ual-theory}

1 The Usual Theory, and a Bit More Notation

For reference and to illustrate our new notation, we will now derive the Euler equation and other standard results for the problem described above. Since we can write value as of the end of the consumption stage as a function of a_t:

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$$\mathrm{v}_{\succ(t)}(a_t) = \beta\mathbb{E}_{\prec(t+1)}[\mathcal{G}_{t+1}^{1-\rho}\mathrm{v}_{\sim(t+1)}(\overbrace{(\mathsf{R}/\mathcal{G}_{t+1})a_t + \theta_{t+1}}^{m_{t+1}})], \tag{1}$$

Derivative notation convention. Throughout this document, a superscript ∂ on a function means its derivative with respect to the relevant state variable at that perch: $\mathrm{v}_{\succ}^{\partial}(a) \equiv \mathrm{dv}_{\succ}/\mathrm{d}a$, $\mathrm{v}_{\sim}^{\partial}(m) \equiv \mathrm{dv}_{\sim}/\mathrm{d}m$. So, the first order condition for (14) with respect to a_t[1] is

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$$\begin{aligned}\mathrm{u}^{\partial}(m_t - a_t) = \mathrm{v}_{\succ(t)}^{\partial}(a_t) &= \mathbb{E}_{\prec(t+1)}[\beta\mathcal{R}_{t+1}\mathcal{G}_{t+1}^{1-\rho}\mathrm{v}_{\sim(t+1)}^{\partial}(m_{t+1})] \\ &= \mathbb{E}_{\prec(t+1)}[\beta\mathsf{R}\quad\mathcal{G}_{t+1}^{-\rho}\mathrm{v}_{\sim(t+1)}^{\partial}(m_{t+1})]\ \mathsf{U}\end{aligned} \tag{2}$$

which illustrates the derivative convention (e.g. v^{∂} is the derivative of v with respect to its argument).[2] For functions of more than one argument, we append the variable name: $\mathrm{v}^{\partial x}$ denotes the partial derivative of v with respect to x.

Because the Envelope theorem tells us that

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$$\mathrm{v}_{\sim(t)}^{\partial}(m_t) = \mathbb{E}_{\prec(t+1)}[\beta\mathsf{R}\mathcal{G}_{t+1}^{-\rho}\mathrm{v}_{\sim(t+1)}^{\partial}(m_{t+1})]\ \mathsf{U} \tag{3}$$

we can substitute the LHS of (3) for the RHS of (2) to get

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$$\mathrm{u}^{\partial}(c_t) = \mathrm{v}_{\sim(t)}^{\partial}(m_t)\ \mathsf{U} \tag{4}$$

and rolling forward one period,

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$$\mathrm{u}^{\partial}(c_{t+1}) = \mathrm{v}_{\sim(t+1)}^{\partial}(a_t\mathcal{R}_{t+1} + \theta_{t+1})\ \mathsf{U} \tag{5}$$

so that substituting the LHS in equation (2) finally gives us the Euler equation for consumption:

{eq:cEuler}

$$\mathrm{u}^{\partial}(c_t) = \mathbb{E}_{\succ(t)}[\beta\mathsf{R}\mathcal{G}_{t+1}^{-\rho}\mathrm{u}^{\partial}(c_{t+1})].\ \mathsf{U} \tag{6}$$

The derivation above used period-qualified subscripts (e.g., $\mathrm{v}_{\succ(t)}$, $\mathrm{v}_{\sim(t+1)}^{\partial}$) because the Euler equation relates objects across periods. We can now restate the problem (14) using the simpler within-stage notation, which drops the period qualifier:

$$\mathrm{v}_{\sim}(m) = \max_{c}\ \ \mathrm{u}(c) + \mathrm{v}_{\succ}(m - c)\ \mathsf{U} \tag{7}$$

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[1] Since $c = m - a$, maximizing over c is equivalent to maximizing over a. Differentiating $\mathrm{u}(m-a) + \mathrm{v}_{\succ}(a)$ with respect to a gives $-\mathrm{u}^{\partial}(c) + \mathrm{v}_{\succ}^{\partial}(a) = 0$, which rearranges to $\mathrm{u}^{\partial}(c) = \mathrm{v}_{\succ}^{\partial}(a)$.

[2] The superscript ∂ plays the same role as the more common prime notation (v'), which we avoid because in dynamic programming contexts the prime conventionally denotes the *next-period* value of a variable (e.g., m' for next-period market resources), creating potential ambiguity.

whose first order condition with respect to c is

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$$\mathrm{u}^{\partial}(c) = \mathrm{v}^{\partial}_{\succ}(m - c)\,\mathsf{U} \tag{8}$$

which is mathematically equivalent to the usual Euler equation for consumption.

We will revert to this formulation when we reach subsection 6.9.