

1 Convergence in Euclidian Space

1.1 Convergence of v_t

Boyd's theorem shows that \mathcal{T} defines a contraction mapping in an F -bounded space. We now show that \mathcal{T} also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator \mathcal{T} , since $v^*(m) = \mathcal{T}v^*(m)$,

$$\|v_{T-n+1} - v^*\|_F \leq \alpha^{n-1} \|v_T - v^*\|_F. \quad (1)$$

On the other hand, $v_T - v^* \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$ and $\kappa = \|v_T - v^*\|_F < \infty$ because v_T and v^* are in $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$. It follows that

$$|v_{T-n+1}(m) - v^*(m)| \leq \kappa \alpha^{n-1} |F(m)|. \quad (2)$$

Then we obtain

$$\lim_{n \rightarrow \infty} v_{T-n+1}(m) = v^*(m). \quad (3)$$

Since $v_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $v_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < v_T(m)$. On the other hand, $v_{T-1} \leq v_T$ means $\mathcal{T}v_{T-1} \leq \mathcal{T}v_T$, in other words, $v_{T-2}(m) \leq v_{T-1}(m)$. Inductively one gets $v_{T-n}(m) \geq v_{T-n-1}(m)$. This means that $\{v_{T-n+1}(m)\}_{n=1}^\infty$ is a decreasing sequence, bounded below by v^* .

1.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{c_{T-n+1}(m)\}_{n=1}^\infty$.

Consider any convergent subsequence $\{c_{T-n(i)}(m)\}$ of $\{c_{T-n+1}(m)\}_{n=1}^\infty$ converging to c^* . By the definition of $c_{T-n}(m)$, we have

$$\begin{aligned} u(c_{T-n(i)}(m)) + \beta \mathbb{E}_{T-n(i)}[\mathcal{G}_{T-n(i)+1}^{1-\rho} v_{T-n(i)+1}(m)] \\ \geq u(c_{T-n(i)}) + \beta \mathbb{E}_{T-n(i)}[\mathcal{G}_{T-n(i)+1}^{1-\rho} v_{T-n(i)+1}(m)], \end{aligned} \quad (4)$$

for any $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$. Now letting $n(i)$ go to infinity, it follows that the left hand side converges to $u(c^*) + \beta \mathbb{E}_t[\mathcal{G}_t^{1-\rho} v(m)]$, and the right hand side converges to $u(c_{T-n(i)}) + \beta \mathbb{E}_t[\mathcal{G}_t^{1-\rho} v(m)]$. So the limit of the preceding inequality as $n(i)$ approaches infinity implies

$$u(c^*) + \beta \mathbb{E}_t[\mathcal{G}_{t+1}^{1-\rho} v(m)] \geq u(c_{T-n(i)}) + \beta \mathbb{E}_t[\mathcal{G}_{t+1}^{1-\rho} v(m)]. \quad (5)$$

Hence, $c^* \in \arg \max_{c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]} \{u(c_{T-n(i)}) + \beta \mathbb{E}_t[\mathcal{G}_{t+1}^{1-\rho} v(m)]\}$. By the uniqueness of $c(m)$, $c^* = c(m)$.