xi}$ such that, if we start at a, $\grave{a} \in \mu_j$. For the moment, let's take $\grave{\mathcal{R}}$ as fixed and consider which values of ξ would lead to some $\grave{a} \in \mu_j$. Because $\mathrm{a}(\bullet)$ is not an analytical function, for any $\grave{\mathcal{R}}(\psi)$ (equivalently, for any ψ) we

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[6]If the consumption rule has a target $\breve{a}$, a natural starting point is to have all of the population mass at $\breve{a}$.

must search numerically for the $\{\dot{\underline{\xi}}(a, \psi, j), \dot{\bar{\xi}}(a, \psi, j)\}$ such that $\dot{a}$ arrives respectively at $\{\underline{\mu}_j, \bar{\mu}_j\}$.[7]

Various characteristics of the a function (continuity; monotonicity; concavity; limits) mean that $\dot{a}$ is monotonically increasing in ξ, so the pmf $\dot{q}(j|a)$ for arriving in state j as a function of ψ is simply:

$$\dot{q}(j|a, \dot{\psi}) = \mathcal{F}^\xi(\bar{\xi}(j|a, \dot{\psi})) - \mathcal{F}^\xi(\underline{\xi}(j|a, \dot{\psi})). \tag{40}$$

so the probability mass for arrival in j is

$$\dot{q}(j|a) = \int_{\underline{\psi}}^{\bar{\psi}} \dot{q}(j|a, \dot{\psi}) d\mathcal{F}^{\dot{\psi}} \tag{41}$$

We now have the pmf of μ_j for any fixed a,[8] so in order to calculate the probability of arriving in μ_j from any point in μ_i all we need to do is integrate the conditional probability over all possible values of $a \in \mu_i$:

$$P_{i,j} = \left(\frac{1}{\mathcal{L}(\mu_i)}\right) \int_{\underline{\mu}_i}^{\bar{\mu}_i} \dot{q}(j|a) d\mathcal{F}^a. \tag{42}$$

Our approximation scheme relies upon the substitution, for each $a \in \mu_i$ a point that is the weighted mean, $\hat{a}_i = \int_{\mu_i} a d\mathcal{F}^a$. But without having the CDF $\mathcal{F}$ of the continuous distribution, we have no way to compute these values.

Fortunately, Esmaeil Zadeh Soudjani and Abate (2014) describe a recursive procedure that is guaranteed to result in bounded errors.

Theorem 1. The functions π_τ satisfy the bound

$$0 \leq \pi_\tau(a) \leq \varepsilon_\tau \quad \forall a \in \mathcal{S} \backslash \Lambda_\tau \tag{43}$$

where the quantities $\{\varepsilon_\tau\}_{\tau=0}^N$ are defined recursively by

$$\varepsilon_{\tau+1} = \epsilon + M_f \varepsilon_\tau, \tag{44}$$

whereas the support sets $\{\Lambda\}_{\tau=0}^N$ are computed as

$$\Lambda_{\tau+1} = \Pi_{\dot{a}} \left(\Gamma \cap (\Lambda \times \mathcal{S}) \right) \tag{45}$$

where $\Pi_{\dot{a}}$ denotes the projection map along the second set of coordinates. This means that we can start with an arbitrary distribution, and if we

π is small for a in $\mathcal{S}$ that are excluded from Λ

bounds grow as you go farther and farther

You expand the size of the support

By the amount given by the outcomes of the stochastic shocks

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[7] See the appendix for a sketch of a procedure for doing this.

[8] We are assuming that $\log \xi \sim \mathcal{N}(\mu, \sigma^2)$, so we can use the fact that: $\mathcal{F}^\xi(\xi) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{\xi - \mu}{\sigma\sqrt{2}}\right)\right)$

iterate long enough we can construct support sets that are large enough to guarantee the errors are bounded arbitrarily tightly.

Concretely, what we do is to begin with a support set Λ_0 defined by the bounds of our discretization of the shock process. Concretely, we construct Λ_0 as $a \in [\underline{\xi} - \mu_\xi, \bar{\xi} - \mu_\xi]$ with pmf identical to the pmf of the shock ξ. Then we proceed by continuing to expand. That is, each of the nodes in Λ_0 yields an array of other points as that location experiences the full distribution of outcomes from the discretized process. So, for instance, the bottommost point in Λ_1 will match the result of having twice drawn the worst possible realization of the shock. We iterate for enough periods that we judge to give a wide enough dispersion of points for the final approximation, then stop.

7 Backwards

We now consider an economy in which there is a last period T.

The main thing that makes this problem different from the former one is that the consumption function now depends on time, a_τ. In the last period $\mathsf{a}_\mathsf{T}(m) = 0$, and for prior periods it will be obtained by solving the dynamic stochastic optimization problem.

Assuming that the distribution of the shocks does not change with the time period,

$$P_{i,j}[\tau] = \int_{\underline{\mu}_i}^{\bar{\mu}_i} \mathring{q}[\tau](j|a)d\mathcal{F}_\tau^a \tag{46}$$

Rewriting (**??**)

$$a_{\tau+1} = \mathsf{a}_{\tau+1}(\mathcal{R}_{\tau+1}a_\tau + \xi_{\tau+1}) \tag{47}$$

we again must solve numerically for $\bar{\xi}[\tau](j|a_\tau, \psi_{\tau+1})$ and $\underline{\xi}[\tau](j|a_\tau, \psi_{\tau+1})$ functions, which will now yield a

$$\mathring{q}[\tau](j|a_\tau, \psi_{\tau+1}) = \mathcal{F}^\xi(\bar{\xi}[\tau](j|a_\tau, \psi_{\tau+1})) - \mathcal{F}^\xi(\underline{\xi}[\tau](j|a_\tau, \psi_{\tau+1})), \tag{48}$$

and the formula for $\mathring{q}$ is again given by (**??**)

8 Intervals

As before, suppose we have a prespecified set of boundaries on the real number line: $\underline{\mu}_i, \bar{\mu}_i$ for $i \in \mathbb{N}_{n+1}$ points that define the boundaries of the intervals we want to use in constructing our Markov chain. Then the transition probabilities are again

$$a_i \equiv \int_{\underline{\mu}_i}^{\bar{\mu}_{i+1}} a d\mathcal{F}^a \tag{49}$$

$$\grave{q}[\tau](j|a_\tau, \psi_{\tau+1}) = \tag{50}$$

We need to figure out how to divide the continuous state space up into appropriately sized intervals for which each of the

9 Anticipation

Per the discussion above, our consumers do not believe themselves to be living in a discretization of their economy; they believe they are living in the original continuous economy.

For any period-τ ending value of a_τ, expected marginal utility will be:

$$\mathfrak{v}'_\tau(a_\tau) \equiv \mathbb{E}_\tau[\mathrm{u}'(\mathrm{c}(\mathcal{R}_{\tau+1}a_\tau + \xi_{\tau+1}))] \tag{51}$$

which as usual yields consumption and beginning-of-period market resources from

$$c_\tau = (\mathrm{u}')^{-1}(\mathfrak{v}'_\tau) \tag{52}$$

$$m_\tau = a_\tau + c_\tau \tag{53}$$

References

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A Numerical Search for ξ

But we do have analytical lower and upper bounds for c, which in turn can be used to give us lower and upper bounds for the search region.

Since $\grave{a}$ is continuous and strictly increasing in the realization of ξ and $\mu_j \in [\underline{\mu}_j, \bar{\mu}_j]$, for any given $\grave{\mathcal{R}}$ we can find the lower bound such that $\grave{a} = \underline{\mu}_j$ and the upper bound such that $\grave{a} = \bar{\mu}_j$

$$\underline{\mu}_j(\grave{\mathcal{R}}) = \grave{\mathcal{R}}a + \underline{\grave{\xi}} - c(\grave{\mathcal{R}}a + \underline{\grave{\xi}}) \tag{54}$$

$$\bar{\mu}_j(\grave{\mathcal{R}}) = \grave{\mathcal{R}}a + \bar{\grave{\xi}} - c(\grave{\mathcal{R}}a + \bar{\grave{\xi}}) \tag{55}$$

Because c is not an analytical expression, this is an implicit function, but for 'small' ξ can be approximated by a first order Taylor expansion:

$$c(\mathcal{R}a + \xi) \approx c(\mathcal{R}a + 1) + (\xi - 1)c'(\mathcal{R}a + 1) \tag{56}$$

and shortening $c'(\mathcal{R}a + 1)$ to c',

$$\breve{a} \approx \left(\grave{\mathcal{R}}a - c(\grave{\mathcal{R}}a + 1)\right) + \grave{\xi} + (\grave{\xi} - 1)c'(\grave{\mathcal{R}}a + 1) \tag{57}$$

$$\breve{a} - (c' + 1)\grave{\xi} \approx \left(\grave{\mathcal{R}}a - (c(\grave{\mathcal{R}}a + 1)\right) - c' \tag{58}$$

$$(c' + 1)\grave{\xi} \approx \breve{a} - \left(\grave{\mathcal{R}}a - (c(\grave{\mathcal{R}}a + 1))\right) + c' \tag{59}$$

$$\grave{\xi} \approx (1 + c')^{-1}\left\{\breve{a} - \left(\grave{\mathcal{R}}a - (c(\grave{\mathcal{R}}a + 1)\right) + c'\right)\right\} \tag{60}$$

which can serve as the starting point for a numerical search for the 'true' value of $\grave{\xi}$ that satisfies (**??**).