1 Unique and Stable Target and Steady State Points

This appendix proves Theorems **??**-**??** and:

Lemma 1. *If $\check{m}$ and $\hat{m}$ both exist, then $\check{m} \leq \hat{m}$.*

1.1 Proof of Theorem **??**

Theorem 2. *For the nondegenerate solution to the problem defined in Section 2.1 when FVAC, WRIC, and GIC-Mod all hold, there exists a unique cash-on-hand-to-permanent-income ratio $\hat{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \hat{m}. \tag{1}$$

Moreover, $\hat{m}$ is a point of 'stability' in the sense that

$$\begin{aligned} \forall m_t \in (0, \hat{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\ \forall m_t \in (\hat{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t. \end{aligned} \tag{2}$$

The elements of the proof of Theorem **??** are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

1.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed sufficient conditions (the WRIC and FVAC; Theorem 1).

Section 2.8 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \boldsymbol{\xi}_t$, even if $\boldsymbol{\xi}_t$ takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

This follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

1.3.1 Existence of m where $\mathbb{E}_t[m_{t+1}/m_t] < 1$

If RIC holds. Logic exactly parallel to that of Section 3.1 leading to equation (39), but dropping the $\mathcal{G}_{t+1}$ from the RHS, establishes that

$$\begin{aligned}\lim_{m_t\uparrow\infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1}}{m_t}\right] \\ &= \mathbb{E}_t[(\mathsf{R}/\mathcal{G}_{t+1})\mathbf{Þ}_{\mathsf{R}}] \\ &= \mathbb{E}_t[\mathbf{Þ}/\mathcal{G}_{t+1}] \\ &< 1\end{aligned} \tag{3}$$

where the inequality reflects imposition of the GIC-Mod (26).

If RIC fails. When the RIC fails, the fact that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$ (see equation (30)) means that the limit of the RHS of (3) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Mod and ~~RIC~~ implies $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m\uparrow\infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

1.3.2 Existence of $m > 1$ where $\mathbb{E}_t[m_{t+1}/m_t] > 1$

Paralleling the logic for c in Section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t\downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\ \boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\ \boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,\end{aligned} \tag{4}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)\left(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1} - m_t\right)\right] \\ &= \bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) - 1.\end{aligned} \tag{5}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (16) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) - 1 < \bar{\mathcal{R}}(1 - \underbrace{(1 - \mathbf{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1$$

$$
\begin{aligned}
&= \bar{\mathcal{R}}\text{\textbf{Þ}}_{\mathsf{R}} - 1 \\
&= \mathbb{E}_t\left[\frac{\mathsf{R}}{\mathcal{G}\boldsymbol{\Psi}}\frac{\text{\textbf{Þ}}}{\mathsf{R}}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\text{\textbf{Þ}}}{\mathcal{G}\boldsymbol{\Psi}}\right]}_{=\text{\textbf{Þ}}_{\underline{\mathcal{G}}}} - 1
\end{aligned}
$$

which is negative because the GIC-Mod says $\text{\textbf{Þ}}_{\underline{\mathcal{G}}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$
\bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) < \bar{\mathcal{R}}
$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (5) is guaranteed to be negative if

$$
\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\mathcal{G}\boldsymbol{\Psi}}\right] < 1. \tag{6}
$$

But the combination of the GIC-Mod holding and the RIC failing can be written:

$$
\overbrace{\mathbb{E}_t\left[\frac{\text{\textbf{Þ}}}{\mathcal{G}\boldsymbol{\Psi}}\right]}^{\text{\textbf{Þ}}_{\underline{\mathcal{G}}}} < 1 < \overbrace{\frac{\text{\textbf{Þ}}}{\mathsf{R}}}^{\text{\textbf{Þ}}_{\mathsf{R}}},
$$

and multiplying all three elements by $\mathsf{R}/\text{\textbf{Þ}}$ gives

$$
\mathbb{E}_t\left[\frac{\mathsf{R}}{\mathcal{G}\boldsymbol{\Psi}}\right] < \mathsf{R}/\text{\textbf{Þ}} < 1
$$

which satisfies our requirement in (6).

1.4 Proof of Theorem ??

Theorem 3. *For the nondegenerate solution to the problem defined in Section 2.1 when FVAC, WRIC, and GIC all hold, there exists a unique pseudo-steady-state cash-on-hand-to-income ratio* $\check{m} > 0$ *such that*

$$
\mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t] = 1 \textit{ if } m_t = \check{m}. \tag{7}
$$

Moreover, $\check{m}$ *is a point of stability in the sense that*

$$
\begin{aligned}
\forall m_t \in (0, \check{m}),\ \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t > \mathcal{G} \\
\forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t < \mathcal{G}.
\end{aligned} \tag{8}
$$

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1} - m_t]$ is monotonically decreasing

1.4.1 Existence and Continuity of the Ratio

Since by assumption $0 < \underline{\boldsymbol{\Psi}} \le \boldsymbol{\Psi}_{t+1} \le \bar{\boldsymbol{\Psi}} < \infty$, our proof in 1.2 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t]$.

1.4.2 Existence of a stable point

Since by assumption $0 < \underline{\boldsymbol{\Psi}} \le \boldsymbol{\Psi}_{t+1} \le \bar{\boldsymbol{\Psi}} < \infty$, our proof in Subsection 1.2 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathcal{G}_{t+1}\left((\mathsf{R}/\mathcal{G}_{t+1})\mathrm{a}(m_t) + \boldsymbol{\xi}_{t+1} \right)/\mathcal{G}}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{(\mathsf{R}/\mathcal{G})\mathrm{a}(m_t) + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \left[\frac{(\mathsf{R}/\mathcal{G})\mathrm{a}(m_t) + 1}{m_t} \right] \\
&= (\mathsf{R}/\mathcal{G})\boldsymbol{\text{Þ}}_{\mathsf{R}} \qquad (9) \\
&= \boldsymbol{\text{Þ}}_{\mathcal{G}} \\
&< 1
\end{aligned}$$

where the last two lines are merely a restatement of the GIC (19).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.4.3 $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}] - m_t$ is monotonically decreasing.

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t] = 1 \qquad (10) \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1} m_{t+1}/m_t] > 1,
\end{aligned}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)\left(\mathcal{R}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - m_t\right)\right] \qquad (11) \\
&= (\mathsf{R}/\mathcal{G})\left(1 - \mathrm{c}'(m_t)\right) - 1.
\end{aligned}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (16) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned} \mathcal{R}\,(1 - \mathrm{c}'(m_t)) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \text{\textbf{Þ}}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\ &= (\mathsf{R}/\mathcal{G})\text{\textbf{Þ}}_{\mathsf{R}} - 1 \end{aligned}$$

which is negative because the GIC says $\text{\textbf{Þ}}_{\mathcal{G}} < 1$.