Calibration and Computation of Household Portfolio Models

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Contents

1 Introduction

This paper discusses issues arising in calibrating household portfolio models and in deriving numerical solutions through computation in a variety of model specifications. Interest in the computational approach has been generated mainly by the difficulties associated with obtaining exact analytical solutions in dynamic, intertemporal models of portfolio choice that allow for uninsurable background earnings risk. We attempt to illustrate the main conceptual, technical, and computational issues that arise in this context, and to explore the portfolio implications of alternative modeling choices that share some common elements.

In the first part, we describe some of the model builder's key choices regarding the ingredients of a household portfolio model. Our first section describes alternative assumptions about household preferences, nesting expected utility, Kreps-Porteus preferences, and rank dependent utility under a common specification. The choice between them is essentially reduced to a choice about the size of certain parameters. Habit formation is also examined as a variant of expected utility preferences. The main purpose here is to explore the extent to which preference specifications reflect elements relevant for portfolio choice, such as risk aversion, elasticity of intertemporal substitution, and excessive concern about bad states of the world. The next section deals with market incompleteness and frictions potentially relevant for portfolio choice. These include nondiversifiable labor income risk, borrowing and short-sales constraints, and fixed costs of entry and participation in the stock market. Earnings risk is a key ingredient if one is to study precautionary motives for asset holding. Quantity constraints on borrowing have important effects on portfolios and on the extent to which these can be adjusted in response to earnings risk. Fixed costs may hold the key to explaining the limited incidence of stockholding among households even in advanced financial systems.

The second part of the paper describes how one might calibrate a household portfolio model. This usually entails approximating continuous stochastic processes that govern labor incomes and stock returns with discrete processes, from simple binomial processes that match their mean and variance to more advanced quadrature approximation methods. In addition, the model builder needs to calibrate preference parameters and to examine sensitivity of solutions to them.

The third part deals with the computation of solutions. Although there are various alternative ways of presenting solutions, we adopt the practice of presenting policy functions for consumption and portfolio components in terms of cash on hand. We start with methods appropriate for versatile, three-period portfolio models that can be used to examine a great variety of model and parameter specifications at small computational cost. We explore the implications of expected utility and various departures from expected utility maximization for the structure of portfolios and for the size of precautionary motives. We also show how risk aversion and positive correlation between stock returns and earnings risk can alter portfolios and discuss effects of income-based borrowing limits. The next section goes to the opposite extreme of considering infinite-horizon portfolio models. It describes alternative methods of solving such models, based either on computation of value functions through grid search or on iteration between first-order conditions, and uses the latter to compute solutions for portfolio models under the assumption that households cannot engage in short sales of either stocks or bonds. This section highlights the role of risk aversion, earnings risk, positive covariance between stock returns and earnings shocks, as well as of entry costs in determining stockholding levels and total wealth holdings. The third section explores the basic implications of a finite-horizon model that mimics the life cycle of the household. This part ends with a brief discussion of the prospects for building general equilibrium models of household portfolio choice with aggregate uncertainty. We then offer some concluding remarks regarding directions for future research.

We wish to state at the outset that we cannot and do not do justice to the full array of existing computational algorithms and approaches to solving intertemporal models of household choice under uncertainty. In order to explore variants that differ only in the sense relevant to each section, we had to write and run numerous computer programs. We cannot claim that they are the only possible ways to solve such models. We describe them and use them because we know them best, and we are reasonably confident that they do not yield materially different solutions from other existing methods in the literature. Wherever possible, we also refer to papers by other authors who follow different techniques than ours. We hope to offer enough information to the readers so that they can experiment with their own models and algorithms.

2 Modeling Choices

2.1 Preferences

Our preference specification is based on a general form proposed by Epstein and Zin (1989), and it can encompass expected utility maximization and a number of departures from it measured in terms of parameter values. A household is assumed to maximize in each period t recursive utility U_t of the form:

$$U_t = W(C_t, \mu(U_{t+1}|I_t)) \tag{1}$$

where W is an "aggregator function". Current intertemporal utility is a function of current consumption and of some certainty equivalent of next period's uncertain utility, based on information up to the current period, I_t. We assume that the aggregator function is:

$$W(C_t, \mu(U_{t+1}|I_t)) = \left[(1-\beta)C_t^{\zeta} + \beta\mu_t^{\zeta}\right]^{\frac{1}{\zeta}}, 0 \neq \zeta < 1 \tag{2}$$

or

$$W(C_t, \mu(U_{t+1}|I_t)) = \left[(1-\beta)\ln C_t + \beta \ln \mu_t\right], \zeta = 0 \tag{3}$$

where $\mu_t(\cdot)$ is an abbreviation for $\mu(\cdot|I_t)$. Our proposed functional form for $\mu_t(\cdot)$ is convenient for nesting alternative preference specifications:

$$\mu(U_{t+1}|I_t) = \left[f_t\left(U_{t+1}^{\alpha}\right)\right]^{\frac{1}{\alpha}}, 0 \neq \alpha < 1 \tag{4}$$

or

$$\ln \mu(U_{t+1}|I_t) = f_t(\ln U_{t+1}), \alpha = 0 \tag{5}$$

where f_t is a linear operator that utilizes information available in period t. The definition of f_t will vary depending on preference type. Suppose that the household chooses at time t some control variable h_{it}, where i indexes control variables (e.g., asset levels). The first order conditions for utility maximization are of the form:

$$C_t^{\zeta-1}\frac{\partial C_t}{\partial h_{it}} + \beta\left[f_t\left(U_{t+1}^{\alpha}\right)\right]^{\frac{\zeta}{\alpha}-1} f_t\left[U_{t+1}^{\alpha-\zeta}C_{t+1}^{\zeta-1}\left(\frac{\partial C_{t+1}}{\partial h_{it}}\right)\right] = 0, \forall i, t. \quad (6)$$

This general form nests a variety of preference specifications.

2.1.1 Expected Utility

In the Epstein-Zin framework, the parameter α is equal to one minus the degree of relative risk aversion and the parameter ζ is equal to one minus the inverse of the elasticity of substitution. Expected utility is a special case of Epstein Zin preferences under two restrictions: (i) $\alpha = \zeta$; and (ii) $f_t \equiv E_t$, i.e., the linear operator f_t is the mathematical expectation operator conditional on information in period t. Then, the first order condition (6) becomes

$$C_t^{\alpha-1}\frac{\partial C_t}{\partial h_{it}} + \beta E_t\left[C_{t+1}^{\alpha-1}\left(\frac{\partial C_{t+1}}{\partial h_{it}}\right)\right] = 0, \forall i, t. \quad (7)$$

which is the condition for expected utility maximization under constant relative risk aversion.

2.1.2 Expected Utility with Habit Formation

Habit formation captures a psychological feature, namely that the repetition of a stimulus diminishes the perception of the stimulus and responses to it. Equivalently, a stock of habits affects current utility; ceteris paribus, for a higher habit level, higher consumption will be necessary to achieve the same utility.[1]

There are two distinct specifications of habit formation, "external" and "internal". With external habit formation, an individual's habit depends on the history of aggregate consumption (this is Abel's (1990) "catching up with the Joneses" formulation or Duesenberry's (1949) "relative income" model). This specification is often easier to work with because the individual decision does not affect directly the history of aggregate consumption, which is taken

[1] Habit formation has a long history in the study of consumption; see Ryder and Heal (1973), Sundaresan (1989), Constantinides (1990) (see Deaton (1992) for a recent overview).

as given. The felicity function is usually specified as $\frac{(C_t - H_t)^{\alpha} - 1}{\alpha}$ where H is the level of the habit. Defining the surplus consumption ratio as $SUR_t = \frac{C_t - H_t}{C_t}$, it is straightforward to show that the local curvature of the utility function equals $\frac{1-\alpha}{SUR_t}$ and is increasing in the level of the habit. Campbell and Cochrane (1998) show that in recessions, the level of consumption approaches the habit, the agent becomes more risk averse and requires a higher return to hold the claim to the risky asset, rationalizing a higher equity premium than the same model with CRRA preferences. The evolution of the surplus consumption ratio is governed by a process that depends on lagged values of consumption. The first order condition now becomes

$$C_t^{\alpha-1} SUR_t^{\alpha-1} (\frac{\partial C_t}{\partial h_{it}}) + \beta E_t C_{t+1}^{\alpha-1} SUR_{t+1}^{\alpha-1} \left(\frac{\partial C_{t+1}}{\partial h_{it}} \right) = 0, \forall i, t.$$

In the internal habit formulation, the habit is determined by past individual consumption (see Constantinides, 1990, for instance). Current individual consumption decisions affect the utility from future consumption as a result of the habit, and this must be taken into account when decisions in the current period are being made.

2.1.3 Kreps-Porteus Preferences

Kreps-Porteus preferences (KP) allow us to disentangle the effects of varying risk aversion for a given elasticity of substitution from those of varying elasticity for a given degree of risk aversion. This is not possible in an expected utility framework which forces one to be the inverse of the other. Under KP, equation (6) becomes

$$C_t^{\zeta-1} \frac{\partial C_t}{\partial h_{it}} + \beta \left[E_t \left(U_{t+1}^{\alpha} \right) \right]^{\frac{\zeta}{\alpha}-1} E_t \left[U_{t+1}^{\alpha-\zeta} C_{t+1}^{\zeta-1} \left(\frac{\partial C_{t+1}}{\partial h_{it}} \right) \right] = 0, \forall i, t. \quad (8)$$

where the linear operator f_t is the expectations operator E_t, as under expected utility, but the risk aversion parameter α is no longer tied to the elasticity parameter ζ. The household portfolio model can be solved for a range of values for α keeping ζ constant (and vice versa), thus tracing the effects of risk aversion (elasticity) alone. By comparing KP results with those from expected utility specifications, one can compute the effects on asset holdings that result from *measurable* departures from expected utility. Departures are measured by the difference between the elasticity of substitution used in the KP model and the value used in the corresponding expected utility model, namely the inverse of risk aversion (see Haliassos and Hassapis, 1997).

2.1.4 Rank-Dependent Utility

Households that maximize expected utility or have KP preferences strive to satisfy first order conditions involving mathematical expectations of marginal utilities. These expectations assign to each state a weight equal to its probability of occurrence. A literature pioneered by Quiggin (1982) and Yaari (1987) argues in favor of specifying weights that depend on the ranking of each state in terms of its desirability for the household.

A simple example involves only two states, "bad" and "good". Suppose that the bad state occurs with probability p and the good state occurs with probability 1-p. First order conditions under expected utility would use p as the weight for the marginal utility of the bad state and 1-p for the good state. Under rank-dependent utility, the bad state obtains a weight equal to p^{γ}, where $\gamma < 1$, and the good state obtains 1-p^{γ}. Given that both p and γ are below unity, this results in overweighting of the bad state relative to expected utility. Households care about the possibility of low return and/or income realizations disproportionately to their probability of occurrence.

When more than two states exist, the formula for assigning a weigth w_j to the state ranked jth in terms of desirability to the household is

$$w_j = \left(\sum_{i=1}^{j} p_i \right)^{\gamma} - \left(\sum_{i=1}^{j-1} p_i \right)^{\gamma} \tag{9}$$

where i indexes states of the world. It is obvious that when $\gamma = 1$, these weights reduce to $w_j = p_j$ exactly as in expected utility (and Kreps-Porteus) models. When we also impose the restriction $\alpha = \zeta$, which is optional in this framework, the shortfall of γ relative to unity is a measure of the departure from expected utility maximization. Finally, when $\gamma < 1$ and $\alpha = 1$, i.e., the degree of relative risk aversion is equal to zero, we have a version of Yaari's "Dual Theory of Choice".

Unlike expected-utility and Kreps-Porteus models, rank-dependent utility yields kinked indifference curves. In portfolio models, the location of these kinks depends on the value of asset holdings. A kink occurs at levels of asset holdings for which the desirability ranking of some states changes. When this happens, previously used weights are no longer valid and are recomputed using equation (9). This alters the objective function itself, thus generating a point of nondifferentiability of the objective and of the indifference curves at the level of asset holdings where the switch in desirability rankings occurs.

The simplest example involves stockholding in the absence of any other type of risk, including labor income risk. When stockholding is positive, the "bad" state is the one that involves low stock returns, and it is assigned a weight equal to its probability of occurrence, p, raised to the power γ. By contrast, when the household engages in short sales of stock, the "bad" state is that which involves high stock returns. This is so because the household has essentially borrowed at the risky stock return rate, and the high realization means that it has to pay more to repay its debt. Thus, the high return state receives a weight of $(1-p)^{\gamma}$ and the low stock return state receives $1-(1-p)^{\gamma}$. This is, of course, different in general from p^{γ}, namely the weight appropriate for the low return state under positive stockholding. In this example, the point of nondifferentiability (kink) in indifference curves occurs at zero stockholding, because the (consumption) outcomes of the high- and low- stock return states are indistinguishable to a household without stocks. As the household contemplates (marginally) positive stockholding, it regards low stock returns as "bad"; the converse is true if it contemplates short sales. Because of this switching of objective functions at zero stockholding, the slope of the maximization objective is not the same on either side of zero stockholding, thus generating a kink in indifference curves. The nature of kinks and the frequency with which solutions lie at kinks can be investigated through computation.[2]

2.2 Market Frictions and Imperfections

2.2.1 Nondiversifiable Labor Income Risk

Labor income risk is nondiversifiable because of moral hazard and adverse selection considerations, and it is not ignored by households concerned about their consumption paths, except under very restrictive conditions. When these restrictions fail, households have incentives to accumulate more wealth in order to buffer consumption from unanticipated income shocks (prudence), and usually to reduce exposure to stockholding risk in view of the risk imposed on their consumption path by the stochastic behavior of labor income (temperance). Conditions under which both of these motives are operative have been discussed in Chapter 1. Analytical solutions for standard household portfolio models with labor income risk are not available for general preference specifications.

Three cases for which analytical solutions have been obtained are the linear, quadratic, and exponential utility functions, but all of them have

[2] For example, Epstein and Zin (1990) and Haliassos and Bertaut (1995) had suggested, among others, that this property of displaying a kink at zero stockholding might enable rank dependent utility models to account for the zero stockholding puzzle. However, Haliassos and Hassapis (1997) subsequently showed that this does not apply to models with background labor income risk, since the kink is no longer located at zero stockholding.

questionable properties for analyses of household portfolios. The first implies risk neutrality and no prudence, since both its second and third derivatives are zero. Quadratic utility is consistent with risk averse behavior, but it exhibits the well-known property of certainty equivalence implying that there is no prudence and, thus, no precautionary wealth accumulation. It also implies increasing absolute and relative risk aversion as a function of wealth. As discussed in Chapter 1, this in turn implies that wealthier households will tend to put a smaller amount (and proportion) of their wealth into the risky asset compared to their poorer counterparts. Finally, exponential utility is consistent both with risk aversion and with prudence and has been used in early analytical studies of precautionary motives. However, it implies constant absolute risk aversion, i.e. an amount of risky investment that is independent of wealth in the atemporal model considered in Chapter 1. Furthermore, marginal utility at zero consumption is equal to unity under exponential utility, and this allows optimal consumption levels to be zero or even negative at various points on the consumption path.

In view of these complications and of the potential importance of precautionary motives for wealth accumulation and portfolio selection, computational methods that allow us to incorporate various stochastic processes for incomes as well as a variety of preference specifications are particularly useful. We discuss next two popular specifications of the labor income process.

Specification I In all calibrated models of this paper, we will adopt the same specification for the exogenous stochastic process followed by individual income of household i, so as to facilitate comparisons:

$$Y_{it} = P_{it}U_{it} \tag{10}$$

$$P_{it} = G_t P_{it-1} N_{it} \tag{11}$$

This process, first used in a nearly identical form by Carroll (1992)[3], is decomposed into a "permanent" component, P_{it}, and a transitory component, U_{it}, where P_{it} is defined as the labor income that would be received if the white noise multiplicative transitory shock U_{it} were equal to its mean of unity. We assume that the $\ln U_{it}$, and $\ln N_{it}$ are each independent and identically (Normally) distributed with mean $-.5 * \sigma_u^2$, $-.5 * \sigma_v^2$, and variances σ_u^2, and

[3]Carroll (1992, 1997) assumes a very small probability (usually 0.5 percent) of an unemployment state with zero labor income.

σ_v^2, respectively. The lognormality of U_{it} and the assumption about the mean of its logarithm imply that

$$EU_{it} = \exp(-.5 * \sigma_u^2 + .5 * \sigma_u^2) = 1 \tag{12}$$

and similarly for EN_{it}. Thus, precautionary wealth and portfolio effects can be computed despite the introduction of lognormally distributed multiplicative shocks. Computation of precautionary effects involves comparison of models in which household i is guaranteed in period t a certain level of income $\overline{Y}_{it}$ versus models in which the same household faces income risk but still has expected income equal to $\overline{Y}_{it}$.

The log of P_{it}, evolves as a random walk with a stochastic drift, $\ln G_t$, assumed to be common to all individuals. Denote the unconditional mean by μ_g and assume a constant unconditional variance, σ_g^2.

Given these assumptions, the growth in individual labor income follows

$$\Delta \ln Y_{it} = \ln G_t + \ln N_{it} + \ln U_{it} - \ln U_{it-1}, \tag{13}$$

where the unconditional mean growth for individual earnings is $\mu_g - .5 * \sigma_v^2$, and the unconditional variance equals $(\sigma_g^2 + \sigma_v^2 + 2\sigma_u^2)$. The last three terms in (13) are idiosyncratic and average to zero over a sufficiently large number of households, implying that per capita aggregate income growth is given by $\ln G_t$. If $\ln G_t$ is either i.i.d. or positively autocorrelated with an MA(1) process, individual income growth in (13) will also follow a first order moving average process. And, as long as the variance of aggregate income represents a sufficiently small fraction of overall individual income variance, this process has a single Wold representation that is equivalent to the MA(1) process for individual income growth estimated using household level data (MaCurdy [1982], Abowd and Card [1989], and Pischke [1995]).[4]

Specification II An alternative specification of income shocks, based on estimated processes for three different education categories of households, has been proposed by Hubbard et al. (1994, 1995). Based on these estimates, multiplicative shocks affect income levels given by the age-earnings profile for each education group; they are lognormally distributed, but their mean is zero; and the persistent shock, though highly serially correlated, does not follow a random walk. Given that estimation on various data sets of interest may well yield such estimated processes, it is instructive to see how these can be handled. Illustrations of how such processes can be incorporated in a

[4] Although these studies generally suggest that individual income changes follow an MA(2), the MA(1) is found to be a close approximation.

variety of small-scale models are given in a series of papers by Bertaut and Haliassos (1997), and Haliassos and Hassapis (1997, 1998).

Suppose that actual incomes are given by

$$Y_{it} = \widehat{P}_{it}\widehat{U}_{it} \tag{14}$$

where $\widehat{P}_{it}$ is now the persistent component of earnings of household i at age t, defined as the product of earnings given by the age-earnings profile (Y_t^*) and the persistent shock. $\widehat{U}_{it}$ is a purely transitory lognormally distributed multiplicative shock. The logarithm of the persistent component follows

$$\ln \widehat{P}_{it} = \ln \widehat{G}_t + \rho \ln \widehat{P}_{it-1} + \ln \widehat{N}_{it} \tag{15}$$

with $|\rho| < 1$, $\ln \widehat{G}_t = \Delta \ln Y_t^*$ and

$$\ln \widehat{N}_{it} \sim N(0, \sigma_n^2). \tag{16}$$

The purely transitory shock follows

$$\ln \widehat{U}_{it} \sim N(0, \sigma_u^2). \tag{17}$$

While the logarithms of these multiplicative shocks have zero mean, their levels have means equal to $\exp(0.5 \cdot \sigma_n^2)$ and $\exp(0.5 \cdot \sigma_u^2)$, which are different from unity. Without any adjustment, it is no longer true that incomes under certainty (i.e. those given by the age-earnings profile) are equal to the expected values of incomes under income risk. In order to restore this equality which is needed for derivation of precautionary effects, income values used in calibration are adjusted using the formula

$$Y_{it}^{ad} = \exp\left[\ln(\widehat{P}_{it}\widehat{U}_{it}) - 0.5 \cdot \left(\sigma_{n,t}^2 + \sigma_{u,t}^2\right)\right] \tag{18}$$

where $\sigma_{n,t}^2$ and $\sigma_{u,t}^2$ are the unconditional variances of the logarithms of persistent and transitory shocks respectively.

For purposes of comparison between the two alternative specifications in this Section, take logarithms of actual incomes and difference them to obtain:

$$\Delta \ln Y_{it} = \ln \widehat{G}_t + (\rho_n - 1) \ln \widehat{P}_{it-1} + \ln \widehat{N}_{it} + \ln \widehat{U}_{it} - \ln \widehat{U}_{it-1} \tag{19}$$

As $\rho \rightarrow 1$, the structure of equations (13) and (19) becomes remarkably similar; the two approaches are identical if $\rho = 1$.

Borrowing and Short-sales Constraints In portfolio models, borrowing may be needed not only for current consumption but also for investment in assets that will enhance future consumption opportunities and facilitate consumption smoothing. End-of-horizon or transversality conditions preclude infinite consumption financed through unbounded borrowing. In addition to those, households are not allowed to adopt consumption-portfolio plans that involve negative consumption in any time period or state of the world. Sometimes, such constraints are endogenously met by the choice of preference specification. For example, the choice of logarithmic or constant-relative-risk aversion utility function of the form

$$U(C_t) = \frac{C_t^\alpha - 1}{\alpha}, 0 \neq \alpha < 1 \tag{20}$$

implies marginal utility equal to $\frac{1}{C_t}$ or $C_t^{\alpha-1}$ respectively, which tend to infinity as C_t tends to zero. Thus, the household will never choose a consumption-portfolio plan that involves zero consumption in any period or state, because it can always increase its utility by transferring consumption from some other period or state to the zero consumption one.

It is often argued that a subset of households face additional borrowing constraints. Computational methods allow us to incorporate such constraints and describe their interaction with saving and portfolio motives. In some cases, constraints facilitate computation by limiting the range of admissible values of borrowing levels and of consumption and asset holdings. Three types of borrowing constraints that can have important portfolio consequences are quantity constraints on borrowing, interest rate wedges, and downpayment requirements.

Quantity constraints specify borrowing limits in the form of restrictions to the allowable extent of short sales of relevant assets, without postulating differences between returns available to households with positive versus negative positions in those assets. In single-asset models of saving, the most frequently imposed restriction of this kind is of the form

$$W_t \geq 0, \forall t \tag{21}$$

essentially preventing households from being net borrowers against their future earnings potential. It is possible to impose the same constraint in portfolio models incorporating N assets, but the interpretation of the borrowing limit is now different. The constraint

$$W_t \equiv \sum_{i=1}^{N} A_{it} \geq 0 \forall t \tag{22}$$

allows borrowing in the form of short sales of any asset, provided that the sum total of such borrowing does not exceed the sum total of positive asset holdings (collateral). A more general form is:

$$W_t \equiv \sum_{i=1}^{N} b_{it} A_{it} \geq 0 \forall t \tag{23}$$

where $0 \leq b_{it} \leq 1$, that allows different collateral requirements depending on the asset used as collateral.

Perhaps the most frequently used quantity constraint in existing portfolio studies is a more extreme version of (22) which imposes no-short-sales restrictions on each asset:

$$A_{it} \geq 0 \forall i, t. \tag{24}$$

For example, in a portfolio model with bonds and stocks, this restriction would preclude borrowing at the riskless rate as well as short sales of stock. In small-scale calibrated models discussed below, this is not only more restrictive than (22) but also computationally a bit more involved, since it introduces N Lagrange multipliers to be solved for. However, in a large-scale model searching for solutions over a grid of values for endogenous variables, constraints of the form of (24) simplify computation by reducing the range of possible values for the endogenous portfolio variables A_{it}.

It is also possible to incorporate borrowing limits that depend on household labor income, perhaps as a signal of the household's ability to meet repayment schedules:

$$-B_t \leq k Y_t, \qquad k \geq 0 \tag{25}$$

where B_t is the amount of riskless asset (bond) holding in period t and the negative of this is borrowing at the riskless rate. The consequences of such constraints have been empirically investigated by Ludvigson (1999) in the context of a single asset model. The saving and portfolio effects of varying the constraint tightness parameter k have been analyzed computationally by Haliassos and Hassapis (1998).

Somewhat less researched are interest rate wedges between loan rates charged to households relative to rates available to them for investment purposes. Such interest rate wedges could be modeled for any asset i by treating positive and negative holdings of i as holdings of two surrogate assets i^+ and i^-, each with its own rate R^+_{it} and R^-_{it}, so that $R^+_{it} < R^-_{it}$ (the interest rate wedge). The model now incorporates two restrictions forcing holdings of each surrogate asset to lie on the proper side of zero:

$$A^+_{it} \geq 0 \forall t \tag{26}$$

and

$$A_{it}^{-} \leq 0 \forall t. \tag{27}$$

A modeling complication here is that the form of first order conditions tends to be sensitive to the source of the interest rate wedge.

Downpayment requirements are a form of borrowing constraints that we know little about. The most obvious example are restrictions governing downpayment for buying a house, which vary widely between countries. While it is possible to buy a house in the United States with a downpayment as low as 5 percent of the value of the house, the corresponding requirement for Italy is of the order of 50 percent. Downpayment requirements are borrowing constraints because they do not allow households to make lumpy purchases of real assets by borrowing the full amount needed for the purchase. They force households to accumulate wealth instead of forcing them to run down their wealth holdings to meet current consumption objectives as is the case with quantity constraints on borrowing. The general form of such a constraint is

$$W_{t+J} \geq \xi \, H_{t+J} \tag{28}$$

where ξ is the downpayment requirement on the value of the real asset in question (say, a house), H_{t+J} is the value of the house at the intended period of purchase, and J is the planned delay for purchasing the house. One would expect that the objective of purchasing a house at a prespecified time might well induce households not only to save more than they otherwise would but also to adopt a more conservative portfolio composition.

Two important complications associated with handling this type of constraints are that real assets enter not only budget constraints but also utility functions, since they provide services of value to the household; and that the horizon J is itself endogenous in principle, and its flexibility depends on the costs and benefits of delaying the purchase as the household accumulates more information. Although this is a fascinating area for future research, it falls outside the scope of the current volume and it will not be dealt with below.[5]

2.2.2 Stock Market Participation Costs

It would perhaps not be unfair to summarize the stockholding implications derived from typical household portfolio models to date by the statement

[5] See Cocco (1998) and Flavin and Yamashita (1998) for interesting recent research on housing.

that these models predict too much stockholding by too many households relative to what is observed in practice. Regarding the latter aspect of this stockholding puzzle, we should note that, despite the substantial expansion of the stockholder base that resulted primarily from privatization efforts and from the introduction of new pension accounts, it is unlikely that the proportion of stockholders among households significantly exceeds 50% in any country, however financially developed. A promising avenue for explaining the limited incidence of stockholding in the population is some form of fixed costs associated with entering the stock market, possibly coupled by subequent recurring costs for continued participation.

Some of these costs may be direct, e.g., in the form of brokerage or membership fees; others may be indirect and household-specific, in the sense that they involve the value of the household's time devoted to keeping up with developments in the stock market and to keeping an eye on brokers and financial advisors. The latter component can justify costs that are positively related to household income. Whatever the objective size of such entry and participation costs, what matters for participation decisions is how the household itself perceives these costs. Misperceptions, ignorance, and even prejudice can contribute to inertia by creating a wedge between the objective size of these costs and the unobservable, subjective estimate that households use for participation decisions.

Rather than attempting to calibrate such unobservable costs directly, calibrated models can be used to compute the minimum size of entry and participation costs required to keep a household with given characteristics out of the stock market. One can introduce either one-time "ticket fees" for entering the stock market, or recurring "membership fees" for continued participation in the stock market. Formally, if we denote the value function associated with participating in the stock market by V_s and the value function when using solely the bond market by V_B, the threshold cost that would make a household indifferent between participating and not participating is a function of a state variable like cash on hand, $K(X)$, such that

$$V_S(X - K(X)) = V_B(X) \tag{29}$$

Value functions are monotonic in the state variable and therefore the value functions can be inverted to derive the cost as

$$K(X) = X - V_S^{-1}(V_B(X)). \tag{30}$$

This function must be greater than zero for all levels of cash on hand since participating in the stock market is like a financial option: the investor has the right (but not the obligation) to participate in the equity market. Using methods described below, we can determine the distribution of cash on hand in the population if households only use bonds as a saving vehicle. This distribution also represents the possible outcomes of cash on hand for a given household over time. If there exists an objective estimate of the entry cost, $\overline{K}$, or a function relating it to cash on hand, $\overline{K}(X)$, it might be tempting to compare it to $K(X)$ to derive the proportion of households that would be deterred from stock market participation. This would be appropriate if these are recurring participation fees that the household needs to pay every period in order to have continued access to the stock market. If, however, costs are one-time ticket fees, then any household will, sooner or later, experience high levels of cash on hand for which the objective ticket fee is not prohibitive.

In order to handle ticket fees and/or the absence of objective measures of membership fees, one can compute an upper bound to the required costs. Using the distribution of cash on hand, one can compute the maximum level of X that any household is likely to experience, $\widehat{X}$, as that which satisfies $\Pr(X \leq \widehat{X}) = 1$. Then a level of costs equal to $K(\widehat{X})$ would make everyone indifferent between participating in or abstaining from the stock market.[6] The lower the levels of such ceilings, the more plausible are entry costs as explanations of the phenomenon. Suitable modifications are needed to handle the more difficult case of finite-horizon models and the evolution of value functions and of threshold costs over a household's horizon.

3 Calibration

3.1 Approximation of Continuous Stochastic Processes

Portfolio models incorporate uncertainty regarding at least asset returns and labor incomes. Both are continuous random variables, but need to be approximated by discrete processes for purposes of numerical computation of solutions. Depending on the desired scale of the model and closeness of approximation, there exist alternative methods for discretization of the state space.

3.1.1 Binomial Models

A simple method is to postulate two possible outcomes, a "high" and a "low" realization, such that their mean and variance match those of the original

[6] Naturally, in a general-equilibrium model, one would also need to take into account any effects of household participation decisions on equilibrium asset returns.

stochastic process. This is accomplished by taking the mean plus or minus one standard deviation as two equiprobable outcomes. Often, the time period considered encompasses more than one periods of empirical observation. This happens, for example, in small-scale models (including overlapping generations setups) in which each time period is thought of as lasting twenty to thirty years. Riskless rates of return are simply compounded over this longer time interval. Risky annual returns can be converted to a binomial process first, with a high and a low realization, as above. Then, one can compute the mean and variance of multi-year compounded returns if annual returns follow this binomial process. Finally, the high (low) realization of multi-year returns can be set equal to this mean plus (minus) their multi-year standard deviation.

When two or more sources of randomness are postulated, any desired degree of positive or negative correlation between them can be induced by altering the relative probabilities of the various states. For example, if we want to explore the implications of positive covariance between earnings and stock returns, we increase the probability of simultaneous realizations of high and of low values for both random variables relative to the probability of experiencing opposite realizations (e.g., high stock returns and low incomes). Given the requirement that high and low realizations of each separate variable remain equiprobable, the only parameter that can be freely chosen to induce the required degree of correlation is the probability of one out of the four possible states.

3.1.2 Quadrature Approximation Methods

In solving nonlinear dynamic rational expectations models, the need often arises to evaluate an integral that usually comes from taking the expectation of a function of a random variable. The problem is to find a good discrete approximation of $I = \int_a^b f(x)w(x)dx$ where $w(x)$ is usually a probability density function. The method generally used is quadrature; I is approximated using $\sum_{i=1}^{i=N} \omega_i f(x_i)$ where the quadrature nodes $\{x_i : i = 1, ..., N\}$ lie in the domain of x and the quadrature weights $\{\omega_i : i = 1, ..., N\}$ are chosen appropriately so as to make the approximation of $\int fw$ a "good" one.[7]

Gauss-Hermite quadrature is widely used to evaluate numerically the integral over a function of a normal variable. For instance, if Y is distributed $N(\mu, \sigma^2)$ and f is a continuous function of Y, then

[7] For a more detailed discussion of the practical issues involved in the numerical evaluation of a definite integral, see Chapter 7 in Judd (1998).

$$
\begin{aligned}
E\{f(Y)\} &= (2\pi\sigma^2)^{-.5} \int f(y) e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy \\
&\approx \pi^{-.5} \sum_{i=1}^{i=N} \omega_i f(\sqrt{2}\sigma x_i + \mu)
\end{aligned}
$$

For $N = 10$, the quadrature nodes and the quadrature weights are given in Judd (1998, Table 7.4).

Tauchen (1986) showed that for univariate problems, a discrete approximation of the underlying random variable over 10 points, for instance, works well in practice. Deaton and Laroque (1995) follow a similar procedure by replacing a standard normal variable with N discrete points $Z = (Z_1, ..., Z_N)$. The Z_i are chosen by first defining an $N+1$ vector $\vartheta = (\vartheta_1, ..., \vartheta_{N+1})$ with $\vartheta_1 = -\infty$ and $\vartheta_{N+1} = \infty$, and satisfying $\Phi(\vartheta_{j+1}) - \Phi(\vartheta_j) = \frac{1}{N}$ for $j = 1, ..., N$ and where Φ is the cdf of the unit normal. The vector ϑ divides the support of the normal distribution into N equiprobable intervals. The N discrete values are then chosen to be the conditional means within each interval $(i = 1, ..., N)$;

$$
Z_i = \frac{\int_{\vartheta_i}^{\vartheta_{i+1}} z d\Phi(z)}{\Phi(\vartheta_{i+1}) - \Phi(\vartheta_i)} = \frac{\phi(\vartheta_i) - \phi(\vartheta_{i+1})}{\Phi(\vartheta_{i+1}) - \Phi(\vartheta_i)} \tag{31}
$$

For $N = 10$, the 10 values are given by (± 1.75498333, ± 1.04463587, ± 0.67730694, ± 0.38649919, ± 0.12599747). Assigning a probability of one tenth for each of these nodes, gives a mean equal to zero and standard deviation equal to .964, whereas if the Gauss-Hermite quadrature is used (with $N = 10$), the mean is again zero but the standard deviation is exactly one. In some instances (especially when estimation is involved) this approximation error is worth paying if a matrix programming language like GAUSS is being used.

Tauchen and Hussey (1991) show how to extend these methods to evaluate expectations of functions of random variables that follow a Markov chain. In particular, they show how to approximate an M-dimensional process $\{y_t\}$ characterized by a conditional density $f(y_{t+1}|y_t)$. Deaton and Laroque (1995) follow a similar procedure.[8] For a serially dependent process, the same Z-values as in (31) are used but the probabilities of moving from a current state to another are affected by the Markov structure of the problem. We assume, as is usually the case, that the underlying random variable follows an

[8]Burnside (1999, pp. 106-107) provides an excellent discussion of the Tauchen and Hussey (1991) proposal and its relationship to the method described in the text.

AR(1). For a continuous distribution with autocorrelation parameter ρ, the probability that the shock in $(t+1)$ lies in the interval $(\vartheta_i, \vartheta_{i+1})$ conditional on lying in $(\vartheta_j, \vartheta_{j+1})$ is given by

$$T_{ij}(\rho) = \frac{\int_{\vartheta_i}^{\vartheta_{i+1}} \int_{\vartheta_j}^{\vartheta_{j+1}} \phi_B(x_1, x_2; \rho) dx_2 dx_1}{\Phi(\vartheta_{j+1}) - \Phi(\vartheta_j)} \tag{32}$$

where $\phi_B(x_1, x_2; \rho)$ is the standard bivariate normal density function with correlation parameter equal to ρ. The integration in (32) can be done using the relevant GAUSS routines. For a given ρ we can therefore construct an N by N transition matrix denoting the probabilities

$$prob(Z_t = z_i | Z_{t-1} = z_j) = T_{ij}(\rho) \tag{33}$$

which can be used to evaluate conditional expectations that will arise in computing the optimal policy / value function.

3.1.3 Interpolation Methods

When using disretization methods, a function is evaluated at, say, 100 grid points. It will often be necessary to evaluate the function at points not on the grid. There are two main interpolation procedures that are commonly used; linear and cubic splines interpolation (see Judd (1998, chapter 6) for more details). Linear interpolation is the simplest method to use and will work well if we are expecting the functions to be interpolated to be well approximated by a piecewise linear specification (this turns out to be the case in a lot of the portfolio choice poblems that we analyze). Cubic splines are slightly more complicated than linear interpolation but are often preferred because they lead to a smoother approximation; in particular, cubic splines are continuously differentiable and have a non-zero third derivative, thus preserving the prudence feature of the utility function. The existence of a second derivative can also be a useful attribute when estimating the model with maximum likelihood, for instance.

3.1.4 Disasters

Disastrous states of the world that result from the confluence of adverse realizations of random economic variables, such as labor incomes and stock returns, can have substantial effects on optimal portfolios even when they have small probability of occurrence. This is obviously true in rank-dependent utility models, where utility in such states receives a weight that is disproportionate to its probability of realization. However, it continues to hold even in expected-utility or Kreps-Porteus frameworks displaying constant relative risk aversion. The reason is that, under constant relative risk aversion utility, marginal utility tends to infinity as consumption tends to zero. Inclusion of such states will induce households to choose portfolios that will not lead to a very low level of consumption even in the small-probability disastrous state. In practice, this means limiting both the extent of borrowing and the exposure to stockholding risk.

In fact, inclusion of such states yields analogous effects on portfolios as postulating quantity constraints on borrowing. For example, the inclusion of a low-probability unemployment state with zero income causes self-restraint in borrowing to maintain solvency in periods of unemployment, even when banks do not impose explicit borrowing limits (see Carroll, 1997).[9] Although this approach is potentially powerful and does away with the need to consider credit market frictions in the form of quantity constraints, it still requires assumptions regarding the institutional and legal framework. For example, would it be possible for households to choose not to repay their loans in such unlikely disastrous states? Alternatively, would it be possible for them to buy unemployment insurance to cover (at least partially) these unlikely events instead of modifying their entire portfolio to accommodate those states? If such unemployment insurance does not exist, then portfolio effects continue to arise from a market failure even though we have not imposed borrowing constraints.

3.2 Preference Parameters

In calibrating non-linear rational expectation models, it has been common to refer to microeconometric studies for guidance in selecting values for different structural parameters like risk aversion, the rate of time preference, the level of idiosyncratic uncertainty faced by households, the intertemporal elasticity of labor substitution and many others. This tradition has been firmly established in real business cycle theory following Kydland and Prescott (1982), but has now spread in other branches of economics as computational methods have become more commonly used. Referring to microeconometric studies to justify a value for a specific structural parameter is no panacea, however. As Hansen and Heckman (1996, p.100) note: "It is simply not true

[9] Rietz (1988) relied on the possibility of a drastic fall in consumption accompanied by sudden drops in dividends to reduce demand for stocks and increase the equilibrium equity premium.

that there is a large shelf of micro estimates already constructed for different economic environments that can be plugged without modification into a new macro model." Hansen and Heckman suggest that "a more productive research program would provide clearly formulated theories that will stimulate more focused microeconomic empirical research." They also note, however, that "sensitivity analyses should be routine in real business cycle simulations." We would argue that, in the current state of affairs, extensive sensitivity analysis is crucial in understanding the implications of a non-linear model, and is arguably the first step in the direction of eventually confronting the model's implications with microeconomic data and, where relevant, with macroeconomic time series.

4 Solution

4.1 Small-scale Models

4.1.1 A Simple Model

Let us start with a simple small-scale model that involves few time periods and states of the world and describe how such versatile setups can be used to gain insight into the main effects of a large number of model variations at small computational cost. Small-scale models can also be used in the future as modules within overlapping generations models that analyze portfolio choice in general equilibrium.

Consider the problem of a household that lives for 3 periods, each lasting for twenty years, has no bequest motive, and its preferences can be represented using the general Epstein-Zin formulation, as formulated in Section 2.1. At the end of the first two twenty-year time periods, the household consumes and chooses portfolios to hold over the second half of working life and during retirement, respectively. Thus, the household solves a problem of the following form in period t:

$$MAX_{\{B_t,S_t\}}\left[U_t = W(C_t, \mu(U_{t+1}|I_t))\right], \tag{34}$$

subject to

$$X_{t+1} = S_t\widetilde{R}_{t+1} + B_tR_f + Y_{t+1}, \quad t = 1, 2 \tag{35}$$

$$C_t + B_t + S_t = X_t, \quad t = 1, 2 \tag{36}$$

$$C_3 = X_3 \tag{37}$$

$$C_t \geq 0, \quad t = 1, 2, 3 \tag{38}$$

$$X_1 = given \tag{39}$$

where E_0 is the mathematical expectations operator, B_t and S_t are real amounts of bonds and stocks held between the end of period t and that of period $t+1$, $\beta \equiv \frac{1}{1+\delta}$ is the constant discount factor, X_t is cash on hand at the end of period t consisting of the value of wealth holdings augmented by labor income, and the inequality budget constraint has been replaced with an equality constraint (36) because of nonsatiation. It is understood that U_4 and $\mu(U_{t+1}|I_t)$ are zero, and so are B_3 and S_3 in the absence of bequest motives. The general form of first-order conditions is given in equation (6). The transition equation (35) and the constraint (36) imply that the first order conditions for bonds and stocks at time t are, respectively:

$$-C_t^{\zeta-1} + \beta \left[f_t \left(U_{t+1}^{\alpha} \right) \right]^{\frac{\zeta}{\alpha}-1} f_t \left[U_{t+1}^{\alpha-\zeta} C_{t+1}^{\zeta-1} R_f \right] = 0, \ t = 1, 2 \tag{40}$$

and

$$-C_t^{\zeta-1} + \beta \left[f_t \left(U_{t+1}^{\alpha} \right) \right]^{\frac{\zeta}{\alpha}-1} f_t \left[U_{t+1}^{\alpha-\zeta} C_{t+1}^{\zeta-1} \widetilde{R}_{t+1} \right] = 0, \ t = 1, 2. \tag{41}$$

In our example, households consume in all three periods, and choose portfolios at the end of the first and of the second time periods in order to hold them over the coming period. Portfolio selection is subject to stock return uncertainty. At the end of the first period, the household in our benchmark model also faces earnings risk regarding its career outcome over the second half of its working life. This risk is uncorrelated with stock returns at the benchmark, but we also examine the effects of positive correlation below. No income risk is present at the end of working life (i.e. at $t = 2$), as retirement income is assumed nonrandom.

4.1.2 Specifying and Solving the Nonlinear Equation System

Solution of a small scale model such as this proceeds by specifying the full system of equations that have to be satisfied at the optimum. This consists of equations (36), (40), and (41) for all relevant time periods and states of the world, using equation (35) to make appropriate substitutions. We can then use any software that solves nonlinear systems to obtain our results with considerable speed and precision without prespecifying a grid or even range of values in which solutions for endogenous variables will lie.[10] These and other computational simplifications and advantages come at some cost, namely that the number of equations increases rapidly as we add time periods and possible states of the world (the dimensionality issue).

For each of the two random variables, we consider two possible states of the world, "high" and "low", yielding four states of the world in the second period. Now, from each of these four states (nodes) emanate two possible third-period states: one with high stock returns and one with low returns, since there is no uncertainty regarding retirement income. Thus, there are $1 + 4 + 8 = 13$ equations for consumption of the form (36), $1 + 4 = 5$ first order conditions for bond holding of the form (40), and another 5 of the form (41) for stock holding.[11] Since all endogenous variables are coded in a way that reveals their position on the event tree, we obtain one solution for each endogenous variable indexed by time period, current state, and the history of past states. It is, therefore, simple to track the evolution of cash on hand and of its various components along each possible path of states, and this helps in deriving policy functions even in more complicated settings.

Borrowing constraints, such as (25), (23) and others discussed in Section 2.2.1, can be incorporated by including Lagrange multipliers in each of the relevant first order conditions, as well as one equation (a complementary slackness condition) for each Lagrange multiplier. The size of Lagrange multipliers that signifies the shadow value of each constraint, is returned as part of the solution of the system. Our experience has shown that accurate guesses as to which constraints are binding are more important for convergence to the solution than accurate guesses as to the levels of other endogenous variables. Speed of solution is particularly helpful in trying alternative configurations of binding constraints, as well as in examining the sensitivity of results to alternative model variants and parameter specifications.

[10] For example, we use the command "fsolve" in the MATLAB optimization toolbox to solve our small-scale models. Although we need to make an initial guess regarding the likely nature of the solution, we have found that solution algorithms will normally converge to the right solution for a variety of guesses and will tend not to converge at all if the solution is wide off the mark.

[11] We need $7 + 3 + 3 = 13$ for the corresponding problem without income risk. Because of the involved nature of utility functions, we found it helpful to code separate equations for utility in each of the four second-period states, instead of substituting into first-order conditions. This brought the total number of equations solved to 27 and 17 respectively.

4.1.3 Calibration

Calibration of Labor and Retirement Incomes For purposes of uniformity, we employ Specification I for annual incomes throughout this paper (see Section 2.2.1). Results for specification II have been obtained in Bertaut and Haliassos (1997), and in Haliassos and Hassapis (1997, 1998). Since we focus on three twenty-year periods, we need to calibrate labor incomes in the first two periods and retirement income in the third. Solutions for consumption and asset stocks are homogeneous of degree one in incomes, and this affords considerable flexibility in specifying the levels of labor and retirement incomes in each period. However, since empirical estimates as well as our discussion in Section 2.2.1 above refer to annual incomes, we prefer to calibrate incomes in each twenty-year period as the present value of annual incomes derived using the relevant annual income processes and the annual riskless rate. Deriving incomes in this way also allows one to examine longer-term implications of changes in the specification of annual income processes, as discussed for example in Bertaut and Haliassos (1997).

In our end-of-period small-scale model, first period income is the present value of labor incomes received between ages 21 and 40, and it is known prior to consumption or portfolio decisions. We can compute the present value for our benchmark household by normalizing initial annual income to be unity and assuming exponential growth of annual incomes. Second-period incomes (from age 41 to 60) are assumed nonstochastic in some model variants, and when this is so, they are derived by extrapolation of this process for the next twenty years. Following specification I, the unconditional mean growth for individual annual earnings is $\mu_g - .5 * \sigma_v^2$. This is the growth rate used in both cases, with $\mu_g = 0.03$ and $\sigma_v = 0.08$.

Our benchmark model assumes that labor incomes in the second period are stochastic. Using specification I, we first simulate 20,000 twenty-year sequences of annual labor incomes, and compute their present values. To derive these annual incomes, we assume that $\ln U_{it}$, and $\ln N_{it}$ are each independent and identically (Normally) distributed with mean $-.5 * \sigma_u^2$, $-.5 * \sigma_v^2$, and standard deviations $\sigma_u = 0.1$, and $\sigma_v = 0.08$, respectively. We then compute the mean and standard deviation of the 20,000 present values. Our high- (low-) income state involves labor income equal to this expected value plus (minus) one standard deviation.[12]

[12] In what follows, results are reported in terms of that level of annual labor income which, if received every year, would yield the same present value. This facilitates comparison with levels of annual incomes used elsewhere in the paper. It is tantamount to dividing each present value by the sum total of the twenty discount factors, and it does not affect solutions normalized by current income.

Third-period (retirement) income is assumed nonstochastic. For the purpose of computing the twenty-year present value, annual retirement income is set to a constant level, equal to 70% of the annual labor income that would be obtained in the last year of working life if annual labor incomes were growing at $\mu_g - .5 * \sigma_v^2$ up to that point.[13]

In what follows, we present solutions for a single household. The specification of stochastic processes for labor incomes also allows one to aggregate across households that differ only in terms of income realizations. Bertaut and Haliassos (1997) find that such exercises do not alter the basic conclusions obtained from single-agent solutions in these small-scale models.

Calibration of Preference Parameters and Asset Returns The benchmark levels of preference parameters, where relevant, are set in our small-scale model at $(\rho, \delta, \sigma, \gamma) = (3, 0.05, 0.5, 0.5)$, where ρ is relative risk aversion, δ is the annual rate of time preference, σ is the intertemporal elasticity of substitution, and γ is the degree of overweighting of inferior states in rank dependent preferences (see Section 2.1). The annual riskless rate is set at 0.02, and the annual equity premium at 0.042, with standard deviation equal to 0.18. The riskless rate and the rate of time preference are simply compounded over twenty-year periods. The high and low values of stock returns over twenty-year periods are computed using the binomial model described in Section 3.1.[14]

Deriving Policy Functions In this paper, we employ policy functions to illustrate the results obtained from a variety of model specifications and techniques. Small-scale models can be used to derive policy functions for each time period considered, as well as to pinpoint the range of cash on hand relevant for each time period and state of the world. In our three-period example, first-period policy functions can be derived by solving the problem repeatedly for a grid of initial cash on hand that can be as wide and as fine as desirable for the model at hand. First-period solutions for real consumption, real stock holdings, and real bond holdings (all normalized by current labor income) can then be plotted against initial cash on hand, as shown in Fig. 1, which employs an expected utility specification without quantity constraints on borrowing. Young expected-utility maximizers choose to hold stocks even at very low levels of cash on hand, because stocks dominate bonds in rate of return and they have zero covariance with the marginal utility of consump-

[13] The labor income levels used in our runs are $[y1, y2h, y2l, y3] = [1.2826, 2.7793, 1.7639, 1.9908]$. Models with income certainty set $y2h = y2l = 2.2716$.

[14] The high and low rates of return on stocks used are 5.2375 and -0.5768, respectively.

tion at zero stockholding (see Haliassos and Bertaut, 1995). At lower levels of initial cash on hand, it is optimal for young households in the model to borrow at the riskless rate, so as to finance consumption above their current cash on hand, as well as the purchase of stocks that offer an equity premium. Stockholding levels increase with cash on hand, but they are less sensitive to initial resources than either consumption or riskless borrowing (or bond holding). As expected, the marginal propensity to consume out of initial cash on hand is less than one, and households with higher initial resources tend to borrow less and to invest more in stocks. However, the model also illustrates a surprising feature that underlies virtually all of the recent work in this literature, and has proved quite robust to a range of preference specifications, length of household horizon, and even to the presence of borrowing constraints. Specifically, the presence of the equity premium makes it optimal for poorer households to hold only stocks in positive net amounts, and to enrich their portfolios with positive net holdings of riskless assets only if their initial cash on hand exceeds a certain threshold (here, four times their initial labor income). Later in this paper, we shall explore the implications of this feature in a variety of setups. Yet, it poses a potentially important empirical puzzle when confronted with household-level portfolio data.

Second-period policy functions can be derived from the output of the same model runs as for first-period functions. Since the model solves for asset holdings in each period and state of the world, it is straightforward to compute second-period cash on hand for each level of first-period resources considered. Consumption and asset holdings in each second-period state can then be plotted against the corresponding level of second-period cash on hand. If these are normalized by current labor income in the relevant state, then solutions for each normalized variable will all be part of the second-period policy function for that variable. Figure 2 provides an example utilizing solutions for only two of the four states, i.e. the "best" state 1 that involves high labor incomes and high stock returns and the "worst" state 4 that involves the corresponding low realizations. By plotting observations for each state separately, one can also visualize the range of normalized cash on hand values observed in each state (indicated by the arrows in the Figure). Not surprisingly, we observe that their union is wider than the range of first-period cash on hand, because of shocks to earnings and to stock returns.

We find that the same basic shape of the policy functions is preserved in the second period, when households choose portfolios to hold over their retirement years. Although the second-period consumption function has a lower intercept than that for the first, comparison with Fig. 1 shows that its marginal propensity to consume (MPC) is higher.[15] Similarly, the bond holding function has a higher intercept but also a higher slope than in the first period. Changes in the stockholding function are hardly noticeable.

The policy function for consumption in the last period of life can be obtained in similar fashion, showing which segment is relevant for each possible life-history of stock return and earnings realizations. This feature of the computational method used to solve small-scale models can be particularly useful in cases where current policy is not simply a function of realized cash on hand but also of the prior portfolio composition (e.g., in models involving differential transactions costs across assets, or capital gains taxation on stocks). Since first- and second-period policy functions for portfolios are quite similar in shape, we focus on policy functions for the young in the remainder of this section on small-scale models.

Sensitivity to Preference Specification Figure 3 shows the effects on the first-period consumption function of adopting different preference specifications. We consider the alternatives of Kreps-Porteus (KP) and Quiggin (Q) preferences described in Sections 2.1.3 and 2.1.4 respectively. The solution method for Q models that involve nondifferentiabilities of the objective function at points that are unknown a priori, was proposed by Haliassos and Hassapis (1997) and is described in detail there. We use the benchmark parameter settings employed there, namely risk aversion equal to 3, elasticity of substitution under KP and Q preferences equal to 0.5, and overweighting of bad states with parameter γ equal to 0.5 (see Section 2.1.4). Fig. 3 shows that the three alternative consumption functions are quite close to each other at low levels of cash on hand, but the marginal propensity to consume is somewhat sensitive to preference specification, being highest for KP and lowest for EU preferences.

More significant effects are obtained for asset holdings, especially as a result of assuming Q preferences (Figs. 4 and 5). The marginal propensity to invest in stocks is somewhat lower for KP preferences at risk aversion of 3 and elasticity of substitution at 0.5 (Fig. 4). Imposing overweighting of bad states through Q preferences dramatically lowers desired stockholding and the marginal propensity to invest in stocks. At level of cash on hand equal to about 3.75 times labor income, the solution gets to a point of nondifferentiability of indifference curves with respect to stockholding, and remains there for the remainder of our range of cash on hand. As a result, optimal

[15] Older households tend to have a higher MPC in view of the shorter remaining lifetime and the more limited earnings uncertainty they face. In our example, there is no uncertainty regarding retirement income.

stockholding is insensitive to cash on hand in this range. Fig. 5 shows that Q preferences can also discourage borrowing considerably, while KP preferences actually encourage borrowing relative to the EU specification for our chosen parameters.[16]

Positive Covariance between Earnings and Stock Returns In view of the relatively high stockholding levels implied by all preference specifications, we also consider the effects of positive correlation between earnings shocks and stock returns. Such correlation enhances the correlation between stock returns and consumption, thus making stockholding less desirable. Recent empirical research suggests that such levels of correlation are relevant especially for highly educated households (see Heaton and Lucas, 1999 and Davis and Willen (1999)), and we will also explore its relevance for models with a large number of periods below.

Positive correlation can be induced by simply raising the probability of states that involve realizations of stock returns and labor incomes on the same side of their respective means (e.g., high returns and high incomes) relative to those that involve realizations on opposite sides. Fig. 6 shows that, when this is done, the policy function for stockholding under EU shifts downwards, and the marginal propensity to invest in stocks is somewhat reduced. The policy function for KP preferences under positive covariance is below that, bearing essentially the same relationship to the EU function as did their counterparts with uncorrelated labor incomes and stock returns. Combination of Q preferences with positive covariance shifts the stockholding function towards the x-axis, implying much lower levels of stockholding at low resources, but not eliminating the desire to hold stocks. Note that solutions for our range of cash on hand do not involve points of nondifferentiability of the objective function, but even if they did, such points would not involve zero stockholding. This claim is confirmed by Fig. 4 and justified in Haliassos and Hassapis (1997).

Interaction of Positive Covariance with Risk Aversion The above results were obtained for models assuming constant relative risk aversion equal to 3. Fig. 7 shows the stockholding implications of positive covariance under a higher degree of risk aversion, equal to 8, while maintaining the same assumptions for elasticity of substitution and the overweighting param-

[16]Note that comparisons between KP and EU policy functions are dependent on parameter choices. For example, had we assumed elasticity of substitution of one third instead of one half under KP preferences, they would coincide with EU preferences that impose an inverse relationship between risk aversion and elasticity of substitution.

eter (each set at .5 where relevant). The figure includes the policy function for EU preferences, risk aversion of 3, and uncorrelated labor incomes and stock returns, for reference. Imposing both higher risk aversion and positive correlation, but continuing to work within the EU framework lowers both the intercept of the stockholding function and its slope. Allowing for KP preferences has small additional effects in the same direction. Effects are small despite the fact that, at risk aversion of 8, the implied elasticity of intertemporal substitution in the EU framework is $\frac{1}{8}$, while our KP calibration assumes a much higher elasticity of $\frac{1}{2}$. More substantial effects, especially on the level of the stockholding function, are obtained when overweighting of inferior states is introduced through our Q specification. The combination of positive covariance between earnings and stock returns, high risk aversion, and overweighting of inferior states induces optimal stockholding to be close to zero at low resource levels, though it remains positive.

Precautionary Effects It is also possible to derive precautionary effects on normalized consumption and asset stocks, as functions of cash on hand. This is done by comparing the policy functions derived above with those for an identical model that simply removes earnings risk and ensures levels of labor income equal to the values that were expected when earnings risk was present. Based on the theoretical literature, we expect that the introduction of earnings risk will induce households to hold additional wealth in order to buffer consumption from shocks to labor incomes. We also expect them to lower at least the portfolio share of risky assets in their portfolios, and probably the level of normalized stockholding when confronted with uncorrelated background earnings risk.

Figures 8-10 show precautionary effects on first-period normalized consumption, stockholding, and borrowing for our original model with risk aversion of 3 and uncorrelated labor incomes and stock returns. Fig. 8 shows that the size of precautionary wealth, normalized by current labor income, is a nonlinear, decreasing function of initial cash on hand for all three preference specifications considered. This accords with intuition: since the marginal propensity to consume out of initial cash on hand is less than one, households with higher initial resources hold a larger amount of total wealth and are able to accommodate future earnings shocks with a smaller precautionary buffer. Introduction of KP preferences, tantamount to allowing the elasticity of substitution to be $\frac{1}{2}$ instead of $\frac{1}{3}$ in our calibration, induces an increase in normalized precautionary wealth by about 0.02, and its size is not particularly sensitive to the level of initial cash on hand. Much bigger positive effects are introduced by overweighting of inferior states in Quiggin preferences (of the order of 0.05). Households that are particularly concerned about utility in low-earnings and low-stock-return states, accumulate a larger precautionary wealth buffer than expected utility maximizers who assign a weight to utility in such states equal to their probability of occurrence.

Figure 9 confirms for our calibration of the EU and KP models that normalized stockholding is discouraged by the presence of uncorrelated background risk, but less so for households with higher initial resources. The sign of the effect is actually reversed for Q preferences. Such households want to accumulate such levels of precautionary wealth buffers that they end up holding more stocks as well as more risky assets (or less riskless borrowing). The spectacular feature of a peak is the consequence of our finding in Fig. 4 that solutions for stockholding in the subsequent range of cash on hand lie at a kink and remain unchanged in this range. Solutions for the corresponding model without risk (not shown) are not at a kink, however, and stockholding continues to increase in this range of cash on hand, lowering the difference with the model of risky earnings. Fig. 10 shows that, in response to this decrease in precautionary accumulation of stocks, households increase their bondholding (or reduce their riskless borrowing) in order to generate the net precautionary wealth buffer shown in Fig. 8.

Effects of Borrowing Constraints It is possible to augment this small scale model by introducing income- and collateral-based borrowing constraints of the type described in Section (2.2.1). Haliassos and Hassapis (1998) exploit the versatility of small-scale models to derive the effects of such constraints under varying degrees of constraint tightness (values of parameters k and b). They employ a model similar to the one presented here, except that it uses specification II of the income process (see Section 2.2.1) and it incorporates a bequest motive. They compute precautionary effects in the presence of borrowing constraints as differences between models with and without earnings risk, when both models incorporate borrowing constraints. They find that binding borrowing constraints of either type reduce precautionary effects on wealth relative to what would have been observed in the absence of constraints and can reduce or even reverse precautionary effects on stockholding. When earnings risk is introduced, prudence induces an increase in wealth holding, which is achieved entirely through stockholding since borrowing is at its limit. In this case, prudence dominates temperance. Such findings suggest that populations which contain a sizeable proportion of borrowing-constrained households are likely to exhibit small or insignificant effects of earnings risk on wealth holding and on the holdings of risky assets.

4.2 Large-scale Models with Infinite Horizons

Economists have traditionally constructed models in which the planning time horizon is infinite as useful benchmarks. This can be done explicitly (Ramsey, 1926) or through the existence of dynastic families that effectively behave as if their horizon is infinite (Barro, 1974). In a portfolio choice problem, the infinite horizon assumption leads to the formulation of the problem as follows:[17]

$$MAX_{\{B_t,S_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t), \tag{42}$$

subject to

$$C_t + B_t + S_t \leq X_t \tag{43}$$

$$X_{t+1} = S_t \widetilde{R}_{t+1} + B_t R_f + Y_{t+1} \tag{44}$$

$$C_t \geq 0 \tag{45}$$

with exactly the same notation as before. Since there is no distinction between work and retirement years, labor income is always given by the two equations first introduced in Section 2.2.1 above:

$$Y_t = P_t U_t \tag{46}$$

$$P_t = G_t P_{t-1} N_t. \tag{47}$$

The two short sales constraints are again:

$$B_t \ \geq 0 \tag{48}$$

$$S_t \geq 0 \tag{49}$$

[17] This section follows closely the analysis in Haliassos and Michaelides (1999).

4.2.1 Solution Methods

The Euler Equation Approach Analytical first order conditions for bonds and for stocks respectively can be written as follows:

$$U'(C_t) = \frac{1+r}{1+\delta} E_t U'(C_{t+1}) + \lambda_B \tag{50}$$

and

$$U'(C_t) = \frac{1}{1+\delta} E_t \left[U'(C_{t+1}) \widetilde{R}_{t+1} \right] + \lambda_S \tag{51}$$

where λ_B and λ_S refer to the Lagrange multipliers for the no short sales constraints on bonds and on stocks. Recalling that the budget constraint in period t is

$$C_t = X_t - B_t - S_t \tag{52}$$

where X_t is cash on hand, a binding short sales constraint on bonds, implies that $C_t = X_t - S_t$ since bond holdings are at a corner of zero. Similarly, when the constraint preventing short sales of stock is binding, (52) implies that $C_t = X_t - B_t$. The Deaton (1991) solution can be generalized to allow for portfolio choice by writing the two Euler equations in the following way:

$$U'(C_t) = MAX \left[U'(X_t - S_t), \frac{1+r}{1+\delta} E_t U'(C_{t+1}) \right] \tag{53}$$

and

$$U'(C_t) = MAX \left[U'(X_t - B_t), \frac{1}{1+\delta} E_t \widetilde{R}_{t+1} U'(C_{t+1}) \right]. \tag{54}$$

The first Euler equation is derived from the optimal choice of bond holdings: when the no borrowing constraint is binding, the maximum possible consumption is $X_t - S_t$. The second Euler equation is derived from the optimal choice of stock holdings; when the short sales constraint is binding, $C_t = X_t - B_t$.

Given the nonstationary process followed by labor income, we normalize asset holdings and cash on hand by the permanent component of earnings P_{it}, denoting the normalized variables by lower case letters (Carroll, 1992). Defining $Z_{t+1} = \frac{P_{t+1}}{P_t}$ and taking advantage of the homogeneity of degree $(-\rho)$ of marginal utility implied by CRRA preferences, we have

$$U'(x_t - s_t - b_t) = MAX\left[U'(x_t - s_t), \frac{1+r}{1+\delta}E_t U'(c_{t+1})Z_{t+1}^{-\rho}\right] \tag{55}$$

and

$$U'(x_t - s_t - b_t) = MAX\left[U'(x_t - b_t), \frac{1}{1+\delta}E_t \widetilde{R}_{t+1} U'(c_{t+1})Z_{t+1}^{-\rho}\right]. \tag{56}$$

The normalized state variable x evolves according to

$$x_{t+1} = (s_t \widetilde{R}_{t+1} + b_t R_f)Z_{t+1}^{-1} + U_{it+1} \tag{57}$$

We use the identity $c_{t+1} = x_{t+1} - b_{t+1} - s_{t+1}$ where both $b_{t+1}$and s_{t+1} will be functions of x_{t+1} to substitute out c_{t+1} on the right hand sides of (55) and (56). Given that the two conditions (given below) that guarantee the above system defines a contraction mapping are satisfied, we can solve simultaneously for $\{s(x), b(x)\}$. Starting with any initial guess (say $s(x) = .1 * x$ and $b(x) = .1 * x$), we use the right hand side of the first Euler equation to get an update for b and continue doing so until b converges to its time invariant solution b_1^* (see Deaton (1991)). We then use the second Euler equation with b_1^* taken as given, to find the solution for the time invariant optimal s, call it s_1^*. We know have two updated functions $\{s_1^*, b_1^*\}$; the process can be repeated until these functions converge to their time invariant solutions.

In order for the algorithm to work, we must make sure that the two functional equations of interest define a contraction mapping. The two conditions that must be satisfied for the individual Euler equations (55) and (56) to define a contraction mapping for $\{b(x), s(x)\}$ respectively are the conditions needed for Theorem 1 in Deaton and Laroque (1992) to hold. For (55) we must have

$$\frac{1+r}{1+\delta}E_t Z_{t+1}^{-\rho} < 1 \tag{58}$$

and for (56) the chosen parameters must satisfy

$$\frac{1}{1+\delta}E_t \widetilde{R}_{t+1} Z_{t+1}^{-\rho} < 1 \tag{59}$$

If these conditions hold simultaneously, there will exist a unique set of optimum policies satisfying the two Euler equations. We next simplify these conditions to gain an intuitive understanding of the economics of the problem. Given that $Z_{t+1} = G_{t+1}N_{t+1}$, with $\{G, N\}$ being log normally distributed and independent of each other, we have $E_t(G_{t+1}N_{t+1})^{-\rho} = \exp(-\rho\mu_g + \frac{\rho^2\sigma_g^2}{2}) * \exp(-\rho\mu_n + \frac{\rho^2\sigma_n^2}{2})$. Assume for now that stock returns are uncorrelated with Z. Then

$$\begin{aligned} E_t\widetilde{R}_{t+1}Z_{t+1}^{-\rho} &= E_t\widetilde{R}_{t+1}E_tZ_{t+1}^{-\rho} \\ &= (1+\mu_r) * \exp(-\rho\mu_g + \frac{\rho^2\sigma_g^2}{2}) * \exp(-\rho\mu_v + \frac{\rho^2\sigma_n^2}{2}) \end{aligned} \tag{60}$$

Taking logs of the two conditions and using the approximation $\log(1+x) \approx x$ for small x, (58) becomes

$$\frac{r-\delta}{\rho} + \frac{\rho}{2}(\sigma_g^2 + \sigma_n^2) < \mu_g + \mu_n \tag{61}$$

which is the condition derived by Deaton (1991) with $\sigma_g^2 = 0$ and $\mu_n = 0$. (59) becomes

$$\frac{\mu_r-\delta}{\rho} + \frac{\rho}{2}(\sigma_g^2 + \sigma_n^2) < \mu_g + \mu_n \tag{62}$$

Note that the two conditions collapse into one when the stock market investment opportunity has the same return characteristics as the risk free rate.

With a positive equity premium ($\mu_r > r$), satisfaction of (62) guarantees (61). Impatience must now be even higher than in the saving model to prevent the accumulation of infinite stocks, since the condition involving $\mu_r - \delta$ must be satisfied. Two other distinct cases can also guarantee the existence of a solution. First, a high expected earnings growth profile (as measured by μ_g) guarantees that the individual will not want to accumulate an infinite amount of stocks or bonds but would rather borrow now, expecting earnings to increase in the future. Second, if the rate of time preference exceeds the expected stock return, more risk averse (higher ρ) individuals will not satisfy the convergence conditions.

The Value Function Approach The value function approach for deriving the optimal policy functions utilizes the indirect utility function from the Bellman equation (see Cocco, Gomes and Maenhout (1999), for instance). We will analyze this approach in more detail in the next section that solves for optimal portfolio choice over the life cycle. We give the main idea (based on backward induction) here though; start by postulating an end of period value function (a common assumption is that consumption equals cash on hand in the terminal period) and then pick the optimal bond and stock holdings of the second to last period that maximize this last period value function. Once this is done, the value function for the second to last period can be determined and the procedure can be iterated backwards in the same way. If the convergence condition for a contraction mapping (see (62)) is satisfied, then the value and policy functions will converge to the infinite horizon solution.

4.2.2 Policy Function Results

We set the rate of time preference, δ, equal to 0.1, and the constant real interest rate, r, equal to 0.02. Carroll (1992) estimates the variances of the idiosyncratic shocks using data from the *Panel Study of Income Dynamics*, and our baseline simulations use values close to those: around 0.1 percent per year for σ_u and 0.08 percent per year for σ_v. We set the mean aggregate labor income growth rate, denoted μ_g, equal to 0.03. The coefficient of relative risk aversion is set equal to 8.

Figures 11, 13, and 14 show respectively consumption, stock holdings, and bond holdings, each normalized by the permanent component of income, as functions of similarly normalized cash on hand. Figure 12 plots the share of financial wealth held in the risky asset for different levels of cash on hand for relative risk aversion coefficients equal to 6, 7, and 8. Figure 11 shows that, at levels of normalized cash on hand below a cutoff x^* (typically around 97% of the permanent component of labor income), the household is bound by both short sales constraints (Figs. 13 and 14). Unable to borrow at the riskless rate, it is even willing to engage in short sales of stock so as to boost consumption, only to come up against the short-sales constraint on stocks.

Figure 11 shows that households with normalized cash on hand above x^* start saving, but they first put all their savings in stocks. This confirms the portfolio specialization result of Heaton and Lucas (1997), for a different earnings process. As shown by Haliassos and Michaelides (1999), the source of this result and its robustness is that under no stockholding and no correlation between earnings and stock returns, we have

$$\frac{1}{1+\delta}E_t\left[U'(C_{t+1})\right]\ E_t\left[\widetilde{R}_{t+1}-R_f\right]=\lambda_B-\lambda_S \qquad (63)$$

Given nonsatiation and an equity premium, the left hand side of (63) is positive, i.e. $\lambda_B > \lambda_S$. Thus, households in the neighborhood of x^* would like to borrow risklessly not only to consume but also to invest in stocks that offer an equity premium and have zero covariance with consumption. Prevented from borrowing, they devote all saving to stocks. Changes in the degree of risk aversion cannot reverse this result, since they do not affect the sign of marginal utility. The same holds for habit persistence. As long as there is an equity premium, its size does not matter, either. As long as we consider earnings processes that are uncorrelated with stock returns, $\lambda_B > \lambda_S$ continues to hold. The robustness of this tendency of portfolio models to predict concentration of asset holding in stocks at low levels of cash on hand (a manifestation of which we already saw in small scale models) is particularly troublesome when model predictions are confronted with household-level data.

Fig. 13 also shows that normalized stock holdings are increasing in risk aversion at levels of normalized cash on hand that justify saving, while Fig. 12 shows that the portfolio share remains unaffected by risk aversion over a range of cash on hand. This result is due to a conflict between risk aversion and "prudence" in the presence of binding short sales constraints. Since prudence is positively related to risk aversion, households want to increase their net wealth beyond x^* (Fig. 11), but none of this increase comes from changes in realized borrowing, which is still at zero because of the binding short sales constraint (Fig. 14). Their desire to increase wealth dominates their motive to reduce exposure to stockholding risk, leading to increased stockholding for higher degrees of risk aversion.

4.2.3 Policy Rules with Positive Correlation Between Stock Market Returns and Labor Income

Positive correlation between labor incomes and stock market returns raises the covariance between the marginal utility of consumption and stock returns at any given level of stockholding. Can the unrealistic portfolio specialization result be eliminated by moderate correlation between stock returns and either transitory or permanent shocks to labor income under short sales constraints?

In unreported experiments, we found that positive correlation between stock returns and transitory earnings shocks is unlikely to be important in reversing the portfolio specialization result. Figures 15 to 18 illustrate the effects of positive correlation between stock returns and permanent shocks to labor income equal to 0.1, 0.3, and 0.5. For correlation of 0.3, the household still enters the stock market first, but the range of cash on hand for which only stocks are used is already severely limited (Fig. 17). At correlation of 0.5, we find that zero stockholding is generated.

How plausible is this level of correlation? Davis and Willen (1999) obtain correlation estimates ranging between .1 and .3 over most of the working life for college educated males and around $-.25$ at all ages for male high school dropouts.[18] Heaton and Lucas (1999a) argue that entrepreneurial risk is positively correlated with stock returns and reaches levels around .2. These numbers appear smaller than needed to explain zero stockholding. Moreover, they are of the opposite sign for these categories; they come close to generating zero stockholding for college graduates or enterpreneurs who in fact tend to hold stocks, and they predict that low education households should actually be holding stocks as a hedging instrument when in fact they tend not to do so.

4.2.4 Deriving Time Series Moments

Individual policy functions are informative about microeconomic behavior; nevertheless, we are very often interested in either the aggregate or the time series implications of a microeconomic model. One usual way of investigating the aggregate or time series implications of non-linear microeconomic models is to simulate individual life histories over time by generating the random shocks from the exogenous distributions of the model and then use the computed policy functions to derive aggregate statistics over time. In the current model, however, normalized cash on hand follows a renewal process and therefore the aggregate or individual time series implications of the model can be derived by computing the time invariant distribution of cash on hand.[19]

To find the time invariant distribution of cash on hand, we first compute the bond and stock policy functions; $b(x)$ and $s(x)$ respectively. Note that the normalized cash on hand evolution equation is

$$\begin{aligned} x_{t+1} &= [b(x_t)R_f + s(x_t)\widetilde{R}_{t+1}]\frac{P_t}{P_{t+1}} + U_{t+1} \quad (64) \\ &= w(x_t|\widetilde{R}_{t+1}, \frac{P_t}{P_{t+1}}) + U_{t+1} \end{aligned}$$

[18] They use the Annual Demographic Files of the March Current Population Survey (CPS) to construct panel data on mean annual earnings between 1963 and 1994.

[19] Computation of the invariant distribution offers high numerical accuracy at a low computational cost; time invariant probabilities are equivalent to simulating an infinite number of invividuals (or a single individual over an infinite number of periods). Deaton and Laroque (1995) use the time invariant distribution to estimate a speculative storage commodity price model. Michaelides and Ng (forthcoming) use the time invariant distribution to assess the simulation bias when simulation is used to compute moments (as opposed to the time invariant distribution).

where $w(x)$ is defined by the last equality and is conditional on $\{\widetilde{R}_{t+1}, \frac{P_t}{P_{t+1}}\}$. Denote the transition matrix of moving from x_j to x_k,[20] as T_{kj}. Let Δ denote the distance between the equally spaced discrete points of cash on hand on the grid. The risky asset return $\widetilde{R}$ and $\frac{P_t}{P_{t+1}}$ are discretized using 10 grid points respectively: $R = \{R_l\}_{l=1}^{l=10}$ and $\frac{P_t}{P_{t+1}} = \{GN_m\}_{m=1}^{m=10}$. $T_{kj} = \Pr(x_{t+1=k}|x_{t=j})$ is found using

$$\sum_{l=1}^{l=10}\sum_{m=1}^{m=10} \Pr(x_{t+1}|x_t, \widetilde{R}_{t+1} = R_l, \frac{P_t}{P_{t+1}} = N_m) * \Pr(\widetilde{R}_{t+1} = R_l) * \Pr(\frac{P_t}{P_{t+1}} = N_m) \tag{65}$$

where both the independence of $(\widetilde{R}_{t+1}, \frac{P_t}{P_{t+1}})$ from x_t and the independence of $\frac{P_t}{P_{t+1}}$ from $\widetilde{R}_{t+1}$ were used. Numerically, this probability is calculated using

$$T_{kjlm} = \Pr(x_k + \frac{\Delta}{2} \geqslant x_{t+1} \geqslant x_k - \frac{\Delta}{2}|x_t = x_j, \frac{P_{it}}{P_{it+1}} = N_m, R_{t+1} = R_l)$$

Making use the approximation that for small values of σ_u^2, $U \sim N(\exp(\mu_u + .5 * \sigma_u^2), (\exp(2 * \mu_u + (\sigma_u^2)) * (\exp(\sigma_u^2) - 1)))$, and denoting the mean of U by $\overline{U}$ and its standard deviation by σ, the transition probability conditional on N_m and R_l then equals

$$\begin{aligned} T_{kjlm} &= \Phi(\frac{x_k + \frac{\Delta}{2} - w(x_t|N_m, R_l) - \overline{U}}{\sigma} \geqslant x_{t+1} \geqslant \frac{x_k - \frac{\Delta}{2} - w(x_t|N_m, R_l) - \overline{U}}{\sigma} \\ |x_t &= x_j, \frac{P_{it}}{P_{it+1}} = N_m, R_{t+1} = R_l) \end{aligned}$$

The unconditional probability from x_j to x_k is then given by

$$T_{kj} = \sum_{l=1}^{l=10}\sum_{m=1}^{m=10} T_{kjlm} \Pr(N_m) \Pr(R_l) \tag{66}$$

Given the matrix T, the probabilities of each of the states are updated by

$$\pi_{kt+1} = \sum_j T_{kj} * \pi_{jt} \tag{67}$$

[20]The normalized grid is discretized between $(x\min, x\max)$ where $x\min$ denotes the minimum point on the equally spaced grid and $x\max$ the maximum point.

so that the invariant distribution can be found by repeatedly multiplying the transition matrix by itself until all its columns stop changing. The invariant distribution π is instead calculated (faster) as the normalized eigenvector of T corresponding to the unit eigenvalue by solving the linear equations

$$\begin{pmatrix} T-I & e \\ e' & 0 \end{pmatrix}\begin{pmatrix} \pi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{68}$$

where e is an M-vector of ones.

Once the limiting distribution of cash on hand is derived, average cash on hand can be computed using

$$\sum_j \pi_j * x_j \tag{69}$$

Similar formulae can be used to compute the mean, median and standard deviations of the variables of interest.

The invariant distribution of normalized cash on hand can be used to show that mean and median bondholding are zero in the infinite horizon model. Consistent with policy functions, mean and median normalized stock holdings are not only positive, but also increasing in risk aversion. Such portfolio behavior by the more risk averse is justified, since it results in smaller standard deviation of normalized consumption, as well as in higher mean normalized consumption.

4.3 Stock Market Mean Reversion

The predictability of the excess return of stocks over Treasury Bills is now considered a stylized fact in finance (see Cochrane (1999)).[21] Samuelson

[21] See Campbell (1987b), Campbell and Shiller (1988), Fama and French (1988, 1989), Flood, Hodrick and Kaplan (1986), Campbell (1991), Hodrick (1992), Lamont (1998) and Campbell, Lo, and MacKinlay (1997), Chapter 7). Some financial indicators found to forecast excess returns over Treasury Bills include the ratios of price to dividends, price to earnings, or dividends to earnings. More recently, Lettau and Ludvigson (1999) argue that transitory deviations of wealth from an estimated trend with aggregate consumption and aggregate labor income can account for a substantial fraction of the variation in future stock market returns in post-war US data. The idea is similar to the Campbell (1987a) "saving for a rainy day" equation but in the context of asset pricing. Campbell (1987a) has shown that higher current saving implies lower expected labor income according to the modern day version of the Permanent Income Hypothesis. In the context of an asset pricing model, Lettau and Ludvigson (1999) show that a lower consumption to wealth ratio implies that consumers expect the higher stock market gains to be reversed in the future.

(1969) and Merton (1969, 1971) have shown that time variation in investment opportunities affects portfolio choice unless investors have unit relative risk aversion. Merton (1973) further emphasized the importance of intertemporal hedging demands when investment opportunities are time varying; risk averse consumers-investors want to hedge against adverse changes in both their consumption and their investment opportunity sets.[22]

Stock market predictability can provide a rationalization of the observed co-existence of bonds and stocks in the portfolio. Michaelides (1999) analyzes a model with stock market mean reversion and undiversifiable labor income risk.[23] Letting $\{r_f, r_t\}$ denote the net risk free rate and the net stock market return respectively and f_t being the factor that predicts future excess returns, we have

$$r_{t+1} - r_f = f_t + z_{t+1} \tag{70}$$

$$f_{t+1} = \mu + \phi(f_t - \mu) + \varepsilon_{t+1} \tag{71}$$

[22] Barberis (1999) and Brennan, Schwartz and Lagnado (1997) use the positive finding of stock market predictability to analyze its implications for portfolio choice (and asset allocation) and find that stock market exposure varies substantially as a response to the predictive factor(s). Campbell and Koo (1997) focus on the consumption-saving decision of an individual facing a time varying real interest rate while Campbell, Cocco, Gomes, Maenhout and Viceira (1998) extend this work to include portfolio choice and solve for optimal consumption and stock market allocation when excess stock market returns exhibit mean reversion; once more the investor is shown to be a very aggressive market timer. Campbell and Viceira (1999) use log linearization techniques to derive analytical expressions for stock market allocations when investment opportunities are time varying and find that market timing can lead to substantial variability in the stock market allocation over time. Given this substantial variability in portfolio allocation, ignoring the signal from the factor predicting future returns leads to substantial welfare losses. Viceira (1999) investigates the effect of labor income variability and investment horizon on asset allocation when excess returns are i.i.d. and rigorously verifies the popular advice (see Malkiel (1996), for instance) in the financial management industry that higher exposure in the stock market be taken during working life with a shift towards safe assets after retirement. Balduzzi and Lynch (1999) examine the loss in utility for a consumer who ignores stock market predictability in the presence of realistic transaction costs, borrowing and short sales constraints and a finite investment horizon. They find that the utility costs of behaving myopically and ignoring predictability can be substantial, unless intermediate consumption is allowed for. They interpret their results as evidence that ignoring predictability might carry a heavier cost for institutional investors who manage assets for the long run, than for individual investors who consume as time goes by.

[23] See Campbell, Cocco, Gomes, Maenhout and Viceira (1998) and Campbell (1999) for the empirical evidence supporting this specification for US returns.

where the two innovations $\{z_{t+1}, \varepsilon_{t+1}\}$ are contemporaneously correlated.

Mean reversion in the stock market is captured by the autoregressive nature of the factor (f_t) predicting stock market returns ($\phi > 0$). The autoregressive nature of the factor is captured by a ten point discretization scheme. Labelling the m factor states $i = 1, ..., m$, there are m bond and stock demand functions defined by the two Euler equations as the solutions to the functional equations

$$\begin{aligned} U'(x - s(x,i) - b(x,i)) &= MAX[U'(x - s(x,i)), \\ &\quad \frac{1+r}{1+\delta} E_t Z_{t+1}^{-\rho} U'(x' - s(x',j) - b(x',j))] \end{aligned} \tag{72}$$

and

$$\begin{aligned} U'(x - s(x,i) - b(x,i)) &= MAX[U'(x - b(x,i)), \\ &\quad \frac{1}{1+\delta} E_t \widetilde{R}_{t+1} Z_{t+1}^{-\rho} U'(x' - s(x',j) - b(x',j))] \end{aligned} \tag{73}$$

where primes are used to denote next period variables[24], j denotes the factor value expected for next period and lower case variables are normalized by P_t. The endogenous state variable x evolves as in (57). A sufficient condition for existence and uniqueness is the generalization of (62), given by

$$\frac{r_f + f_t - \delta}{\rho} + \frac{\rho}{2}\sigma_n^2 < \mu_g + \mu_n \tag{74}$$

Figures 19-22 depict the resulting policy functions.[25] When the factor predicting stock returns follows an AR(1) process, there is an incentive for the individual to "time the stock market". The market timing component of the demand for stocks is illustrated by comparing the policy functions from

[24] U' denotes marginal utility, all other variables with primes denote next period variables.

[25] $\delta = 0.12, r = 0.01$, $\sigma_u = 0.1, \sigma_n = 0.08$, $\mu_g = 0.03, \rho = 3$. The high discount rate is chosen to accomodate the convergence conditions (74) for all factor realizations. The parameters describing the evolution of stock market returns are selected from Campbell (1999, Table 2C) who reports parameter estimates for a VAR model based on annual US data between 1891 and 1994. They are $\mu = .042, \phi = .798, \sigma_z^2 = .0319, \sigma_\varepsilon^2 = .9^2 * .001$, and $\sigma_{z,\varepsilon} = -.0039$. Campbell (1999, Table 2C) estimates r_f to be .0199 and $\sigma_\varepsilon = .001$. We decrease both quantities so that the convergence condition (74) can be satisfied for all factor state realizations. Calibrating the model over the life cycle need not satisfy the condition and can therefore allow richer experimentation with different parameter values.

this case with the i.i.d. model. The policy functions are plotted in figure 19 for $\rho = 3$. A few observations can be made about the shape of the policy functions. First, a high current factor realization signifying higher future returns induces an increase in saving to take advantage of more favorable future investment opportunities while a very low factor realization makes saving less desirable and induces an increase in consumption. The increase in saving to take advantage of higher future returns can be thought of as a speculative demand for saving. Second, the composition of savings is substantially changed conditional on the factor realization. A high factor realization this period signifies higher future returns, and therefore generates additional demand for stocks compared to the i.i.d. case. With current realizations above the mean[26] (five cases in total in the discretization scheme chosen), the stock market allocation is higher than in the i.i.d. model (see figure 21) due to the increase in total saving (see figure 19). Nevertheless, the borrowing constraint provides an upper bound on the ability of market timing to generate additional demand for stocks; the maximum amount that can be invested in the stock market is total savings on account of the borrowing constraint. Since the borrowing constraint is already binding in the i.i.d. model, the additional demand for stocks comes only from the increase in saving. The share of wealth invested in the stock market stays the same as in the i.i.d. model, therefore, and equals one; the consumer would like to borrow to invest in the stock market but is unable to do so (figure 20).

On the other hand, for the five cases where the current factor realization is below its mean, the demand for stocks (relative to the i.i.d. model) falls, since the factor is signalling lower returns in the future. There are now substantial portfolio allocation effects since the borrowing constraint does not prevent the individual from lowering the proportion of stocks in the portfolio and indeed the individual aggressively lowers the stock market exposure (figure 20). Moreover, market timing becomes so important that for the two lowest realizations of the factor (signalling very low future stock market returns) the investor allocates savings completely in the bond market (figure 22). On account of the future low stock market returns, the investor is now even willing to short the stock market position, but is prevented from doing so from the short sales constraint.

To see why this is happening, we must go back to the Euler equations. For

[26]The i.i.d. model policy functions have been computed by setting the mean equity premium equal to 4.2 percent; the factor also has the same unconditional mean in the AR(1) model. In the AR(1) model, the conditional expectation of next period returns (conditional at time t information) equals f_t (see (70)). As a result, a realization of the current factor above the mean equity premium (.042) generates additional demand for stocks relative to what would happen in the i.i.d. economy.

the two lowest realizations of the factor, the consumer is saving everything in the bond market. For this to be the case, the normalized versions of (53) and (54) imply that[27]

$$\frac{1+r}{1+\delta}E_t\{Z_{t+1}^{-\rho}U'(c_{t+1})\} > \frac{1}{1+\delta}E_t\{\widetilde{R}_{t+1}Z_{t+1}^{-\rho}U'(c_{t+1})\} \tag{75}$$

with equality holding when neither constraint is binding.[28] When $s_t = 0$ and stock returns are uncorrelated with labor income shocks, (75) is equivalent to[29]

$$\frac{1+r}{1+\delta}E_t\{Z_{t+1}^{-\rho}U'(c_{t+1})\} > \frac{1}{1+\delta}E_t\{\widetilde{R}_{t+1}\}E_t\{Z_{t+1}^{-\rho}U'(c_{t+1})\} \tag{76}$$

The constraint ($s_t = 0$) will therefore continue to bind for as long as $(1+r) > E_t\widetilde{R}_{t+1} = 1 + r + f_t$ (see (70)). This is indeed the case for the two lowest realizations of the factor state ($f_t = -.04, -.01$ respectively). Intuitively, the expected next period return on the risky asset conditional on the factor realization at time t is lower than the risk free rate and therefore the riskless asset dominates the stock market investment as a saving vehicle.

Introducing positive correlation between labor income innovations and stock market returns increases the hedging demand for bonds. Time series moments (either through simulation or the use of the time invariant distribution) confirm the co-existence of bonds and stocks in portfolios; see Michaelides (1999). Nevertheless, median stockholding (counterfactually) remains equal to one, while the volatility of stock market trading that arises from the market timing activity is very high.

4.4 Large-scale Models with Finite Horizons

Let us now turn to large-scale portfolio models that incorporate many time periods and analyze the time series patterns of consumption, saving, and portfolio choices for each household over the life cycle. The methods presented for handling small-scale models are not suitable in general for this case, because the number of periods and possible states of the world in each period soon result in a very large system of simultaneous equations that

[27] For notational convenience I suppress the dependence of consumption on the current factor (f_t).

[28] $c_{t+1} = g(Z_{t+1}^{-1}b_tR_f + U_{t+1})$ when $s_t = 0$, where g is a non-decreasing, non-linear function.

[29] This step utilizes the fact that $s_t = 0$ and that $\widetilde{R}_{t+1}$ is uncorrelated with $Z_{t+1}^{-\rho}U'(c_{t+1})$.

is cumbersome to program and difficult to solve. Different approaches are needed, which have their advantages and disadvantages compared to solution of small-scale models. Consider the following problem of a household that lives for $T+1$ periods:

$$MAX_{\{S_{it},B_{it}\}_{t=1}^{T}} E_1 \sum_{t=1}^{T} \beta^{t-1} \{\Pi_{j=0}^{t-1} p_j\} U(C_{it}), \tag{77}$$

subject to

$$C_t + B_t + S_t \leq X_t \tag{78}$$

$$X_{t+1} = S_t \widetilde{R}_{t+1} + B_t R_f + Y_{t+1} \tag{79}$$

$$C_t \geq 0 \tag{80}$$

where E_1 is the mathematical expectations operator, B_t and S_t are real amounts of bond and stock holding respectively, $\beta \equiv \frac{1}{1+\delta}$ is the constant discount factor, and X_t is cash on hand at the beginning of period t, consisting of the value of wealth holdings augmented by labor income. We allow for uncertainty in T in the manner of Hubbard, Skinner and Zeldes (1995). The probability that a consumer/investor is alive at time $(t+1)$ conditional on being alive at time t is denoted by p_t $(p_0 = 1)$. Bequests are not left at the end of life: the numerical solution can easily accomodate a bequest motive. During working years, $1 \leq t \leq T-k-1$, labor income is given by the following two equations, as discussed in Section 2.2.1 above:

$$Y_t = P_t U_t \tag{81}$$

$$P_t = G_t P_{t-1} N_t. \tag{82}$$

In the k retirement years, $T-k < t \leq T$, pension income is a fraction c of permanent income

$$Y_t = c P_t \tag{83}$$

where c lies between zero and one. We introduce two constraints precluding short sales of either stocks or bonds, as discussed in Section 2.2.1:

$$B_t \geq 0 \tag{84}$$

$$S_t \geq 0 \tag{85}$$

4.4.1 The Value Function Approach

The "value function" approach involves the repeated use of backward induction on the value function. Let us specialize the problem in Section **??** to the case of constant relative risk aversion. Then, the Bellman equation associated with the problem is[30]

$$V_t(X_t, P_t) = MAX_{\{S_t, B_t\}} \left[\frac{C_t^{1-\rho}}{1-\rho} + \beta E_t V_{t+1} \left(\left[S_t \widetilde{R}_{t+1} + B_t R_f + Y_{t+1} \right], P_{t+1} \right) \right] \tag{86}$$

where $V_t(\cdot)$ denotes the value function which depends on the age of the individual and thus has a time subscript, and the first argument of $V_{t+1}(X_{t+1}, P_{t+1})$ has been substituted using equation (79). Cocco, Gomes and Maenhout (1999) use backward induction on (86) to derive the optimal policy functions. In the last period and without a bequest motive ($C_T = X_T$), the value function corresponds to the indirect utility function. This value function can be used to compute the policy rules for the previous period and given these, obtain the corresponding value function. The procedure is then iterated backwards to the beginning of working life. Standard grid search can be used to avoid the danger of choosing local optima; this can be done by discretizing the set of admissible values for the decision variables using equally spaced grids. For a given level of cash on hand (which is also discretized), the decision variables that maximize the value function are chosen by evaluating the value function at all possible decision variable nodes, and picking the maximands. Quadrature methods can be used to take expectations of random variables and interpolation can be used to evaluate the value function for points not on the grid, as previously discussed.

A slightly different approach can utilize the fact that the value function is homogeneous of degree $(1-\rho)$, a property that it inherits from the isoelastic felicity function.[31] This property can be used to reduce the number of state variables from three (X_t, P_t, Age_t) to two (the ratio $x_t \equiv \frac{X_t}{P_t}$ and Age_t) rewriting the Bellman equation (86) as

[30] See Cocco, Gomes and Maenhout (1999).

[31] In fact, Merton has shown that the value functions for problems with HARA felicity functions inherit the functional form of the felicity function. Homogeneity follows from the same arguments as in proposition 4 and lemma 1 in Koo (1995). Viceira (1998) uses a similar normalization (divides by the level of earnings).

$$MAX_{\{s_t(x_t),b_t(x_t)\}}\left[\frac{c_t^{1-\rho}}{1-\rho}+\beta E_t V_{t+1}\{\frac{P_t}{P_{t+1}}\}^{1-\rho}\left(\left[s_t\widetilde{R}_{t+1}+b_t R_f\right]\frac{P_t}{P_{t+1}}+U_{t+1}\right)\right] \tag{87}$$

where $s_t=\frac{S_t}{P_t}$, $b_t=\frac{B_t}{P_t}$ are the normalized holdings of stocks and bonds respectively, and U_{t+1} is the transitory earnings shock which enters as the ratio of Y_{t+1} to the normalization factor P_{t+1}. The short sales constraints (84) and (85) bound b_t and s_t, respectively, from below at zero.

Backward induction on the value function proceeds by (again) making a assumptions about the last period of life, deriving from them last period optimal consumption and maximized utility, and then solving the problem backwards to the beginning of life. This induction method produces the value functions, $V_t(x_t)$, and the policy functions, $b_t(x_t)$ and $s_t(x_t)$, for each period. For example, assuming that there is no bequest motive and the time of death (T) is known with certainty, $b_T(x_T)=s_T(x_T)=0$, and therefore $c_T(x_T)=x_T$. Thus the value function in the last period of life is simply:

$$V_T(x_T)=\frac{c_T^{1-\rho}}{1-\rho}=\frac{x_T^{1-\rho}}{1-\rho}. \tag{88}$$

Using this last period value function, we can maximize (87) over $\{s_{T-1},b_{T-1}\}$ and then evaluate the expression to get $V_{T-1}(x_{T-1})$:

$$V_{T-1}(x_{T-1})=MAX_{\{s_{T-1}(x_{T-1}),b_{T-1}(x_{T-1})\}}\left[\frac{c_{T-1}^{1-\rho}}{1-\rho}+\beta E_{T-1}\{\frac{P_{T-1}}{P_T}\}^{1-\rho}\left(\frac{x_T^{1-\rho}}{1-\rho}\right)\right], \tag{89}$$

where $c_{T-1}=x_{T-1}-b_{T-1}-s_{T-1}$ and $x_T=\left[s_{T-1}\widetilde{R}_T+b_{T-1}R_f\right]\frac{P_{T-1}}{P_T}+U_T$. Maximization is achieved by discretizing the state variable x. The two choice variables (b and s) are discretized over two grids comprised of n_b and n_s points respectively. Expectations are taken using quadrature. Given V_T , we can then evaluate the maximand in (89) for all possible pairs (on our grid) of (b,s) and then simply pick the pair (b,s) that achieves this maximization. Evaluating expression (89) at this pair of values (b,s), we get $V_{T-1}(x_{T-1})$. This procedure can then be repeated backwards.

4.4.2 The Euler Equation Approach

The model can also be solved using the first-order conditions. Analytical first order conditions for bonds and stocks respectively can be written as in the infinite horizon setup, but are now age dependent:

$$U'(C_t) = MAX\left[U'(X_t - S_t), \frac{1+r}{1+\delta}E_tU'(C_{t+1})\right] \quad (90)$$

and

$$U'(C_t) = MAX\left[U'(X_t - B_t), \frac{1}{1+\delta}E_t\widetilde{R}_{t+1}U'(C_{t+1})\right]. \quad (91)$$

where t now denotes the age of the individual, a new (relative to the infinite horizons case) state variable.

Given the nonstationary process followed by labor income, we normalize asset holdings and cash on hand by the permanent component of earnings Y_t^p, denoting the normalized variables by lower case letters. Assuming constant relative risk aversion and taking advantage of the homogeneity of degree $-\rho$ of the marginal utility function, we have

$$U'(x_t - s_t - b_t) = MAX\left[U'(x_t - s_t), \frac{1+r}{1+\delta}E_tU'(c_{t+1})Z_{t+1}^{-\rho}\right] \quad (92)$$

and

$$U'(x_t - s_t - b_t) = MAX\left[U'(x_t - b_t), \frac{1}{1+\delta}E_t\widetilde{R}_{t+1}U'(c_{t+1})Z_{t+1}^{-\rho}\right]. \quad (93)$$

with $x_{t+1} = (s_t\widetilde{R}_{t+1} + b_tR_f)Z_{t+1}^{-1} + U_{it+1}$.

Equations (92) and (93) comprise a system with two unknowns, $s(x_t)$ and $b(x_t)$, once a functional form for $c_{t+1}(x_{t+1})$ is given. An assumption must therefore be made about what happens in the last period of life. For example, if there is no bequest motive, it is natural for the household to consume all of its resources in the last period before its death. In this case, $c_T = x_T$, and the functional form is determined for period T. Using this, we can begin solving simultaneously this system of Euler equations using backward induction, as is customary for dynamic programming problems.

Two questions arise: (a) Do solutions for $\{s(x_t), b(x_t)\}$ that satisfy (92) and (93) exist? (b) Are these solutions unique? If we assume that c_{t+1} is an increasing function of cash on hand, then one can easily show that given $s(x_t)$ the right hand side of (92) is decreasing in $b(x_t)$ while the left hand side is increasing in $b(x_t)$ guaranteeing existence and uniqueness of a solution for

$b(x_t)$ from the bond Euler equation. The argument works in exactly the same fashion for $s(x_t)$ by symmetry (given $b(x_t)$ now).

The proposed algotrithm takes the following form: (1) Given an initial guess about $s(x_t)$, find $b(x_t)$ from (92) using a standard bisection algorithm.[32] (2) Given $b(x_t)$ from (1), find $s(x_t)$ from (93) using the bisection algorithm. (3) If the maximum of the absolute differences between the initial $s(x_t)$ and its update from (2) is less than a convergence criterion (say .0001), then the policy functions for normalized bonds and stocks that depend on normalized cash on hand are determined. The policy function for normalized consumption can also determined using $c_t = x_t - b_t - s_t$. Once the policy function for (normalized) consumption in period t is determined, we apply the same routine to period $t-1$, until we reach the first period of the model.

As this description suggests, in order to solve the problem for each period t we need to have previously determined the policy function $c_{t+1}(x_{t+1})$. This policy function is determined numerically, as a set of consumption levels each of which corresponds to a grid point for normalized cash on hand. The researcher decides this set of grid points when solving the problem for period $t+1$, prior to solving the problem for period t.

4.4.3 Portfolio Choice over the Life Cycle

We first check that for parameter configurations that respect the contraction mapping condition, the backward recursion converges to the infinite horizon solution derived earlier using a different method. This is confirmed in figures 23-26 that report nromalized consumption and the share of wealth in stocks both during retirement and working life.[33] The infinite horizon policy rule corresponding to these parameters is indeed the policy function for the younger agents (age 25 in figure 25). Thus, infinite horizon models appear to be a good approximation to the behavior of younger consumers in a population.

The low level of prudence ($\rho = 3$) and the equity premium continue to generate complete portfolio specialization in stocks during working life (see figure 26), illustrating that the puzzle is not unique to the infinite horizons model. Figure 25 illustrates how saving rises as one ages (this is saving for retirement, see Gourinchas and Parker (1999)). Moreover, older people tend to enter the stock market first (figure 26) and this occurs because they have a higher saving rate than their impatient, younger counterparts who, faced with a growing labor income process, would rather consume than save. In

[32] For more details on bisection, see Judd (1998, pp. 147-150).

[33] $\delta = 0.1$, $r = 0.02$, $\sigma_u = 0.1, \sigma_v = 0.08$, $\mu_g = 0.03, \rho = 3, c = .7$.

retirement, cash on hand is gradually depleted (figure 23) and stock market exposure is reduced with age.

Policy functions for $\delta = .5$ must also be reported. This is currently work in progress. With more patient households, there is a higher wealth accumulation over the lifecycle and therefore the range of normalized cash on hand over which the policy functions are solved for must increase to accomodate this change in the wealth profile; see the next subsection for the method chosen to determine the grid range.

4.4.4 Weatlh Accumulation over the Life Cycle

Given age specific policy functions, the evolution of wealth over the life cycle can be analyzed either via simulation or through the use of the transition distribution of normalized wealth in the economy. We choose to focus on time series simulations at this point. Figures 27-30 report the average normalized consumption, bond and stock holdings of 10000 simulated life histories for the benchmark parameters used in the previous subsection. High impatience results in very low wealth accumulation and consumption is very close to mean normalized labor income during working life (equal to one) and very close or equal to the normalized pension benefit (0.7) during retirement. Higher prudence generates higher wealth accumulation (compare figure 28 to 27 and figure 30 to 29), while a higher expected growth rate acts as a higher discount factor and reduces weatlh accumulation (compare figure 29 to 27 and figure 30 to 28).

In order to consider how wealth accumulation evolves for more patient households, we need to solve for the policy functions. To do so, we must determine the relevant range of possible cash on hand; we therefore are currently experimenting with the following iteration to choose the grid range. Start with a particular grid range and compute the policy functions. Use the policy functions to simulate individual life histories and check whether the simulated data fall within the bounds imposed. If not, extend the grid range and redo the experiment. This is work in progress.

4.5 Towards General Equilibrium Models

4.5.1 An Overlapping Generations Model

The small-scale optimization models we described could in principle be embedded in general equilibrium, overlapping generations models, in which each generation lives for three periods. A pioneering overlapping generations model of this type has been proposed by Constantinides, Donaldson, and

Mehra (1998), in order to study the likely effects of borrowing constraints on the equity premium. In their model, as in ours, the young are faced with earnings risk in the second period of their lives, and they would like to borrow in order to invest in stocks and thus take advantage of the equity premium. The middle-aged know that their income in old age will depend on their holdings of stocks and bonds, and they choose to hold positive amounts of both. A borrowing constraint on the young prevents them from financing stockholding through borrowing. This lowers the aggregate demand for stocks in the economy and raises the equity premium relative to what would be obtained in the absence of borrowing constraints. The result shows that equity premia are likely to be higher when "junior can't borrow" and highlights the importance of undestanding the portfolio behavior of young households.

4.5.2 The Role of the Wealth and Income Distribution

The large-scale optimization models we examined are candidate building blocks for a different type of general equilibrium models. Aiyagari (1994) and Huggett (1993) were the first to study the implications for the equilibrium interest rate in heterogeneous agents models with labor income uncertainty and borrowing constraints but without aggregate uncertainty. Den Haan (1996), Rios-Rull (1996) and Krussell and Smith (henceforth KS, 1998) extend this line of research by solving general equilibrium models with aggregate uncertainty. In models with heterogeneous agents and aggregate uncertainty, the wealth distribution becomes an endogenously evolving state variable. Given that this is an infinite dimensional state variable, the problem becomes intractable. A potential solution would be to approximate the evolution of this distribution with its important moments, for instance. Consistency of rational expectations would then require that agents' expectations about the future evolution of the weatlh distribution moments materialize in reality. In a non-linear model, however, there is no guarantee that the predicted evolution of the wealth distribution will match the actual wealth distribution, and there is no obvious choice of which moments should be followed. Surprisingly, Krussell and Smith find that (in the first order Markov equilibrium they examine) the mean of the wealth distribution and the aggregate productivity shock are sufficient statistics for the evolution of the wealth distribution. These statistics are sufficient in the sense that they can generate an "approximate" rational expectations equilibrium where the actual values of the wealth distribution are very close to the predicted values agents use to solve their individual optimization problems.

Storeslettern, Telmer and Yaron (1998) analyze the implications of the Krussell-Smith (1998) economy for the equity premium using similar techniques and find that their model can explain part of the equity premium. We think that the general equilibrium research agenda holds great promise in analyzing important public policy and asset pricing questions. Nevertheless, we feel that the partial equilibrium models cannot yet replicate basic observed data regularities, especially with regard to risky asset holdings. Given that it is far easier to work with partial equilibrium models, perhaps a useful approach to building general equilibrium models with plausible empirical predictions is to first look for a partial equilibrium model that can generate observed portfolio patterns, before embedding this mosel in the general equilibrium framework.

5 Concluding Remarks

This paper has presented a set of computational techniques that can be used to solve models of household portfolio choice. This is an exciting area of research with many still unanswered questions. Most of the existing models have explored the choice between risky and riskless financial assets and have enhanced our understanding of the portfolio implications of our standard choice models. However, it is fair to say that we do not yet have models that can account simultaneously for the limited incidence of stockholding in the population, the pattern of observed stockholding among different age, education, and other relevant demographic groups, and for the positive (concave?) relationship between risky portfolio shares and cash on hand typically observed in country data.

A second fruitful avenue for future research is the interaction between portfolios of financial assets and holdings of real assets, such as private businesses and housing. Importantly, real assets have implications for overall wealth and consumption risk faced by the household, they can provide collateral for loans, reasons for accumulation of downpayments or startup funds, and sometimes consumption services.

A third area for potential research lies in the analysis of tax effects on portfolios, in view of the tax treatment of labor income, interest and dividend income, capital gains, bequests and inheritances. Particularly exciting from a computational perspective is the analysis of capital gains taxation, in view of the requirement that capital gains be taxed at realization and of special tax provisions for capital gains bequeathed to descendants, such as “step up of basis” clauses.

A fourth dimension deals with the privatization of pension systems, new types of retirement accounts and their tax treatment, as well as the implications of such schemes for the rest of a household’s portfolio.

In all of these areas, there is considerable scope for moving from partial-equilibrium to general-equilibrium models in an effort to account simultaneously for portfolio puzzles and for asset return puzzles, such as the equity premium. This move will be much smoother once we have a clear understanding of the most appropriate ways to model portfolio behavior in view of the empirically observed stochastic processes for asset returns. When asset returns are also successfuly endogenized, the stage will have been set for computer simulations of artificial economies with optimizing agents in which portfolio behavior and the wealth distribution play a key role. Perhaps this is as close as we are likely to get to controlled experiments in Economics.

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Figure 1: First-Period Policy Functions for Consumption, Stock- and Bond Holding
as a Function of Cash on Hand (Expected Utility Specification)

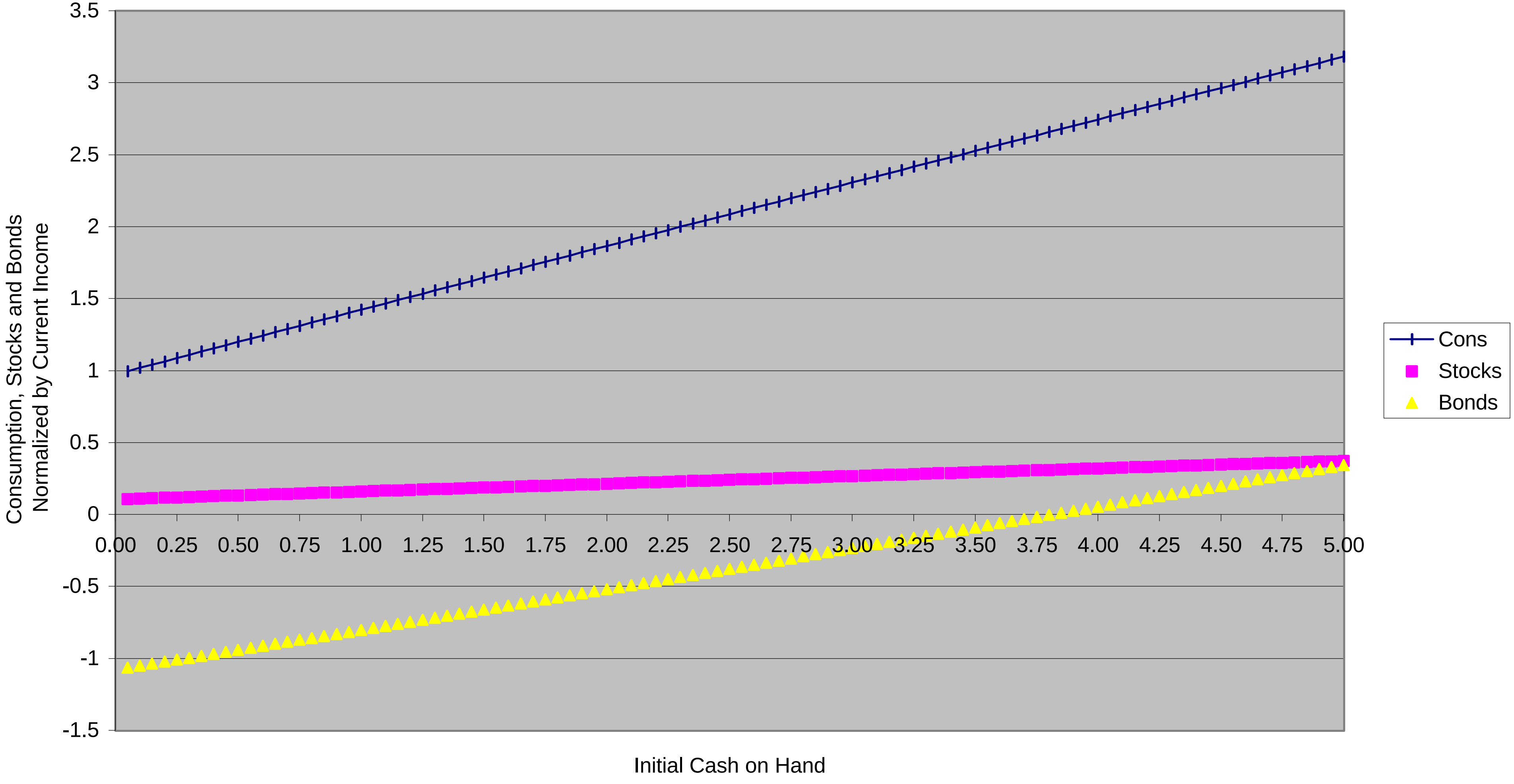


Figure 2: Second-Period Policy Functions for Consumption, Stockholding, and Bondholding
(Expected Utility Specification)

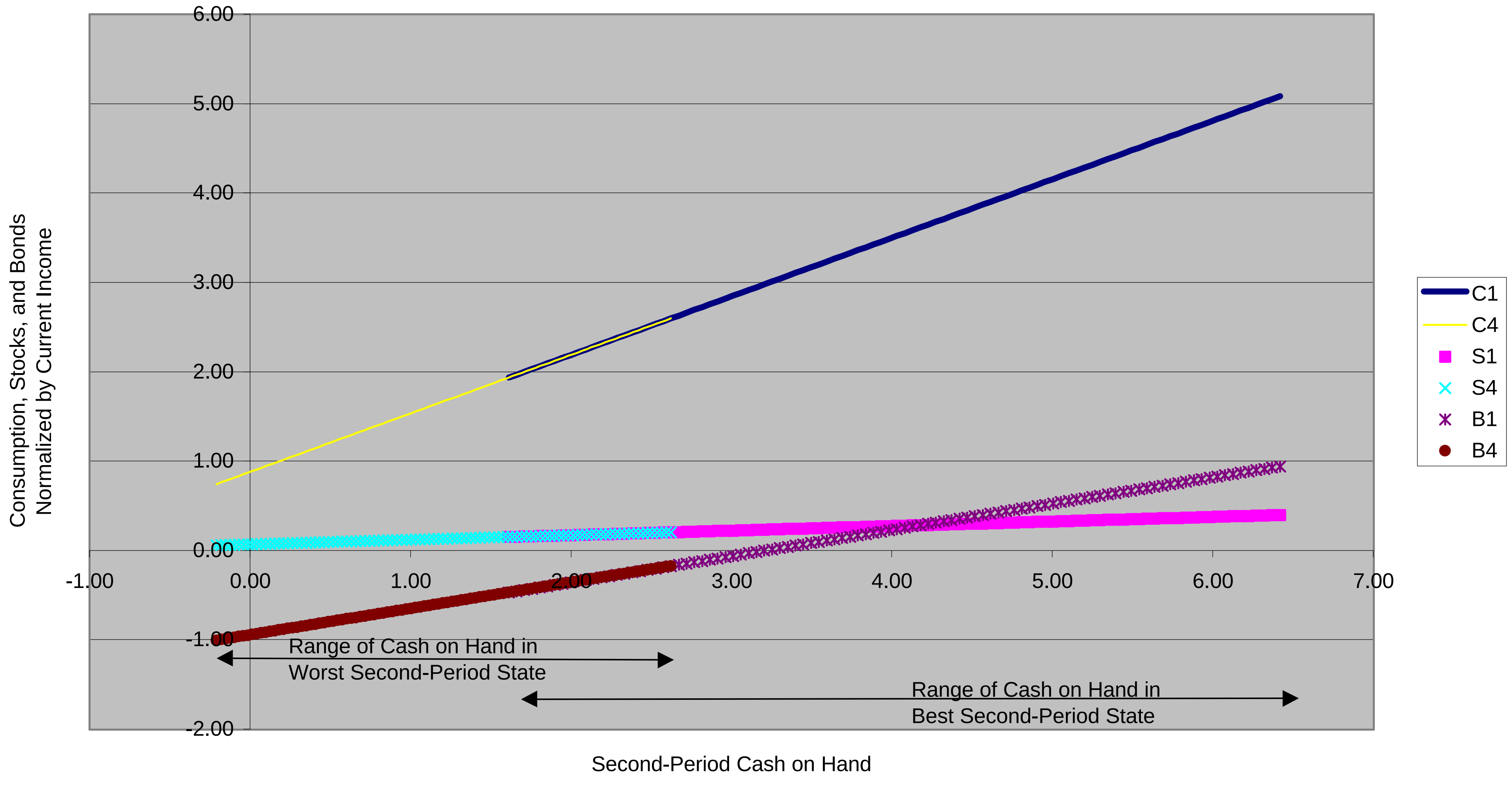


Figure 3: Policy Functions for Consumption Under Alternative Preference Specifications:
Expected Utility (EU), Kreps-Porteus (KP), and Quiggin (Q) Preferences

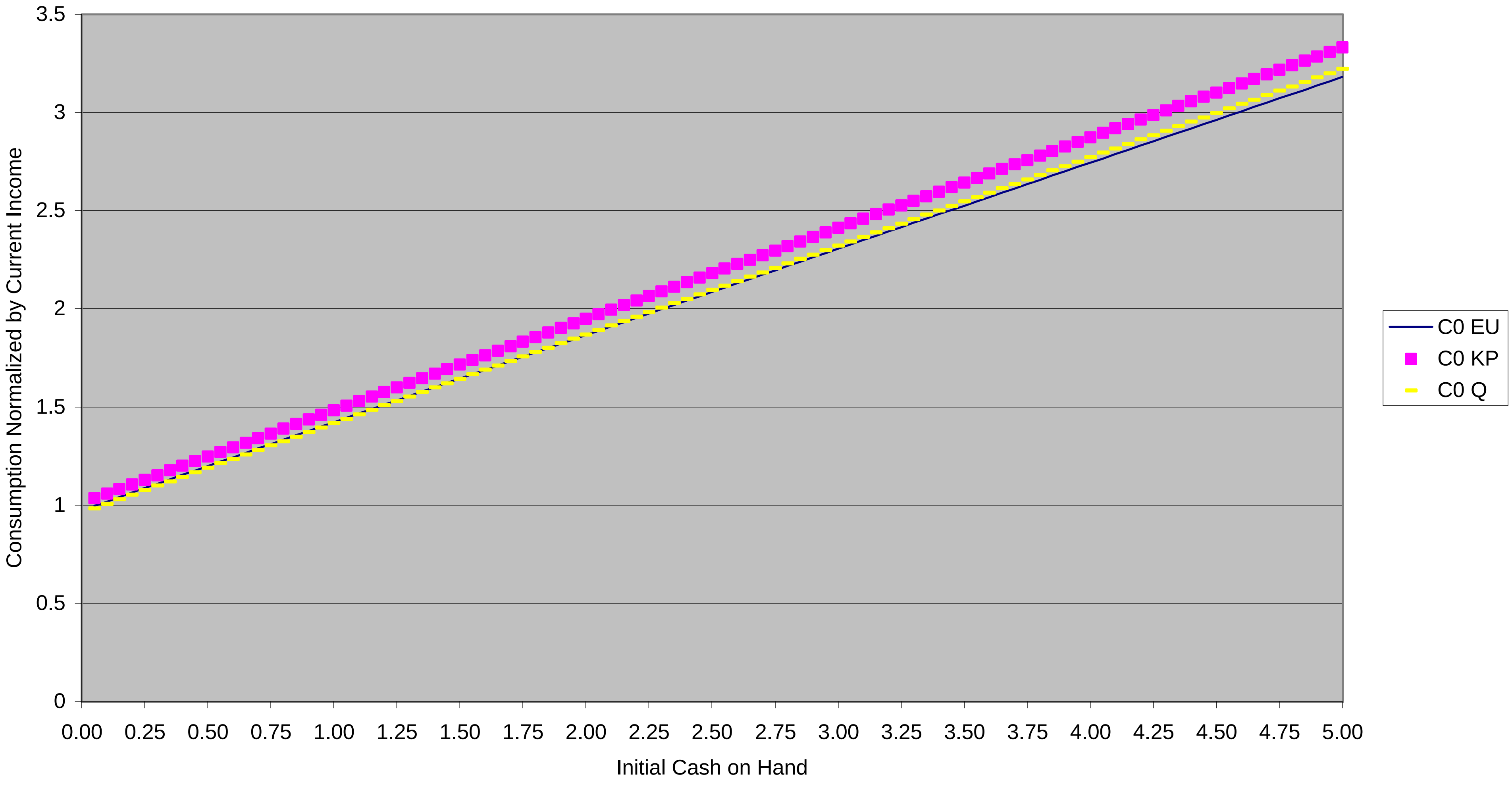


Figure 4: Policy Functions for Stockholding Under Alternative Preference Specifications:
Expected Utility (EU), Kreps-Porteus (KP), and Quiggin (Q) Preferences

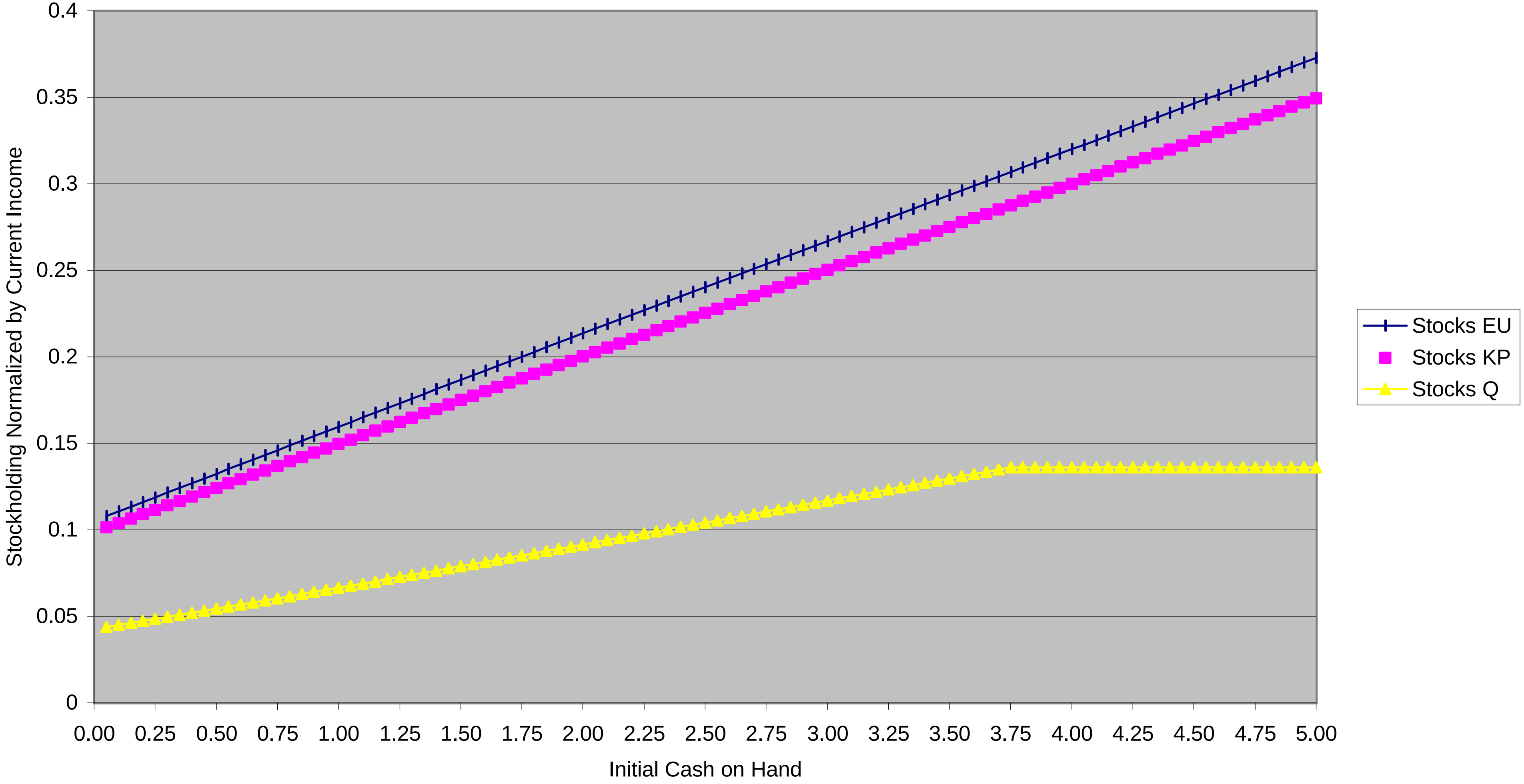


Figure 5: Policy Functions for Bondholding Under Alternative Preference Specifications:
Expected Utility (EU), Kreps-Porteus (KP), and Quiggin (Q) Preferences

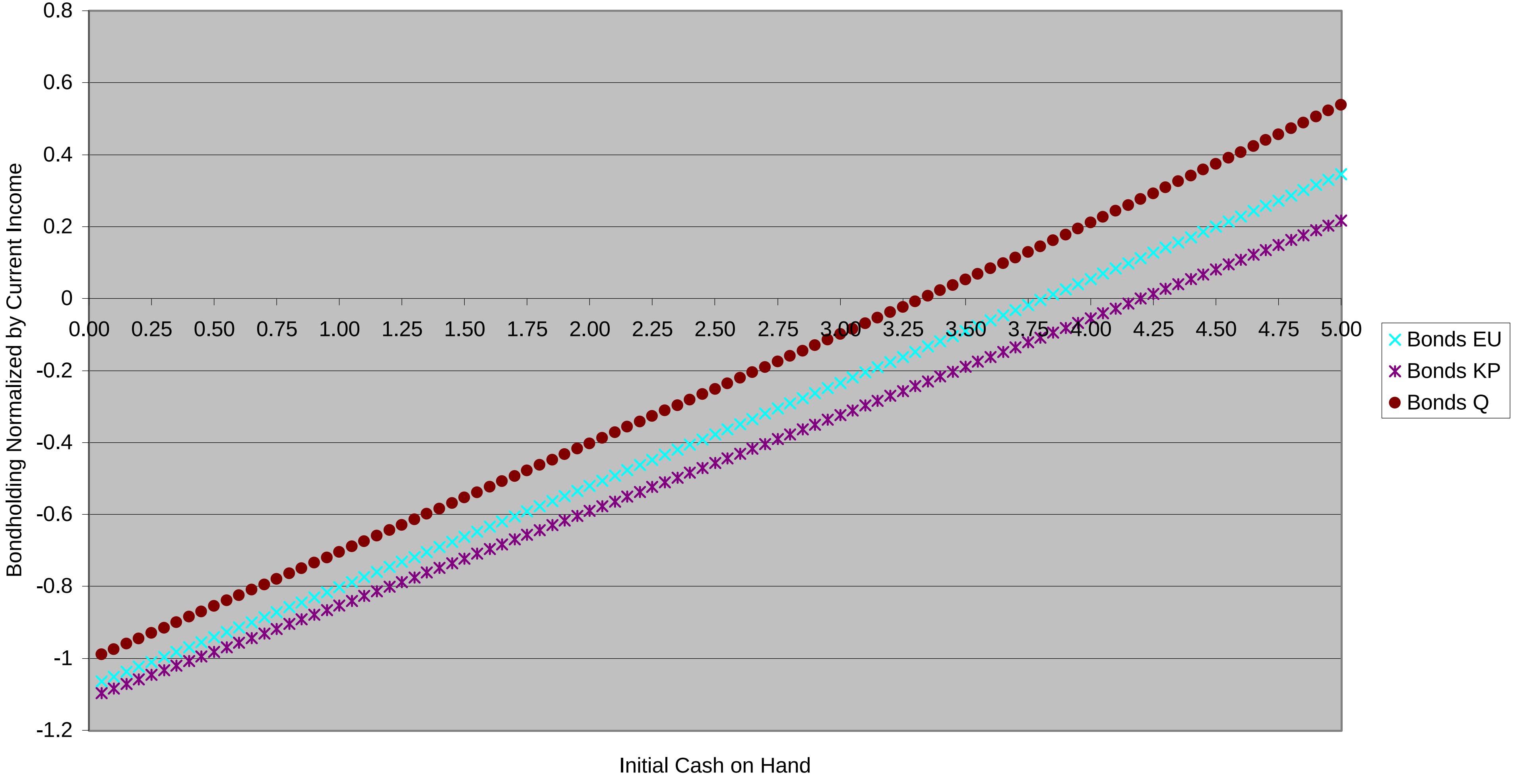


Figure 6: Effects on Stockholding of Positive Covariance Between Stock Returns and Labor Incomes
Under Expected Utility (EU), Kreps-Porteus (KP), and Quiggin Preferences

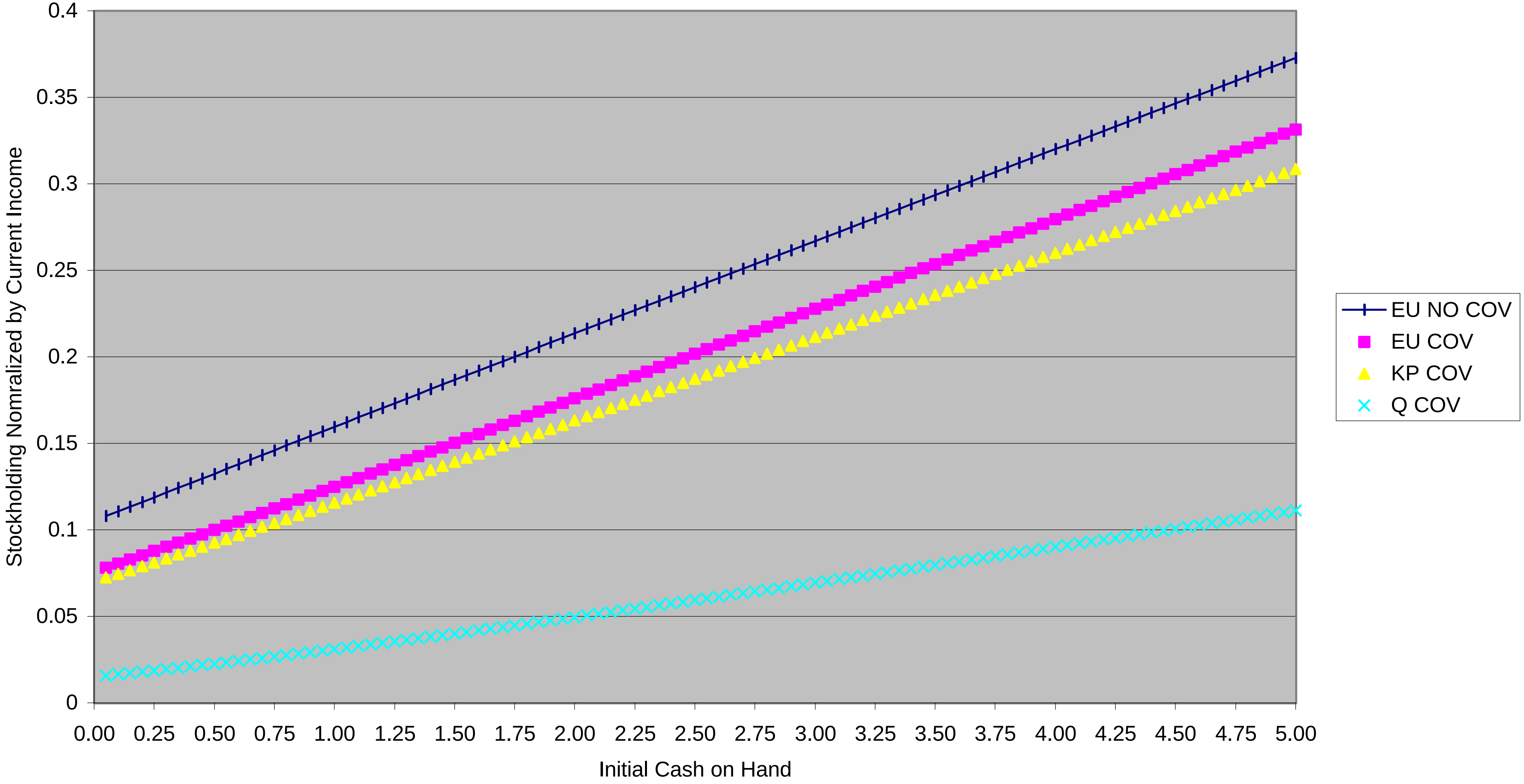


Figure 7: Effects of Risk Aversion and Positive Covariance between Earnings and Stock Returns on Stockholding
Under Expected Utility (EU), Kreps-Porteus (KP), and Quiggin Preferences

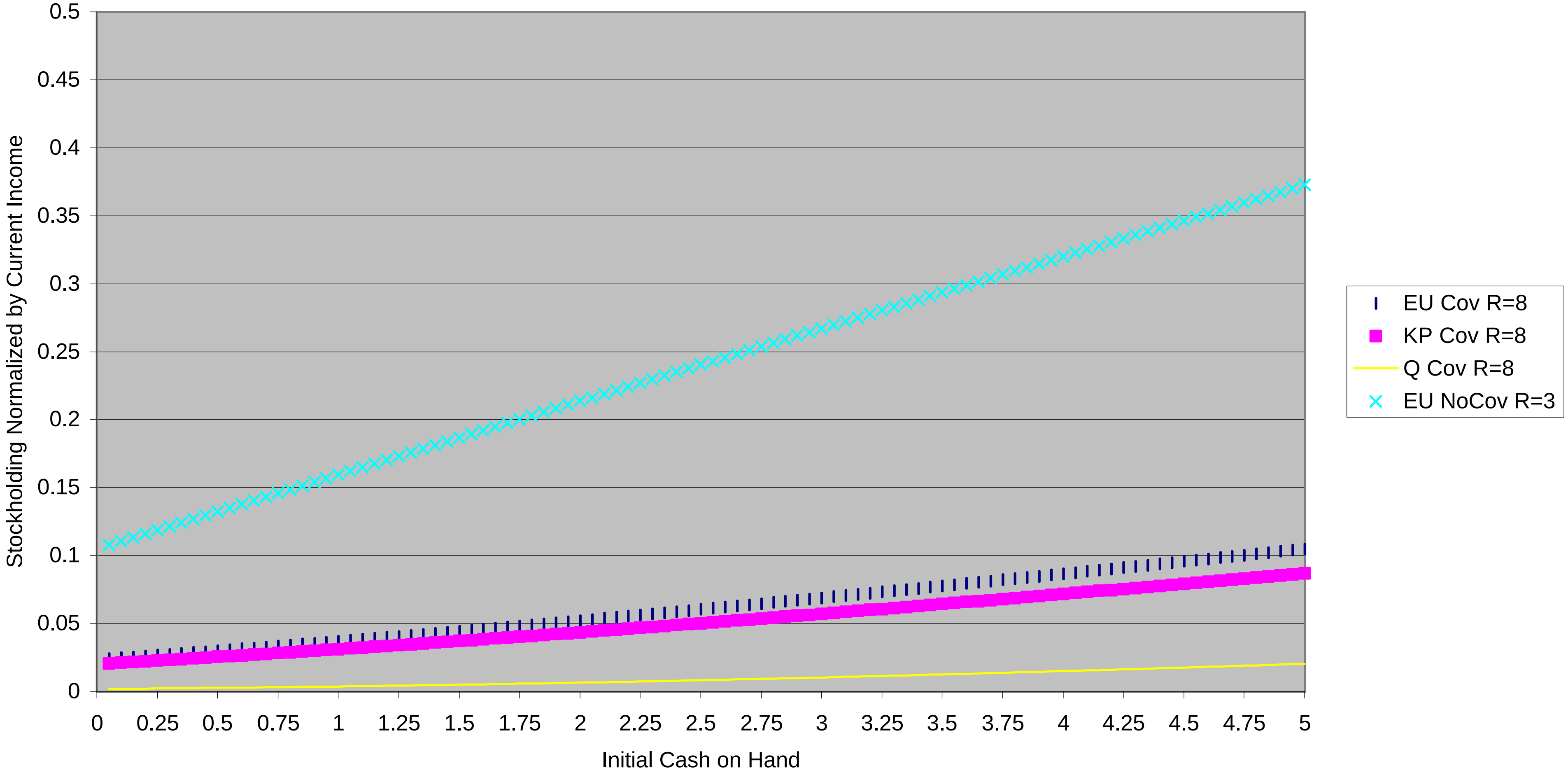


Figure 8: Precautionary Wealth Under Alternative Preference Specifications:
Expected Utility (EU), Kreps-Porteus (KP), and Quiggin (Q) Preferences

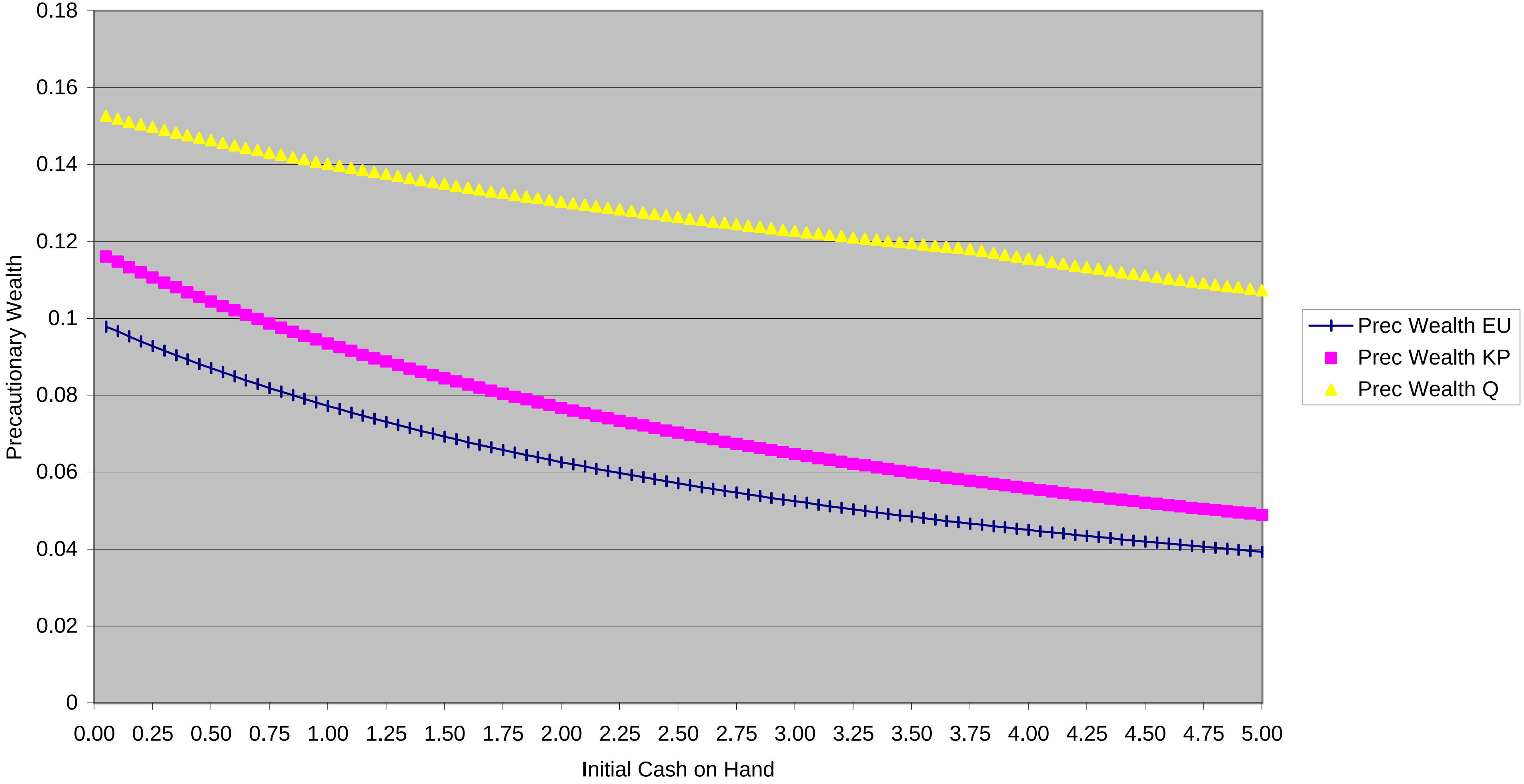


Figure 9: Precautionary Effects on Stockholding Under Alternative Preference Specifications:
Expected Utility (EU), Kreps-Porteus (KP), and Quiggin (Q) Preferences

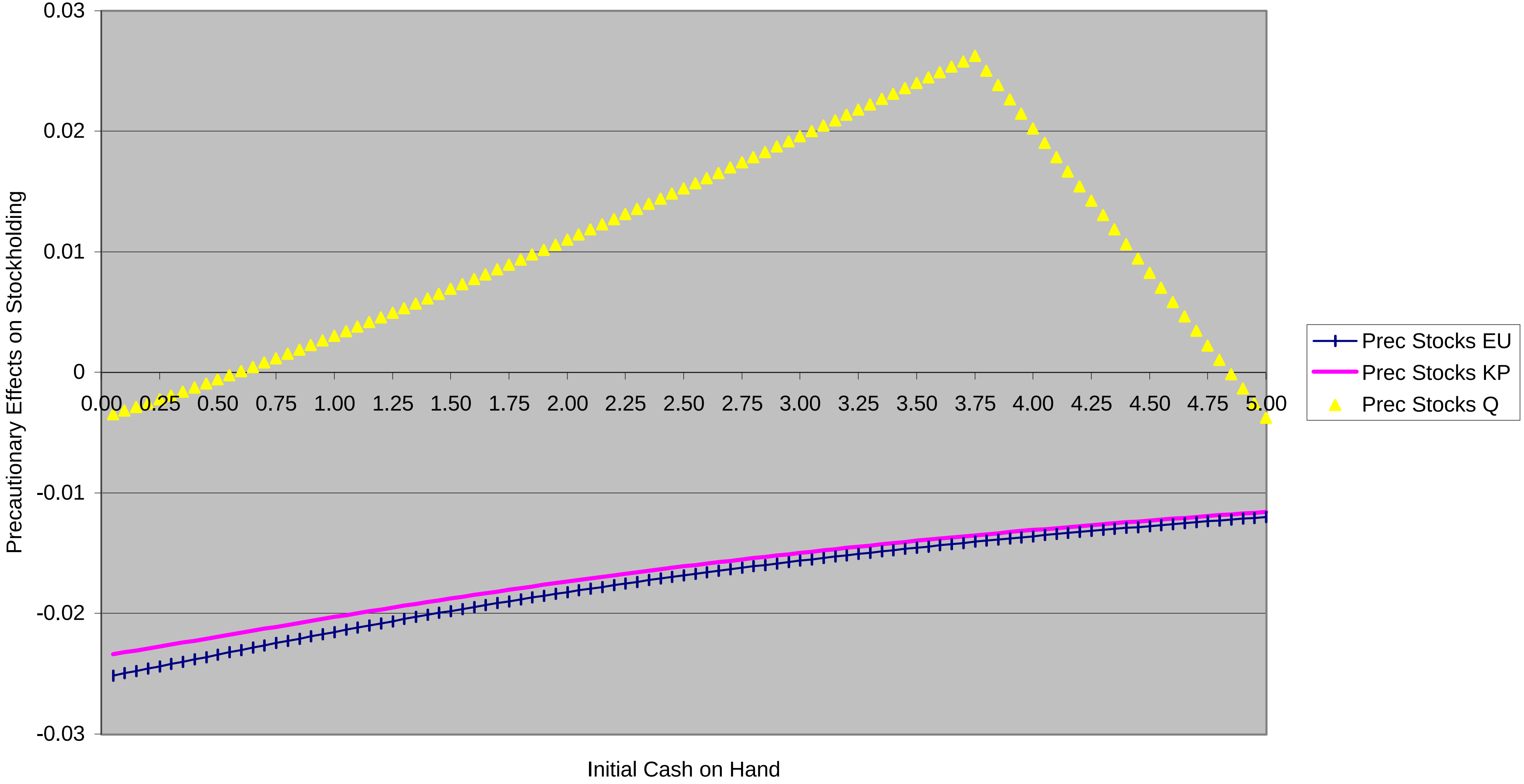


Figure 10: Precautionary Effects on Bondholding Under Alternative Preference Specifications:
Expected Utility (EU), Kreps-Porteus (KP), and Quiggin (Q) Preferences

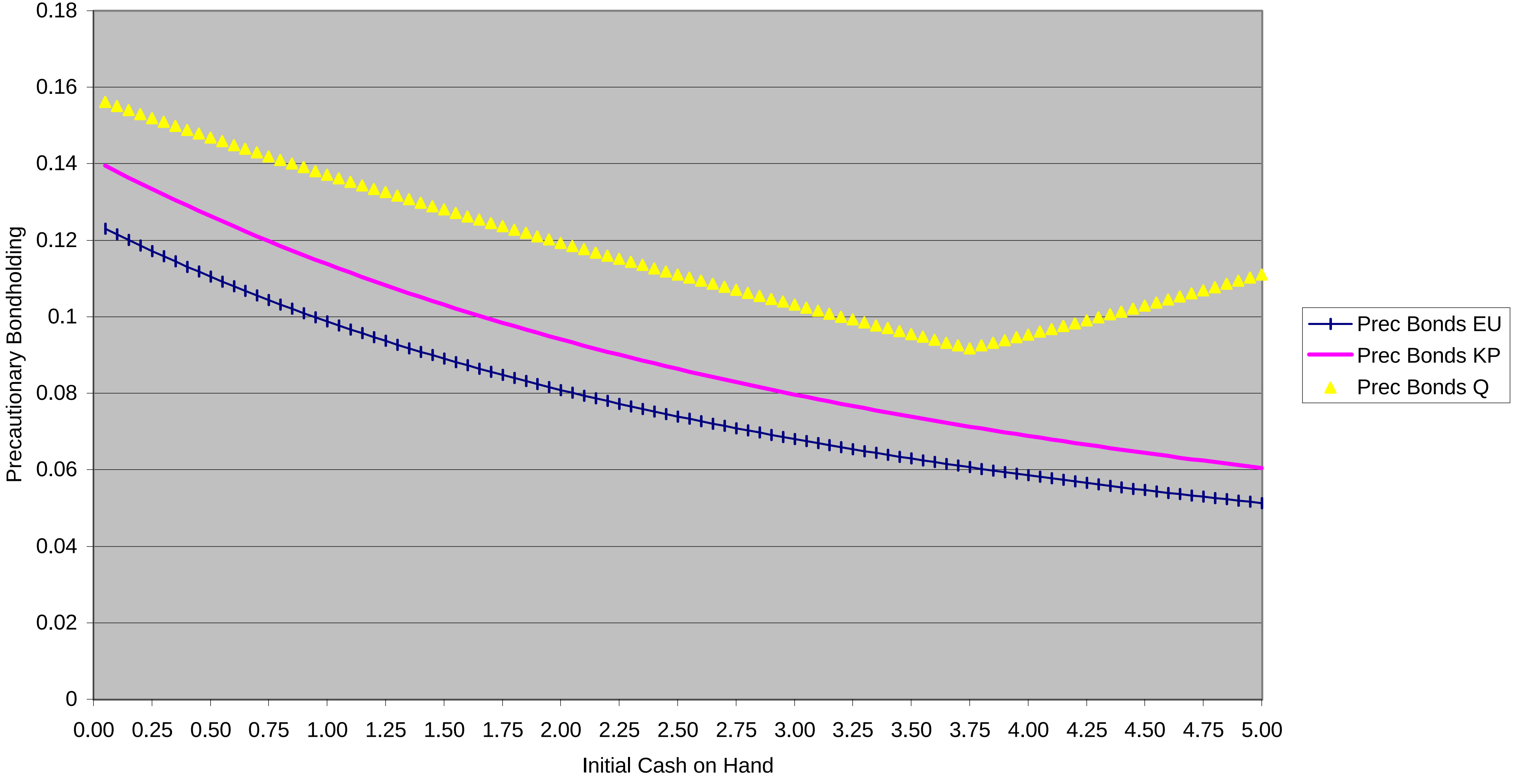


Fig.11 : Normalized Consumption (varying rho)

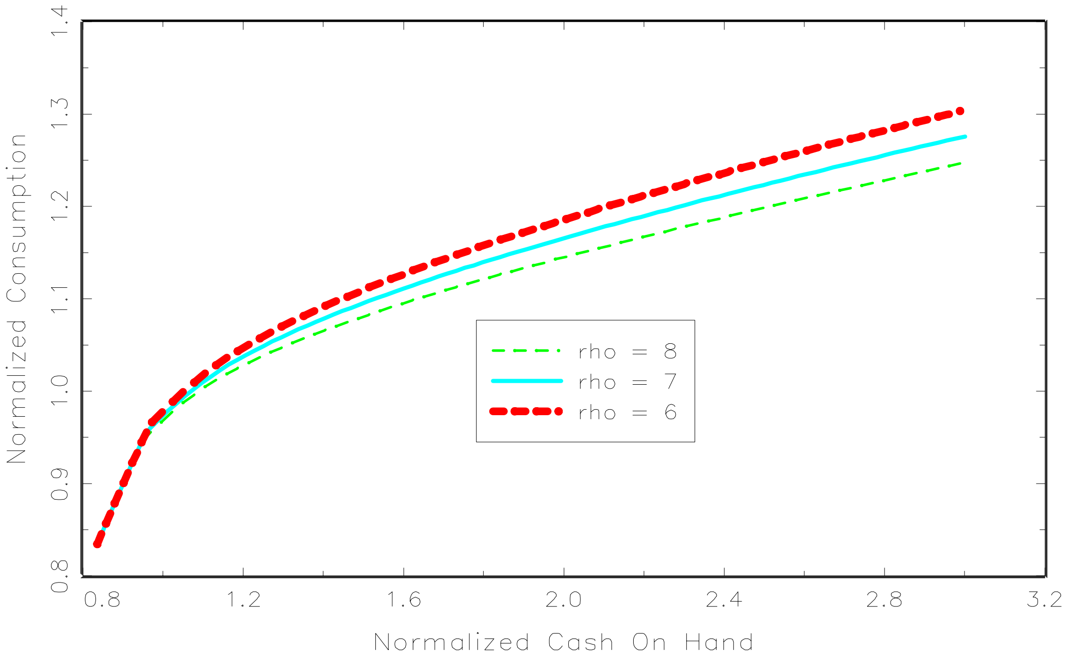


Fig.12 : Share of Wealth in Stocks (varying rho)

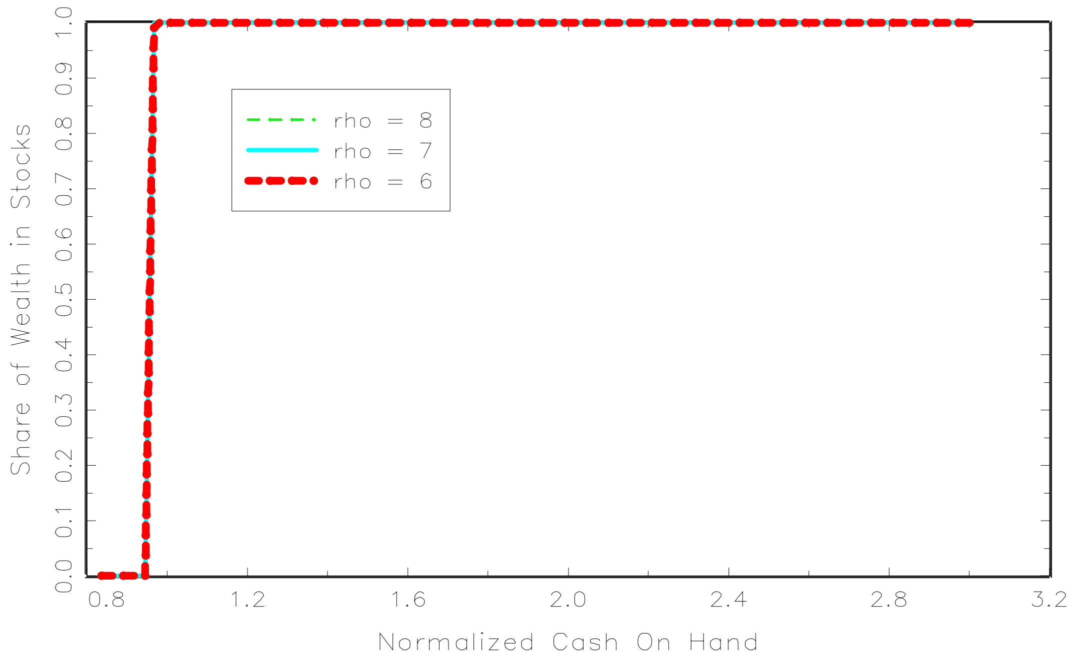


Fig.13 : Normalized Stock Holdings (varying rho)

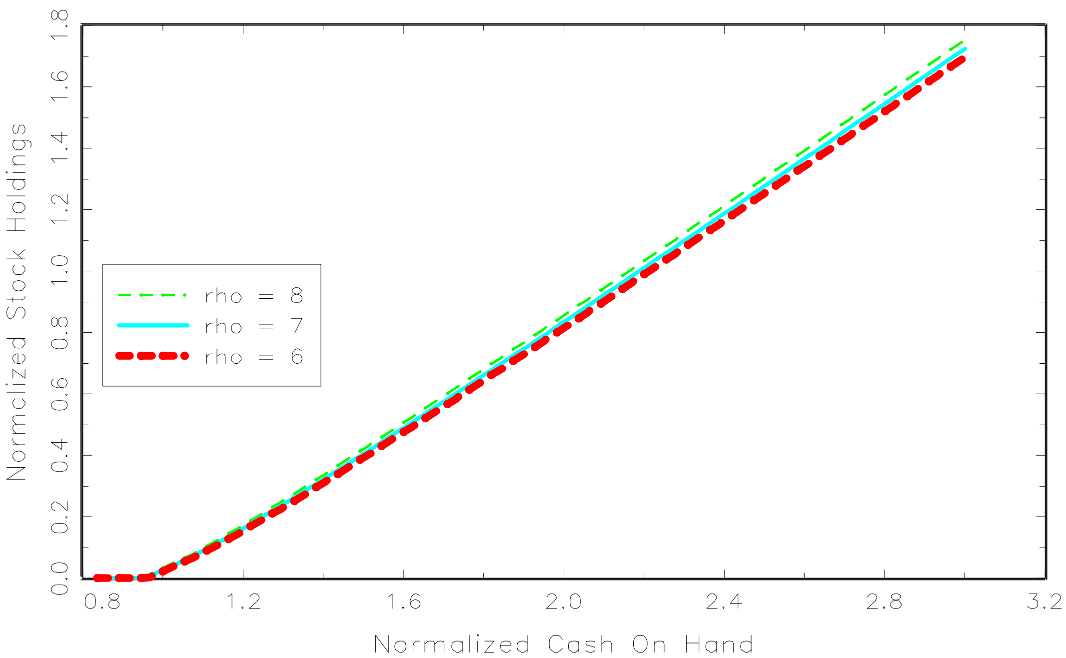


Fig.14 : Normalized Bond Holdings (varying rho)

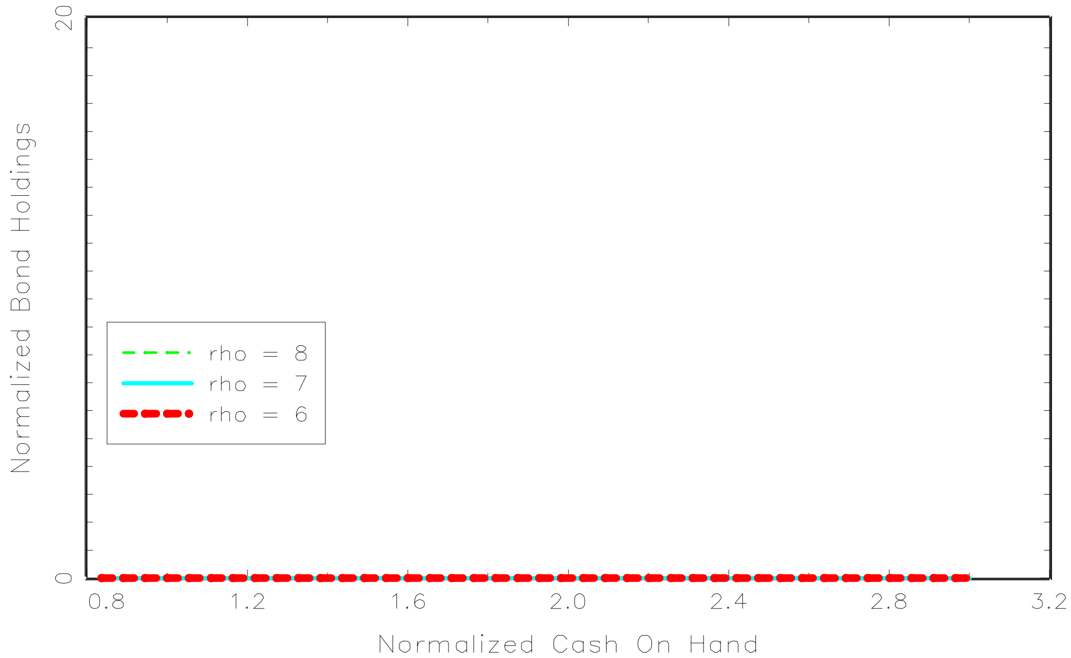


Fig.15 : Normalized Consumption

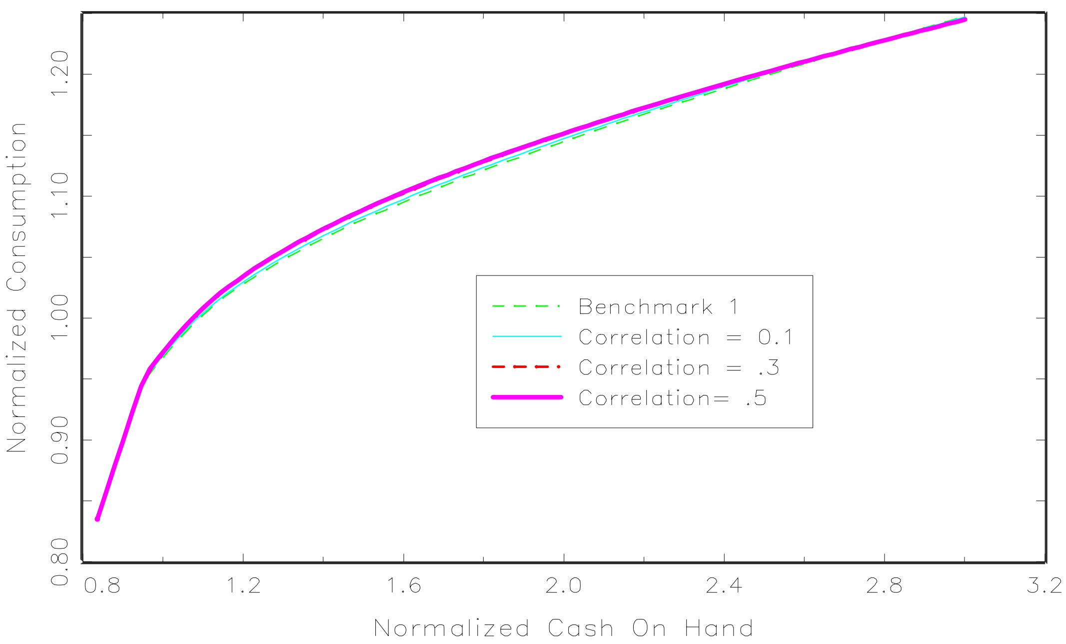


Fig.16 : Share of Wealth in Stocks

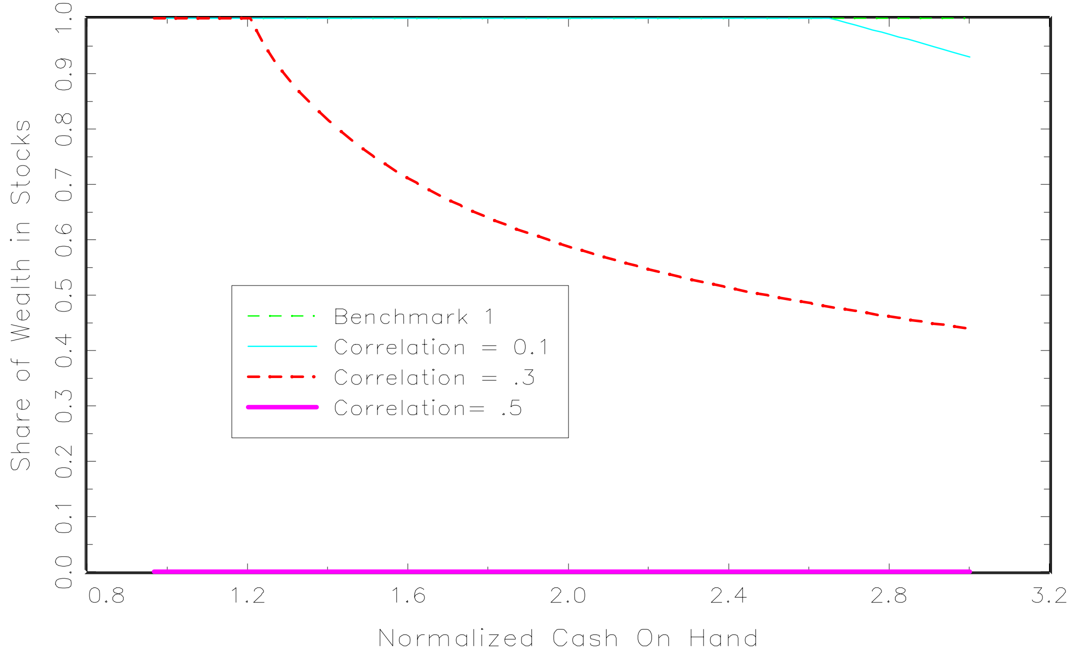


Fig.17 : Normalized Stock Holdings

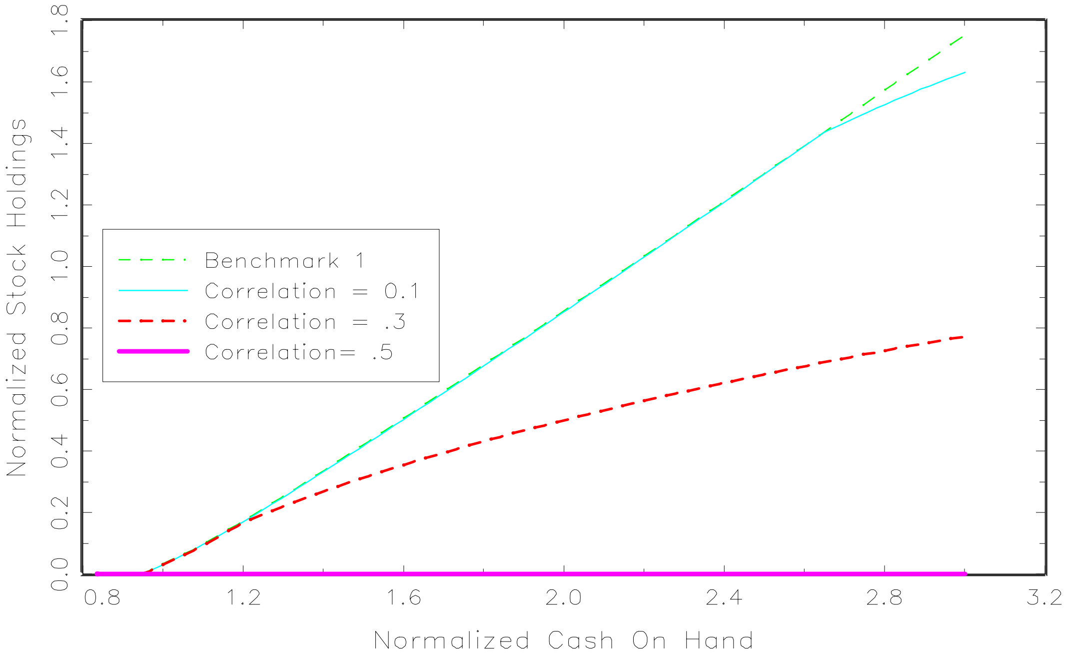
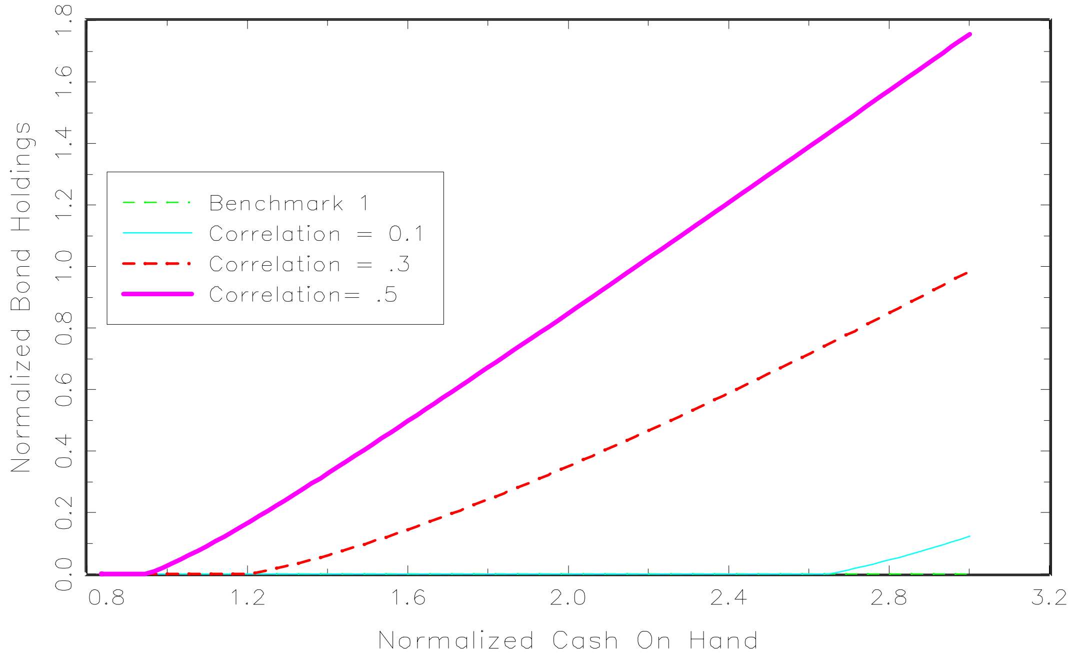


Fig.18 : Normalized Bond Holdings



Market Timing vs No Market Timing, rho=3, $\rho_{\varepsilon,z}=0$ and $\rho_{n,z}=0$

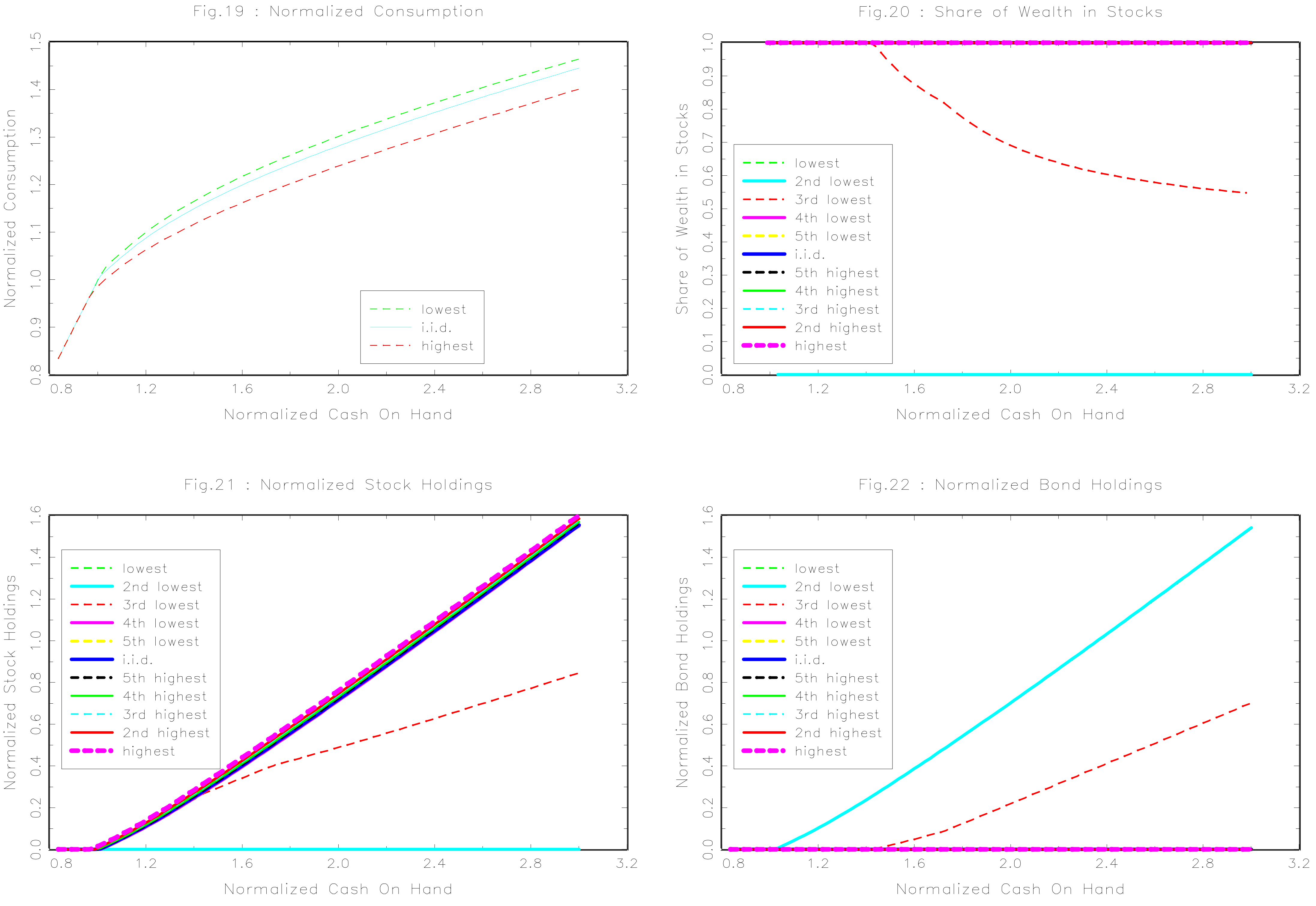


Fig.23 : Normalized Consumption (Retirement)

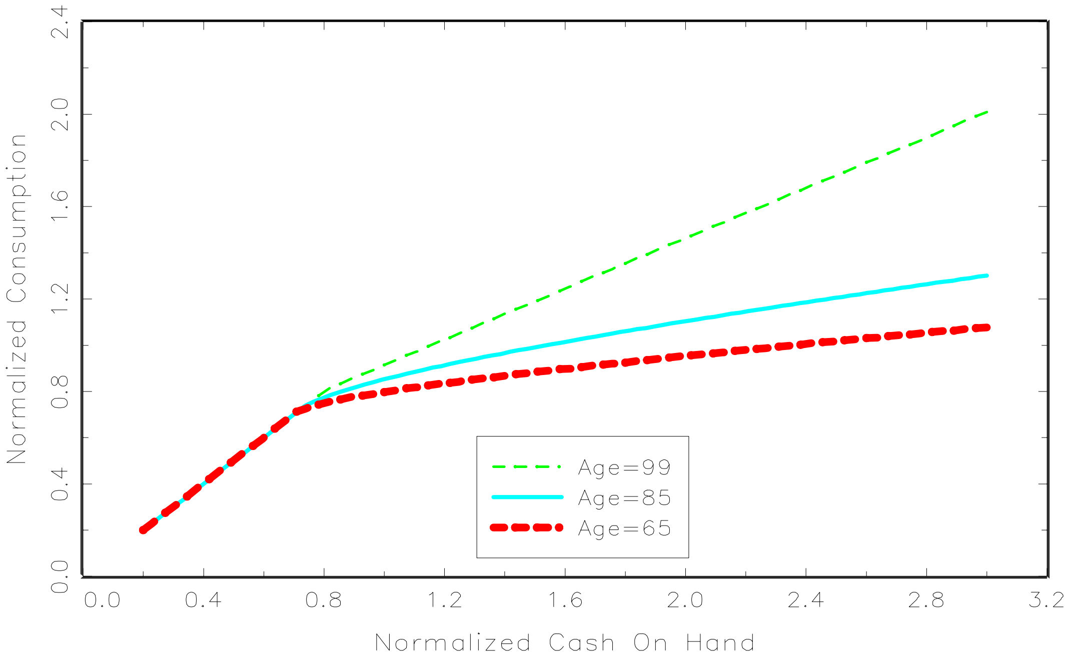


Fig.24 : Share of Wealth in Stocks (Retirement)

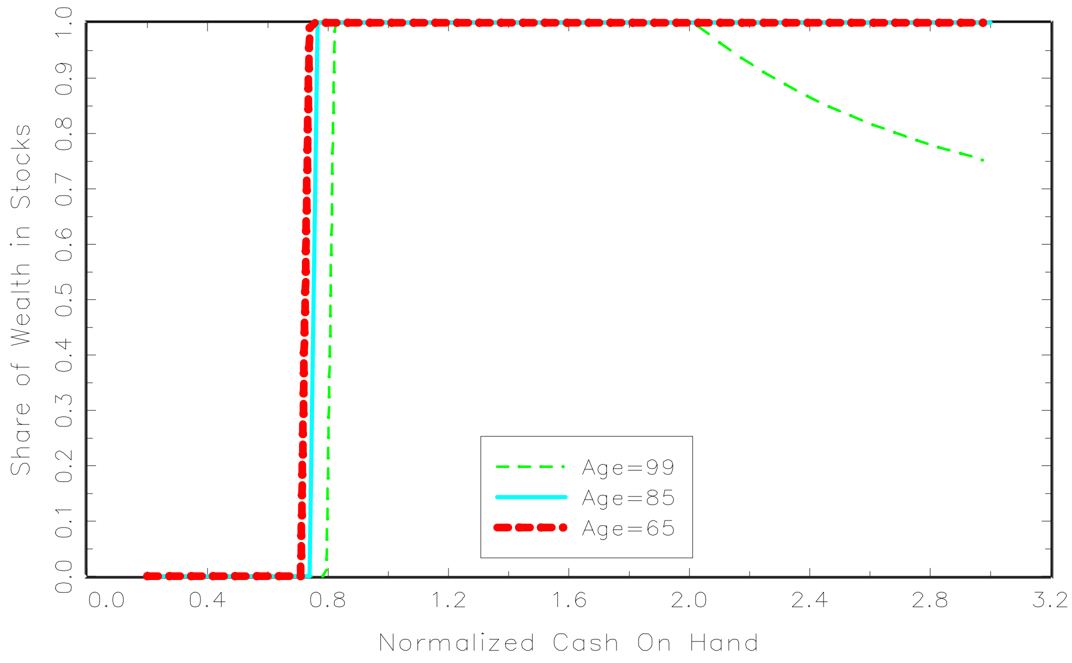


Fig.25 : Normalized Consumption (Working Life)

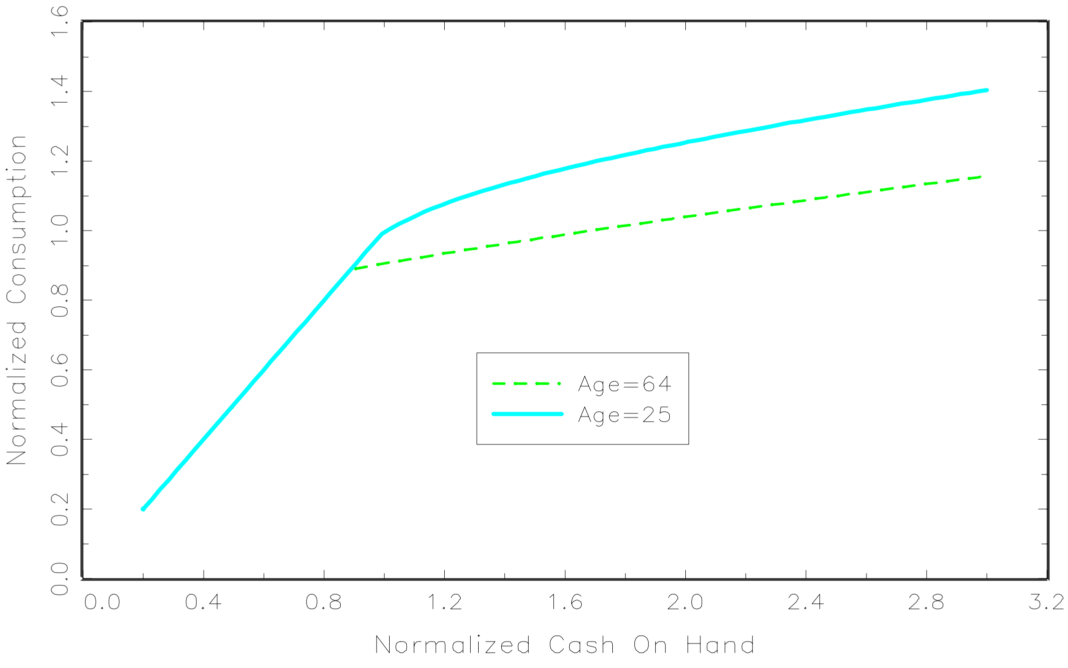


Fig.26 : Share of Wealth in Stocks (Working Life)

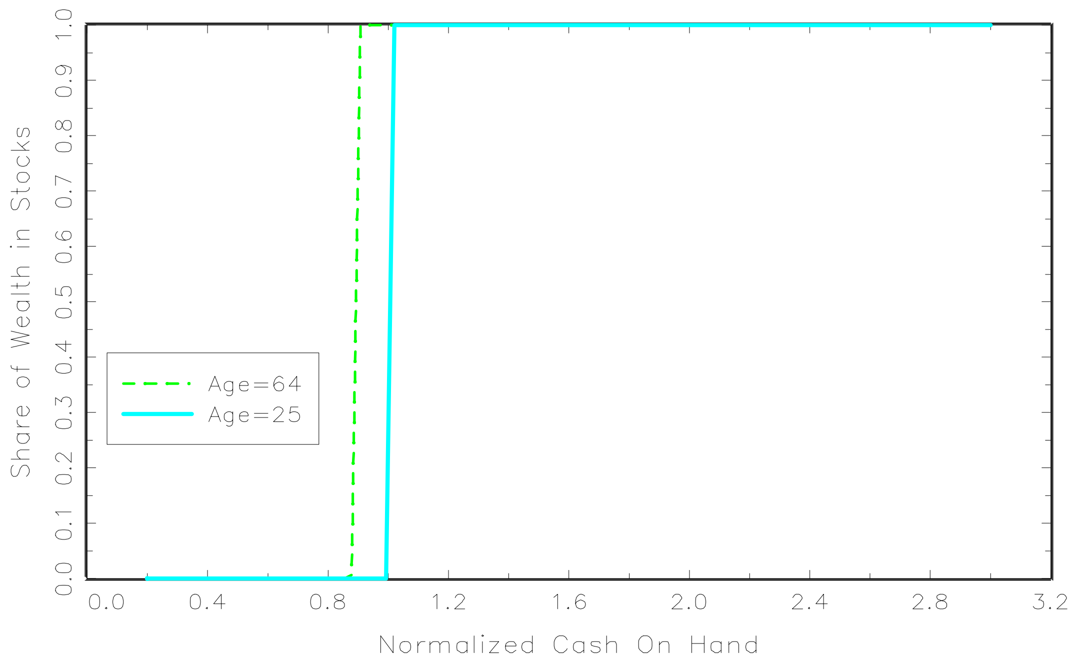


Fig.27 : Stocks, Bonds and Consumption: g=.03, delta=.1, rho=3

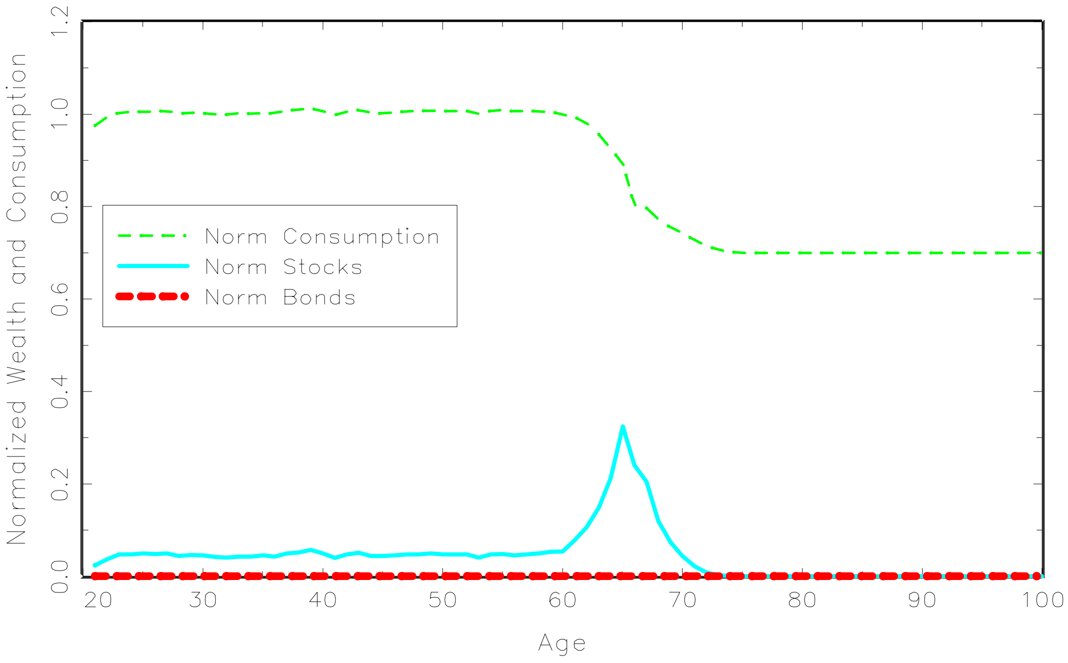


Fig.28 : Stocks, Bonds and Consumption: g=.03, delta=.1, rho=5

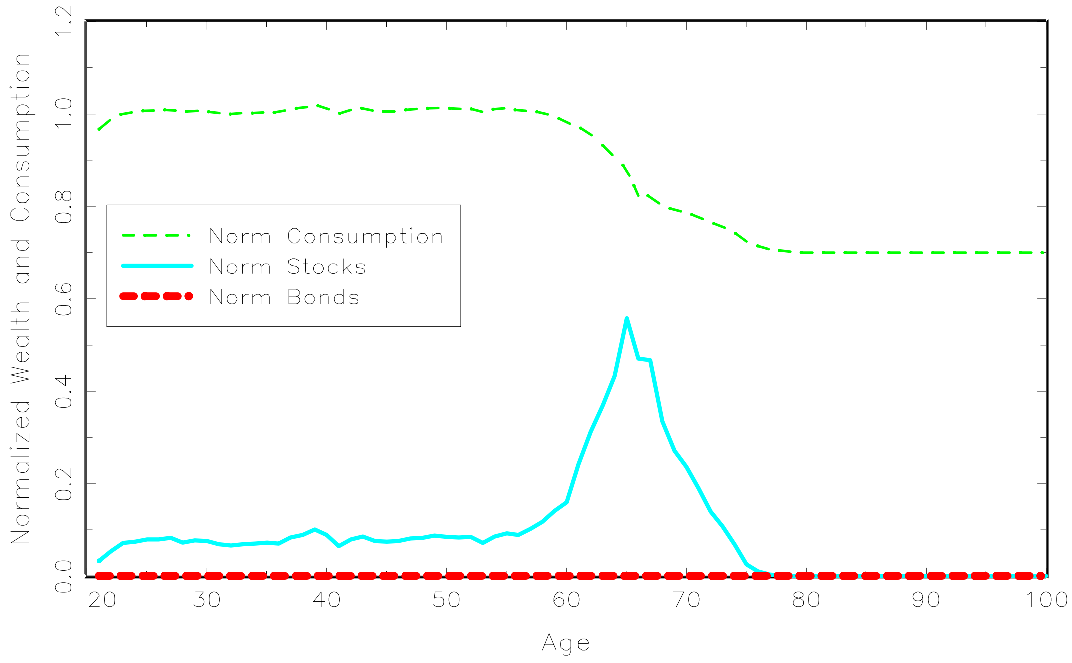


Fig.29 : Stocks, Bonds and Consumption: g=.01, delta=.1, rho=3

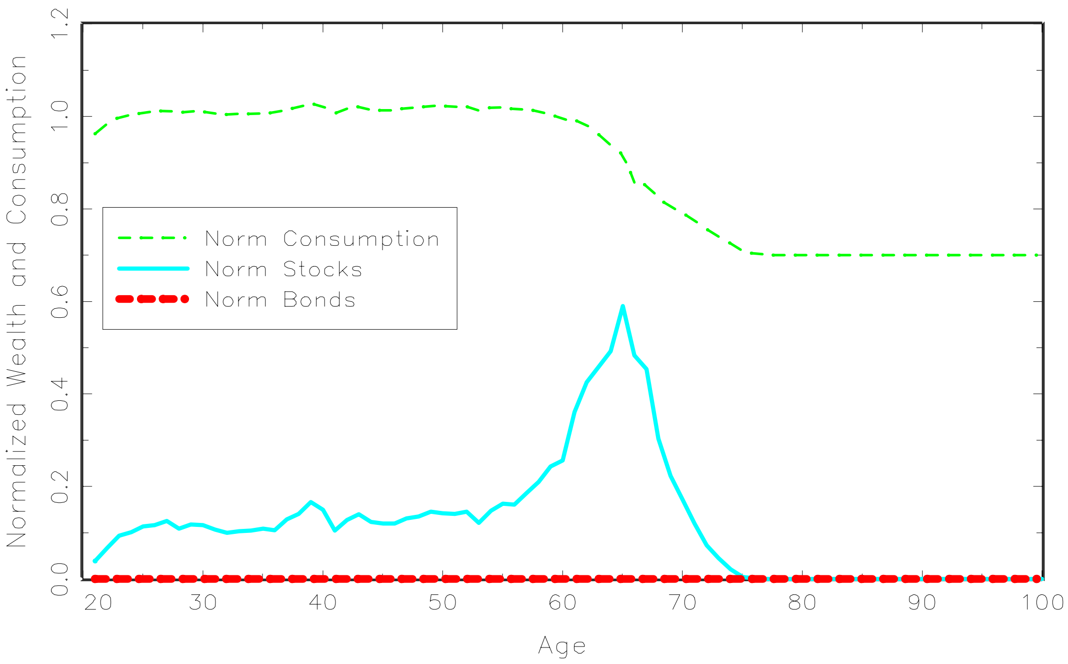


Fig.30 : Stocks, Bonds and Consumption: g=.01, delta=.1, rho=5