# Handout on Merton (1969)-Samuelson (1969) Model (Consumption Out of Risky Assets)

This handout presents the solution to the optimal consumption/saving problem when there is uncertainty about the rate of return on savings, and the consumer receives no labor income.

Thus, the only deviation from the standard structure of the optimization problem is in the DBC, which now does not contain a term for labor income:

$$x_{t+1} = R_{t+1}[x_t - c_t]. \tag{1}$$

Start with the standard Euler equation for consumption:

$$1 = \beta E_t \left[ \tilde{R}_{t+1} \left( \frac{c_{t+1}}{c_t} \right)^{-\rho} \right] \tag{2}$$

Postulate a solution of the form $c_t = \gamma x_t$.

$$1 = \beta E_t \left[ \tilde{R}_{t+1} \left( \frac{\gamma x_{t+1}}{\gamma x_t} \right)^{-\rho} \right] \tag{3}$$

$$= \beta E_t \left[ \tilde{R}_{t+1} \left( \frac{\tilde{R}_{t+1}[x_t - c_t]}{x_t} \right)^{-\rho} \right] \tag{4}$$

$$= \beta E_t \left[ \tilde{R}_{t+1} \left( \frac{\tilde{R}_{t+1}(1-\gamma)x_t}{x_t} \right)^{-\rho} \right] \tag{5}$$

$$= \beta E_t \left[ \tilde{R}_{t+1} \left( \tilde{R}_{t+1}(1-\gamma) \right)^{-\rho} \right] \tag{6}$$

$$= \beta E_t \left[ \tilde{R}_{t+1}^{1-\rho}(1-\gamma)^{-\rho} \right] \tag{7}$$

$$(1-\gamma)^\rho = \beta E_t[\tilde{R}_{t+1}^{1-\rho}] \tag{8}$$

$$(1-\gamma) = \left[ \beta E_t[\tilde{R}_{t+1}^{1-\rho}] \right]^{1/\rho} \tag{9}$$

$$\gamma = 1 - \left[ \beta E_t[\tilde{R}_{t+1}^{1-\rho}] \right]^{1/\rho} \tag{10}$$

Before we proceed we need the following fact:

**Fact 1** *If* $\log Z \sim N(\mu, \sigma^2)$ *then* $\log E[Z] = E[\log Z] + \sigma^2/2 = \mu + \sigma^2/2$.

Now assume $\log \tilde{R}_{t+1} \sim N(\bar{r} - \sigma_r^2/2, \sigma_r^2)$, which implies that

$$\log E_t[\tilde{R}_{t+1}] = E_t[\log \tilde{R}_{t+1}] + \sigma_r^2/2 \tag{11}$$

$$= \bar{r} - \sigma_r^2/2 + \sigma_r^2/2 \tag{12}$$

$$= \bar{r} \tag{13}$$

Since $\log \tilde{R}_{t+1}^{1-\rho} = (1-\rho)\log \tilde{R}_{t+1}$, this implies that

$$\log E_t[\tilde{R}_{t+1}^{1-\rho}] = (1-\rho)(\bar{r} - \sigma_r^2/2) + \left[\frac{(1-\rho)^2}{2}\right]\sigma_r^2 \tag{14}$$

$$= (1-\rho)\bar{r} - (1-\rho)\sigma_r^2/2 + \left[\frac{(1-\rho)}{2}\right]\sigma_r^2 - \frac{\rho(1-\rho)}{2}\sigma_r^2 \tag{15}$$

$$E_t[\tilde{R}_{t+1}^{1-\rho}] = \exp[(1-\rho)\bar{r} - \rho(1-\rho)\sigma_r^2/2] \tag{16}$$

Substituting in (10):

$$\gamma = 1 - \beta^{1/\rho}\exp[(1/\rho - 1)\bar{r} - (1-\rho)\sigma_r^2/2] \tag{17}$$

Now use the following approximations:

$$\beta^{1/\rho} = \left(\frac{1}{1+\delta}\right)^{1/\rho} \tag{18}$$

$$\approx 1 - (1/\rho)\delta \tag{19}$$

$$\approx \exp(-(1/\rho)\delta) \tag{20}$$

$$\exp(\eta) \approx 1 + \eta \tag{21}$$

which hold if $\eta$ and $(1/\rho)\delta$ are close to zero. Substituting these into (17) gives

$$\gamma \approx 1 - (1 + (1/\rho)(\bar{r} - \delta) + (\rho - 1)\sigma_r^2/2) \tag{22}$$

$$= \bar{r} - (1/\rho)(\bar{r} - \delta) - \left(\frac{\rho - 1}{2}\right)\sigma_r^2. \tag{23}$$

Note that this equation implies the plausible result that as uncertainty goes up ($\sigma_r^2$ rises) the level of consumption falls (because $\rho > 1$), reflecting the precautionary saving motive.

# References

MERTON, ROBERT C. (1969): "Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case," *Review of Economics and Statistics*, 50, 247–257.

SAMUELSON, PAUL A. (1969): "Lifetime Portfolio Selection By Dynamic Stochastic Programming," *Review of Economics and Statistics*, 51, 239–46.