The Lucas Asset Pricing Model

1 Introduction/Setup

(See Lucas-Asset-Price-Model for a notebook that solves the model numerically).

Lucas [1978] considers an economy populated by infinitely many[1] identical individual consumers, in which the only assets are a set of identical infinitely-lived trees. Aggregate output is the fruit that falls from the trees, and cannot be stored (it would rot!); because $u'(c) > 0 \ \forall \ c > 0$, the fruit is all eaten:

$$c_t L_t = d_t K_t \tag{1}$$

where c_t is consumption of fruit per person, L_t is the population, K_t measures the stock of trees, and d_t is the exogenous output of fruit that *d*rops from each tree. In a given year, each tree produces exactly the same amount of fruit as every other tree, but d_t varies from year to year depending on the weather.[2]

An economy like this, in which output arrives without any deliberate actions on the part of residents, is called an 'endowment' economy (or, sometimes, an 'exchange' economy).[3]

2 The Market for Trees

If there is a perfect capital market for trees, the price of trees P_t must be such that, each period, each (identical) consumer does not want either to increase or to decrease their holding of trees.[4]

If a tree is sold, the sale is assumed to occur after the existing owner receives that period's fruit (P_t is the 'ex-dividend' price). Total resources available to consumer i in period t are the sum of the fruit received from the trees owned, $d_t k_t^i$, plus the potential proceeds if the consumer were to sell all his stock of trees, $\mathsf{P}_t k_t^i$. Total resources are divided into two uses: Current consumption c_t^i and the purchase of trees for next period

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[1] As in `Aggregation`.

[2] d_t must always be strictly positive because $u(0) = -\infty$ and $u'(0) = \infty$.

[3] The alternative is a 'production' economy, in which factors of production – labor, capital, maybe land, maybe knowledge – combine somehow to generate the output. There is no consensus about the best way to construct production economies with asset pricing.

[4] If, at a hypothesized equilibrium price, every identical consumer wanted (say) to increase their holdings, that price could not be an equilibrium price, because with a fixed supply of trees everyone cannot increase their holding of trees at once!

k_{t+1}^i at price P_t,

$$\overbrace{k_{t+1}^i \mathsf{P}_t + c_t^i}^{\text{Uses of resources}} = \overbrace{d_t k_t^i + \mathsf{P}_t k_t^i}^{\text{Total resources}} \tag{2}$$
$$k_{t+1}^i = (1 + d_t/\mathsf{P}_t)k_t^i - c_t^i/\mathsf{P}_t.$$

3 The Problem of an Individual Consumer

Consumer i maximizes

$$v(m_t^i) = \max \ \mathbb{E}_t^i \left[\sum_{n=0}^{\infty} \beta^n u(c_{t+n}^i) \right]$$

$$\text{s.t.}$$
$$k_{t+1}^i = (1 + d_t/\mathsf{P}_t)k_t^i - c_t^i/\mathsf{P}_t$$
$$m_{t+1}^i = (\mathsf{P}_{t+1} + d_{t+1})k_{t+1}^i.$$

Rewriting in the form of Bellman's equation,

$$v(m_t^i) = \max_{\{c_t^i\}} \ u(c_t^i) + \beta \, \mathbb{E}_t^i \left[v(m_{t+1}^i) \right],$$

the first order condition tells us that

$$0 = u'(c_t^i) + \beta \, \mathbb{E}_t^i \left[v'(m_{t+1}^i) \frac{\partial}{\partial c_t^i} \left(\overbrace{(\mathsf{P}_{t+1} + d_{t+1}) \underbrace{\left((1 + d_t/\mathsf{P}_t)k_t^i - c_t^i/\mathsf{P}_t\right)}_{k_{t+1}^i}}^{m_{t+1}^i} \right) \right] \tag{3}$$

so

$$u'(c_t^i) = \beta \, \mathbb{E}_t^i \left[v'(m_{t+1}^i) \left(\underbrace{\frac{\mathsf{P}_{t+1} + d_{t+1}}{\mathsf{P}_t}}_{\equiv \mathbf{R}_{t+1}} \right) \right] \tag{4}$$
$$= \beta \, \mathbb{E}_t^i \left[\mathbf{R}_{t+1} v'(m_{t+1}^i) \right]$$

where $\mathbf{R}_{t+1}$ is the return factor that measures the resources in period $t+1$ that are the reward for owning a unit of trees at the end of t.

The `Envelope` theorem tells us that $v'(m_{t+1}^i) = u'(c_{t+1}^i)$, so (4) becomes

$$u'(c_t^i) = \beta \, \mathbb{E}_t^i \left[u'(c_{t+1}^i) \left(\frac{\mathsf{P}_{t+1} + d_{t+1}}{\mathsf{P}_t} \right) \right]$$
$$\mathsf{P}_t = \beta \, \mathbb{E}_t^i \left[\left(\frac{u'(c_{t+1}^i)}{u'(c_t^i)} \right) (\mathsf{P}_{t+1} + d_{t+1}) \right]. \tag{5}$$

4 Aggregation

Since all consumers are identical, $c_t^i = c_t^j \; \forall \; i, j$, so henceforth we just call consumption per capita c_t. Since aggregate consumption must equal aggregate production because fruit cannot be stored, normalizing the population to $L_t = 1 \; \forall \; t$ and the stock of trees to $K_t = 1 \; \forall \; t$, equation (1) becomes:

$$c_t = d_t. \tag{6}$$

Substituting c_t and c_{t+1} for c_t^i and c_{t+1}^i in (5) and then substituting d_t for c_t we get

$$\mathsf{P}_t = \beta \, \mathbb{E}_t \left[\left(\frac{u'(d_{t+1})}{u'(d_t)} \right) (\mathsf{P}_{t+1} + d_{t+1}) \right]. \tag{7}$$

We can rewrite this more simply if we define

$$M_{t,t+n} = \beta^n \left(\frac{u'(d_{t+n})}{u'(d_t)} \right) \tag{8}$$

$M_{t,t+n}$ is called the 'stochastic discount factor' because (a) it is stochastic (thanks to the shocks between t and $t+n$ that determine the value of d_{t+n}); and (b) it measures the rate at which all agents in this economy in period t will discount a dividend received in a future period, e.g. $t+1$:

$$\mathsf{P}_t = \mathbb{E}_t \left[M_{t,t+1} (\mathsf{P}_{t+1} + d_{t+1}) \right]. \tag{9}$$

A corresponding equation will hold in period $t+1$ (and in period $t+2$ and beyond):

$$\mathsf{P}_{t+1} = \mathbb{E}_{t+1} \left[M_{t+1,t+2} (\mathsf{P}_{t+2} + d_{t+2}) \right] \tag{10}$$

so we can use repeated substitution, e.g. of (10) into (9), and so on to get

$$\mathsf{P}_t = \mathbb{E}_t \left[M_{t,t+1} d_{t+1} \right] + \mathbb{E}_t [M_{t,t+1} \, \mathbb{E}_{t+1} [M_{t+1,t+2} d_{t+2}]] + \dots . \tag{11}$$

The 'law of iterated expectations' says that $\mathbb{E}_t[\mathbb{E}_{t+1}[\mathsf{P}_{t+2}]] = \mathbb{E}_t[\mathsf{P}_{t+2}]$; given this, and noting that $M_{t,t+2} = M_{t,t+1} M_{t+1,t+2}$, (11) becomes:

$$\mathsf{P}_t = \mathbb{E}_t \left[M_{t,t+1} d_{t+1} + M_{t,t+2} d_{t+2} + M_{t,t+3} d_{t+3} + \dots \right]. \tag{12}$$

So, the price of the asset is the present discounted value of the stream of future 'dividends,' where the stochastic factor by which (potentially stochastic) dividends received in $t+n$ are discounted back to t is $M_{t,t+n}$.[5]

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[5]The assumption that all consumers are identical here is important; heterogeneity in much of anything (wealth, income processes, along with an assumption that each consumer has only a finite amount of wealth and a finite horizon, will destroy the implication that there is a single unique SDF that all agents use to discount the future).

5 Specializing the Model

This is as far as we can go without making explicit assumptions about the structure of utility. If utility is CRRA, $u(c) = (1 - \rho)^{-1} c^{1-\rho}$, substituting $u'(d) = d^{-\rho}$ into (7) yields

$$\mathsf{P}_t = \beta d_t^{\rho} \, \mathbb{E}_t \left[d_{t+1}^{-\rho} (\mathsf{P}_{t+1} + d_{t+1}) \right]$$

$$\frac{\mathsf{P}_t}{d_t^{\rho}} = \beta \, \mathbb{E}_t \left[\frac{\mathsf{P}_{t+1}}{d_{t+1}^{\rho}} + d_{t+1}^{1-\rho} \right] \tag{13}$$

The particularly special case of logarithmic utility (which Lucas emphasizes) corresponds to $\rho = 1$, implying $d_{t+1}^{1-\rho} = 1$ which (again using the law of iterated expectations) allows us to simplify the second version of (13) to

$$\frac{\mathsf{P}_t}{d_t} = \beta \left(1 + \mathbb{E}_t \left[\frac{\mathsf{P}_{t+1}}{d_{t+1}} \right] \right)$$

$$= \beta \left(1 + \beta \left(1 + \mathbb{E}_t \left[\frac{\mathsf{P}_{t+2}}{d_{t+2}} \right] \right) \right)$$

$$= \frac{\beta}{1 - \beta} + \beta \, \mathbb{E}_t \left\{ \lim_{n \to \infty} \beta^{n-1} \left[\frac{\mathsf{P}_{t+n}}{d_{t+n}} \right] \right\}.$$

If the price is bounded (it cannot ever go, for example, to a value such that it would cost more than the economy's entire output to buy a single tree), it is possible to show that the lim term in this equation goes to zero. Using the usual definition of the time preference factor as $\beta = 1/(1 + \vartheta)$ where ϑ is the time preference rate, the equilibrium price is:

$$\mathsf{P}_t = d_t \left(\frac{\beta}{1 - \beta} \right)$$

$$= d_t \left(\frac{1}{1/\beta - 1} \right)$$

$$= d_t \left(\frac{1}{1 + \vartheta - 1} \right)$$

$$= \frac{d_t}{\vartheta}$$

or, equivalently, the 'dividend-price ratio' is always $d_t / \mathsf{P}_t = \vartheta$.

It may surprise you that the equilibrium price of trees today does not depend on the expected level of fruit output in the future. If the weather was bad this year, but is expected to return to normal next year (and, by definition, is expected to be equal to normal in subsequent years), you might think that the price today would mostly reflect the 'normal' value of fruit prodution that the trees produce, not the (temporarily low) value that happens to obtain today.

The above derivation says that intuition is wrong: Today's price depends only on today's output.

Nevertheless, the logic (higher future output is a reason for higher current prices) is

not wrong; but it is (exactly) counterbalanced by another, and subtler, fact: Since future consumption will equal future fruit output, higher expected fruit output means lower marginal utility of consumption in that future period of (more) abundant fruit (basically, people are starving today, which reduces the attractiveness of cutting their consumption to buy trees that will produce more in a period when they expect *not* to be starving). These two forces are the manifestation of the (pure) income effect and substitution effect in this model (there is no human wealth, and therefore no human wealth effect). In our assumed special case of logarithmic utility, income and substitution effects are of the same size and opposite sign so the two forces exactly offset.

6 The 'Rate of Return' in a Lucas Model

We can decompose the return factor attributable to ownership of a share of capital (cf. (4)) by adding and subtracting P_t in the numerator:

$$\boldsymbol{r}_{t+1} = \left(\frac{\mathsf{P}_{t+1} + \mathsf{P}_t - \mathsf{P}_t + d_{t+1}}{\mathsf{P}_t} \right)$$

$$= \left(1 + \frac{\Delta \mathsf{P}_{t+1}}{\mathsf{P}_t} + \frac{d_{t+1}}{\mathsf{P}_t} \right)$$

so the 'rate of return' is

$$\boldsymbol{r}_{t+1} = \frac{\Delta \mathsf{P}_{t+1}}{\mathsf{P}_t} + \frac{d_{t+1}}{\mathsf{P}_t} \tag{14}$$

which is a useful decomposition because the two components have natural interpretations: The first is a 'capital gain' (or loss), and the second can plausibly identified as 'the interest rate' paid by the asset (because it corresponds to income received regardless of whether the asset is liquidated).

In models that do not explicitly discuss asset pricing, the implicit assumption is usually that the price of capital is constant (which might be plausible if capital consists mostly of reproducible items like machines,[6] rather than Lucas trees). In this case

$$\mathbf{R}_{t+1} = \left(1 + \frac{d_{t+1}}{\mathsf{P}_t} \right)$$

says that the only risk in the rate of return is attributable to unpredictable variation in the size of dividend/interest payments. Indeed, if additional assumptions are made (e.g., perfect capital markets) that yield the conclusion that the interest rate matches the marginal product of capital, then such models generally imply that variation in returns (at least at high frequencies) is very small, because aggregate capital typically is very stable from one period to the next; if the aggregate production function is stable, this implies great stability in the marginal product of capital.

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[6]The key insights below remain true even if there is a gradual trend in the real price of capital goods, as has in fact been true.

7 Aggregate Returns Versus Individual Returns

One of the subtler entries in Aristotle [350 BC]'s catalog of common human reasoning errors was the 'fallacy of composition,' in which the reasoner supposes that if a proposition is true of each element of a whole, then it must be true of the whole.

The Lucas model provides a counterexample. From the standpoint of any individual (atomistic) agent, it is quite true that a decision to save one more unit will yield greater future resources, in the amount $\mathbf{R}_{t+1}$. But from the standpoint of the society as a whole, if everyone decided to do the same thing (save one more unit), there would be no effect on aggregate resources in period $t + 1$. Put another way, for any individual agent, it appears that the 'marginal product of capital' is $\mathbf{R}_{t+1}$, but for the society as a whole the marginal product of capital is zero.

The proposition that the return for society as a whole must be the same as the return that is available to individuals is an error because it implicitly assumes that there are no general equilibrium effects of a generalized desire to save more (or, more broadly, there is no interaction between the decisions one person makes and the decisions of another person). The Lucas model provides a counterexample in which, if everyone's preferences change (e.g., ϑ goes down for everyone), the price of the future asset is affected – indeed, it is affected in a way that is sufficient to exactly counteract the increased desire for ownership of future dividends (since there is a fixed supply of assets to be owned, the demand must be reconciled with that preexisting supply).

Aristotle was a smart guy!

8 A Surprise

In the case where dividends are identically individually lognormally distributed, $\log d_{t+1} \sim \mathcal{N}(-\sigma_{\mathbf{r}}^2/2, \sigma_{\mathbf{r}}^2)$, the appendix shows that

$$\log \mathsf{P}_t = \rho \log d_t + \rho(\rho - 1)\sigma_{\mathbf{r}}^2/2 - \vartheta \tag{15}$$

and thus the variances obey

$$\mathrm{var}(\log \mathsf{P}) = \rho^2 \, \mathrm{var}(\log d). \tag{16}$$

Given that $\rho > 1$, this derivation yields some interesting insights:

1. (the log of) asset prices will be more volatile than (the log of) dividends

   - This would be even more true if ϑ were stochastic, as is often assumed in the asset pricing literature

2. An increase in risk aversion ρ *increases* the price P_t (because $\rho(\rho-1)\sigma_{\mathbf{r}}^2/2 > 0$ and an increase in ρ increases its size)

The second point is surprising; ChatGPT correctly summarizes the usual received wisdom about risk aversion by saying "In general, an increase in risk aversion can cause

a decrease in the overall stock market ..."[7] The reason for this prediction is intuitive: an increase in risk aversion makes people want the risky asset less, and if they want it less one would think that the price should be lower. But no: an *increase* in risk aversion *increases* the price of the risky asset. The resolution of this conflict between ChatGPT and Lucas comes from realizing that the usual model (ChatGPT will always summarize 'the usual model') is one in which investors have access to a safe asset as well as a risky one, while in the Lucas model presented here the *only* asset available is risky.

9 Analytical and Numerical Solutions

The appendices derive various results about the solution to the model under different assumptions. But, unfortunately, the model has analytical solutions (like, $P = d/\vartheta$) or approximate analytical solutions only in special circumstances. The accompanying DemARK notebook shows how to solve the model numerically for a simple case where there is no such analytical solution (the case where dividends follow an AR(1) process), and also shows how the numerical solution compares with the approximate analytical solution in the CRRA utility case.

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[7] *"When risk aversion increases, investors tend to be more cautious and prefer safer investments over riskier ones, such as stocks. As a result, there is a decrease in demand for stocks, which leads to a decrease in their prices. In general, an increase in risk aversion can cause a decrease in the overall stock market, as investors sell off their shares to move their money into safer investments, such as bonds or cash. This can result in a downward trend in the stock market indices. However, it's important to note that the impact of risk aversion on the stock market can be complex and multifaceted, and there are many other factors that can influence the stock market as well. For example, government policies, economic data, and news events can all have a significant impact on the stock market, regardless of the level of risk aversion among investors."* – ChatGPT version 2022-02-13

Appendix: Analytical Solutions in CRRA Utility Case

A When Dividends are IID

When $\rho > 1$, we can rewrite (13) by multiplying the second term on the right by $\mathsf{P}_{t+1}/\mathsf{P}_{t+1}$, yielding

$$
\begin{aligned}
\left(\frac{\mathsf{P}_t}{d_t^\rho}\right) &= \beta\,\mathbb{E}_t\left[\left(\frac{\mathsf{P}_{t+1}}{d_{t+1}^\rho} + \frac{\mathsf{P}_{t+1}}{d_{t+1}^\rho}\frac{d_{t+1}}{\mathsf{P}_{t+1}}\right)\right] \\
&= \beta\,\mathbb{E}_t\left[\frac{\mathsf{P}_{t+1}}{d_{t+1}^\rho}\left(1 + \frac{d_{t+1}}{\mathsf{P}_{t+1}}\right)\right] \\
&= \beta\,\mathbb{E}_t\left[\frac{\mathsf{P}_{t+1}}{d_{t+1}^\rho}\left(1 + \frac{d_{t+1}d_{t+1}^{-\rho}d_{t+1}^\rho}{\mathsf{P}_{t+1}}\right)\right] \\
&= \beta\,\mathbb{E}_t\left[\frac{\mathsf{P}_{t+1}}{d_{t+1}^\rho} + d_{t+1}^{1-\rho}\right]
\end{aligned}
\tag{17}
$$

and we can hypothesize that there is a solution with a constant ratio $\delta = d^\rho/\mathsf{P}$. In that case this equation simplifes to

$$
\delta^{-1} = \beta\,\mathbb{E}_t\left[\delta^{-1} + d_{t+1}^{1-\rho}\right]
\tag{18}
$$

Suppose d_{t+n} is identically individually distributed in every future period, so that its expectation as of t is the same for any date $n > 0$:

$$
\grave{d} \equiv \mathbb{E}_t[d_{t+n}^{1-\rho}].
\tag{19}
$$

Now note that (18) can be rewritten as

$$
\begin{aligned}
\frac{\mathsf{P}_t}{d_t^\rho} &= \beta\left(\grave{d} + \mathbb{E}_t\left[\frac{\mathsf{P}_{t+1}}{d_{t+1}^\rho}\right]\right) \\
&= \beta\grave{d}\left(1 + \beta + \beta\,\mathbb{E}_t\left[\frac{\mathsf{P}_{t+2}}{d_{t+2}^\rho}\right]\right) \\
&= \beta\grave{d}\left(1 + \beta + \beta^2 + \ldots + \underbrace{\mathbb{E}_t\left[\lim_{n\to\infty}\beta^{n-1}\left[\frac{\mathsf{P}_{t+n}}{d_{t+n}^\rho}\right]\right]}_{\text{assume goes to zero}}\right) \\
&= \left(\frac{\beta\grave{d}}{1-\beta}\right) \\
&= \left(\frac{\grave{d}}{\beta^{-1}-1}\right)
\end{aligned}
\tag{20}
$$

To make further progress, suppose that the iid process for the stochastic component of dividends is a mean-one lognormal: $\log d_{t+n} \sim \mathcal{N}(-\sigma^2/2, \sigma^2)\ \forall\,n$ so that $\mathbb{E}_t[d_{t+n}] = 1\ \forall\,n$

(see [ELogNormMeanOne]), in which case [ELogNormTimes] can be used to show that[8]

$$\grave{d} = e^{\rho(\rho-1)\sigma_d^2/2} \tag{21}$$

and if we define the discount factor as $\beta = 1/(1+\vartheta)$ then $\beta^{-1} = 1+\vartheta$; substituting into (20),

$$\frac{\mathsf{P}_t}{d_t^\rho} = \left(\frac{e^{\rho(\rho-1)\sigma_d^2/2}}{\vartheta} \right)$$

$$\mathsf{P}_t = \left(\frac{d_t^\rho e^{\rho(\rho-1)\sigma_d^2/2}}{\vartheta} \right) \tag{22}$$

So the log is

$$\log \mathsf{P}_t = \rho \log d_t + \rho(\rho-1)\sigma_d^2/2 - \vartheta \tag{23}$$

as asserted in the main text.

B When Dividends Follow a Random Walk

The polar alternative to IID shocks would be for dividends to follow a random walk: $\log(d_{t+1}/d_t) \sim \mathcal{N}(-\sigma_d^2/2, \sigma_d^2)$. (Recall that this assumption implies that the expected arithmetic growth rate for dividends is zero: $\mathbb{E}_t[d_{t+1}/d_t] = \exp(-\sigma_d^2/2 + \sigma_d^2/2) = e^0 = 1$; later we will consider the case in which dividends have a positive trend growth rate).

Now divide both sides of (13) by d_t, and rewrite the object inside the expectations operator by multiplying the first term by d_{t+1} and dividing the second term by d_{t+1}, yielding

$$\mathsf{P}_t = \beta d_t^\rho \, \mathbb{E}_t \left[d_{t+1}^{-\rho}(\mathsf{P}_{t+1} + d_{t+1}) \right]$$

$$\left(\frac{\mathsf{P}_t}{d_t} \right) = \beta d_t^{-(1-\rho)} \, \mathbb{E}_t \left[d_{t+1}^{1-\rho} \left(\frac{\mathsf{P}_{t+1}}{d_{t+1}} + 1 \right) \right] \tag{24}$$

$$= \beta \, \mathbb{E}_t \left[\left(\frac{d_{t+1}}{d_t} \right)^{1-\rho} \left(\frac{\mathsf{P}_{t+1}}{d_{t+1}} + 1 \right) \right].$$

Note that our assumption here about the distribution of d_{t+1}/d_t is identical to the assumption about d_{t+1} above, so the expectation will be the same $\grave{d}$. Now hypothesize that there will be a solution under which the price-dividend ratio is a constant; call it

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[8]The derivation is identical to the one in Portfolio-CRRA for the portfolio-weighted return in the case of lognormally distributed risky returns, setting the mean log return to $\mathbf{r} = 0$ and the portfolio share ς to 1.

$\boldsymbol{r}^{-1}$:

$$\boldsymbol{r}^{-1} = \beta\left[\grave{d}(\boldsymbol{r}^{-1} + 1)\right]$$
$$1 = \beta\grave{d}(1 + \boldsymbol{r}) \tag{25}$$
$$\frac{1}{\beta\grave{d}} = 1 + \boldsymbol{r}$$

but $\log \grave{d} = \rho(\rho - 1)\sigma_d^2/2$, while `OverPlus` says that $\log \beta \approx -\vartheta$ and `ExpEps` says $\log(1 + \boldsymbol{r}) \approx \boldsymbol{r}$, so taking the log of both sides of this equation therefore yields

$$\vartheta - \rho(\rho - 1)\sigma_d^2/2 \approx \boldsymbol{r} \tag{26}$$

In the case of logarithmic utility ($\rho = 1$), this equation confirms our earlier conclusion in the main text that the arithmetic interest rate must be equal to the time preference rate in order for the economy to be in equilibrium.

But with $\rho > 1$, the expression subtracted from ϑ must be positive. That is simply saying that if consumers have risk aversion higher than that of logarithmic utility, the equilibrium price-dividend ratio must be such that consumers expect a higher $\boldsymbol{r}$ to make them willing to hold the asset. For example, in the case where $\rho = 2$, the expression becomes

$$\boldsymbol{r} \approx \vartheta - \sigma_d^2, \tag{27}$$

which indicates that the arithmetic rate of return must be lower than the time preference rate by the amount σ_d^2 in order to induce consumers to hold the risky asset. Remembering that, for a given dividend payout, a lower d/P must be accomplished by a higher price, this says that the price of the risky asset will be higher in the economy with $\rho > 1$.

This equation seems puzzing because with a low enough time preference rate, and with $\rho > 1$ and $\sigma_d^2 > 0$, it could possibly imply that $d_t/\mathsf{P}_t = \boldsymbol{r} < 0$. If prices are positive, this must mean that dividends are negative - which was ruled out by assumption in the statement of the model (because aggregate consumption is equal to aggregate dividends, and negative c_t would yield undefined utility).

What this reveals is that the model has no solution unless people are sufficiently impatient. (This is another impatience condition like those articulated in `PerfForesightCRRA` and `TractableBufferStock` – in this case, you must be impatient enough to want to hold the risky asset despite its riskiness).

Recall our earlier assumption that $\grave{d} = 1$; that is, the arithmetic growth rate of dividends is zero. If, instead, dividends had a positive growth factor $\Gamma = e^\gamma \approx 1 + \gamma$, the consequence is that a further subtraction from ϑ is $\rho\gamma$:

$$\boldsymbol{r} \approx \vartheta - \rho\gamma - \rho(\rho - 1)\sigma_d^2/2 \tag{28}$$

This tightens the impatience condition to

$$\vartheta \gtrsim \rho\gamma + \rho(\rho - 1)\sigma_d^2/2 \tag{29}$$

and note that by reducing the required rate of return, for any given level of dividends

the price must be higher than in the case where $\gamma = 0$. That makes sense: People should be more eager to own shares when dividends are rising than when they are constant.

All of this finally puts us in position to calculate the price from the dividend, which we can do using an updated version of (22) which incorporates growth:

$$\frac{\mathsf{P}_t}{d_t^\rho} = \left(\frac{e^{\rho(\gamma + (\rho-1)\sigma_d^2/2)}}{\vartheta} \right) \tag{30}$$

$$\log \mathsf{P}_t = \rho \log d_t + \rho(\gamma + (\rho-1)\sigma_d^2/2) - \log \vartheta$$

from which we can directly read off the following propositions:

For any given d_t, the price P_t is higher when:

1. Relative risk aversion ρ is higher

2. Dividend growth γ is higher

3. The time preference rate ϑ is smaller (people are more patient)

   - Remember that ϑ is a small number (far less than 1), so $\log \vartheta < 0$.
   - Decreasing it makes it a larger negative number and thus $-\log \vartheta$ becomes more positive

References

Aristotle. *On Sophistical Refutations*. The Wikipedia Foundation, 350 BC.

Robert E. Lucas. Asset prices in an exchange economy. *Econometrica*, 46:1429–1445, December 1978. URL http://www.jstor.org/stable/1913837. Available at http://www.jstor.org/stable/1913837.