Theoretical Foundations of Buffer Stock Saving

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Abstract

"Buffer-stock" models of saving are now common in the consumption literature. This paper builds theoretical foundations for rigorous understanding of the main features of such models, including the existence of a target wealth ratio and the proposition that aggregate consumption growth equals aggregate income growth in a small open economy populated by buffer stock savers.

Keywords Precautionary saving, buffer stock saving, marginal propensity to consume, permanent income hypothesis

JEL codes D81, D91, E21

PDF: `http://econ.jhu.edu/people/ccarroll/papers/BufferStockTheory.pdf`
Slides: `http://econ.jhu.edu/people/ccarroll/papers/BufferStockTheory-Slides.pdf`
Web: `http://econ.jhu.edu/people/ccarroll/papers/BufferStockTheory/`
Github: `http://github.com/llorracc/BufferStockTheory`
(See `/Code` tools for solving and simulating the model)

And CLICK HERE for an interactive Jupyter Notebook that uses the Econ-ARK/HARK toolkit to produce all of the paper's figures

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1 Introduction

Spurred by the success of Modigliani and Brumberg's (1954) Life Cycle model and Friedman's (1957) Permanent Income Hypothesis, a vast literature in the 1960s and 1970s formalized the idea that household spending can be modeled as reflecting optimal intertemporal choice. Famous papers by Schectman and Escudero (1977) and Bewley (1977) capped this literature, providing key building blocks for the ascendancy of dynamic stochastic optimizing models in economics.

Given this pedigree, it is surprising that the standard method for analyzing such problems, contraction mapping theory, has not yet established some basic properties of the solution to the benchmark consumption problem with unbounded (e.g. constant relative risk aversion) utility, uncertainty about permanent and transitory income *a la* Friedman (1957), and no liquidity constraints (nor has any other method established such results). The gap exists because (except in a few special cases) standard theorems from the contraction mapping literature (including those in Stokey et. al. (1989) and up through the recent work of Matkowski and Nowak (2011)) cannot be used for this problem (for reasons explained below).

This paper fills that gap, deriving the conditions that must be satisfied for this standard problem to have a nondegenerate solution.

The reader could be forgiven for not having noticed a gap. A large literature solving precisely such problems has emerged following Zeldes (1989), fueled by advances in numerical solution methods. But numerical solutions are a 'black box': They make it possible to use (or misuse) a model without really understanding it. Without theoretical underpinnings, the analyst often has little intuition for how results might change with changes in the structure (or even just the calibration) of the model. Indeed, without such theory, it can be difficult even to check whether a computational solution is correct.

For example, numerical solutions typically imply the existence of a target level of nonhuman wealth ('cash' for short). Carroll (1992; 1997) showed that target saving behavior arises under plausible parameter values for both infinite and finite horizon models. Gourinchas and Parker (2002) estimate the model using household data and conclude that for the mean household the buffer-stock phase of life lasts from age 25 until around age 40-45; using the same model with different data Cagetti (2003) finds target saving behavior into the 50s for the median household. But none of these papers provides a rigorous delineation of the circumstances under which target saving will emerge; nor has any other paper derived those conditions.

This paper derives the conditions necessary for a target to exist, and shows that they differ interestingly from the conditions required for the problem to define a contraction mapping. The paper also provides analytical foundations for other results that have become familiar from the numerical literature. All theoretical conclusions are paired with numerically computed illustrations (using software available both on the author's website and incorporated into the set of replications available at Econ-ARK project), providing an integrated framework for understanding buffer-stock saving.

The paper proceeds in three parts.

The first part specifies the conditions required for the problem to define a unique limiting consumption function. The conditions turn out to strongly resemble those required for the liquidity constrained perfect foresight model to have a solution; that parallel is explored and explained. Next, some limiting properties are derived for the consumption function as cash approaches infinity and as it approaches its lower bound, and the theorem asserting that the problem defines a contraction mapping is proven. Finally, a related class of commonly-used models (exemplified by Deaton (1991)) is shown to constitute a particular limit of this paper's more general model.

The next section examines five key properties of the model. First, as cash approaches infinity the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as cash approaches zero the expected growth rate of consumption approaches infinity, and the MPC approaches a simple analytical limit. Third, if the consumer is sufficiently 'impatient' (in a particular sense), a unique target cash-to-permanent-income ratio will exist. Fourth, at the target cash ratio, the expected growth rate of consumption is slightly less than the expected growth rate of permanent noncapital income. Finally, the expected growth rate of consumption is declining in the level of cash. The first four propositions are proven under general assumptions about parameter values; the last is shown to hold if there are no transitory shocks, but may fail in extreme cases if there are both transitory and permanent shocks.

2 The Problem

2.1 Setup

The consumer solves an optimization problem from the current period t until the end of life at T defined by the objective

$$\max\ \mathbb{E}_t \left[\sum_{n=0}^{T-t} \beta^n u(c_{t+n}) \right] \tag{1}$$

where $u(\bullet) = \bullet^{1-\rho}/(1-\rho)$ is a constant relative risk aversion utility function with $\rho > 1$.[1,2] The consumer's initial condition is defined by market resources m_t (what Deaton (1991) calls 'cash-on-hand') and permanent noncapital income $\boldsymbol{p}_t$. (This will henceforth be called a 'Friedman/Buffer Stock' (FBS) income process because its definition corresponds reasonably well to the descriptions in Friedman (1957) and because such a process has been widely used in the numerical buffer stock saving literature.)

In the usual treatment, a dynamic budget constraint (DBC) simultaneously incorporates all of the elements that determine next period's m given this period's choices; but for the detailed analysis here, it will be useful to disarticulate the steps so that individual

[1] The main results also hold for logarithmic utility which is the limit as $\rho \to 1$ but incorporating the logarithmic special case in the proofs is cumbersome and therefore omitted.

[2] We will define the infinite horizon solution as the limit of the finite horizon problem as the horizon $T-t$ approaches infinity.

ingredients can be separately examined:

$$
\begin{aligned}
a_t &= m_t - c_t \\
b_{t+1} &= a_t \mathsf{R} \\
\boldsymbol{p}_{t+1} &= \boldsymbol{p}_t \underbrace{\Gamma \psi_{t+1}}_{\equiv \Gamma_{t+1}} \\
m_{t+1} &= b_{t+1} + \boldsymbol{p}_{t+1} \xi_{t+1},
\end{aligned}
\tag{2}
$$

where a_t indicates the consumer's assets at the end of period t, which grow by a fixed interest factor $\mathsf{R} = (1 + \mathsf{r})$ between periods, so that b_{t+1} is the consumer's financial ('bank') balances before next period's consumption choice;[3] m_{t+1} ('market resources' or 'money') is the sum of financial wealth b_{t+1} and noncapital income $\boldsymbol{p}_{t+1}\xi_{t+1}$ (permanent noncapital income $\boldsymbol{p}_{t+1}$ multiplied by a mean-one iid transitory income shock factor ξ_{t+1}; from the perspective of period t, all future transitory shocks are assumed to satisfy $\mathbb{E}_t[\xi_{t+n}] = 1 \; \forall \; n \geq 1$). Permanent noncapital income in period $t + 1$ is equal to its previous value, multiplied by a growth factor Γ, modified by a mean-one iid shock ψ_{t+1}, $\mathbb{E}_t[\psi_{t+n}] = 1 \; \forall \; n \geq 1$ satisfying $\psi \in [\underline{\psi}, \bar{\psi}]$ for $0 < \underline{\psi} \leq 1 \leq \bar{\psi} < \infty$ where $\underline{\psi} = \bar{\psi} = 1$ is the degenerate case with no permanent shocks.[4] (Hereafter for brevity we occasionally drop time subscripts, e.g. $\mathbb{E}[\psi^{-\rho}]$ signifies $\mathbb{E}_t[\psi_{t+1}^{-\rho}]$.)

In future periods $t + n \; \forall \; n \geq 1$ there is a small probability $\wp$ that income will be zero (a 'zero-income event'),

$$
\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/\not\wp & \text{with probability } \not\wp \equiv (1 - \wp) \end{cases}
\tag{3}
$$

where θ_{t+n} is an iid mean-one random variable ($\mathbb{E}_t[\theta_{t+n}] = 1 \; \forall \; n > 0$) that has a distribution satisfying $\theta \in [\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} \leq 1 \leq \bar{\theta} < \infty$ (degenerately $\underline{\theta} = \bar{\theta} = 1$). Call the cumulative distribution functions $\mathcal{F}_\psi$ and $\mathcal{F}_\theta$ (and $\mathcal{F}_\xi$ is derived trivially from (3) and $\mathcal{F}_\theta$). Permanent income and cash start out strictly positive, $\{\boldsymbol{p}_t, m_t\} \in (0, \infty)$ and the consumer cannot die in debt,

$$
c_T \leq m_T. \tag{4}
$$

The model looks more special than it is. In particular, the assumption of a positive probability of zero-income events may seem objectionable. However, it is easy to show that a model with a nonzero minimum value of ξ (motivated, for example, by the existence of unemployment insurance) can be redefined by capitalizing the PDV of minimum income into current market assets,[5] analytically transforming that model back

[3] Allowing a stochastic interest factor is straightforward but adds little insight. The effects are more interesting for analysis of the invariant distribution (Szeidl (2012)).

[4] It is useful to emphasize that permanent noncapital income as defined here differs from what Deaton (1992) calls permanent income (which is often adopted in the macro literature). Deaton defines permanent income as the amount that a perfect foresight consumer could spend while leaving total (human and nonhuman) wealth constant. Relatedly, we refer to m_t as 'cash-on-hand' or 'market resources' rather than as wealth to avoid any confusion for readers accustomed to thinking of the discounted value of future noncapital income as a part of wealth. The 'market resources' terminology is motivated by the model's assumption that human wealth cannot be capitalized, an implication of anti-slavery laws.

[5] So long as this PDV is a finite number and unemployment benefits are proportional to $\boldsymbol{p}_t$; see the discussion in section 2.11.

into the model analyzed here. Also, the assumption of a positive point mass (as opposed to positive density) for the worst realization of the transitory shock is inessential, but simplifies and clarifies the proofs and is a powerful aid to intuition.

This model differs from Bewley's (1977) classic formulation in several ways. The CRRA utility function does not satisfy Bewley's assumption that $u(0)$ is well defined, or that $\mathrm{u}'(0)$ is well defined and finite, so neither the value function nor the marginal value function will be bounded. It differs from Schectman and Escudero (1977) in that they impose liquidity constraints and positive minimum income. It differs from both of these in that it permits permanent growth, and also permanent shocks to income, which a large empirical literature finds are to be quantitatively important in micro data (MaCurdy (1982); Abowd and Card (1989); Carroll and Samwick (1997); Jappelli and Pistaferri (2000); Storesletten, Telmer, and Yaron (2004); Blundell, Low, and Preston (2008)) and which the theory since Friedman (1957) suggests are far more consequential for household welfare than are transitory fluctuations. (The incorporation of permanent shocks also rules out application of the tools of Matkowski and Nowak (2011) and the extensive literature cited therein). It differs from Deaton (1991) because liquidity constraints are absent; there are separate transitory and permanent shocks (*a la* Muth (1960)); and the transitory shocks here can occasionally cause income to reach zero.[6] Finally, it differs from models found in Stokey et. al. (1989) because neither liquidity constraints nor bounds on utility or marginal utility are imposed.[7]

2.2 The Problem Can Be Rewritten in Ratio Form

We establish a bit more notation by reviewing the standard result that in problems of this class (CRRA utility, permanent shocks) the number of relevant state variables can be reduced from two ($\boldsymbol{m}$ and $\boldsymbol{p}$) to one ($m = \boldsymbol{m}/\boldsymbol{p}$) as follows. Defining nonbold variables as the boldface counterpart normalized by $\boldsymbol{p}_t$ (as with m_t), assume that value in the last period of life is $u(m_T)$, and consider the problem in the second-to-last period,

$$\begin{aligned}
\mathbf{v}_{T-1}(\boldsymbol{m}_{T-1},\boldsymbol{p}_{T-1}) &= \max_{\boldsymbol{c}_{T-1}} \ u(\boldsymbol{c}_{T-1}) + \beta\,\mathbb{E}_{T-1}[u(\boldsymbol{m}_T)] \\
&= \max_{c_{T-1}} \ u(\boldsymbol{p}_{T-1}c_{T-1}) + \beta\,\mathbb{E}_{T-1}[u(\boldsymbol{p}_T m_T)] \\
&= \boldsymbol{p}_{T-1}^{1-\rho}\left\{\max_{c_{T-1}} \ u(c_{T-1}) + \beta\,\mathbb{E}_{T-1}[u(\Gamma_T m_T)]\right\}. \qquad (5)
\end{aligned}$$

Now consider the related problem

$$\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{\{\mathrm{c}\}_t^T} \ u(\mathrm{c}_t) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \qquad (6) \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
b_{t+1} &= (\mathsf{R}/\Gamma_{t+1})a_t \ = \ \mathcal{R}_{t+1}a_t
\end{aligned}$$

[6] Below it will become clear that the Deaton model is a particular limit of this paper's model.

[7] Similar restrictions to those in the cited literature are made in the well known papers by Scheinkman and Weiss (1986) and Clarida (1987). See Toche (2005) for an elegant analysis of a related but simpler continuous-time model.

$$m_{t+1} = b_{t+1} + \xi_{t+1}$$

where $\mathcal{R}_{t+1} \equiv (\mathsf{R}/\Gamma_{t+1})$ is a 'growth-normalized' return factor, and the problem's first order condition is

$$c_t^{-\rho} = \mathsf{R}\beta\,\mathbb{E}_t[\Gamma_{t+1}^{-\rho} c_{t+1}^{-\rho}]. \tag{7}$$

Since $\mathrm{v}_T(m_T) = u(m_T)$, defining $\mathrm{v}_{T-1}(m_{T-1})$ from (6) for $t = T-1$, (5) reduces to

$$\mathbf{v}_{T-1}(\boldsymbol{m}_{T-1}, \boldsymbol{p}_{T-1}) = \boldsymbol{p}_{T-1}^{1-\rho}\mathrm{v}_{T-1}(\boldsymbol{m}_{T-1}/\boldsymbol{p}_{T-1}).$$

This logic induces to all earlier periods, so that if we solve the normalized one-state-variable problem specified in (6) we will have solutions to the original problem for any $t < T$ from:[8]

$$\begin{aligned}
\mathbf{v}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) &= \boldsymbol{p}_t^{1-\rho}\mathrm{v}_t(\overbrace{\boldsymbol{m}_t/\boldsymbol{p}_t}^{m_t}),\\
\mathbf{c}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) &= \boldsymbol{p}_t \mathrm{c}_t(m_t).
\end{aligned}$$

2.3 Definition of a Nondegenerate Solution

We say that a consumption problem has a nondegenerate solution if it defines a unique limiting consumption function whose optimal c satisfies

$$0 < c < \infty \tag{8}$$

for every $0 < m < \infty$. ('Degenerate' limits will be cases where the limiting consumption function is either $\mathrm{c}(m) = 0$ or $\mathrm{c}(m) = \infty$.)

2.4 Perfect Foresight Benchmarks

The analytical solution to the perfect foresight specialization of the model, obtained by setting $\wp = 0$ and $\underline{\theta} = \bar{\theta} = \underline{\psi} = \bar{\psi} = 1$, is well known, but deriving it provides a useful reference point and allows us to define some remaining notation and terminology.

2.4.1 Human Wealth

The dynamic budget constraint, strictly positive marginal utility, and the can't-die-in-debt condition (4) imply an exactly-holding intertemporal budget constraint (IBC)

$$\mathrm{PDV}_t(\boldsymbol{c}) = \overbrace{\boldsymbol{m}_t - \boldsymbol{p}_t}^{\boldsymbol{b}_t} + \overbrace{\mathrm{PDV}_t(\boldsymbol{p})}^{\boldsymbol{h}_t}, \tag{9}$$

where $\boldsymbol{h}_t$ is 'human wealth,' the discounted value of noncapital income, and with a constant $\mathcal{R} \equiv \mathsf{R}/\Gamma$, human wealth will be

$$\boldsymbol{h}_t = \boldsymbol{p}_t + \mathcal{R}^{-1}\boldsymbol{p}_t + \mathcal{R}^{-2}\boldsymbol{p}_t + \dots + \mathcal{R}^{t-T}\boldsymbol{p}_t$$

[8] v is an exception to the notational convention that boldfaced variables are the nonbold version multiplied by $\boldsymbol{p}$; the appropriate scaling factor for value is $\boldsymbol{p}^{1-\rho}$.

$$= \underbrace{\left(\frac{1-\mathcal{R}^{-(T-t+1)}}{1-\mathcal{R}^{-1}}\right)}_{\equiv h_t} \boldsymbol{p}_t. \tag{10}$$

(10) makes plain that in order for $h \equiv \lim_{n\to\infty} h_{T-n}$ to be finite, we must impose the Finite Human Wealth Condition ('FHWC')

$$\underbrace{\Gamma/\mathsf{R}}_{\mathcal{R}^{-1}} < 1. \tag{11}$$

Intuitively, for human wealth to be finite, the growth rate of noncapital income must be smaller than the interest rate at which that income is being discounted.

2.4.2 Unconstrained Solution

In our unconstrained problem, the consumption Euler equation holds in every period; with $\mathrm{u}'(c) = c^{-\rho}$, this says

$$c_{t+1}/c_t = (\mathsf{R}\beta)^{1/\rho} \equiv \text{Þ} \tag{12}$$

where the Old English letter 'thorn' represents what we will call the 'absolute patience factor' $(\mathsf{R}\beta)^{1/\rho}$.[9] The sense in which Þ captures patience is that if the 'absolute impatience condition' (AIC) holds,

$$\text{Þ} < 1, \tag{13}$$

the consumer will choose to spend an amount too large to sustain indefinitely (the level of consumption must fall over time). We say that such a consumer is 'absolutely impatient' (this is the key condition in Bewley (1977)).

We next define a 'return patience factor' that relates absolute patience to the return factor:

$$\text{Þ}_{\mathsf{R}} \equiv \text{Þ}/\mathsf{R} \tag{14}$$

so that

$$\begin{aligned}
\mathrm{PDV}_t(c) &= \left(1 + \text{Þ}_{\mathsf{R}} + \text{Þ}_{\mathsf{R}}^2 + \ldots + \text{Þ}_{\mathsf{R}}^{T-t}\right) c_t \\
&= \left(\frac{1-\text{Þ}_{\mathsf{R}}^{T-t+1}}{1-\text{Þ}_{\mathsf{R}}}\right) c_t
\end{aligned}$$

from which the IBC (9) implies

$$c_t = \overbrace{\left(\frac{1-\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}^{T-t+1}}\right)}^{\equiv \underline{\kappa}_t} (b_t + \boldsymbol{h}_t) \tag{15}$$

[9]Impatience conditions of one kind or another have figured in intertemporal optimization problems since they were first formalized, most notably by Ramsey (1928). Discussion of these issues was prominent in the literature of the 1960s and 1970s, and no brief citations here could do it justice, so I refrain from the attempt.

which defines a normalized finite-horizon perfect foresight consumption function

$$\bar{\mathrm{c}}_{T-n}(m_{T-n}) \;=\; (\overbrace{m_{T-n}-1}^{=b_{T-n}}+h_{T-n})\underline{\kappa}_{T-n} \tag{16}$$

where $\underline{\kappa}_t$ is the marginal propensity to consume (MPC) because it answers the question 'if the consumer had an extra unit of wealth, how much more would he spend.' Equation (15) makes plain that for the limiting MPC to be strictly positive as $n = T-t$ goes to infinity we must impose the condition

$$\text{Þ}_{\mathsf{R}} \;<\; 1, \tag{17}$$

so that

$$0 < \underline{\kappa} \;\equiv\; 1 - \text{Þ}_{\mathsf{R}} = \lim_{n\to\infty} \underline{\kappa}_{T-n}. \tag{18}$$

Equation (17) thus imposes a second kind of 'impatience:' The consumer cannot be so pathologically patient as to wish, in the limit as the horizon approaches infinity, to spend nothing today out of an increase in current wealth. This rules out the degenerate limiting solution $\bar{\mathrm{c}}(m) = 0$, which motivates us to call equation (17) the 'return impatience condition' or RIC; we will say that a consumer who satisfies the condition is 'return impatient.'

Given that the RIC holds, and defining limiting objects by the absence of a time subscript (e.g., $\bar{\mathrm{c}}(m) = \lim_{n\uparrow\infty} \bar{\mathrm{c}}_{T-n}(m)$), the limiting consumption function will be

$$\bar{\mathrm{c}}(m_t) \;=\; (m_t + h - 1)\underline{\kappa}, \tag{19}$$

and we now see that in order to rule out the degenerate limiting solution $\bar{\mathrm{c}}(m_t) = \infty$ we need h to be finite so we must impose the finite human wealth condition (11).

A final useful point is that since the perfect foresight growth factor for consumption is Þ, using $u(xy) = x^{1-\rho}u(y)$ yields the following expression for value:

$$\begin{aligned} v_t &= u(c_t) + \beta u(c_t\text{Þ}) + \beta^2 u(c_t\text{Þ}^2) + ... \\ &= u(c_t)\left(1 + \beta\text{Þ}^{1-\rho} + (\beta\text{Þ}^{1-\rho})^2 + ...\right) \\ &= u(c_t)\left(\frac{1-(\beta\text{Þ}^{1-\rho})^{T-t+1}}{1-\beta\text{Þ}^{1-\rho}}\right) \end{aligned}$$

which asymptotes to a finite value as $n = T-t$ approaches $+\infty$ if $\beta\text{Þ}^{1-\rho} < 1$; with a bit of algebra, this requirement can be shown to be equivalent to the RIC.[10] Thus, the same conditions that guarantee a nondegenerate limiting consumption function also guarantee a nondegenerate limiting value function (this will not be true in the version of the model that incorporates uncertainty).

[10]

$$\begin{aligned} \beta((\mathsf{R}\beta)^{1/\rho})^{1-\rho} &< 1 \\ \beta(\mathsf{R}\beta)^{1/\rho}/\mathsf{R}\beta &< 1 \\ (\mathsf{R}\beta)^{1/\rho}/\mathsf{R} &< 1 \end{aligned}$$

.

2.4.3 Constrained Solution

If the liquidity constraint is ever to be relevant, it must be relevant at the lowest possible level of market resources, $m_t = 1$, which obtains for a consumer who enters period t with $b_t = 0$. The constraint is 'relevant' if it prevents the choice that would otherwise be optimal; at $m_t = 1$ the constraint is relevant if the marginal utility from spending all of today's resources $c_t = m_t = 1$, exceeds the marginal utility from doing the same thing next period, $c_{t+1} = 1$; that is, if such choices would violate the Euler equation (7):

$$1^{-\rho} \quad > \quad \mathsf{R}\beta(\Gamma)^{-\rho}1^{-\rho}. \tag{20}$$

By analogy to the return patience factor, we therefore define a 'perfect foresight growth patience factor' as

$$\text{Þ}_{\Gamma} \quad = \quad \text{Þ}/\Gamma, \tag{21}$$

and define a 'perfect foresight growth impatience condition' (PF-GIC)

$$\text{Þ}_{\Gamma} \quad < \quad 1 \tag{22}$$

which is equivalent to (20) (exponentiate both sides by $1/\rho$).

If the RIC and the FHWC hold, appendix A shows that an unconstrained consumer behaving according to (19) would choose $c < m$ for all $m > m_{\#}$ for some $0 < m_{\#} < 1$. The solution to the constrained consumer's problem in this case is simple: For any $m \geq m_{\#}$ the constraint does not bind (and will never bind in the future) and so the constrained consumption function is identical to the unconstrained one. In principle, if the consumer were somehow[11] to arrive at an $m < m_{\#} < 1$ the constraint would bind and the consumer would have to consume $c = m$. We use the $\circ$ accent to designate the limiting constrained consumption function:

$$\mathring{\mathrm{c}}(m) = \begin{cases} m & \text{if } m < m_{\#} \\ \bar{\mathrm{c}}(m) & \text{if } m \geq m_{\#}. \end{cases} \tag{23}$$

More useful is the case where the perfect foresight growth and impatience conditions both hold. In this case appendix A shows that the limiting constrained consumption function is piecewise linear, with $\mathring{\mathrm{c}}(m) = m$ up to a first 'kink point' at $m_{\#}^{1} > 1$, and with discrete declines in the MPC at successively increasing kink points $\{m_{\#}^{1}, m_{\#}^{2}, ...\}$. As $m \uparrow \infty$ the constrained consumption function $\mathring{\mathrm{c}}(m)$ approaches arbitrarily close to the unconstrained $\bar{\mathrm{c}}(m)$, and the marginal propensity to consume function $\mathring{\boldsymbol{\kappa}}(m) \equiv \mathring{\mathrm{c}}'(m)$ limits to $\underline{\kappa}$. Similarly, the value function $\mathring{\mathrm{v}}(m)$ is nondegenerate and limits into the value function of the unconstrained consumer. Surprisingly, this logic holds even when the finite human wealth condition fails (denoted ~~FHWC~~). A solution exists because the constraint prevents the consumer from borrowing against infinite human wealth to finance infinite current consumption. Under these circumstances, the consumer who starts with any amount of resources $b_t > 1$ will run those resources down over time so

[11] "Somehow" because such values of m are of questionable relevance because they could only be obtained by entering the period with $b < 0$ which the constraint rules out).

that by some finite number of periods n in the future the consumer will reach $b_{t+n} = 0$, and thereafter will set $c = m = 1$ for eternity, a policy that will yield value of

$$\begin{aligned} v_{t+n} &= u(\boldsymbol{p}_{t+n})\left(1 + \beta\Gamma^{1-\rho} + (\beta\Gamma^{1-\rho})^2 + ...\right) \\ &= \Gamma^{n(1-\rho)}u(\boldsymbol{p}_t)\left(\frac{1-(\beta\Gamma^{1-\rho})^{T-(t+n)+1}}{1-\beta\Gamma^{1-\rho}}\right), \end{aligned}$$

which will be a finite number whenever

$$\overbrace{\beta\Gamma^{1-\rho}}^{\equiv\beth} < 1 \tag{24}$$

$$\beta\mathsf{R}\Gamma^{-\rho} < \mathsf{R}/\Gamma$$

$$\text{Þ}_\Gamma < (\mathsf{R}/\Gamma)^{1/\rho} \tag{25}$$

which we call the Perfect Foresight Finite Value of Autarky Condition, PF-FVAC, because it guarantees that a consumer who always spends all his permanent income will have finite value (the consumer has 'finite autarky value'). Note that the version of the PF-FVAC in (25) implies the PF-GIC $\text{Þ}_\Gamma < 1$ whenever ~~PF-FHWC~~ $\mathsf{R} < \Gamma$ holds. So, if ~~PF-FHWC~~ holds, value for any finite m will be the sum of two finite numbers: The component due to the unconstrained consumption choice made over the finite horizon leading up to $b_{t+n} = 0$, and the finite component due to the value of consuming all income thereafter. The consumer's value function is therefore nondegenerate.

The most peculiar possibility occurs when the RIC fails. Remarkably, the appendix shows that although under these circumstances the FHWC must also fail, the constrained consumption function is nondegenerate even in this case. While it is true that $\lim_{m\uparrow\infty} \mathring{\boldsymbol{\kappa}}(m) = 0$, nevertheless the limiting constrained consumption function $\mathring{\mathrm{c}}(m)$ is strictly positive and strictly increasing in m. This result interestingly reconciles the conflicting intuitions from the unconstrained case, where ~~RIC~~ would suggest a dengenerate limit of $\mathring{\mathrm{c}}(m) = 0$ while ~~FHWC~~ would suggest a degenerate limit of $\mathring{\mathrm{c}}(m) = \infty$.

Tables 2 and 3 (and appendix table 4) codify the key points to help the reader keep them straight (and to facilitate upcoming comparisons with the surprisingly parallel results in the presence of uncertainty but the absence of liquidity constraints (also tabulated for comparison)).

2.5 Uncertainty-Modified Conditions

2.5.1 Impatience

When uncertainty is introduced, the expectation of b_{t+1} can be rewritten as:

$$\mathbb{E}_t[b_{t+1}] = a_t\,\mathbb{E}_t[\mathcal{R}_{t+1}] = a_t\mathcal{R}\,\mathbb{E}_t[\psi_{t+1}^{-1}] \tag{26}$$

where Jensen's inequality guarantees that the expectation of the inverse of the permanent shock is strictly greater than one. It will be convenient to define the object

$$\acute{\psi} \equiv (\mathbb{E}[\psi^{-1}])^{-1}$$

because this permits us to write expressions like the RHS of (26) compactly as, e.g., $a_t \mathcal{R} \acute{\psi}^{-1}$.[12] We refer to this as the 'return compensated' permanent shock, because it compensates for the effect of uncertainty on the expected growth-normalized return (in the sense implicitly defined in (26)). Note that Jensen's inequality implies that $\acute{\psi} < 1$ for nondegenerate ψ (since $\mathbb{E}[\psi] = 1$ by assumption).

Using this definition, we can transparently generalize the PF-GIC (22) by defining a 'compensated growth factor'

$$\underline{\Gamma} = \Gamma \acute{\psi} \tag{27}$$

and a compensated growth patience factor

$$\text{Þ}_{\acute{\Gamma}} = \text{Þ}/\underline{\Gamma} \tag{28}$$

and a straightforward derivation using some of the results below yields the conclusion that

$$\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] = \text{Þ}_{\acute{\Gamma}},$$

which implies that if we wish to prevent m from heading to infinity (that is, if we want m to be guaranteed to be expected to fall for some large enough value of m) we must impose a generalized version of the Perfect Foresight Growth Impatience Conditon (22) which we call simply the 'growth impatience condition' (GIC):[13]

$$\text{Þ}_{\acute{\Gamma}} < 1 \tag{29}$$

which is stronger than the perfect foresight version (22) because $\underline{\Gamma} < \Gamma$.

2.5.2 Value

A consumer who spent his permanent income every period would have value

$$\begin{aligned} v_t &= \mathbb{E}_t \left[u(\boldsymbol{p}_t) + \beta u(\boldsymbol{p}_t \Gamma_{t+1}) + ... + \beta^{T-t} u(\boldsymbol{p}_t \Gamma_{t+1} ... \Gamma_T) \right] \\ &= u(\boldsymbol{p}_t) \left(1 + \beta\, \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}] + ... + \beta^{T-t}\, \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}] ... \mathbb{E}_t[\Gamma_T^{1-\rho}] \right) \\ &= u(\boldsymbol{p}_t) \left(\frac{1 - (\beta \Gamma^{1-\rho}\, \mathbb{E}[\psi^{1-\rho}])^{T-t+1}}{1 - \beta \Gamma^{1-\rho}\, \mathbb{E}[\psi^{1-\rho}]} \right) \end{aligned}$$

which invites the definition of a utility-compensated equivalent of the permanent shock,

$$\underline{\underline{\psi}} = (\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)}$$

---

[12] One way to think of $\acute{\psi}$ is as a particular kind of a 'certainty equivalent' of the shock; this captures the intuition that mean-one shock renders a given mean level of income less valuable than if the shock did not exist, so that $\acute{\psi} < 1$.

[13] Equation (29) is a bit easier to satisfy than the similar condition imposed by Deaton (1991): $\left(\mathbb{E}[\psi^{-\rho}]\right)^{1/\rho} \text{Þ}_{\Gamma} < 1$ to guarantee that his problem defined a contraction mapping.

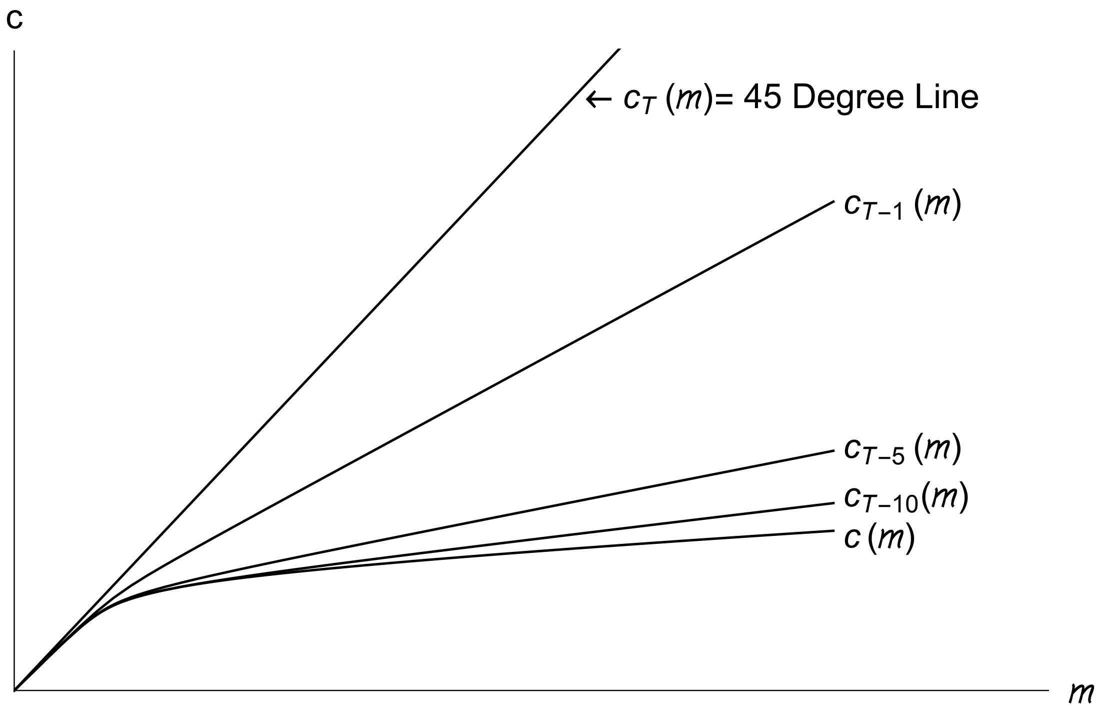


Figure 1 Convergence of the Consumption Rules

which will satisfy $\underline{\underline{\psi}} < 1$ for $\rho > 1$ and nondegenerate ψ (and $\underline{\underline{\psi}} < \acute{\psi}$ for the preferred (though not required) case of $\rho > 2$); defining $\underline{\underline{\Gamma}} = \Gamma\underline{\underline{\psi}}$ we can see that v_t will be finite as T approaches ∞ if

$$\begin{aligned} \overbrace{\beta\underline{\underline{\Gamma}}^{1-\rho}}^{\equiv \underline{\underline{\beth}}} &< 1 \\ \beta &< \underline{\underline{\Gamma}}^{\rho-1} \end{aligned} \tag{30}$$

which we call the 'finite value of autarky' condition (FVAC) because it is the value obtained by always consuming permanent income, and which for nondegenerate ψ is stronger (harder to satisfy in the sense of requiring lower β) than the perfect foresight version (24) because $\underline{\underline{\Gamma}} < \Gamma$.

2.6 The Baseline Numerical Solution

Figure 1 depicts the successive consumption rules that apply in the last period of life ($\mathrm{c}_T(m)$), the second-to-last period, and various earlier periods under the baseline parameter values listed in Table 1. (The 45 degree line is labelled as $\mathrm{c}_T(m) = m$ because in the last period of life it is optimal to spend all remaining resources.)

In the figure, the consumption rules appear to converge as the horizon recedes (below we show that this appearance is not deceptive); we call the limiting infinite-horizon

Table 1 Microeconomic Model Calibration

Calibrated Parameters			
Description	Parameter	Value	Source
Permanent Income Growth Factor	Γ	1.03	PSID: Carroll (1992)
Interest Factor	R	1.04	Conventional
Time Preference Factor	β	0.96	Conventional
Coefficient of Relative Risk Aversion	ρ	2	Conventional
Probability of Zero Income	$\wp$	0.005	PSID: Carroll (1992)
Std Dev of Log Permanent Shock	σ_{ψ}	0.1	PSID: Carroll (1992)
Std Dev of Log Transitory Shock	σ_{θ}	0.1	PSID: Carroll (1992)

Model Characteristics Calculated From Parameters		
Description	Symbol and Formula	Approximate Calculated Value
Finite Human Wealth Measure	$\mathcal{R}^{-1} \equiv \Gamma/\mathsf{R}$	0.990
PF Finite Value of Autarky Measure	$\beth \equiv \beta\Gamma^{1-\rho}$	0.932
Growth Compensated Permanent Shock	$\acute{\psi} \equiv (\mathbb{E}[\psi^{-1}])^{-1}$	0.990
Uncertainty-Adjusted Growth	$\underline{\Gamma} \equiv \Gamma\acute{\psi}$	1.020
Utility Compensated Permanent Shock	$\underline{\underline{\psi}} \equiv (\mathbb{E}_t[\psi^{1-\rho}])^{1/(1-\rho)}$	0.990
Utility Compensated Growth	$\underline{\underline{\Gamma}} \equiv \Gamma\underline{\underline{\psi}}$	1.020
Absolute Patience Factor	$\textbf{Þ} \equiv (\mathsf{R}\beta)^{1/\rho}$	0.999
Return Patience Factor	$\textbf{Þ}_{\mathsf{R}} \equiv \mathsf{R}^{-1}(\mathsf{R}\beta)^{1/\rho}$	0.961
PF Growth Patience Factor	$\textbf{Þ}_{\Gamma} \equiv \Gamma^{-1}(\mathsf{R}\beta)^{1/\rho}$	0.970
Growth Patience Factor	$\textbf{Þ}_{\acute{\Gamma}} \equiv \underline{\Gamma}^{-1}(\mathsf{R}\beta)^{1/\rho}$	0.980
Finite Value of Autarky Measure	$\underline{\underline{\beth}} \equiv \beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho}$	0.941

consumption rule

$$\mathrm{c}(m) \quad \equiv \quad \lim_{n \to \infty} \mathrm{c}_{T-n}(m). \tag{31}$$

2.7 Concave Consumption Function Characteristics

A precondition for the main proof is that the maximization problem (6) defines a sequence of continuously differentiable strictly increasing strictly concave[14] functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, ...\}$.[15] The proof of this precondition is straightforward but tedious, and so is relegated to appendix B. For present purposes, the most important point is the following intuition: $\mathrm{c}_t(m) < m$ for all periods $t < T$ because if the consumer spent all available resources, he would arrive in period $t+1$ with balances b_{t+1} of zero, then might earn zero noncapital income for the rest of his life (an unbroken series of zero-income events is unlikely but possible). In such a case, the budget constraint and the can't-die-in-debt condition mean that the consumer would be forced to spend zero, incurring negative infinite utility. To avoid this disaster, the consumer never spends everything. (This is an example of the 'natural borrowing constraint' induced by a precautionary motive (Zeldes (1989)).)[16]

2.8 Bounds for the Consumption Functions

The consumption functions depicted in Figure 1 appear to have limiting slopes as $m \downarrow 0$ and as $m \uparrow \infty$. This section confirms that impression and derives those slopes, which also turn out to be useful in the contraction mapping proof.

Assume (as discussed above) that a continuously differentiable concave consumption function exists in period $t+1$, with an origin at $\mathrm{c}_{t+1}(0) = 0$, a minimal MPC $\underline{\kappa}_{t+1} > 0$, and maximal MPC $\bar{\kappa}_{t+1} \leq 1$. (If $t+1 = T$ these will be $\underline{\kappa}_T = \bar{\kappa}_T = 1$; for earlier periods they will exist by recursion from the following arguments.)

For $m_t > 0$ we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$ and the Euler equation (7) can be rewritten

$$\mathrm{e}_t(m_t)^{-\rho} \quad = \quad \beta \mathsf{R}\, \mathbb{E}_t \left[\left(\mathrm{e}_{t+1}(m_{t+1}) \left(\frac{\overbrace{\mathsf{R}\mathrm{a}_t(m_t) + \Gamma_{t+1}\xi_{t+1}}^{=m_{t+1}\Gamma_{t+1}}}{m_t} \right) \right)^{-\rho} \right] \tag{32}$$

$$\begin{aligned} &= \quad \not\wp \beta \mathsf{R} m_t^{\rho}\, \mathbb{E}_t \left[(\mathrm{e}_{t+1}(m_{t+1}) m_{t+1} \Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0 \right] \\ &\quad + \wp \beta \mathsf{R}^{1-\rho}\, \mathbb{E}_t \left[\left(\mathrm{e}_{t+1}(\mathcal{R}_{t+1} \mathrm{a}_t(m_t)) \frac{m_t - \mathrm{c}_t(m_t)}{m_t} \right)^{-\rho} \mid \xi_{t+1} = 0 \right]. \end{aligned} \tag{33}$$

[14] There is one obvious exception: $\mathrm{c}_T(m)$ is a linear (and so only weakly concave) function.

[15] Carroll and Kimball (1996) proved concavity but not the other desired properties.

[16] It would perhaps be better to call it the 'utility-induced borrowing constraint' as it follows from the assumptions on the utility function (in particular, $\lim_{c \downarrow 0} u(c) = -\infty$); for example, no such constraint arises if utility is of the (implausible) Constant Absolute Risk Aversion form.

Consider the first conditional expectation in (33), recalling that if $\xi_{t+1} > 0$ then $\xi_{t+1} \equiv \theta_{t+1}/\wp$. Since $\lim_{m \downarrow 0} \mathrm{a}_t(m) = 0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0]$ is contained within bounds defined by $(\mathrm{e}_{t+1}(\underline{\theta}/\wp)\Gamma\underline{\psi}\underline{\theta}/\wp)^{-\rho}$ and $(\mathrm{e}_{t+1}(\bar{\theta}/\wp)\Gamma\bar{\psi}\bar{\theta}/\wp)^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by $\wp$ goes to zero as m_t^ρ goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\bar{\kappa}_t$ satisfies $\bar{\kappa}_t^{-\rho} = \beta\wp\mathsf{R}^{1-\rho}\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. We can conclude that

$$
\begin{aligned}
\bar{\kappa}_t &= \wp^{-1/\rho}(\beta\mathsf{R})^{-1/\rho}\mathsf{R}(1-\bar{\kappa}_t)\bar{\kappa}_{t+1} \\
\underbrace{\wp^{1/\rho}\overbrace{\mathsf{R}^{-1}(\beta\mathsf{R})^{1/\rho}}^{\text{Þ}_{\mathsf{R}}}}_{\equiv\wp^{1/\rho}\text{Þ}_{\mathsf{R}}}\bar{\kappa}_t &= (1-\bar{\kappa}_t)\bar{\kappa}_{t+1} \qquad (34)
\end{aligned}
$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$
\begin{aligned}
(\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1} &= 1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1}. \qquad (35)
\end{aligned}
$$

Then $\left\{\bar{\kappa}_{T-n}^{-1}\right\}_{n=0}^{\infty}$ is a decreasing convergent sequence if

$$0 \leq \wp^{1/\rho}\text{Þ}_{\mathsf{R}} < 1, \qquad (36)$$

a condition that we dub the 'Weak Return Impatience Condition' (WRIC) because with $\wp < 1$ it will hold more easily (for a larger set of parameter values) than the RIC ($\text{Þ}_{\mathsf{R}} < 1$).

Since $\bar{\kappa}_T = 1$, iterating (35) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n\to\infty}\bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1-\wp^{1/\rho}\text{Þ}_{\mathsf{R}} \qquad (37)$$

and we will therefore call $\bar{\kappa}$ the 'limiting maximal MPC.'

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$. If the FHWC holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving ξ_{t+1} in (32) can be neglected, leading to a revised limiting Euler equation

$$(m_t\mathrm{e}_t(m_t))^{-\rho} = \beta\mathsf{R}\,\mathbb{E}_t\left[(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1})\,(\mathsf{R}\mathrm{a}_t(m_t)))^{-\rho}\right]$$

and we know from L'Hôpital's rule that $\lim_{m_t\to\infty}\mathrm{e}_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t\to\infty}\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) = \underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$
\begin{aligned}
(m_t\underline{\kappa}_t)^{-\rho} &= \beta\mathsf{R}\left(\underline{\kappa}_{t+1}\mathsf{R}(1-\underline{\kappa}_t)m_t\right)^{-\rho} \\
\underbrace{\mathsf{R}^{-1}\text{Þ}}_{\equiv\text{Þ}_{\mathsf{R}}=(1-\underline{\kappa})}\underline{\kappa}_t &= (1-\underline{\kappa}_t)\underline{\kappa}_{t+1}
\end{aligned}
$$

and the same sequence of derivations used above yields the conclusion that if the RIC $0 \leq \text{Þ}_{\mathsf{R}} < 1$ holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} \quad = \quad 1 + \underline{\kappa}_{t+1}^{-1} \text{Þ}_{\mathsf{R}} \tag{38}$$

so that $\{\underline{\kappa}_{T-n}^{-1}\}_{n=0}^{\infty}$ is also an increasing convergent sequence, with $\underline{\kappa}$ being the 'limiting minimal MPC.' If the RIC does *not* hold, then $\lim_{n\to\infty} \underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$.

We are now in position to observe that the optimal consumption function must satisfy

$$\underline{\kappa}_t m_t \ \leq \ \ \mathrm{c}_t(m_t) \ \ \leq \ \bar{\kappa}_t m_t \tag{39}$$

because consumption starts at zero and is continuously differentiable (as argued above), is strictly concave (Carroll and Kimball (1996)), and always exhibits a slope between $\underline{\kappa}_t$ and $\bar{\kappa}_t$ (the formal proof is provided in appendix D).

2.9 Conditions Under Which the Problem Defines a Contraction Mapping

To prove that the consumption rules converge, we need to show that the problem defines a contraction mapping. This cannot be proven using the standard theorems in, say, Stokey et. al. (1989), which require marginal utility to be bounded over the space of possible values of m, because the possibility (however unlikely) of an unbroken string of zero-income events for the remainder of life means that as m approaches zero c must approach zero (see the discussion in 2.7); thus, marginal utility is unbounded. Although a recent literature examines the existence and uniqueness of solutions to Bellman equations in the presence of 'unbounded returns' (see Matkowski and Nowak (2011) for a recent contribution), the techniques in that literature cannot be used to solve the problem here because the required conditions are violated by a problem that involves permanent shocks.[17]

Fortunately, Boyd (1990) provides a weighted contraction mapping theorem that can be used. To use Boyd's theorem we need

Definition 1. *Consider any function $\bullet \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ where $\mathcal{C}(\mathcal{A}, \mathcal{B})$ is the space of continuous functions from $\mathcal{A}$ to $\mathcal{B}$. Suppose $F \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ with $\mathcal{B} \subseteq \mathbb{R}$ and $F > 0$. Then $\bullet$ is F-bounded if the F-norm of $\bullet$,*

$$\| \bullet \|_F = \sup_m \left[\frac{| \bullet (m)|}{F(m)} \right], \tag{40}$$

is finite.

For $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ defined as the set of functions in $\mathcal{C}(\mathcal{A}, \mathcal{B})$ that are F-bounded; w, x, y, and z as examples of F-bounded functions; and using $\mathbf{0}(m) = 0$ to indicate the function that returns zero for any argument, Boyd (1990) proves the following.

[17] See Yao (2012) for a detailed discussion of the reasons the existing literature up through Matkowski and Nowak (2011) cannot handle the problem described here.

Boyd's Weighted Contraction Mapping Theorem. *Let* $\mathsf{T} : \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B}) \to \mathcal{C}(\mathcal{A}, \mathcal{B})$ *such that*[18,19]

$$\begin{array}{ll} 1) & \mathsf{T} \textit{ is non-decreasing, i.e. } \mathrm{x} \leq \mathrm{y} \Rightarrow \{\mathsf{T}\mathrm{x}\} \leq \{\mathsf{T}\mathrm{y}\} \\ 2) & \{\mathsf{T}\mathbf{0}\} \in \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B}) \\ 3) & \textit{There exists some real } 0 < \xi < 1, \textit{ such that} \\ & \{\mathsf{T}(\mathrm{w} + \zeta\digamma)\} \leq \{\mathsf{T}\mathrm{w}\} + \zeta\xi\digamma \quad \textit{holds for all real } \zeta > 0 \ . \end{array}$$

Then T *defines a contraction with a unique fixed point.*

For our problem, take $\mathcal{A}$ as $\mathbb{R}_{++}$ and $\mathcal{B}$ as $\mathbb{R}$, and define

$$\{\mathsf{E}\mathrm{z}\}(a_t) \;=\; \mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{z}(a_t\mathcal{R}_{t+1} + \xi_{t+1})\right].$$

Using this, we introduce the mapping $\mathfrak{T} : \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B}) \to \mathcal{C}(\mathcal{A}, \mathcal{B})$,[20]

$$\{\mathfrak{T}\mathrm{z}\}(m_t) \;=\; \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} u(c_t) + \beta\left(\{\mathsf{E}\mathrm{z}\}(m_t - c_t)\right). \tag{41}$$

We can show that our operator $\mathfrak{T}$ satisfies the conditions that Boyd requires of his operator T, if we impose two restrictions on parameter values. The first restriction is the WRIC necessary for convergence of the maximal MPC, equation (36) above. A more serious restriction is the utility-compensated Finite Value of Autarky condition, equation (30). (We discuss the interpretation of these restrictions in detail in section 2.11 below.) Imposing these restrictions, we are now in position to state the central theorem of the paper.

Theorem 1. $\mathfrak{T}$ *is a contraction mapping if the restrictions on parameter values* (36) *and* (30) *are true.*

The proof is cumbersome, and therefore relegated to appendix D. Given that the value function converges, appendix D.3 shows that the consumption functions converge.

2.10 The Liquidity Constrained Solution as a Limit

This section shows that a related problem commonly considered in the literature (e.g. with a simpler income process by Deaton (1991)), with a liquidity constraint and a positive minimum value of income, is the limit of the problem considered here as the probability $\wp$ of the zero-income event approaches zero.

Formally, suppose we change the description of the problem by making the following two assumptions:

$$\begin{aligned} \wp &= 0 \\ c_t &\leq m_t, \end{aligned}$$

[18] We will usually denote the function that results from the mapping as, e.g., $\{\mathsf{T}\mathrm{w}\}$.

[19] To non-theorists, this notation may be slightly confusing; the inequality relations in 1) and 3) are taken to mean 'for any specific element $\bullet$ in the domain of the functions in question' so that, e.g., $\mathrm{x} \leq \mathrm{y}$ is short for $\mathrm{x}(\bullet) \leq \mathrm{y}(\bullet)\ \forall\ \bullet \in \mathcal{A}$. In this notation, $\zeta\xi\digamma$ in 3) is a *function* which can be applied to any argument $\bullet$ (because $\digamma$ is a function).

[20] Note that the existence of the maximum is assured by the continuity of $\{\mathsf{E}\mathrm{z}\}(a_t)$ (it is continuous because it is the sum of continuous $\digamma$-bounded functions z) and the compactness of $[\underline{\kappa} m_t, \bar{\kappa} m_t]$.

and we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed $\wp$ as $\mathrm{c}_t(m;\wp)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and $\wp$, which is not. The proposition we wish to demonstrate is

$$\lim_{\wp \downarrow 0} \mathrm{c}_t(m;\wp) = \grave{\mathrm{c}}_t(m).$$

We will first examine the problem in period $T-1$, then argue that the desired result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = \mathsf{R} = \Gamma = 1$, and there are no permanent shocks, $\psi = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

$$\grave{\mathrm{a}}^*_{T-1}(m) = \arg\max_a \left\{ u(m-a) + \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{v}_T(a+\theta) d\mathcal{F}_\theta \right\}. \tag{42}$$

As usual, the envelope theorem tells us that $\mathrm{v}'_T(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{u}'(\mathrm{c}_T(a+\theta)) d\mathcal{F}_\theta,$$

and the solution to (42) will satisfy

$$\mathrm{u}'(m-a) = \grave{\mathfrak{v}}'_{T-1}(a). \tag{43}$$

$\grave{\mathrm{a}}^*_{T-1}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $\wp > 0$.) The restrained consumer's actual asset position will be

$$\grave{\mathrm{a}}_{T-1}(m) = \max[0, \grave{\mathrm{a}}^*_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (1991)) that

$$m^1_{\#} = \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\rho}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T-1$.

Analogously to (43), defining

$$\mathfrak{v}'_{T-1}(a;\wp) \equiv \left[\wp a^{-\rho} + \not\wp \int_{\underline{\theta}}^{\bar{\theta}} (\mathrm{c}_T(a+\theta/\not\wp))^{-\rho}\, d\mathcal{F}_\theta\right], \tag{44}$$

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\rho} = \mathfrak{v}'_{T-1}(a;\wp) \tag{45}$$

with solution $\mathrm{a}^*_{T-1}(m;\wp)$. Now note that for any fixed $a > 0$, $\lim_{\wp\downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (43) and (45) are identical, this means that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. That is, for any fixed value of $m > m^1_{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $\wp \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^1_{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (44) is $\lim_{a\downarrow 0} \wp a^{-\rho} = \infty$, while $\lim_{a\downarrow 0}(m-a)^{-\rho}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^1_{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{\wp\downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}^*_{T-1}(m) < 0$, and we showed earlier that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^1_{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{\wp\downarrow 0} \mathrm{c}_{T-1}(m;\wp) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (42). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (37) for the maximal marginal propensity to consume satisfies

$$\lim_{\wp\downarrow 0} \bar{\kappa} = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by our definitions of 'constrained' and 'restrained.'

2.11 Discussion of Parametric Restrictions

2.11.1 The RIC

In the perfect foresight unconstrained problem (section 2.4.2), the RIC was required for existence of a nondegenerate solution. It is surprising, therefore, that in the presence of uncertainty, the RIC is neither necessary nor sufficient for a nondegenerate solution to exist. We thus begin our discussion by asking what features the problem must exhibit (given the FVAC) if the RIC fails (that is, $\mathsf{R} < (\mathsf{R}\beta)^{1/\rho}$):

$$\begin{aligned}
\mathsf{R} &< \overbrace{(\mathsf{R}\beta)^{1/\rho} < (\mathsf{R}(\Gamma\underline{\underline{\psi}})^{\rho-1})^{1/\rho}}^{\text{implied by FVAC}} \\
\mathsf{R} &< (\mathsf{R}/\Gamma)^{1/\rho}\Gamma\underline{\underline{\psi}}^{1-1/\rho} \\
\mathsf{R}/\Gamma &< (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \\
(\mathsf{R}/\Gamma)^{1-1/\rho} &< \underline{\underline{\psi}}^{1-1/\rho}
\end{aligned} \tag{46}$$

but since $\underline{\underline{\psi}} < 1$ and $0 < 1 - 1/\rho < 1$ (because we have assumed $\rho > 1$), this can hold only if $\mathsf{R}/\Gamma < 1$; that is, given the FVAC, the RIC can fail only if human wealth is unbounded. Unbounded human wealth is permitted here, as in the perfect foresight liquidity constrained problem. But, from equation (38), an implication of $\cancel{\text{RIC}}$ is that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Thus, interestingly, the presence of uncertainty both permits unlimited human wealth and at the same time prevents that unlimited wealth from resulting in infinite consumption. That is, in the presence of uncertainty, pathological patience (which in the perfect foresight model with finite wealth results in consumption of zero) plus infinite human wealth (which the perfect foresight model rules out because it leads to infinite consumption) combine here to yield a unique finite limiting level of consumption for any finite value of m. Note the close parallel to the conclusion in the perfect foresight liquidity constrained model in the {PF-GIC,$\cancel{\text{RIC}}$} case (for detailed analysis of this case see the appendix). There, too, the tension between infinite human wealth and pathological patience was resolved with a nondegenerate consumption function whose limiting MPC was zero.

2.11.2 The WRIC

The 'weakness' of the additional requirement for contraction, the weak RIC, can be seen by asking 'under what circumstances would the FVAC hold but the WRIC fail?' Algebraically, the requirement is

$$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho} \; < \; 1 \; < \; (\wp\beta)^{1/\rho}/\mathsf{R}^{1-1/\rho}. \tag{47}$$

If there were no conceivable parameter values that could satisfy both of these inequalities, the WRIC would have no force; it would be redundant. And if we require $\mathsf{R} \geq 1$, the WRIC is indeed redundant because now $\beta < 1 < \mathsf{R}^{\rho-1}$, so that the RIC (and WRIC) must hold.

But neither theory nor evidence demands that we assume $\mathsf{R} \geq 1$. We can therefore approach the question of the WRIC's relevance by asking just how low R must be for the condition to be relevant. Suppose for illustration that $\rho = 2$, $\underline{\underline{\psi}}^{1-\rho} = 1.01$, $\Gamma^{1-\rho} = 1.01^{-1}$ and $\wp = 0.10$. In that case (47) reduces to

$$\beta \; < 1 < \; (0.1\beta/\mathsf{R})^{1/2}$$

but since $\beta < 1$ by assumption, the binding requirement is that

$$\mathsf{R} \ < \ \beta/10$$

so that for example if $\beta = 0.96$ we would need $\mathsf{R} < 0.096$ (that is, a perpetual riskfree rate of return of worse than -90 percent a year) in order for the WRIC to bind. Thus, the relevance of the WRIC is indeed "Weak."

Perhaps the best way of thinking about this is to note that the space of parameter values for which the WRIC is relevant shrinks out of existence as $\wp \rightarrow 0$, which section 2.10 showed was the precise limiting condition under which behavior becomes arbitrarily close to the liquidity constrained solution (in the absence of other risks). On the other hand, when $\wp = 1$, the consumer has no noncapital income (so that the FHWC holds) and with $\wp = 1$ the WRIC is identical to the RIC; but the RIC is the only condition required for a solution to exist for a perfect foresight consumer with no noncapital income. Thus the WRIC forms a sort of 'bridge' between the liquidity constrained and the unconstrained problems as $\wp$ moves from 0 to 1.

2.11.3 The GIC

If both the GIC and the RIC hold, the arguments above establish that the limiting consumption function asymptotes to the consumption function for the perfect foresight unconstrained function. The more interesting case is where the GIC fails. A solution that satisfies the combination FVAC and ~~GIC~~ is depicted in Figure 2. The consumption function is shown along with the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus that identifies the 'sustainable' level of spending at which m is expected to remain unchanged. The diagram suggests a fact that is confirmed by deeper analysis: Under the depicted configuration of parameter values (see the software archive for details), the consumption function never reaches the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus; indeed, when the RIC holds but the GIC does not, the consumption function's limiting slope $(1-\text{\textbf{Þ}}/\mathsf{R})$ is shallower than that of the sustainable consumption locus $(1-\underline{\Gamma}/\mathsf{R})$,[21] so the gap between the two actually increases with m in the limit. That is, although a nondegenerate consumption function exists, a target level of m does not (or, rather, the target is $m = \infty$), because no matter how wealthy a consumer becomes, he will always spend less than the amount that would keep m stable (in expectation).

For the reader's convenience, Tables 2 and 3 present a summary of the connections between the various conditions in the presence and the absence of uncertainty.

[21] This is because $\mathbb{E}_t[m_{t+1}] = \mathbb{E}_t[\mathcal{R}_{t+1}(m_t - c_t)] + 1$; solve $m = (m - c)\mathcal{R}\acute{\psi}^{-1} + 1$ for c and differentiate.

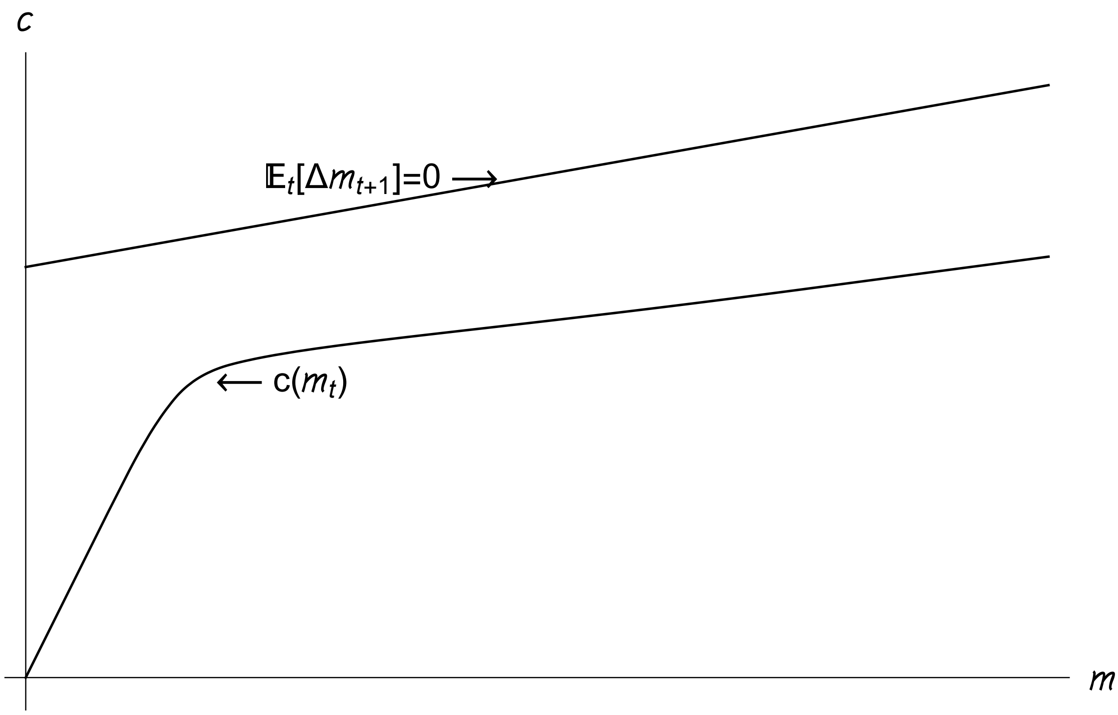


Figure 2 Example Solution when FVAC Holds but GIC Does Not

Table 2 Definitions and Comparisons of Conditions

Perfect Foresight Versions	Uncertainty Versions
Finite Human Wealth Condition (FHWC)	
$\Gamma/\mathsf{R} < 1$ The growth factor for permanent income Γ must be smaller than the discounting factor R, for human wealth to be finite.	$\Gamma/\mathsf{R} < 1$ The model's risks are mean-preserving spreads, so the PDV of future income is unchanged by their introduction.
Absolute Impatience Condition (AIC)	
$\mathbf{Þ} < 1$ The unconstrained consumer is sufficiently impatient that the level of consumption will be declining over time: $c_{t+1} < c_t$	$\mathbf{Þ} < 1$ If wealth is large enough, the expectation of consumption next period will be smaller than this period's consumption: $\lim_{m_t \to \infty} \mathbb{E}_t[c_{t+1}] < c_t$
Return Impatience Conditions	
Return Impatience Condition (RIC)	Weak RIC (WRIC)
$\mathbf{Þ}/\mathsf{R} < 1$ The growth factor for consumption $\mathbf{Þ}$ must be smaller than the discounting factor R, so that the PDV of current and future consumption will be finite: $\mathrm{c}'(m) = 1 - \mathbf{Þ}/\mathsf{R} < 1$	$\wp^{1/\rho}\mathbf{Þ}/\mathsf{R} < 1$ If the probability of the zero-income event is $\wp = 1$ then income is always zero and the condition becomes identical to the RIC. Otherwise, weaker. $\mathrm{c}'(m) < 1 - \wp^{1/\rho}\mathbf{Þ}/\mathsf{R} < 1$
Growth Impatience Conditions	
PF-GIC	GIC
$\mathbf{Þ}/\Gamma < 1$ Guarantees that for an unconstrained consumer, the ratio of consumption to permanent income will fall over time. For a constrained consumer, guarantees the constraint will eventually be binding.	$\mathbf{Þ}\,\mathbb{E}[\psi^{-1}]/\Gamma < 1$ By Jensen's inequality, stronger than the PF-GIC. Ensures consumers will not expect to accumulate m unboundedly. $\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] = \mathbf{Þ}_{\acute{\Gamma}}$
Finite Value of Autarky Conditions	
PF-FVAC	FVAC
$\beta\Gamma^{1-\rho} < 1$ equivalently $\mathbf{Þ}/\Gamma < (\mathsf{R}/\Gamma)^{1/\rho}$ The discounted utility of constrained consumers who spend their permanent income each period should be finite.	$\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}] < 1$ By Jensen's inequality, stronger than the PF-FVAC because for $\rho > 1$ and nondegenerate ψ, $\mathbb{E}[\psi^{1-\rho}] > 1$.

Table 3 Sufficient Conditions for Nondegenerate‡ Solution

Model	Conditions	Comments
PF Unconstrained	RIC, FHWC°	RIC $\Rightarrow \lvert \mathrm{v}(m)\rvert < \infty$; FHWC $\Rightarrow 0 < \lvert \mathrm{v}(m)\rvert$
		RIC prevents $\bar{\mathrm{c}}(m) = 0$
		FHWC prevents $\bar{\mathrm{c}}(m) = \infty$
PF Constrained	PF-GIC*	If RIC, $\lim_{m\to\infty} \mathring{c}(m) = \bar{c}(m), \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		If $\cancel{\text{RIC}}$, $\lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = 0$
Buffer Stock Model	FVAC, WRIC	FHWC $\Rightarrow \lim_{m\to\infty} \mathring{c}(m) = \bar{c}(m), \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		$\cancel{\text{FHWC}}$+RIC $\Rightarrow \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		$\cancel{\text{FHWC}}$+$\cancel{\text{RIC}}$ $\Rightarrow \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = 0$
		GIC guarantees finite target wealth ratio
		FVAC is stronger than PF-FVAC
		WRIC is weaker than RIC

‡For feasible m, limiting consumption function defines unique value of c satisfying $0 < c < \infty$. °RIC, FHWC are necessary as well as sufficient. *Solution also exists for ~~PF-GIC~~ and RIC, but is identical to the unconstrained model's solution for feasible $m \geq 1$.

3 Analysis of the Converged Consumption Function

Figures 3 and 4a,b capture the main properties of the converged consumption rule when the RIC, GIC, and FHWC all hold.[22] Figure 3 shows the expected consumption growth factor $\mathbb{E}_t[c_{t+1}/c_t]$ for a consumer behaving according to the converged consumption rule, while Figures 4a,b illustrate theoretical bounds for the consumption function and the marginal propensity to consume.

Five features of behavior are captured, or suggested, by the figures. First, as $m_t \uparrow \infty$ the expected consumption growth factor goes to $\text{\textbf{Þ}}$, indicated by the lower bound in Figure 3, and the marginal propensity to consume approaches $\underline{\kappa} = (1 - \text{\textbf{Þ}}_{\mathsf{R}})$ (Figure 4), the same as the perfect foresight MPC.[23] Second, as $m_t \downarrow 0$ the consumption growth factor approaches ∞ (Figure 3) and the MPC approaches $\bar{\kappa} = (1 - \wp^{1/\rho}\text{\textbf{Þ}}_{\mathsf{R}})$ (Figure 4). Third (Figure 3), there is a target cash-on-hand-to-income ratio $\check{m}$ such that if $m_t = \check{m}$ then $\mathbb{E}_t[m_{t+1}] = m_t$, and (as indicated by the arrows of motion on the $\mathbb{E}_t[c_{t+1}/c_t]$ curve), the model's dynamics are 'stable' around the target in the sense that if $m_t < \check{m}$ then cash-on-hand will rise (in expectation), while if $m_t > \check{m}$, it will fall (in expectation). Fourth (Figure 3), at the target m, the expected rate of growth of consumption is slightly less than the expected growth rate of permanent noncapital income. The final proposition suggested by Figure 3 is that the expected consumption growth factor is declining in the level of the cash-on-hand ratio m_t. This turns out to be true in the absence of permanent shocks, but in extreme cases it can be false if permanent shocks are present.[24]

[22]These figures reflect the converged rule corresponding to the parameter values indicated in Table 1.

[23]If the RIC fails, the limiting minimal MPC is 0; see appendix.

[24]Throughout the remaining analysis I make a final assumption that is not strictly justified by the foregoing. We have seen that the finite-horizon consumption functions $\mathrm{c}_{T-n}(m)$ are twice continuously differentiable and strictly concave,

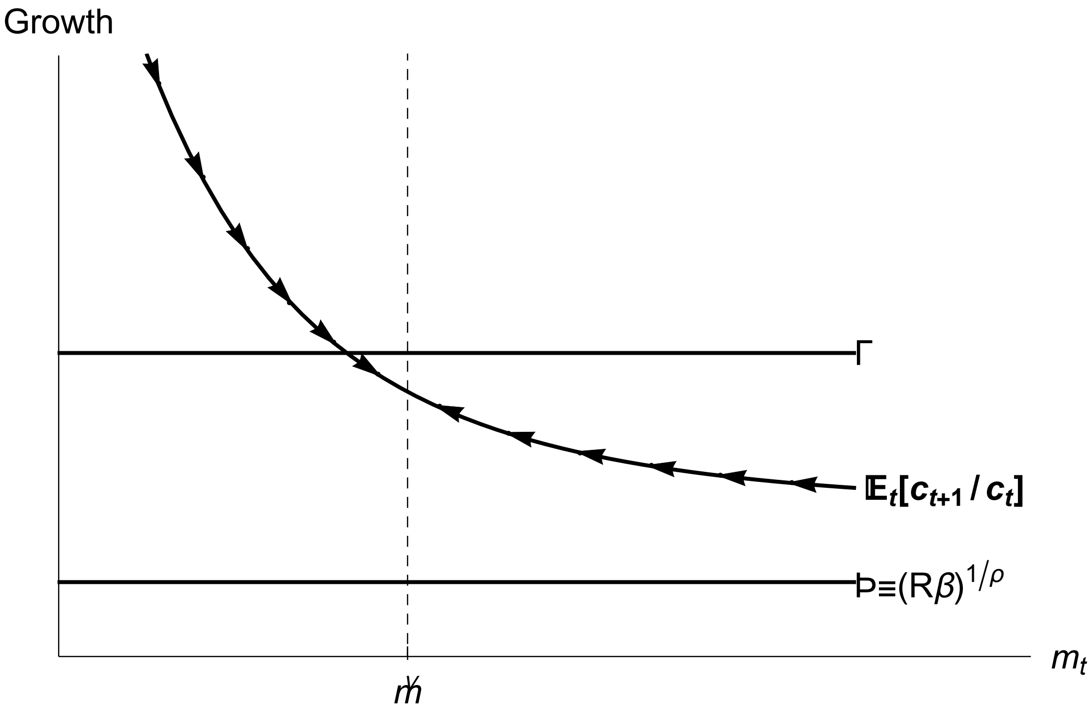


Figure 3 Target m, Expected Consumption Growth, and Permanent Income Growth

3.1 Limits as $m_t \rightarrow \infty$

Define

$$\underline{\mathrm{c}}(m) \quad = \quad \underline{\kappa} m$$

which is the solution to an infinite-horizon problem with no noncapital income ($\xi_{t+n} = 0 \ \forall \ n \geq 1$); clearly $\underline{\mathrm{c}}(m) < \mathrm{c}(m)$, since allowing the possibility of future noncapital income cannot reduce current consumption.[25]

Assuming the FHWC holds, the infinite horizon perfect foresight solution (19) constitutes an upper bound on consumption in the presence of uncertainty, since Carroll and Kimball (1996) show that the introduction of uncertainty strictly decreases the level of consumption at any m.

Thus, we can write

$$\begin{aligned} \underline{\mathrm{c}}(m) < & \quad \mathrm{c}(m) & < \bar{\mathrm{c}}(m) \\ 1 < & \quad \mathrm{c}(m)/\underline{\mathrm{c}}(m) & < \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m). \end{aligned}$$

and that they converge to a continuous function $\mathrm{c}(m)$. It does not strictly follow that the limiting function $\mathrm{c}(m)$ is twice continuously differentiable, but I will assume that it is.

[25]We will assume the RIC holds here and subsequently so that $\underline{\kappa} > 0$; the situation is a bit more complex when the RIC does not hold. In that case the bound on consumption is given by the spending that would be undertaken by a consumer who faced binding liquidity constraints. Detailed analysis of this special case is not sufficiently interesting to warrant inclusion in the paper.

But

$$\lim_{m\to\infty} \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) = \lim_{m\to\infty} (m-1+h)/m$$
$$= 1,$$

so as $m \to \infty, \mathrm{c}(m)/\underline{\mathrm{c}}(m) \to 1$, and the continuous differentiability and strict concavity of $\mathrm{c}(m)$ therefore implies

$$\lim_{m\to\infty} \mathrm{c}'(m) = \underline{\mathrm{c}}'(m) = \bar{\mathrm{c}}'(m) = \underline{\kappa}$$

because any other fixed limit would eventually lead to a level of consumption either exceeding $\bar{\mathrm{c}}(m)$ or lower than $\underline{\mathrm{c}}(m)$.

Figure 4 confirms these limits visually. The top plot shows the converged consumption function along with its upper and lower bounds, while the lower plot shows the marginal propensity to consume.

Next we establish the limit of the expected consumption growth factor as $m_t \to \infty$:

$$\lim_{m_t\to\infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] = \lim_{m_t\to\infty} \mathbb{E}_t[\Gamma_{t+1} c_{t+1}/c_t].$$

But

$$\mathbb{E}_t[\Gamma_{t+1}\underline{c}_{t+1}/\bar{c}_t] \le \mathbb{E}_t[\Gamma_{t+1}c_{t+1}/c_t] \le \mathbb{E}_t[\Gamma_{t+1}\bar{c}_{t+1}/\underline{c}_t]$$

and

$$\lim_{m_t\to\infty} \Gamma_{t+1}\underline{\mathrm{c}}(m_{t+1})/\bar{\mathrm{c}}(m_t) = \lim_{m_t\to\infty} \Gamma_{t+1}\bar{\mathrm{c}}(m_{t+1})/\underline{\mathrm{c}}(m_t) = \lim_{m_t\to\infty} \Gamma_{t+1}m_{t+1}/m_t,$$

while

$$\lim_{m_t\to\infty} \Gamma_{t+1}m_{t+1}/m_t = \lim_{m_t\to\infty} \left(\frac{\mathsf{R}\mathrm{a}(m_t) + \Gamma_{t+1}\xi_{t+1}}{m_t}\right)$$
$$= (\mathsf{R}\beta)^{1/\rho} = \boldsymbol{\text{Þ}}$$

because $\lim_{m_t\to\infty} \mathrm{a}'(m) = \boldsymbol{\text{Þ}}_{\mathsf{R}}$[26] and $\Gamma_{t+1}\xi_{t+1}/m_t \le (\Gamma\bar{\psi}\bar{\theta}/\cancel{\wp})/m_t$ which goes to zero as m_t goes to infinity.

Hence we have

$$\boldsymbol{\text{Þ}} \le \lim_{m_t\to\infty} \mathbb{E}_t[\boldsymbol{c}_{t+1}/\boldsymbol{c}_t] \le \boldsymbol{\text{Þ}}$$

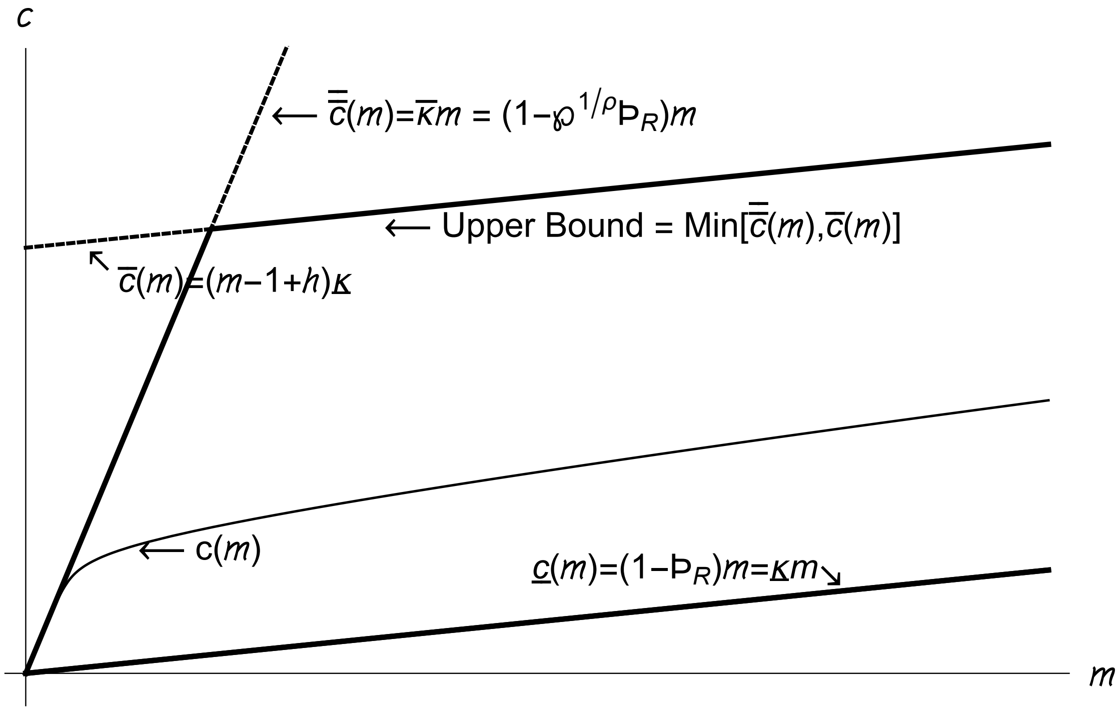
so as cash goes to infinity, consumption growth approaches its value $\boldsymbol{\text{Þ}}$ in the perfect foresight model.

This argument applies equally well to the problem of the restrained consumer, because as m approaches infinity the constraint becomes irrelevant (assuming the FHWC holds).

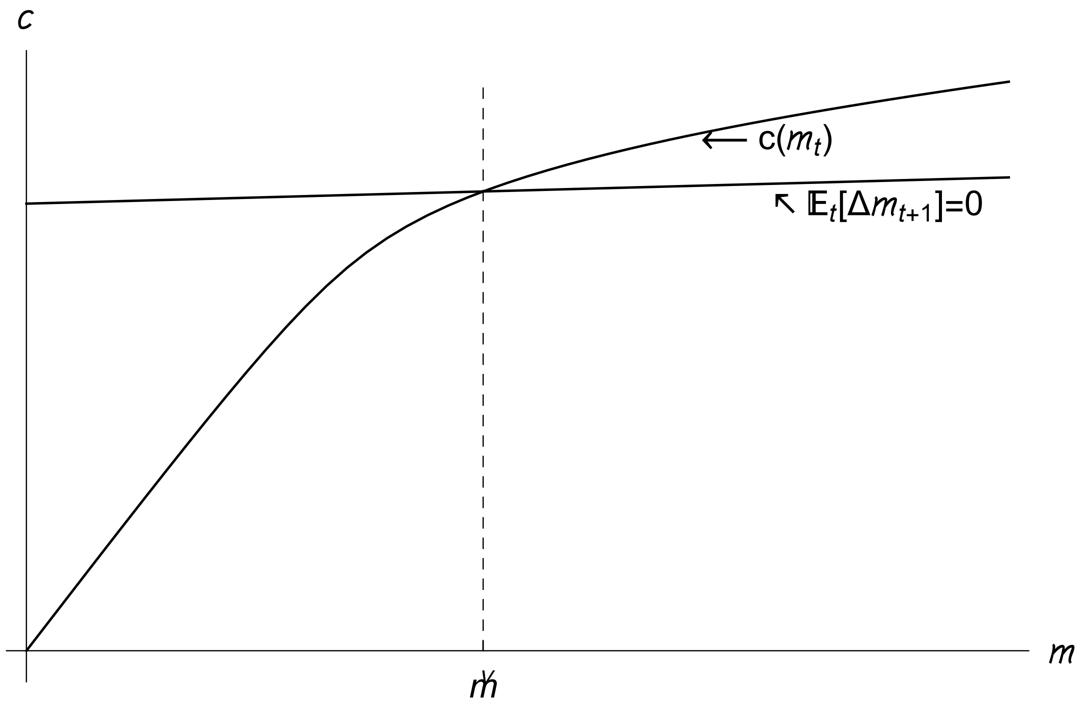
3.2 Limits as $m_t \to 0$

Now consider the limits of behavior as m_t gets arbitrarily small.

[26] This is because $\lim_{m_t\to\infty} \mathrm{a}(m_t)/m_t = 1 - \lim_{m_t\to\infty} \mathrm{c}(m_t)/m_t = 1 - \lim_{m_t\to\infty} \mathrm{c}'(m_t) = \boldsymbol{\text{Þ}}_{\mathsf{R}}$.



(a) Bounds



(b) Target m

Figure 4 The Consumption Function

Equation (37) shows that the limiting value of $\bar{\kappa}$ is

$$\bar{\kappa} \quad = \quad 1 - \mathsf{R}^{-1}(\wp \mathsf{R}\beta)^{1/\rho}.$$

Defining $\mathrm{e}(m) = \mathrm{c}(m)/m$ as before we have

$$\lim_{m \downarrow 0} \mathrm{e}(m) \quad = \quad (1 - \wp^{1/\rho}\text{\TH}_{\mathsf{R}}) = \bar{\kappa}.$$

Now using the continuous differentiability of the consumption function along with L'Hôpital's rule, we have

$$\lim_{m \downarrow 0} \mathrm{c}'(m) \quad = \quad \lim_{m \downarrow 0} \mathrm{e}(m) = \bar{\kappa}.$$

Figure 4 confirms that the numerical solution method obtains this limit for the MPC as m approaches zero.

For consumption growth, as $m \downarrow 0$ we have

$$\begin{aligned}
\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right) \Gamma_{t+1} \right] &> \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \wp \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&\quad + \not\wp \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \theta_{t+1}/\not\wp)}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&> \not\wp \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\theta_{t+1}/\not\wp)}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \infty
\end{aligned}$$

where the second-to-last line follows because $\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.[27]

3.3 There Exists Exactly One Target Cash-on-Hand Ratio, which is Stable

Define the target cash-on-hand-to-income ratio $\check{m}$ as the value of m such that

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{48}$$

where the $\vee$ accent is meant to invoke the fact that this is the value that other m's 'point to.'

We prove existence by arguing that $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous on $m_t > 0$, and takes on values both above and below 1, so that it must equal 1 somewhere by the intermediate value theorem.

Specifically, the same logic used in section 3.2 shows that $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}/m_t] = \infty$.

The limit as m_t goes to infinity is

$$\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] \quad = \quad \lim_{m_t \to \infty} \mathbb{E}_t \left[\frac{\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1}}{m_t} \right]$$

[27] The same arguments establish $\lim_{m \downarrow 0} \mathbb{E}_t[c_{t+1}/c_t] = \infty$ for the problem of the restrained consumer.

$$
\begin{aligned}
&= \mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})\text{\textbf{Þ}}_{\mathsf{R}}] \\
&= \mathbb{E}_t[\text{\textbf{Þ}}/\Gamma_{t+1}] \\
&< 1
\end{aligned}
$$

where the last line is guaranteed by our imposition of the GIC (29).

Stability means that in a local neighborhood of $\check{m}$, values of m_t above $\check{m}$ will result in a smaller ratio of $\mathbb{E}_t[m_{t+1}/m_t]$ than at $\check{m}$. That is, if $m_t > \check{m}$ then $\mathbb{E}_t[m_{t+1}/m_t] < 1$. This will be true if

$$
\left(\frac{d}{dm_t}\right)\mathbb{E}_t[m_{t+1}/m_t] \;<\; 0
$$

at $m_t = \check{m}$. But

$$
\begin{aligned}
\left(\frac{d}{dm_t}\right)\mathbb{E}_t[m_{t+1}/m_t] &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)[\mathcal{R}_{t+1}(1-\mathrm{c}(m_t)/m_t)+\xi_{t+1}/m_t]\right] \\
&= \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}(\mathrm{c}(m_t)-\mathrm{c}'(m_t)m_t)-\xi_{t+1}}{m_t^2}\right]
\end{aligned}
$$

which will be negative if its numerator is negative. Define $\boldsymbol{\zeta}(m_t)$ as the expectation of the numerator,

$$
\boldsymbol{\zeta}(m_t) = \underbrace{\mathbb{E}_t[\mathcal{R}_{t+1}]}_{\equiv\bar{\mathcal{R}}}(\mathrm{c}(m_t)-\mathrm{c}'(m_t)m_t)-1. \tag{49}
$$

The target level of market resources is the m such that if $m_t = \check{m}$ then $\mathbb{E}_t[m_{t+1}] = \check{m}$.

$$
\begin{aligned}
\mathbb{E}_t[m_{t+1}] &= \mathbb{E}_t[\mathcal{R}_{t+1}(m_t-c_t)+\xi_{t+1}] \\
\check{m} &= \bar{\mathcal{R}}(\check{m}-\mathrm{c}(\check{m}))+1 \\
\bar{\mathcal{R}}\mathrm{c}(\check{m}) &= 1+(\bar{\mathcal{R}}-1)\check{m}. \qquad (50)
\end{aligned}
$$

At the target, equation (49) is

$$
\boldsymbol{\zeta}(\check{m}) \;=\; \bar{\mathcal{R}}\mathrm{c}(\check{m})-\bar{\mathcal{R}}\mathrm{c}'(\check{m})\check{m}-1.
$$

Substituting for the first term in this expression using (50) gives

$$
\begin{aligned}
\boldsymbol{\zeta}(\check{m}) &= 1+(\bar{\mathcal{R}}-1)\check{m}-\bar{\mathcal{R}}\mathrm{c}'(\check{m})\check{m}-1 \\
&= \check{m}\left(\bar{\mathcal{R}}-1-\bar{\mathcal{R}}\mathrm{c}'(\check{m})\right) \\
&= \check{m}\left(\bar{\mathcal{R}}(1-\mathrm{c}'(\check{m}))-1\right) \\
&< \check{m}\left(\bar{\mathcal{R}}(1-(1-\mathsf{R}^{-1}(\mathsf{R}\beta)^{1/\rho}))-1\right) \\
&= \check{m}\left(\bar{\mathcal{R}}\text{\textbf{Þ}}_{\mathsf{R}}-1\right) \\
&= \check{m}\left(\underbrace{\mathbb{E}_t[\text{\textbf{Þ}}/\Gamma_{t+1}]}_{<1\text{ from (29)}}-1\right) \\
&< 0
\end{aligned}
$$

where the step introducing the inequality imposes the fact that $\mathrm{c}' > \text{\textbf{Þ}}_{\mathsf{R}}$ which is an implication of the concavity of the consumption function.

We have now proven that some target $\check{m}$ must exist, and that at any such $\check{m}$ the solution is stable. Nothing so far, however, rules out the possibility that there will be multiple values of m that satisfy the definition (48) of a target.

Multiple targets can be ruled out as follows. Suppose there exist multiple targets; these can be arranged in ascending order and indexed by an integer superscript, so that the target with the smallest value is, e.g., $\check{m}^1$. The argument just completed implies that since $\mathbb{E}_t[m_{t+1}/m_t]$ is continuously differentiable there must exist some small ϵ such that $\mathbb{E}_t[m_{t+1}/m_t] < 1$ for $m_t = \check{m}^1 + \epsilon$. (Continuous differentiability of $\mathbb{E}_t[m_{t+1}/m_t]$ follows from the continuous differentiability of $\mathrm{c}(m_t)$.)

Now assume there exists a second value of m satisfying the definition of a target, $\check{m}^2$. Since $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, it must be approaching 1 from below as $m_t \rightarrow \check{m}^2$, since by the intermediate value theorem it could not have gone above 1 between $\check{m}^1 + \epsilon$ and $\check{m}^2$ without passing through 1, and by the definition of $\check{m}^2$ it cannot have passed through 1 before reaching $\check{m}^2$. But saying that $\mathbb{E}_t[m_{t+1}/m_t]$ is approaching 1 from below as $m_t \rightarrow \check{m}^2$ implies that

$$\left(\frac{d}{dm_t}\right) \mathbb{E}_t[m_{t+1}/m_t] \;\; > \;\; 0 \tag{51}$$

at $m_t = \check{m}^2$. However, we just showed above that, under our assumption that the GIC holds, precisely the opposite of equation (51) must hold for any m that satisfies the definition of a target. Thus, assuming the existence of more than one target implies a contradiction.

The foregoing arguments rely on the continuous differentiability of $\mathrm{c}(m)$, so the arguments do not directly go through for the restrained consumer's problem in which the existence of liquidity constraints can lead to discrete changes in the slope $\mathrm{c}'(m)$ at particular values of m. But we can use the fact that the restrained model is the limit of the baseline model as $\wp \downarrow 0$ to conclude that there is likely a unique target cash level even in the restrained model.

If consumers are sufficiently impatient, the limiting target level in the restrained model will be $\check{m} = \mathbb{E}_t[\xi_{t+1}] = 1$. That is, if a consumer starting with $m = 1$ will save nothing, $\mathrm{a}(1) = 0$, then the target level of m in the restrained model will be 1; if a consumer with $m = 1$ would choose to save something, then the target level of cash-on-hand will be greater than the expected level of income.

3.4 Expected Consumption Growth at Target m Is Less than Expected Permanent Income Growth

In Figure 3 the intersection of the target cash-on-hand ratio locus at $\check{m}$ with the expected consumption growth curve lies below the intersection with the horizontal line representing the growth rate of expected permanent income. This can be proven as follows.

Strict concavity of the consumption function implies that if $\mathbb{E}_t[m_{t+1}] = \check{m} = m_t$ then

$$\begin{aligned}
\mathbb{E}_t\left[\frac{\Gamma_{t+1}\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)}\right] &< \mathbb{E}_t\left[\left(\frac{\Gamma_{t+1}(\mathrm{c}(\check{m})+\mathrm{c}'(\check{m})(m_{t+1}-\check{m}))}{\mathrm{c}(\check{m})}\right)\right] \\
&= \mathbb{E}_t\left[\Gamma_{t+1}\left(1+\left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)(m_{t+1}-\check{m})\right)\right] \\
&= \Gamma + \left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)\mathbb{E}_t\left[\Gamma_{t+1}\left(m_{t+1}-\check{m}\right)\right] \\
&= \Gamma + \left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)\left[\mathbb{E}_t[\Gamma_{t+1}]\underbrace{\mathbb{E}_t[m_{t+1}-\check{m}]}_{=0}+\mathrm{cov}_t(\Gamma_{t+1},m_{t+1})\right] \qquad (52)
\end{aligned}$$

and since $m_{t+1} = (\mathsf{R}/\Gamma_{t+1})\mathrm{a}(\check{m})+\xi_{t+1}$ and $\mathrm{a}(\check{m}) > 0$ it is clear that $\mathrm{cov}_t(\Gamma_{t+1}, m_{t+1}) < 0$ which implies that the entire term added to Γ in (52) is negative, as required.

3.5 Expected Consumption Growth Is a Declining Function of m_t (or Is It?)

Figure 3 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$\boldsymbol{\Upsilon}(m_t) \equiv \Gamma_{t+1}\mathrm{c}(\mathcal{R}_{t+1}\mathrm{a}(m_t)+\xi_{t+1})/\mathrm{c}(m_t) = c_{t+1}/c_t$$

and the proposition in which we are interested is

$$(d/dm_t)\,\mathbb{E}_t[\underbrace{\boldsymbol{\Upsilon}(m_t)}_{\equiv\boldsymbol{\Upsilon}_{t+1}}] < 0$$

or differentiating through the expectations operator, what we want is

$$\mathbb{E}_t\left[\Gamma_{t+1}\left(\frac{\mathrm{c}'(m_{t+1})\mathcal{R}_{t+1}\mathrm{a}'(m_t)\mathrm{c}(m_t)-\mathrm{c}(m_{t+1})\mathrm{c}'(m_t)}{\mathrm{c}(m_t)^2}\right)\right] < 0. \qquad (53)$$

Henceforth indicating appropriate arguments by the corresponding subscript (e.g. $\mathrm{c}'_{t+1} \equiv \mathrm{c}'(m_{t+1})$), since $\Gamma_{t+1}\mathcal{R}_{t+1} = \mathsf{R}$, the portion of the LHS of equation (53) in brackets can be manipulated to yield

$$\begin{aligned}
\mathrm{c}_t\boldsymbol{\Upsilon}'_{t+1} &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\Gamma_{t+1}\mathrm{c}_{t+1}/\mathrm{c}_t \\
&= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}. \qquad (54)
\end{aligned}$$

Now differentiate the Euler equation with respect to m_t:

$$\begin{aligned}
1 &= \mathsf{R}\beta\,\mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho}] \\
0 &= \mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}\boldsymbol{\Upsilon}'_{t+1}] \\
&= \mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}]\,\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] + \mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1},\boldsymbol{\Upsilon}'_{t+1}) \\
\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] &= -\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1},\boldsymbol{\Upsilon}'_{t+1})/\,\mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}] \qquad (55)
\end{aligned}$$

but since $\boldsymbol{\Upsilon}_{t+1} > 0$ we can see from (55) that (53) is equivalent to

$$\text{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}'_{t+1}) \quad > \quad 0$$

which, using (54), will be true if

$$\text{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}) \quad > \quad 0$$

which in turn will be true if both

$$\text{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}) \quad > \quad 0$$

and

$$\text{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}_{t+1}) \quad < \quad 0.$$

The latter proposition is obviously true under our assumption $\rho > 1$. The former will be true if

$$\text{cov}_t\left((\Gamma\psi_{t+1}\mathrm{c}(m_{t+1}))^{-\rho-1}, \mathrm{c}'(m_{t+1})\right) \quad > \quad 0.$$

The two shocks cause two kinds of variation in m_{t+1}. Variations due to ξ_{t+1} satisfy the proposition, since a higher draw of ξ both reduces $c_{t+1}^{-\rho-1}$ and reduces the marginal propensity to consume. However, permanent shocks have conflicting effects. On the one hand, a higher draw of ψ_{t+1} will reduce m_{t+1}, thus increasing both $c_{t+1}^{-\rho-1}$ and c'_{t+1}. On the other hand, the $c_{t+1}^{-\rho-1}$ term is multiplied by $\Gamma\psi_{t+1}$, so the effect of a higher ψ_{t+1} could be to decrease the first term in the covariance, leading to a negative covariance with the second term. (Analogously, a lower permanent shock ψ_{t+1} can also lead a negative correlation.)

4 The Aggregate and Idiosyncratic Relationship Between Consumption Growth and Income Growth

This section examines the behavior of large collections of buffer-stock consumers with identical parameter values. Such a collection can be thought of as either a subset of the population within a single country (say, members of a given education or occupation group), or as the whole population in a small open economy.[28]

Formally, we assume a continuum of *ex ante* identical households on the unit interval, with constant total mass normalized to one and indexed by $i \in [0, 1]$, all behaving according to the model specified above.[29]

[28]We will continue to take the aggregate interest rate as exogenous and constant. It is also possible, and only slightly more difficult, to solve for the steady-state of a closed-economy version of the model where the interest rate is endogenous.

[29]One inconvenient aspect of the model as specified is that it does not exhibit a stationary distribution of idiosyncratic permanent noncapital income; the longer the economy lasts, the wider is the distribution. This problem can be remedied by assuming a constant probability of death, and replacing deceased households with newborns whose initial idiosyncratic permanent income matches the mean idiosyncratic permanent income of the population. For a fully worked-out general equilibrium version of such a model, see Carroll, Slacalek, and Tokuoka (2011).

4.1 Convergence of the Cross-Section Distribution

Szeidl (2012) proves that such a population will be characterized by an invariant distribution of m that induces invariant distributions for c and a; designate these $\mathcal{F}^m$, $\mathcal{F}^a$, and $\mathcal{F}^c$.[30]

Szeidl's proof, however, does not yield any sense of how quickly convergence occurs, which in principle depends on all of the parameters of the model as well as the initial conditions. To build intuition, Figure 5 supplies an example in which a population begins with a particularly simple distribution that is far from the invariant one:

$$m_{1,i} = \xi_{1,i},$$

which would characterize a population in which all assets had been wiped out immediately before the receipt of period 1's noncapital income.[31]

The figure plots the distributions of a (for technical reasons, this is slightly better than plotting m) at the ends of 1, 4, 10, and 40 periods.[32]

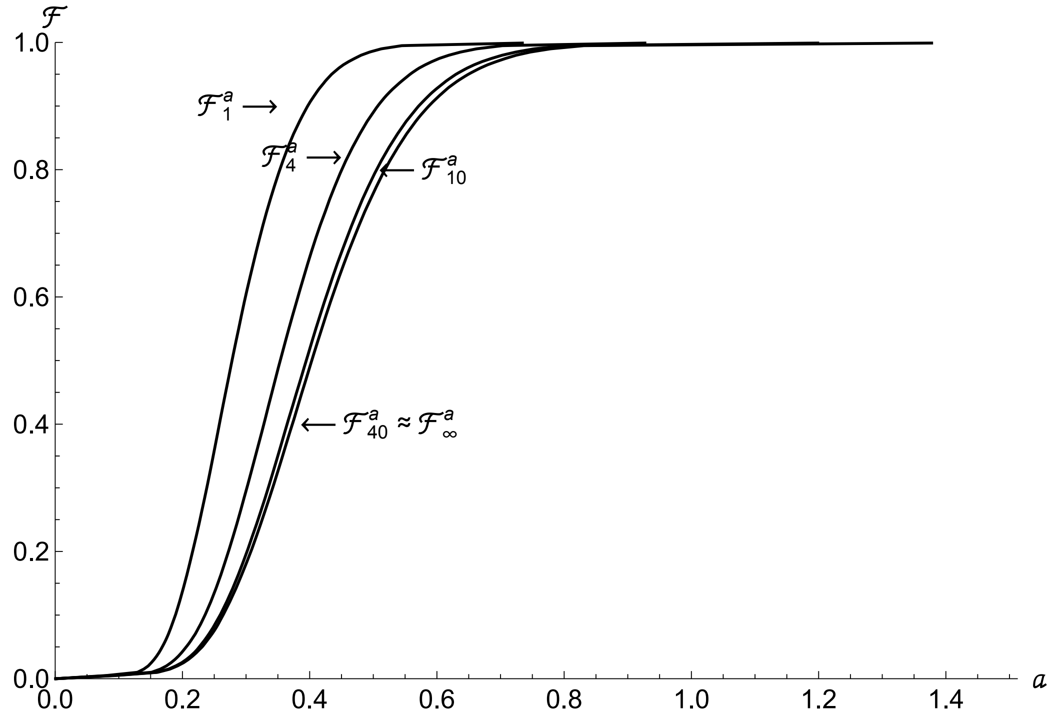


Figure 5 Convergence of $\mathcal{F}^a$ to Invariant Distribution

[30] Szeidl's proof supplants simulation evidence of ergodicity that appeared in an earlier version of this paper.

[31] We assume that $\log \psi_{t+1}$ and $\log \xi_{t+1}$ are normally distributed with means $-\sigma^2_\psi/2$ and $-\sigma^2_\xi/2$ and variances σ^2_ψ and σ^2_ξ (so that $\mathbb{M}[\psi_{t+1}] = \mathbb{M}[\xi_{t+1}] = 1$), where $\mathbb{M}[]$ is the mean operator defined below.

[32] The figure reflects results for the calibration detailed in Table 1, which are representative of the micro literature which has mainly focused on matching behavior of median or "typical" households, who hold little liquid wealth. A higher time preference factor would be necessary to match the behavior of richer households who hold much of the aggregate capital stock. However, for these richer households, the precautionary motives highlighted in this model may be less relevant than other motives which are less well understood.

The figure illustrates the fact that, under these parameter values, convergence to the invariant distribution has largely been accomplished within 10 periods. By 40 periods, the distribution is indistinguishable from the invariant distribution.

4.2 Consumption and Income Growth at the Household Level

It is useful to define the operator $\mathbb{M}[\bullet]$ which yields the mean value of its argument in the population, as distinct from the expectations operator $\mathbb{E}[\bullet]$ which represents beliefs about the future.

An economist with a microeconomic dataset could calculate the average growth rate of idiosyncratic consumption, and would find

$$\begin{aligned}
\mathbb{M}[\Delta \log c_{t+1}] &= \mathbb{M}[\log c_{t+1}\boldsymbol{p}_{t+1} - \log c_t\boldsymbol{p}_t] \\
&= \mathbb{M}[\log \boldsymbol{p}_{t+1} - \log \boldsymbol{p}_t + \log c_{t+1} - \log c_t] \\
&= \mathbb{M}[\log \boldsymbol{p}_{t+1} - \log \boldsymbol{p}_t] + \mathbb{M}[\log c_{t+1} - \log c_t] \\
&= \gamma - \sigma_\psi^2/2,
\end{aligned}$$

where $\gamma = \log \Gamma$ and the last equality follows because the invariance of $\mathcal{F}^c$ means that $\mathbb{M}[\log c_{t+1}] = \mathbb{M}[\log c_t]$.[33]

5 Conclusions

This paper provides theoretical foundations for many characteristics of buffer stock saving models that have heretofore been observed in simulations but not proven. Perhaps the most important such proposition is the existence of a target cash-to-permanent-income ratio toward which actual cash will tend.

Another contribution is provision a set of tools for numerical solution and simulation (available on the author's web page) that confirm and illustrate the theoretical propositions. These programs demonstrate how the incorporation of the paper's theoretical results can make numerical solution algorithms more efficient and simpler. A goal of the paper has been to make these tools accessible and easy to use while incorporating the full rigor of the theoretical results in the structure.

[33] Papers in the simulation literature have observed an approximate equivalence between the average growth rates of idiosyncratic consumption and permanent income, but formal proof was not possible until Szeidl's proof of ergodicity.

Appendices

A Perfect Foresight Liquidity Constrained Solution

This appendix taxonomizes the characteristics of the limiting consumption function $\mathring{c}(m)$ under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$ under various conditions. Results are summarized in table 4.

A.1 If PF-GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails ($\cancel{\text{PF-GIC}}$, $1 < \text{Þ}/\Gamma$). Under $\cancel{\text{PF-GIC}}$ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathsf{R}\beta)^{1/\rho}/\Gamma$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[34]

$$
\begin{aligned}
1 &< (\mathsf{R}\beta)^{1/\rho}\Gamma^{-1} && (56)\\
1 &< \mathsf{R}\beta\Gamma^{-\rho} && (57)\\
u'(1) &< \mathsf{R}\beta u'(\Gamma). && (58)
\end{aligned}
$$

Similar logic shows that under these circumstances the constraint will never bind for an unconstrained consumer with a finite horizon of n periods, so such a consumer's consumption function will be the same as for the unconstrained case examined in the main text.

If the RIC fails ($1 < \text{Þ}_{\mathsf{R}}$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$\mathring{c}_{T-n}(m) = 0(b_t + h). \tag{59}$$

If the RIC fails and the FHWC fails, human wealth limits to $h = \infty$ so the consumption function limits to either $\mathring{c}_{T-n}(m) = 0$ or $\mathring{c}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞.[35]

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying $\cancel{\text{PF-GIC}}$ we must impose the RIC (and the FHWC can be shown to be a consequence of $\cancel{\text{PF-GIC}}$ and RIC). In this case, the consumer's optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer would choose $c = m$ from (19):

$$
\begin{aligned}
m_{\#} &= (m_{\#} - 1 + h)\underline{\kappa} && (60)\\
m_{\#}(1 - \underline{\kappa}) &= (h - 1)\underline{\kappa} && (61)
\end{aligned}
$$

[34] The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $u'(c_{\#}) = \mathsf{R}\beta u'(\Gamma)$ which is $c_{\#} = \Gamma/(\mathsf{R}\beta)^{1/\rho}$ and the consumption function will be defined by $\mathring{c}(m) = \min[m, c_{\#} + (m - c_{\#})\underline{\kappa}]$.

[35] The knife-edge case is where $\text{Þ} = \Gamma$, in which case the two quantites counterbalance and the limiting function is $\mathring{c}(m) = \min[m, 1]$.

$$m_{\#} \;=\; (h-1)\left(\frac{\underline{\kappa}}{1-\underline{\kappa}}\right) \tag{62}$$

which (under these assumptions) satisfies $0 < m_{\#} < 1$.[36] For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m; for such m, the constrained consumer is obliged to choose $\mathring{c}(m) = m$.[37] For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{c}(m)$.

A.2 If PF-GIC Holds

Imposition of the PF-GIC reverses the inequality in (56)-(58), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t, but perpetually thereafter.

Now define $b_{\#}^n$ as the b_t such that an unconstrained consumer holding $b_t = b_{\#}^n$ would behave so as to arrive in period $t+n$ with $b_{t+n} = 0$ (with $b_{\#}^0$ trivially equal to 0); for example, a consumer with $b_{t-1} = b_{\#}^1$ was on the 'cusp' of being constrained in period $t-1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the PF-GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m_{\#}^0 = 1 + b_{\#}^0 = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c_{\#}^0 = 1$.

We can construct the entire 'prehistory' of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must have been growing according to Þ_Γ, so consumption n periods in the past must have been

$$c_{\#}^n \;=\; \text{Þ}_\Gamma^{-n} c_t = \text{Þ}_\Gamma^{-n}. \tag{63}$$

The PDV of consumption from $t-n$ until t can thus be computed as

$$\begin{aligned}
\mathbb{C}_{t-n}^t &= c_{t-n}(1 + \text{Þ}/\mathsf{R} + \ldots + (\text{Þ}/\mathsf{R})^n) \\
&= c_{\#}^n(1 + \text{Þ}_{\mathsf{R}} + \ldots + \text{Þ}_{\mathsf{R}}^n) \\
&= \text{Þ}_\Gamma^{-n}\left(\frac{1-\text{Þ}_{\mathsf{R}}^{n+1}}{1-\text{Þ}_{\mathsf{R}}}\right)
\end{aligned} \tag{64}$$

and note that the consumer's human wealth between $t-n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post-t income) is

$$h_{\#}^n \;=\; 1 + \ldots + \mathcal{R}^{-n} \tag{65}$$

while the intertemporal budget constraint says

$$\mathbb{C}_{t-n}^t \;=\; b_{\#}^n + h_{\#}^n$$

[36] Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by ~~PF-GIC~~.

[37] As an illustration, consider a consumer for whom $\text{Þ} = 1$, $\mathsf{R} = 1.01$ and $\Gamma = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\Gamma < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

from which we can solve for the $b_{\#}^{n}$ such that the consumer with $b_{t-n} = b_{\#}^{n}$ would unconstrainedly plan (in period $t-n$) to arrive in period t with $b_t = 0$:

$$b_{\#}^{n} = \mathbb{C}_{t-n}^{t} - \overbrace{\left(\frac{1-\mathcal{R}^{-(n+1)}}{1-\mathcal{R}^{-1}}\right)}^{h_{\#}^{n}}. \tag{66}$$

Defining $m_{\#}^{n} = b_{\#}^{n} + 1$, consider the function $\mathring{c}(m)$ defined by linearly connecting the points $\{m_{\#}^{n}, c_{\#}^{n}\}$ for integer values of $n \geq 0$ (and setting $\mathring{c}(m) = m$ for $m < 1$). This function will return, for any value of m, the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes, because for infinitesimal ϵ the MPC of a consumer with assets $m = m_{\#}^{n} - \epsilon$ is discretely higher than for a consumer with assets $m = m_{\#}^{n} + \epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (66) for the entire domain of positive real values of b, we need $b_{\#}^{n}$ to become arbitrarily large with n. That is, we need

$$\lim_{n\to\infty} b_{\#}^{n} = \infty. \tag{67}$$

A.2.1 If FHWC Holds

The FHWC requires $\mathcal{R}^{-1} < 1$, in which case the second term in (66) limits to a constant as $n \uparrow \infty$, and (67) reduces to a requirement that

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n} - (\text{Þ}_{\mathsf{R}}/\text{Þ}_{\Gamma})^{n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n} - \mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty.
\end{aligned}$$

Given the PF-GIC $\text{Þ}_{\Gamma}^{-1} > 1$, this will hold iff the RIC holds, $\text{Þ}_{\mathsf{R}} < 1$. But given that the FHWC $\mathsf{R} > \Gamma$ holds, the PF-GIC is stronger (harder to satisfy) than the RIC; thus, FHWC and the PF-GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because as the date at which the constraint binds becomes arbitrarily distant, the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m\to\infty} \mathring{c}(m) - \bar{c}(m) = 0. \tag{68}$$

A.2.2 If FHWC Fails

If the FHWC fails, matters are a bit more complex. Given failure of FHWC, (67) requires

$$\lim_{n\to\infty}\left(\frac{\mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}-\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right)+\left(\frac{1-\mathcal{R}^{-(n+1)}}{\mathcal{R}^{-1}-1}\right) = \infty$$

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\mathsf{R}}}{\text{Þ}_{\mathsf{R}}-1}-\frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right)\mathcal{R}^{-n}-\left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right) = \infty$$

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\mathsf{R}}(\mathcal{R}^{-1}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)}-\frac{\mathcal{R}^{-1}(\text{Þ}_{\mathsf{R}}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)}\right)\mathcal{R}^{-n}-\left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right) = \infty. \quad (69)$$

If RIC Holds. When the RIC holds, rearranging (69) gives

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right)-\mathcal{R}^{-n}\left(\frac{\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}+\frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right) = \infty$$

and for this to be true we need

$$\begin{aligned}\text{Þ}_{\Gamma}^{-1} &> \mathcal{R}^{-1}\\ \Gamma/\text{Þ} &> \Gamma/\mathsf{R}\\ 1 &> \text{Þ}/\mathsf{R}\end{aligned}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n\to\infty}\kappa_{\#}^{n} = \lim_{n\to\infty}\left(\frac{c_{\#}^{n}}{b_{\#}^{n}}\right) \quad (70)$$

which with a few lines of algebra can be shown to asymptote to the MPC in the perfect foresight model:[38]

$$\lim_{m\to\infty}\mathring{\boldsymbol{\kappa}}(m) = 1-\text{Þ}_{\mathsf{R}}. \quad (71)$$

If RIC Fails. Consider now the $\cancel{\text{RIC}}$ case, $\text{Þ}_{\mathsf{R}} > 1$. In this case the constant multiplying $\mathcal{R}^{-n}$ in (69) will be positive if

$$\begin{aligned}\text{Þ}_{\mathsf{R}}\mathcal{R}^{-1}-\text{Þ}_{\mathsf{R}} &> \mathcal{R}^{-1}\text{Þ}_{\mathsf{R}}-\mathcal{R}^{-1}\\ \mathcal{R}^{-1} &> \text{Þ}_{\mathsf{R}}\\ \Gamma &> \text{Þ}\end{aligned}$$

which is merely the PF-GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving $\mathcal{R}^{-n}$ goes to $+\infty$ faster than the term involving $-\text{Þ}_{\Gamma}^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned}\mathcal{R}^{-1} &> \text{Þ}_{\Gamma}^{-1}\\ \Gamma/\mathsf{R} &> \Gamma/\text{Þ}\\ \text{Þ}/\mathsf{R} &> 1\end{aligned}$$

which merely confirms the starting assumption that the RIC fails. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the RIC fails. It

[38] For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the software archive.

includegraphics./Figures/PFGICHoldsFHWCFailsRICFails

Figure 6 Nondegenerate Consumption Function with ~~FHWC~~ and ~~RIC~~

remains true that ~~RIC~~ implies a limiting MPC of zero,

$$\lim_{m \to \infty} \mathring{\boldsymbol{\kappa}}(m) \;=\; 0, \tag{72}$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\mathring{c}(m) = 0$. (Figure 6 presents an example for $\rho = 2$, $\mathsf{R} = 0.98$, $\beta = 0.99$, $\Gamma = 1.0$).

We can summarize as follows. Given that the PF-GIC holds, the interesting question is whether the FHWC holds. If so, the RIC automatically holds, and the solution limits into the solution to the unconstrained problem as $m \uparrow \infty$. But even if the FHWC fails, the problem has a well-defined solution, whether or not the RIC holds.

B Existence of a Concave Consumption Function

To show that (6) defines a sequence of continuously differentiable strictly increasing concave functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, ..., \mathrm{c}_{T-k}\}$, we start with a definition. We will say that a function $\mathrm{n}(z)$ is 'nice' if it satisfies

1. $\mathrm{n}(z)$ is well-defined iff $z > 0$
2. $\mathrm{n}(z)$ is strictly increasing
3. $\mathrm{n}(z)$ is strictly concave
4. $\mathrm{n}(z)$ is $\mathbf{C}^3$ (its first three derivatives exist)
5. $\mathrm{n}(z) < 0$
6. $\lim_{z \downarrow 0}\ \mathrm{n}(z) = -\infty$.

(Notice that an implication of niceness is that $\lim_{z \downarrow 0} \mathrm{n}'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = u(m)$ and $u(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $\mathfrak{v}_t(a)$ as

$$\mathfrak{v}_t(a) = \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(\mathcal{R}_{t+1}a + \xi_{t+1})\right]. \tag{73}$$

Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a \downarrow 0} \mathfrak{v}_t(a) = -\infty$ and $\lim_{a \downarrow 0} \mathfrak{v}_t'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchi (2003).)

Next define $\underline{\mathrm{v}}_t(m, c)$ as

$$\underline{\mathrm{v}}_t(m, c) = u(c) + \mathfrak{v}_t(m - c) \tag{74}$$

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (6) can be written as

$$\mathrm{v}_t(m) \;=\; \max_{c}\; \underline{\mathrm{v}}_t(m,c). \tag{75}$$

$\underline{\mathrm{v}}_t$ is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c\downarrow 0}\underline{\mathrm{v}}_t(m,c) = \lim_{c\uparrow m}\underline{\mathrm{v}}_t(m,c) = -\infty$, $\frac{\partial^2 \underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c\downarrow 0}\frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c\uparrow m}\frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$. It follows that the $\mathrm{c}_t(m)$ defined by

$$\mathrm{c}_t(m) \;=\; \operatorname*{arg\,max}_{0<c<m}\; \underline{\mathrm{v}}_t(m,c) \tag{76}$$

exists and is unique, and (6) has an internal solution that satisfies

$$u'(\mathrm{c}_t(m)) = \mathfrak{v}'_t(m - \mathrm{c}_t(m)). \tag{77}$$

Since both u and $\mathfrak{v}_t$ are strictly concave, both $\mathrm{c}_t(m)$ and $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t$ are three times continuously differentiable, using (77) we can conclude that $\mathrm{c}_t(m)$ is continuously differentiable and

$$\mathrm{c}'_t(m) \;=\; \frac{\mathfrak{v}''_t(\mathrm{a}_t(m))}{u''(\mathrm{c}_t(m)) + \mathfrak{v}''_t(\mathrm{a}_t(m))}. \tag{78}$$

Similarly we can easily show that $\mathrm{c}_t(m)$ is twice continuously differentiable (as is $\mathrm{a}_t(m)$) (See Appendix C.) This implies that $\mathrm{v}_t(m)$ is nice, since $\mathrm{v}_t(m) = u(\mathrm{c}_t(m)) + \mathfrak{v}_t(\mathrm{a}_t(m))$.

C $\mathrm{c}_t(m)$ is Twice Continuously Differentiable

First we show that $\mathrm{c}_t(m)$ is $\mathbf{C}^1$. Define y as $y \equiv m + dm$. Since $u'(\mathrm{c}_t(y)) - u'(\mathrm{c}_t(m)) = \mathfrak{v}'_t(\mathrm{a}_t(y)) - \mathfrak{v}'_t(\mathrm{a}_t(m))$ and $\frac{\mathrm{a}_t(y)-\mathrm{a}_t(m)}{dm} = 1 - \frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}$,

$$\frac{\mathfrak{v}'_t(\mathrm{a}_t(y)) - \mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)} = \left(\frac{u'(\mathrm{c}_t(y)) - u'(\mathrm{c}_t(m))}{\mathrm{c}_t(y) - \mathrm{c}_t(m)} + \frac{\mathfrak{v}'_t(\mathrm{a}_t(y)) - \mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)}\right)\frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim\limits_{dm\to+0}\frac{u'(\mathrm{c}_t(y))-u'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} < 0$ and $\lim\limits_{dm\to+0}\frac{\mathfrak{v}'_t(\mathrm{a}_t(y))-\mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ are satisfied. Then $\frac{u'(\mathrm{c}_t(y))-u'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}'_t(\mathrm{a}_t(y))-\mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ for sufficiently small dm. Hence we obtain a well-defined equation:

$$\frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm} = \frac{\frac{\mathfrak{v}'_t(\mathrm{a}_t(y))-\mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}{\frac{u'(\mathrm{c}_t(y))-u'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}'_t(\mathrm{a}_t(y))-\mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}.$$

This implies that the right-derivative, $\mathrm{c}'^{+}_t(m)$ is well-defined and

$$\mathrm{c}'^{+}_t(m) = \frac{\mathfrak{v}''_t(\mathrm{a}_t(m))}{u''(\mathrm{c}_t(m)) + \mathfrak{v}''_t(\mathrm{a}_t(m))}.$$

Similarly we can show that $\mathrm{c}_t'^{+}(m) = \mathrm{c}_t'^{-}(m)$, which means $\mathrm{c}_t'(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, $\mathrm{c}_t'(m)$ exists and is continuous. $\mathrm{c}_t'(m)$ is differentiable because $\mathfrak{v}_t''$ is $\mathbf{C}^1$, $\mathrm{c}_t(m)$ is $\mathbf{C}^1$ and $u''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m)) < 0$. $\mathrm{c}_t''(m)$ is given by

$$\mathrm{c}_t''(m) = \frac{a_t'(m)\mathfrak{v}_t'''(a_t)\left[u''(c_t) + \mathfrak{v}_t''(a_t)\right] - \mathfrak{v}_t''(a_t)\left[c_t'u'''(c_t) + a_t'\mathfrak{v}_t'''(a_t)\right]}{\left[u''(c_t) + \mathfrak{v}_t''(a_t)\right]^2}. \tag{79}$$

Since $\mathfrak{v}_t''(\mathrm{a}_t(m))$ is continuous, $\mathrm{c}_t''(m)$ is also continuous.

D Proof that $\mathfrak{T}$ Is a Contraction Mapping

We must show that our operator $\mathfrak{T}$ satisfies all of Boyd's conditions.

Boyd's operator T maps from $\mathcal{C}_{F}(\mathcal{A},\mathcal{B})$ to $\mathcal{C}(\mathcal{A},\mathcal{B})$. A preliminary requirement is therefore that $\{\mathfrak{T}\mathrm{z}\}$ be continuous for any $F-$bounded z, $\{\mathfrak{T}\mathrm{z}\} \in \mathcal{C}(\mathbb{R}_{++},\mathbb{R})$. This is not difficult to show; see Hiraguchi (2003).

Consider condition 1). For this problem,

$$\begin{aligned}
\{\mathfrak{T}\mathrm{x}\}(m_t) \quad &\text{is} \quad \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{u(c_t) + \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{x}\left(m_{t+1}\right)\right]\right\} \\
\{\mathfrak{T}\mathrm{y}\}(m_t) \quad &\text{is} \quad \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{u(c_t) + \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{y}\left(m_{t+1}\right)\right]\right\},
\end{aligned}$$

so $\mathrm{x}(\bullet) \le \mathrm{y}(\bullet)$ implies $\{\mathfrak{T}\mathrm{x}\}(m_t) \le \{\mathfrak{T}\mathrm{y}\}(m_t)$ by inspection.[39]

Condition 2) requires that $\{\mathfrak{T}\mathbf{0}\} \in \mathcal{C}_{F}\left(\mathcal{A},\mathcal{B}\right)$. By definition,

$$\{\mathfrak{T}\mathbf{0}\}(m_t) = \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{\left(\frac{c_t^{1-\rho}}{1-\rho}\right) + \beta 0\right\}$$

the solution to which is patently $u(\bar{\kappa} m_t)$. Thus, condition 2) will hold if $(\bar{\kappa} m_t)^{1-\rho}$ is F-bounded. We use the bounding function

$$F(m) \quad = \quad \eta + m^{1-\rho}, \tag{80}$$

for some real scalar $\eta > 0$ whose value will be determined in the course of the proof. Under this definition of F, $\{\mathfrak{T}\mathbf{0}\}(m_t) = u(\bar{\kappa} m_t)$ is clearly F-bounded.

Finally, we turn to condition 3), $\{\mathfrak{T}(\mathrm{z} + \zeta F)\}(m_t) \le \{\mathfrak{T}\mathrm{z}\}(m_t) + \zeta\xi F(m_t)$. The proof will be more compact if we define $\check{\mathrm{c}}$ and $\check{\mathrm{a}}$ as the consumption and assets functions[40] associated with $\mathfrak{T}\mathrm{z}$ and $\hat{\mathrm{c}}$ and $\hat{\mathrm{a}}$ as the functions associated with $\mathfrak{T}(\mathrm{z} + \zeta F)$; using this notation, condition 3) can be rewritten

$$u(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z} + \zeta F)\}(\hat{\mathrm{a}}) \quad \le \quad u(\check{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\check{\mathrm{a}}) + \zeta\xi F.$$

Now note that if we force the $\smile$ consumer to consume the amount that is optimal for the $\wedge$ consumer, value for the $\smile$ consumer must decline (at least weakly). That is,

$$u(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) \quad \le \quad u(\check{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\check{\mathrm{a}}).$$

---

[39] For a fixed m_t, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

[40] Section 2.7 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

Thus, condition 3) will certainly hold under the stronger condition

$$\begin{aligned}
u(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z} + \zeta F)\}(\hat{\mathrm{a}}) &\leq u(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\xi F \\
\beta\{\mathsf{E}(\mathrm{z} + \zeta F)\}(\hat{\mathrm{a}}) &\leq \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\xi F \\
\beta\zeta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &\leq \zeta\xi F \\
\beta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &\leq \xi F \\
\beta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &< F.
\end{aligned}$$

Using $F(m) = \eta + m^{1-\rho}$ and defining $\hat{a}_t = \hat{\mathrm{a}}(m_t)$, this condition is

$$\beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}] - m_t^{1-\rho} \quad < \quad \eta(1 - \underbrace{\beta\,\mathbb{E}_t\,\Gamma_{t+1}^{1-\rho}}_{=\beth})$$

which by imposing the PF-FVAC (24) $\beth < 1$ can be rewritten as:

$$\eta > \frac{\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}\right] - m_t^{1-\rho}}{1 - \beth}. \tag{81}$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing that the numerator of (81) is bounded from above:

$$\begin{aligned}
& \not\wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1} + \theta_{t+1}/\not\wp)^{1-\rho}\right] + \wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1})^{1-\rho}\right] - m_t^{1-\rho} \\
\leq\ & \not\wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1} + \theta_{t+1}/\not\wp)^{1-\rho}\right] + \wp\beta\mathsf{R}^{1-\rho}((1-\bar{\kappa})m_t)^{1-\rho} - m_t^{1-\rho} \\
=\ & \not\wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1} + \theta_{t+1}/\not\wp)^{1-\rho}\right] + m_t^{1-\rho}\left(\wp\beta\mathsf{R}^{1-\rho}\left(\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)^{1-\rho} - 1\right) \\
=\ & \not\wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1} + \theta_{t+1}/\not\wp)^{1-\rho}\right] + m_t^{1-\rho}\left(\underbrace{\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}}_{<1\text{ by WRIC}} - 1\right) \\
<\ & \not\wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\underline{\theta}/\not\wp)^{1-\rho}\right] = \beth\not\wp^{\rho}\underline{\theta}^{1-\rho}.
\end{aligned} \tag{82}$$

We can thus conclude that equation (81) will certainly hold for any:

$$\eta > \underline{\eta} = \frac{\beth\not\wp^{\rho}\underline{\theta}^{1-\rho}}{1 - \beth} \tag{83}$$

which is a positive finite number under our assumptions.

The proof that $\mathfrak{T}$ defines a contraction mapping under the conditions (36) and (30) is now complete.

D.1 $\mathfrak{T}$ and v

In defining our operator $\mathfrak{T}$ we made the restriction $\underline{\kappa}m_t \leq c_t \leq \bar{\kappa}m_t$. However, in the discussion of the consumption function bounds, we showed only (in (39)) that $\underline{\kappa}_t m_t \leq \mathrm{c}_t(m_t) \leq \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (6) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof in 2.9, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a stronger condition. The place where these conditions have force is in the step at (82). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$\begin{aligned}
\wp\beta(\mathsf{R}(1-\bar{\kappa}_{T-1}))^{1-\rho} &< 1 \\
(\wp\beta)^{1/(1-\rho)}(1-\bar{\kappa}_{T-1}) &< 1 \\
(\wp\beta)^{1/(1-\rho)}(1-(1+\wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}})^{-1}) &< 1
\end{aligned}$$

where we have used (35) for $\bar{\kappa}_{T-1}$. For small values of $\wp$ this expression can be further simplified using $(1+\wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}})^{-1} \approx 1-\wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}}$ so that it becomes

$$\begin{aligned}
(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}} &< 1 \\
(\wp\beta)\wp^{(1-\rho)/\rho}\mathbf{Þ}_{\mathsf{R}}^{1-\rho} &< 1 \\
\beta\wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}}^{1-\rho} &< 1
\end{aligned}$$

which for small values of $\wp$ is plainly easy to satisfy.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique $\mathrm{v}(m)$. But since $\lim_{n\to\infty} \underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $\mathrm{v}(m)$ toward which these v_{T-n}'s are converging is the *same* $\mathrm{v}(m)$ that was the endpoint of the contraction defined by our operator $\mathcal{T}$. Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (6) converge, they converge to the same unique v defined by $\mathcal{T}$.[41]

D.2 Convergence of v_t in Euclidian Space

Boyd's theorem shows that $\mathcal{T}$ defines a contraction mapping in a F-bounded space. We now show that $\mathcal{T}$ also defines a contraction mapping in Euclidian space.

Since $\mathrm{v}^*(m) = \mathcal{T}\mathrm{v}^*(m)$,

$$\left\|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)\right\|_F \leq \xi^{n-1}\left\|\mathrm{v}_T(m) - \mathrm{v}^*(m)\right\|_F. \tag{84}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_F(\mathcal{A},\mathcal{B})$ and $\kappa = \left\|\mathrm{v}_T(m) - \mathrm{v}^*(m)\right\|_F < \infty$ because v_T and v^* are in $\mathcal{C}_F(\mathcal{A},\mathcal{B})$. It follows that

$$\left|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)\right| \leq \kappa\xi^{n-1}\left|F(m)\right|. \tag{85}$$

Then we obtain

$$\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{86}$$

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathcal{T}\mathrm{v}_{T-1} \leq \mathcal{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets

[41] It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

$\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

D.3 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

We start by showing that

$$\mathrm{c}(m) = \underset{c_t \in [\underline{\kappa} m, \bar{\kappa} m]}{\arg\max} \left\{ u(c_t) + \beta\, \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}(m_{t+1}) \right] \right\} \tag{87}$$

is uniquely determined. We show this by contradiction. Suppose there exist c_1 and c_2 that both attain the supremum for some m, with mean $\tilde{c} = (c_1 + c_2)/2$. c_i satisfies

$$\mathcal{T}\mathrm{v}(m) = u(c_i) + \beta \underbrace{\mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}(\mathrm{m}_{t+1}(m, c_i)) \right]}_{\equiv \mathfrak{v}} \tag{88}$$

where $\mathrm{m}_{t+1}(m, c_i) = (m - c_i)\mathcal{R}_{t+1} + \xi_{t+1}$ and $i = 1, 2$. $\mathcal{T}\mathrm{v}$ is concave for concave $\mathfrak{v}$. Since the space of continuous and concave functions is closed, $\mathfrak{v}$ is also concave and satisfies

$$\frac{1}{2} \sum_{i=1,2} \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}(\mathrm{m}_{t+1}(m, c_i)) \right] \leq \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}(\mathrm{m}_{t+1}(m, \tilde{c})) \right]. \tag{89}$$

On the other hand, $\frac{1}{2}\{u(c_1) + u(c_2)\} < u(\tilde{c})$. Then one gets

$$\mathcal{T}\mathrm{v}(m) < u(\tilde{c}) + \beta\, \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}(\mathrm{m}_{t+1}(m, \tilde{c})) \right]. \tag{90}$$

Since $\tilde{c}$ is a feasible choice for c_i, the LHS of this equation cannot be a maximum, which contradicts the definition.

Using uniqueness of $\mathrm{c}(m)$ we can now show

$$\lim_{n \to \infty} \mathrm{c}_{T-n+1}(m) = \mathrm{c}(m). \tag{91}$$

Suppose this does not hold for some $m = m^*$. In this case, $\{\mathrm{c}_{T-n+1}(m^*)\}_{n=1}^{\infty}$ has a subsequence $\left\{c_{T-n(i)}(m^*)\right\}_{i=1}^{\infty}$ that satisfies $\lim_{i\to\infty} \mathrm{c}_{T-n(i)}(m^*) = c^*$ and $c^* \neq \mathrm{c}(m^*)$. Now define $c^*_{T-n+1} = \mathrm{c}_{T-n+1}(m^*)$. $c^* > 0$ because $\lim_{i\to\infty} \mathrm{v}_{T-n(i)+1}(m^*) \leq \lim_{i\to\infty} u(c^*_{T-n(i)})$. Because $\mathrm{a}(m^*) > 0$ and $\psi \in [\underline{\psi}, \bar{\psi}]$ there exist $\{\underline{m}^*_+, \bar{m}^*_+\}$ satisfying $0 < \underline{m}^*_+ < \bar{m}^*_+$ and $m_{T-n+1}(m^*, c^*_{T-n+1}) \in \left[\underline{m}^*_+, \bar{m}^*_+\right]$. It follows that $\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}(m)$ and the convergence is uniform on $m \in \left[\underline{m}^*_+, \bar{m}^*_+\right]$. (Uniform convergence is obtained from Dini's theorem.[42]) Hence for any $\delta > 0$, there exists an n_1 such that

$$\beta\, \mathbb{E}_{T-n} \left[\Gamma_{T-n+1}^{1-\rho} \left| \mathrm{v}_{T-n+1}(m_{T-n+1}(m^*, c^*_{T-n+1})) - \mathrm{v}(m_{T-n+1}(m^*, c^*_{T-n+1})) \right| \right] \; < \; \delta$$

for all $n \geq n_1$. It follows that if we define

$$\mathrm{w}(m^*, z) = u(z) + \beta\, \mathbb{E}_{T-n} \left[\Gamma_{T-n+1}^{1-\rho} \mathrm{v}(m_{T-n+1}(m^*, z)) \right] \tag{92}$$

[42][**Dini's theorem**] For a monotone sequence of continuous functions $\{\mathrm{v}_n(m)\}_{n=1}^{\infty}$ which is defined on a compact space and satisfies $\lim_{n\to\infty} \mathrm{v}_n(m) = \mathrm{v}(m)$ where $\mathrm{v}(m)$ is continuous, convergence is uniform.

then $\mathrm{v}_{T-n}(m^*)$ satisfies

$$\lim_{n\to\infty}\left|\mathrm{v}_{T-n}(m^*)-\mathrm{w}(m^*,c^*_{T-n+1})\right|=0. \tag{93}$$

On the other hand, there exists an $i_1\in\mathbb{N}$ such that

$$\left|\mathrm{v}(\mathrm{m}_{T-n(i)}(m^*,c^*_{T-n(i)}))-\mathrm{v}(\mathrm{m}_{T-n(i)}(m^*,c^*))\right|\leq\delta \text{ for all } i\geq i_1 \tag{94}$$

because v is uniformly continuous on $[\underline{m}^*_+,\bar{m}^*_+]$. $\lim_{i\to\infty}\left|\mathrm{c}_{T-n(i)}(m^*)-c^*\right|=0$ and

$$\left|\mathrm{m}_{T-n(i)}(m^*,c^*_{T-n(i)})-\mathrm{m}_{T-n(i)}(m^*,c^*)\right|\leq\frac{\mathsf{R}}{\Gamma\underline{\psi}}\left|c^*_{T-n(i)}-c^*\right|. \tag{95}$$

This implies

$$\lim_{i\to\infty}\left|\mathrm{w}(m^*,c^*_{T-n(i)+1})-\mathrm{w}(m^*,c^*)\right|=0. \tag{96}$$

From (93) and (96), we obtain $\lim_{i\to\infty}\mathrm{v}_{T-n(i)}(m^*)=\mathrm{w}(m^*,c^*)$ and this implies $\mathrm{w}(m^*,c^*)=\mathrm{v}(m^*)$. This implies that $\mathrm{c}(m)$ is not uniquely determined, which is a contradiction.

Thus, the consumption functions must converge.

E Equality of Aggregate Consumption Growth and Income Growth with Transitory Shocks

The text asserted that in the absence of permanent shocks it is possible to prove that the growth factor for aggregate consumption approaches that for aggregate permanent income. This section establishes that result.

Suppose the population starts in period t with an arbitrary value for $\mathrm{cov}_t(a_{t+1,i},\boldsymbol{p}_{t+1,i})$. Then if $\breve{m}$ is the invariant mean level of m we can define a 'mean MPS away from $\breve{m}$' function

$$\acute{\mathrm{a}}(\Delta) \;=\; \Delta^{-1}\int_{\breve{m}}^{\breve{m}+\Delta}\mathrm{a}'(z)dz$$

and since $\psi_{t+1,i}=1$, $\mathcal{R}_{t+1,i}$ is a constant at $\mathcal{R}$ we can write

$$a_{t+1,i} \;=\; \mathrm{a}(\breve{m})+(m_{t+1,i}-\breve{m})\acute{\mathrm{a}}(\overbrace{\mathcal{R}a_{t,i}+\xi_{t+1,i}}^{m_{t+1,i}}-\breve{m})$$

so

$$\mathrm{cov}_t(a_{t+1,i},\boldsymbol{p}_{t+1,i}) \;=\; \mathrm{cov}_t\left(\acute{\mathrm{a}}(\mathcal{R}a_{t,i}+\xi_{t+1,i}-\breve{m}),\Gamma\boldsymbol{p}_{t,i}\right).$$

But since $\mathsf{R}^{-1}(\wp\mathsf{R}\beta)^{1/\rho}<\acute{\mathrm{a}}(m)<\text{Þ}_{\mathsf{R}}$,

$$|\mathrm{cov}_t((\wp\mathsf{R}\beta)^{1/\rho}a_{t+1,i},\boldsymbol{p}_{t+1,i})|<|\mathrm{cov}_t(a_{t+1,i},\boldsymbol{p}_{t+1,i})|<|\mathrm{cov}_t(\text{Þ}a_{t+1,i},\boldsymbol{p}_{t+1,i})|$$

and for the version of the model with no permanent shocks the GIC says that $\text{Þ}<\Gamma$, which implies

$$|\mathrm{cov}_t(a_{t+1,i}, \boldsymbol{p}_{t+1,i})| < \Gamma|\mathrm{cov}_t(a_{t,i}, \boldsymbol{p}_{t,i})|.$$

This means that from any arbitrary starting value, the relative size of the covariance term shrinks to zero over time (compared to the $A\Gamma^n$ term which is growing steadily by the factor Γ). Thus, $\lim_{n\to\infty} A_{t+n+1}/A_{t+n} = \Gamma$.

This logic unfortunately does not go through when there are permanent shocks, because the $\mathcal{R}_{t+1,i}$ terms are not independent of the permanent income shocks.

To see the problem clearly, define $\check{\mathcal{R}} = \mathbb{M}[\mathcal{R}_{t+1,i}]$ and consider a first order Taylor expansion of $\acute{\mathrm{a}}(m_{t+1,i})$ around $\check{m}_{t+1,i} = \check{\mathcal{R}}a_{t,i} + 1$,

$$\acute{\mathrm{a}}_{t+1,i} \approx \acute{\mathrm{a}}(\check{m}_{t+1,i}) + \acute{\mathrm{a}}'(\check{m}_{t+1,i})\,(m_{t+1,i} - \check{m}_{t+1,i})\,.$$

The problem comes from the $\acute{\mathrm{a}}'$ term. The concavity of the consumption function implies convexity of the a function, so this term is strictly positive but we have no theory to place bounds on its size as we do for its level $\acute{\mathrm{a}}$. We cannot rule out by theory that a positive shock to permanent income (which has a negative effect on $m_{t+1,i}$) could have an unboundedly positive effect on $\acute{\mathrm{a}}'$ (as for instance if it pushes the consumer arbitrarily close to the self-imposed liquidity constraint).

F Endogenous Gridpoints Solution Method

The model is solved using an extension of the method of endogenous gridpoints (Carroll (2016)): A grid of possible values of end-of-period assets $\vec{a}$ is defined (`aVec` in the software), and at these points, marginal end-of-period-t value is computed as the discounted next-period expected marginal utility of consumption (which the Envelope theorem says matches expected marginal value). The results are then used to identify the corresponding levels of consumption at the beginning of the period:[43]

$$\begin{aligned} \mathrm{u}'(\mathfrak{c}_t(\vec{a})) &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}'(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a} + \xi_{t+1}))] \qquad (97) \\ \vec{c}_t \equiv \mathfrak{c}_t(\vec{a}) &= \left(\mathsf{R}\beta\,\mathbb{E}_t\left[(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a} + \xi_{t+1}))^{-\rho}\right]\right)^{-1/\rho}. \end{aligned}$$

The dynamic budget constraint can then be used to generate the corresponding m's:

$$\vec{m}_t = \vec{a} + \vec{c}_t.$$

An approximation to the consumption function could be constructed by linear interpolation between the $\{\vec{m}, \vec{c}\}$ points. But a vastly more accurate approximation can be made (for a given number of gridpoints) if the interpolation is constructed so that it also matches the marginal propensity to consume at the gridpoints. Differentiating (97) with respect to a (and dropping policy function arguments for simplicity) yields a marginal propensity to *have consumed* $\mathfrak{c}^a$ at each gridpoint:

$$\mathrm{u}''(\mathfrak{c}_t)\mathfrak{c}_t^a = \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\Gamma_{t+1}\mathrm{c}_{t+1}^m\mathcal{R}_{t+1}]$$

[43] The software can also solve a version of the model with explicit liquidity constraints, where the Envelope condition does not hold.

$$
\begin{aligned}
&= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}^m_{t+1}] \\
\mathfrak{c}^a_t &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}^m_{t+1}]/\mathrm{u}''(\mathfrak{c}_t)
\end{aligned} \tag{98}
$$

and the marginal propensity to consume at the beginning of the period is obtained from the marginal propensity to have consumed by noting that, if we define $\mathfrak{m}(a) = \mathfrak{c}(a) - a$,

$$
\begin{aligned}
c &= \mathfrak{m} - a \\
\mathfrak{c}^a + 1 &= \mathfrak{m}^a
\end{aligned}
$$

which, together with the chain rule $\mathfrak{c}^a = \mathrm{c}^m \mathfrak{m}^a$, yields the MPC from

$$
\begin{aligned}
\mathrm{c}^m(\mathfrak{c}^a + 1) &= \mathfrak{c}^a \\
\mathrm{c}^m &= \mathfrak{c}^a/(1+\mathfrak{c}^a)
\end{aligned}
$$

and we call the vector of MPC's at the $\vec{m}_t$ gridpoints $\vec{\kappa}_t$.

G The Terminal/Limiting Consumption Function

For any set of parameter values that satisfy the conditions required for convergence, the problem can be solved by setting the terminal consumption function to $\mathrm{c}_T(m) = m$ and constructing $\{\mathrm{c}_{T-1}, \mathrm{c}_{T-2}, ...\}$ by time iteration (a method that will converge to $\mathrm{c}(m)$ by standard theorems). But $\mathrm{c}_T(m) = m$ is very far from the final converged consumption rule $\mathrm{c}(m)$,[44] and thus many periods of iteration will likely be required to obtain a candidate rule that even remotely resembles the converged function.

A natural alternative choice for the terminal consumption rule is the solution to the perfect foresight liquidity constrained problem, to which the model's solution converges (under specified parametric restrictions) as all forms of uncertainty approach zero (as discussed in the main text). But a difficulty with this idea is that the perfect foresight liquidity constrained solution is 'kinked:' The slope of the consumption function changes discretely at the points $\{m^1_{\#}, m^2_{\#}, ...\}$. This is a practical problem because it rules out the use of derivatives of the consumption function in the approximate representation of $\mathrm{c}(m)$, thereby preventing the enormous increase in efficiency obtainable from a higher-order approximation.

Our solution is simple: The formulae in appendix A that identify kink points on $\mathring{\mathrm{c}}(m)$ for integer values of n (e.g., $c^n_{\#} = \text{\textbf{Þ}}_{\Gamma}^{-n}$) are continuous functions of n; the conclusion that $\mathring{\mathrm{c}}(m)$ is piecewise linear between the kink points does not require that the *terminal consumption rule* (from which time iteration proceeds) also be piecewise linear. Thus, for values $n \geq 0$ we can construct a smooth function $\check{\mathrm{c}}(m)$ that matches the true perfect foresight liquidity constrained consumption function at the set of points corresponding to integer periods in the future, but satisfies the (continuous, and greater at non-kink

[44]Unless $\beta \approx +0$.

points) consumption rule defined from the appendix's formulas by noninteger values of n at other points.[45]

This strategy generates a smooth limiting consumption function – except at the remaining kink point defined by $\{m^0_\#, c^0_\#\}$. Below this point, the solution must match $\mathrm{c}(m) = m$ because the constraint is binding. At $m = m^0_\#$ the MPC discretely drops (that is, $\lim_{m \uparrow m^0_\#} \mathrm{c}'(m) = 1$ while $\lim_{m \downarrow m^0_\#} \mathrm{c}'(m) = \kappa^0_\# < 1$).

Such a kink point causes substantial problems for numerical solution methods (like the one we use, described below) that rely upon the smoothness of the limiting consumption function.

Our solution is to use, as the terminal consumption rule, a function that is identical to the (smooth) continuous consumption rule $\check{\mathrm{c}}(m)$ above some $n \geq \underline{n}$, but to replace $\check{\mathrm{c}}(m)$ between $m^0_\#$ and $m^{\underline{n}}_\#$ with the unique polynomial function $\hat{\mathrm{c}}(m)$ that satisfies the following criteria:

1. $\hat{\mathrm{c}}(m^0_\#) = c^0_\#$
2. $\hat{\mathrm{c}}'(m^0_\#) = 1$
3. $\hat{\mathrm{c}}'(m^{\underline{n}}_\#) = (d\mathrm{c}^n_\#/dn)(d\mathrm{m}^n_\#/dn)^{-1}|_{n=\underline{n}}$
4. $\hat{\mathrm{c}}''(m^{\underline{n}}_\#) = (d^2\mathrm{c}^n_\#/dn^2)(d^2\mathrm{m}^n_\#/dn^2)^{-1}|_{n=\underline{n}}$

where $\underline{n}$ is chosen judgmentally in a way calculated to generate a good compromise between smoothness of the limiting consumption function $\check{\mathrm{c}}(m)$ and fidelity of that function to the $\mathring{\mathrm{c}}(m)$ (see the actual code for details).

We thus define the terminal function as[46]

$$\mathrm{c}_T(m) = \begin{cases} 0 < m \leq m^0_\# & m \\ m^0_\# < m < m^{\underline{n}}_\# & \check{\mathrm{c}}(m) \\ m^{\underline{n}}_\# < m & \mathring{\mathrm{c}}(m) \end{cases} \tag{99}$$

Since the precautionary motive implies that in the presence of uncertainty the optimal level of consumption is below the level that is optimal without uncertainty, and since $\check{\mathrm{c}}(m) \geq \mathring{\mathrm{c}}(m)$, implicitly defining $m = e^\mu$ (so that $\mu = \log m$), we can construct

$$\chi_t(\mu) \quad = \quad \log(1 - \mathrm{c}_t(e^\mu)/\mathrm{c}_T(e^\mu)) \tag{100}$$

which must be a number between $-\infty$ and $+\infty$ (since $0 < \mathrm{c}_t(m) < \check{\mathrm{c}}(m)$ for $m > 0$). This function turns out to be much better behaved (as a numerical observation; no formal proof is offered) than the level of the optimal consumption rule $\mathrm{c}_t(m)$. In particular, $\chi_t(\mu)$ is well approximated by linear functions both as $m \downarrow 0$ and as $m \uparrow \infty$.

Differentiating with respect to μ and dropping consumption function arguments yields

$$\chi'_t(\mu) \quad = \quad \left(\frac{-\left(\frac{\mathrm{c}'_t\mathrm{c}_T - \mathrm{c}_t\mathrm{c}'_T}{\mathrm{c}_T^2} e^\mu\right)}{1 - \mathrm{c}_t/\mathrm{c}_T} \right) \tag{101}$$

[45]In practice, we calculate the first and second derivatives of $\check{\mathrm{c}}$ and use piecewise polynomial approximation methods that match the function at these points.

[46]For further details see the archive file `./CoreCode/Documentation/cTerminal.nb`.

which can be solved for

$$\mathrm{c}_t' \;=\; (\mathrm{c}_t \mathrm{c}_T' / \mathrm{c}_T) - ((\mathrm{c}_T - \mathrm{c}_t)/m)\chi_t'. \tag{102}$$

Similarly, we can solve (100) for

$$\mathrm{c}_t(m) \;=\; \left(1 - e^{\chi_t(\log m)}\right) \mathrm{c}_T(m). \tag{103}$$

Thus, having approximated χ_t, we can recover from it the level and derivative(s) of c_t.[47]

[47] See `./CoreCode/Documentation/Derivations.nb` for implementation details and further derivations.

Table 4 Taxonomy of Liquidity Constrained Model Outcomes

Name	Condition			Outcome/Comments
$\cancel{\text{PF-GIC}}$		$1 <$	$\text{\thorn}/\Gamma$	Constraint never binds for $m \geq 1$
RIC	$\text{\thorn}/\mathsf{R}$	< 1		FHWC holds ($\mathsf{R} > \Gamma$)
				$\mathring{\mathrm{c}}(m) = \bar{\mathrm{c}}(m)$ for $m \geq 1$
$\cancel{\text{RIC}}$		$1 <$	$\text{\thorn}/\mathsf{R}$	$\mathring{\mathrm{c}}(m)$ is degenerate
PF-GIC	$\text{\thorn}/\Gamma$	< 1		Constraint binds in finite time for any m
RIC	$\text{\thorn}/\mathsf{R}$	< 1		FHWC may or may not hold
				$\lim_{m\uparrow\infty} \bar{\mathrm{c}}(m) - \mathring{\mathrm{c}}(m) = 0$
				$\lim_{m\uparrow\infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
$\cancel{\text{RIC}}$		$1 <$	$\text{\thorn}/\mathsf{R}$	$\cancel{\text{FHWC}}$
				$\lim_{m\uparrow\infty} \mathring{\boldsymbol{\kappa}}(m) = 0$

Conditions are applied from left to right; for example, the second and third rows indicate conclusions in the case where ~~PF-GIC~~ and RIC both hold, while the fourth row indicates that when the PF-GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the PF-GIC holds, the constraint will bind in finite time.

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