1 Multiple Control Variables and Modularity

We now consider problems with multiple control variables. Section 1.1 presents the joint consumption-and-portfolio optimization as a canonical example: we derive the simultaneous first-order conditions and show that direct numerical solution of the multidimensional problem is computationally expensive—but that decomposing it into a sequence of single-control problems is much faster. Section 1.2 develops the stage-based machinery for such decompositions, building on the modular notation from section 4: the discounting stage (disc), the standalone portfolio optimization (section 1.2.2), and the general-purpose portable returns stage that unifies portfolio choice and shock realization. Section 1.2.5 combines these building blocks into three canonical period types.

1.1 The Joint Optimization Problem

Our canonical example of a multi-control problem is the case where the agent has both a consumption choice and a decision about ς (archaic Greek stigma): the share of assets in the risky asset. Given a risky-asset return factor $\boldsymbol{R}$ and a riskless return factor R, the realized portfolio return is

$$\mathfrak{R}(\varsigma) = \mathsf{R} + (\boldsymbol{R} - \mathsf{R})\varsigma. \tag{1}$$

Written as a single combined decision (substituting the budget constraint):

$$\mathrm{v}_t(m) = \max_{\{c,\varsigma\}} \ \mathrm{u}(c) + \mathbb{E}_{\sim}[\beta \mathrm{v}_{t+1}(\overbrace{(m-c)\mathfrak{R}(\varsigma)+\theta}^{m_{t+1}})]\ \mathsf{U} \tag{2}$$

Whether the stochastic variables $\boldsymbol{R}$ and θ are revealed at the end of the current period or the start of the next does not affect this equation. The first-order conditions for this joint problem are:

$$\begin{aligned} \mathrm{u}^{\partial}(c) &= \mathrm{v}_{\succ}^{\partial a}(m-c,\varsigma) \\ 0 &= \mathrm{v}_{\succ}^{\partial \varsigma}(m-c,\varsigma)\ \mathsf{U} \end{aligned} \tag{3}$$

Direct simultaneous solution of these two conditions is computationally expensive; the remainder of this section develops modular stage-based machinery that decomposes the problem into a sequence of simpler single-control optimizations.

1.2 Decomposing Into Modular Stages

The single-stage-per-period formulation from sections 2–6 is equivalent to a sequence of simpler stages, each with a single job. We decompose it in that way so that adding portfolio choice later requires no change to the consumption-stage code—only the stage list and the Connectors between stages change (see section 4). The decomposition yields the same value functions, consumption function, and Euler equation.

1.2.1 The Discounting Stage (disc)

disc-stage}

The simplest decomposition isolates discounting into a standalone stage that applies for the entire period. We call it the disc stage (control-name β). From equation (15), $\mathrm{v}_{\succ(t)} := \beta \mathrm{v}_{\prec(t+1)}$; the factor β was implicitly inside the backward builder $\mathsf{B}^{\leftarrow}_{\mathrm{prd}}$ that created $\mathrm{v}_{\succ}$. Instead, henceforth at the end of every period, we put the stub disc stage which has no choice, no shocks, and a trivial decision value function $\mathrm{v}_{\sim} = \beta \cdot \mathrm{v}_{\succ}$, and we set the discount factor to 1 for every prior stage in the period (leaving all the discounting to the disc stage).[1]

Table 1 Discounting Stage (β) Perches

prchs-disc}

Perch	Indicator	State	value functions	Explanation
Arrival	$\prec$	$\bullet$	$\mathrm{v}_{\prec} = \mathrm{v}_{\sim}$	no shocks
Decision(s)	$\sim$	$\bullet$	$\mathrm{v}_{\sim} = \beta \mathrm{v}_{\succ}$	apply β
Continuation	$\succ$	$\bullet$	$\mathrm{v}_{\succ}$	value at exit

n.b.: $\bullet$ is a generic passthrough state whose type (k-type or m-type) is inherited from the predecessor stage's Continuation state. For example, when disc follows cons-noshocks, $\bullet$ stands for a (k-type); when it follows portable, $\bullet$ stands for $\check{m}$ (m-type).

The two-stage period has stage list [cons-with-shocks, disc] (more compactly, $[\mathrm{c}, \beta]$). The convention that every non-disc stage is undiscounted and every period ends with disc will mean we do not have to rearrange discounting as we rearrange stages within the period.

1.2.2 The Standalone Portfolio Problem

c:MCTheory}

Consider the standalone problem of an 'investor' choosing the portfolio share ς. The state variable at the start is k (capital available for investment); the Decision chooses ς; stochastic shocks $(\boldsymbol{R}, \theta)$ are then realized, yielding the Continuation state $\check{m} = \mathfrak{R}(\varsigma)k + \theta$. The portfolio share must be chosen *before* the return shock $\boldsymbol{R}$ is realized. We write $\check{m}$ rather than m for the post-shock state because a strict rule of our framework is the prohibition of multiple timings of a given variable within a period. Note that $\check{m}$ is m-type (market resources after shocks), matching m; the $\check{}$ decoration distinguishes the two timings while preserving the type (see section 4.3).

The first-order condition with respect to ς is:

eq:FOCport}

$$0 = \mathbb{E}_{\sim}\left[(\boldsymbol{R} - \mathsf{R})\mathrm{v}^{\partial}_{\succ}(\check{m})\right] \; \mathsf{U} \tag{4}$$

[1] Equations in sections 2–5, which predate the modular decomposition, show β inside the continuation value of the consumption problem (e.g., (12)). That formulation is equivalent: the β that appears there is the same discount factor, housed inside a composite continuation value rather than in a separate stage. The modular decomposition separates what those equations combine.

where $\mathrm{v}_{\succ}^{\partial}$ denotes the derivative of the continuation value function with respect to $\check{m}$.

The Decision equation yields the portfolio share function:

hrDecision}

$$\varsigma(k) = \arg\max_{\varsigma} \; \mathbb{E}_{\sim}\left[\mathrm{v}_{\succ}\left((\mathsf{R} + (\boldsymbol{R} - \mathsf{R})\varsigma)k + \theta\right)\right], \mathsf{U} \tag{5}$$

c:portable}

1.2.3 The portable Stage

The portfolio optimization problem above can be generalized into a single configurable *returns-and-shocks* stage by making the portfolio share an *optional parameter* rather than a control variable that must always be optimized. We call this the portable stage ("*port*folio-*able*"): it is "able" to perform portfolio optimization or not, depending on a parameter $\bar{\varsigma}$.

The stage's Arrival state includes the investable capital k and an optional portfolio share $\bar{\varsigma} \in [0,1] \cup \{\perp\}$, where $\perp$ means "absent—optimize." The behavior of the stage depends on $\bar{\varsigma}$:

- **Optimize mode** ($\bar{\varsigma} = \perp$): The stage solves the portfolio optimization problem from section 1.2.2, choosing ς optimally via (5) with first-order condition (4).
- **Fixed-share mode** ($\bar{\varsigma} \in [0,1]$): The portfolio share is prescribed; no optimization occurs. The value function is $\mathrm{v}_{\prec}(k,\bar{\varsigma}) = \mathbb{E}_{\prec}[\mathrm{v}_{\succ}(\mathfrak{R}(\bar{\varsigma})k + \theta)]$.
- **No-risky-investment mode** ($\bar{\varsigma} = 0$): A special case of fixed-share mode. All assets earn the riskless return: $\check{m} = \mathsf{R}k + \theta$. The risky return $\boldsymbol{R}$ is irrelevant. The value function reduces to $\mathrm{v}_{\prec}(k,0) = \mathbb{E}_{\prec}[\mathrm{v}_{\succ}(\mathsf{R}k + \theta)]$.

The perches of this stage are:

Table 2 portable Stage Perches

s-portable}

Perch	Indicator	State	value functions	Explanation
Arrival	$\prec$	$(k,\bar{\varsigma})$	$\mathrm{v}_{\prec} = \mathrm{v}_{\sim}$	no pre-decision shocks
Decision(s)	$\sim$	$(k,\bar{\varsigma})$	depends on $\bar{\varsigma}$ (see above)	choose ς or use $\bar{\varsigma}$
Continuation	$\succ$	$\check{m}$	$\mathrm{v}_{\succ}$	post-shock value

When $\bar{\varsigma} = \perp$, the Decision value function is $\mathrm{v}_{\sim} = \max_{\varsigma} \mathbb{E}_{\sim}[\mathrm{v}_{\succ}]$. When $\bar{\varsigma} \in [0,1]$, it is $\mathrm{v}_{\sim} = \mathbb{E}[\mathrm{v}_{\succ}]$ with ς fixed at $\bar{\varsigma}$.

Argument notation. When a stage can be constructed with a fixed parameter, we write the value of that argument in parentheses. Thus $\varsigma(0)$ denotes the portable stage with $\bar{\varsigma} = 0$ (the shocks-only configuration described below), and $\varsigma(\perp)$ denotes portable in optimize mode.

1.2.4 Separating Shocks from Choices

ons-decomp}

The cons-with-shocks stage in section 4.2 combines the $k \to m$ shock realization and the $m \to a$ consumption decision. We split these into two stages.

The shocks-only stage (portable with $\bar{\varsigma} = 0$). Calling the portable stage with $\bar{\varsigma} = 0$ produces a stage whose only function is to draw the shocks. We call this configuration shocks-only. It handles the transition from capital k to (post-shock) market resources $\check{m}$: with $\varsigma = 0$, the transition is $\check{m} = \mathcal{R}k + \theta$—precisely the $k \to \check{m}$ mapping from the original problem (the next stage receives this as m). No optimization occurs: The

Table 3 shocks-only (portable with $\bar{\varsigma} = 0$) Stage Perches

hocks-only}

Perch	Indicator	State	value functions	Explanation
Arrival	$\prec$	k	$\mathrm{v}_{\prec} = \mathbb{E}_{\prec}[\mathrm{v}_{\succ}]$	pre-shock value
Decision(s)	$\sim$	k	(none)	no choice
Continuation	$\succ$	$\check{m}$	$\mathrm{v}_{\succ}$	post-shock value

Arrival value function takes the expectation over the shocks: $\mathrm{v}_{\prec}(k) = \mathbb{E}_{\prec}[\mathrm{v}_{\succ}(\mathcal{R}k + \theta)]$. The Continuation state $\check{m} = \mathcal{R}k + \theta$ is fully determined (non-stochastic) once the shocks are realized.

re-problem}

The shock-free consumption stage (cons-noshocks). With shocks in the preceding stage, the consumption stage has Arrival state m and no shocks between Arrival and Decision, so $\mathrm{v}_{\prec} = \mathrm{v}_{\sim}$. We call this the cons-noshocks stage (control-name c):

Table 4 cons-noshocks Stage Perches

-shockfree}

Perch	Indicator	State	value functions	Explanation
Arrival	$\prec$	m	$\mathrm{v}_{\prec} = \mathrm{v}_{\sim}$	no shocks; identity
Decision(s)	$\sim$	m	$\mathrm{v}_{\sim}(m) = \max_c \mathrm{u}(c) + \mathrm{v}_{\succ}(m - c)$	choose consumption
Continuation	$\succ$	a	$\mathrm{v}_{\succ}$	value at exit

The Decision equation for this stage defines the consumption function:

s-decision}

$$\mathrm{c}_{\mathrm{c},\sim}(m) = \underset{c}{\arg\max} \quad \mathrm{u}(c) + \mathrm{v}_{\mathrm{c},\succ}(\underbrace{m - c}_{a}). \text{U} \tag{6}$$

The Decision is unchanged from the single-stage formulation. But this stage is a 'stub': It could not be the only stage in a period because its continuation state a is of a different type than its arrival state m. A full period would require it to be paired with another stage that produced the transition from a to m.

The three-stage period. The consumption-only period from sections 2–6 is defined by the stage list [shocks-only, $\mathtt{C}(\check{m} \leftrightarrow m)$, cons-noshocks, disc], or equivalently [$\varsigma(0)$, $\mathtt{C}(\check{m} \leftrightarrow m)$, c, β] in control-name form (recall $\varsigma(0)$ denotes portable with $\bar{\varsigma} = 0$; see section 1.2.3). Explicitly:

Element	Transition	Action
shocks-only	$k \to \check{m}$	shocks realize (no choice)
$\mathtt{C}(\check{m} \leftrightarrow m)$	$\check{m} \to m$	rename
cons-noshocks	$m \to a$	choose c
disc		apply β

This is functionally identical to the single-stage formulation. Each stage resolves its stochastic content *internally*; the value at a stage's exit is therefore non-stochastic.

Multi-stage notation. Once our Pile $\mathtt{P}$ has accumulated multiple periods and stages, we address any perch-specific object (like a value function) by comma-separated subscripts, ordered from outermost to innermost—period, stage, perch:

$$\mathrm{v}_{t,\varsigma,\succ}(a).$$

We drop the period when considering a stage from a context inside a period ($\mathrm{v}_{\varsigma,\succ}(a)$).

riod-types}

1.2.5 Three Period Types

Using the portable stage (with $\bar{\varsigma} = 0$ or $\perp$), the cons-noshocks stage, and the disc stage at the end of every period, we can construct three kinds of periods. A period type is defined by its stage list (section 4), written in square brackets:

riod-types}

Table 5 Three Period Types

Period type	Transition	Description
shocksonly–consnoshocks	$k \to a$	No portfolio choice
portable–consnoshocks	$k \to a$	Beginning-of-period returns
consnoshocks–portable	$m \to \check{m}$	End-of-period returns

We now describe these three variants in detail. We present the consnoshocks–portable (end-of-period returns) variant first, because it makes the need for the $\check{m}$ notation most transparent.

ng-returns}

The consnoshocks–portable period (end-of-period returns). If the portfolio share choice is made and stochastic shocks are realized at the *end* of the period, the shock-free consumption stage comes first and the ς stage follows: stage list by control-name [c, $\mathtt{C}(a \leftrightarrow k)$, ς, β]. The flow is:

Element	Transition	Action
cons-noshocks	$m \to a$	choose c
$\mathtt{C}(a \leftrightarrow k)$	$a \to k$	rename
portable	$k \to \check{m}$	choose ς, shocks realize
disc		apply β

The within-period Connector $\mathtt{C}(a \leftrightarrow k)$ relabels the consumption stage's output as the ς stage's input; the remaining Connectors are catalogued in section 1.2.6. The reason we used $\check{m}$ rather than m for the post-portfolio state now becomes evident: this prevents two different values of m from coexisting within the same period. The ˇ decoration is appropriate because $\check{m}$ remains m-type—it is market resources, differing from m only in timing.

ng-returns}

The portable–consnoshocks period (beginning-of-period returns). The beginning-returns problem places the ς stage *before* the shock-free consumption stage: stage list by control-name [ς, $\mathtt{C}(\check{m} \leftrightarrow m)$, c, β]. The first-order condition for the portfolio-choice stage is (4).

The flow through a single period is:

Element	Transition	Action
portable	$k \to \check{m}$	choose ς, shocks realize
$\mathtt{C}(\check{m} \leftrightarrow m)$	$\check{m} \to m$	rename
cons-noshocks	$m \to a$	choose c
disc		apply β

Since all stochastic shocks are realized *inside* the ς stage, the value $\check{m}$ that exits it is fully determined. The within-period Connector $\mathtt{C}(\check{m} \leftrightarrow m)$ therefore simply relabels $\check{m}$ as m for the consumption stage (see section 1.2.6 for the full catalogue).

The shocksonly–consnoshocks period. This is the three-stage decomposition from above: [$\varsigma(0)$, $\mathtt{C}(\check{m} \leftrightarrow m)$, c, β]. A life cycle model in which portfolio choice is available at some ages but not others is now trivially constructed: the modeler simply sets $\bar{\varsigma} = \perp$ for ages with active portfolio choice and $\bar{\varsigma} = 0$ for ages without.

1.2.6 Connectors for Each Period Type

connectors}

To construct a Pile $\mathtt{P}$ that repeats instances of a given period type, the backward builder $\mathtt{B}^{\leftarrow}_{\mathrm{prd}}$ must know the *between-period* Connector that ties the last stage of one period to the first stage of the next (see section 4.3). Within-period Connectors are already specified in the period tables above.

period type	Between-period Connector
shocksonly–consnoshocks	$\mathcal{C}(a \leftrightarrow k)$
portable–consnoshocks	$\mathcal{C}(a \leftrightarrow k)$
consnoshocks–portable	$\mathcal{C}(\check{m} \leftrightarrow m)$

Each of these Connectors is a pure rename that respects the state-variable type constraint from section 4.3: a and k are both k-type (capital before returns), while $\check{m}$ and m are both m-type (market resources after returns). The connector is determined by the predecessor's exit state and the successor's first-stage arrival state. For portable–consnoshocks, the period ends after disc (passthrough), so the exit state is a; the next period's portable stage arrives with k, hence $\mathcal{C}(a \leftrightarrow k)$. For shocksonly–consnoshocks the same logic gives $\mathcal{C}(a \leftrightarrow k)$. For consnoshocks–portable, the period exits with $\check{m}$; the Connector relabels it as m for the next period's cons-noshocks stage.

1.2.7 Numerical Solution

Following the sequential approach outlined in section 1.1, we solve the portfolio stage numerically for the optimal ς at a vector of $\mathbf{k}$ and construct an approximated optimal portfolio share function $\grave{\varsigma}(k)$ as the interpolating function among the members of the $\{\mathbf{k}, \boldsymbol{\varsigma}\}$ mapping. Having done this, we calculate a vector of values and marginal values at that grid:

eq:vShrEnd}

$$\begin{aligned} \mathbf{v} &= \mathrm{v}_{\varsigma,\prec}(\mathbf{k}) \\ \mathbf{v}^{\partial} &= \mathrm{v}^{\partial}_{\varsigma,\prec}(\mathbf{k}).\,\mathsf{U} \end{aligned} \tag{7}$$

With the $\mathbf{v}^{\partial}$ approximation in hand, we construct our approximation to the consumption function using *exactly the same EGM procedure* that we used in solving the problem *without* a portfolio choice (see (37)):

q:cVecPort}

$$\mathbf{c} \equiv \left(\mathbf{v}^{\partial}\right)^{-1/\rho}, \mathsf{U} \tag{8}$$

which, following the procedure in subsection 6.9, yields an approximated consumption function $\grave{c}_t(m)$.

1.3 Implementation

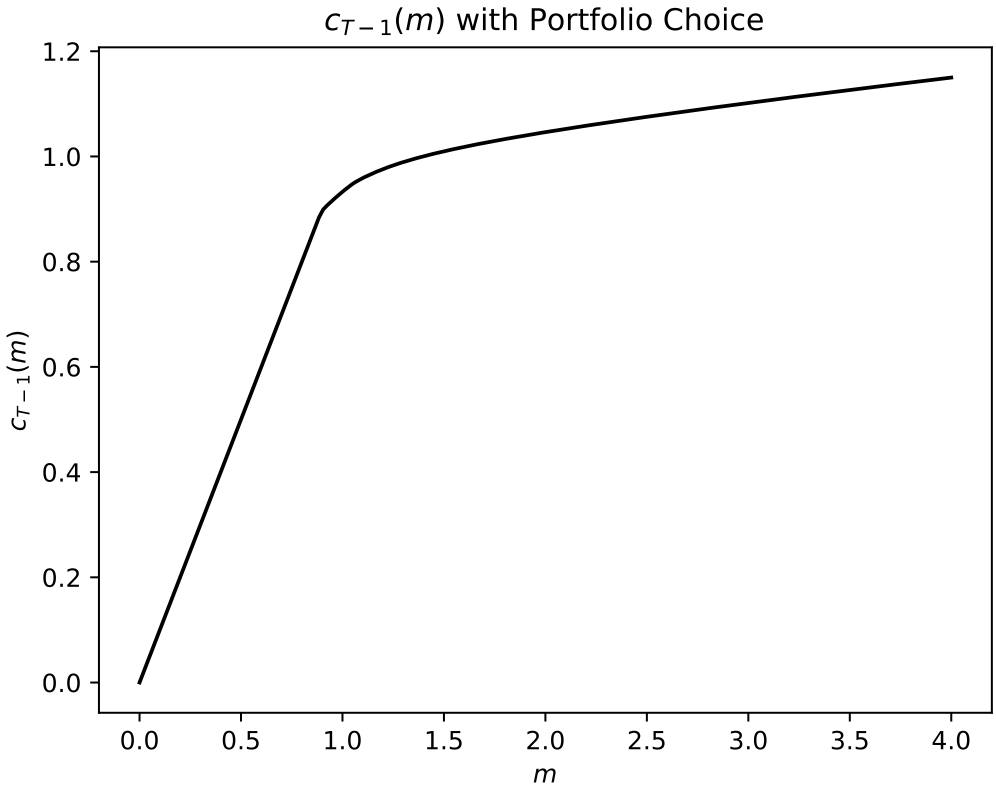
Following the discussion from section 1.1, to provide a numerical solution to the problem with multiple control variables, we must define expressions that capture the expected marginal value of end-of-period assets with respect to the level of assets and the share invested in risky assets. This is addressed in "Multiple Control Variables."

1.4 Results With Multiple Controls

e-controls}

Figure 1 plots the $t-1$ consumption function generated by the program; qualitatively it does not look much different from the consumption functions generated by the program without portfolio choice.

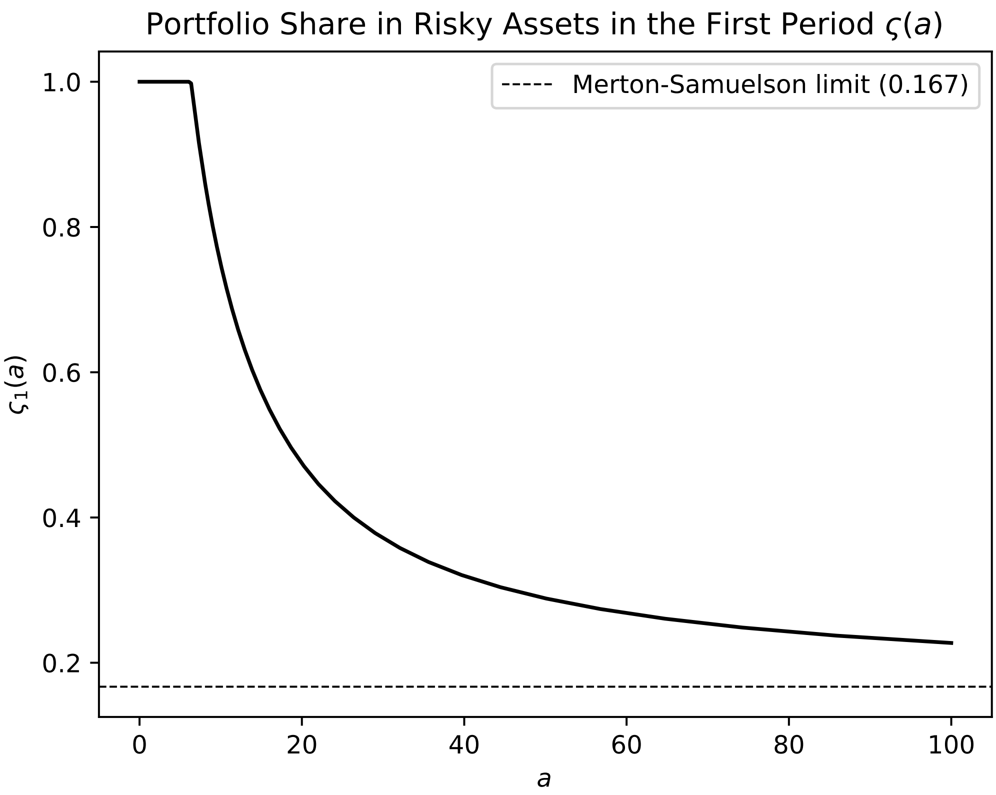
But Figure 2 which plots the optimal portfolio share as a function of the level of assets, exhibits several interesting features. First, even with a coefficient of relative risk aversion of 6, an equity premium of only 4 percent, and an annual standard deviation in equity returns of 15 percent, the optimal choice at values of a_t less than about 2 is for the agent to invest a proportion 1 (100 percent) of the portfolio in stocks (instead of the



tMultContr}

Figure 1 c(m_1) With Portfolio Choice

safe bank account with riskless return R). Second, the proportion of the portfolio kept in stocks is *declining* in the level of wealth - i.e., the poor should hold all of their meager assets in stocks, while the rich should be cautious, holding more of their wealth in safe bank deposits and less in stocks. This seemingly bizarre (and highly counterfactual – see Carroll (2002)) prediction reflects the nature of the risks the consumer faces. Those consumers who are poor in measured financial wealth will likely derive a high proportion of future consumption from their labor income. Since by assumption labor income risk is uncorrelated with rate-of-return risk, the covariance between their future consumption and future stock returns is relatively low. By contrast, persons with relatively large wealth will be paying for a large proportion of future consumption out of that wealth, and hence if they invest too much of it in stocks their consumption will have a high covariance with stock returns. Consequently, they reduce that correlation by holding some of their wealth in the riskless form.



SharetOfat}

Figure 2 Portfolio Share in Risky Assets $\varsigma_{T-1}(a)$