What Does The Classical Theory Have to Say about Household Portfolios?[1]

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1 Introduction

The objective of this chapter is to examine the mechanisms by which households determine the optimal structure of their portfolio. Because risky assets generate larger expected returns than the risk free asset, risk-averse households must determine the best compromise between risk and expected return. In a static framework, this problem is usually formalized by introducing a concave utility function on final consumption. Under the standard axioms of decision under uncertainty (von Neumann-Morgenstern (1944)), households will select the portfolio which maximizes the expected utility of their final consumption. The degree of concavity of the utility, as defined by Arrow (1971) and Pratt (1964), characterizes their degree of absolute risk aversion. It can be measured by answering to questionnaires related to risk choices. When this information is combined with the specific distribution of returns of financial assets, one can compute the optimal portfolio of the household. An increase in the absolute risk aversion reduces the demand for risky assets. This mechanism is well understood and is included in most textbooks in market finance.

I am tempted to say that this is also the only mechanism that is well understood in this field. For example, important progresses have been made in the late sixties by introducing time into the portfolio strategies. Usually, households have long term objectives as retirement when they invest in financial markets. How does long horizon affect the optimal structure of the portfolio? Said differently, how does the option to purchase stocks tomorrow modify the attitude towards portfolio risk today? The pathbreaking papers of Mossin (1968), Merton (1969) and Samuelson (1969) did answer to this question in a very simple but intriguing way: the intrinsically dynamic structure of portfolio management has no effect the solution to the problem. Under their assumptions, the sequence of portfolio structures that are statically optimal is also dynamically optimal. In other words, households should do as if each period of investment is the last period before retirement. Myopia is optimal. The econometrician should not use age as an explanatory variable for households' portfolios. But it is often forgot that this result holds only under very restrictive conditions on the utility function. Namely, myopia is optimal only if the utility function exhibits constant relative risk aversion. In this paper, we explain why this is the case and under which condition households who can invest longer in risky assets should invest more in them.

Household portfolios do not only serve long term objectives as financing retirement. An important share of savings are invested in liquid funds that play the role of buffer stocks. When they have been lucky on their previous investments, they can reduce their saving rate to cash immediately the benefits of the larger-than-expected portfolio returns. They can also increase their saving rate in case of an adverse shock on portfolio returns. This means that households do not bear the accumulated lifetime portfolio risk at the time of retirement. Rather, they can disseminate this risk over their lifetime consumption pattern. This allows for some forms of time diversification through consumption smoothing. This is a strong incentive to accept riskier portfolios. The implications for the econometrician is a testable hypothesis: households who can adapt over time their contributions to their pension funds do select riskier portfolio structures.

Two other factors will have important implications on household portfolios. First, the presence of a liquidity constraint on the consumption-saving strategy should be taken into account to design portfolio strategies. Indeed, this liquidity constraint, if it binds, will eliminate the possibility to transfer capital risks over time through consumption smoothing. It will induce a more conservative portfolio management for those households that are more likely to face a binding liquidity constraint in the future. Second, most risk borne by households are not portfolio risks but uninsurable risks affecting their human capital. The intuition suggests that households bearing a larger risk on their human capital should invest less in risky assets.

We examine all these factors influencing the optimal portfolio composition. We present each factor in isolation in a simplified model which excludes the other factors. We hope that this will help the reader to understand the nature of this factor without hiding the "big picture". The analysis of how these factors can be combined is let for the calibrations that will be presented in various chapters of this book. Section 2 is devoted to the presentation of the static portfolio problem under uncertainty. It will be useful for the remaining of the paper to observe that this problem is symmetric to the dynamic consumption problem under certainty. In section 3, we derive the main properties of the optimal static portfolio. The effect of background risk is examined in section 4, whereas the repeated nature of portfolio management is examined in section 5. The notion of time diversification is developed in section 6. Finally, we examine the effect of the liquidity constraint in section 7 before providing some concluding remarks.

2 The basic model and its applications

Most of the paper deals with the properties of the following model:

$$\max_{C_1,...,C_N} \quad \sum_{i=1}^{N} p_i u(C_i) \tag{1}$$

$$s.t. \quad \sum_{i=1}^{N} p_i \pi_i C_i = X, \tag{2}$$

where $(p_1, ..., p_N)$ and $(\pi_1, ..., \pi_N)$ are two vectors of nonnegative scalars and u is a real-valued, increasing and concave function. Notice that the only specificity of this model with respect to the very basic problem of consumer theory under certainty is the additivity of the objective function. Under the concavity of u, the necessary and sufficient condition for program (1) under constraint (2) is written as

$$u'(C_i) = \xi \pi_i, \qquad i = 1, ..., N, \tag{3}$$

where ξ is the Lagrangian multiplier associated to the constraint.

There are three standard applications of this decision problem in economics. The first one is related to the efficient allocation of risk in a pool of N risk-averse agents. This topic is not directly related to portfolio decisions and will not be covered in this paper. The second application of program (1) is the static portfolio problem of a risk-averse investor in an Arrow-Debreu economy. More specifically, consider an economy in which investors live for one period. At the beginning of the period, the investor under scrutiny is endowed with a sure wealth X. He does not know the state of the world that will prevail at the end of the period. There are N states of the world indexed by i, $i = 1, ..., N$. The uncertainty is described by the probability p_i that state i occurs, with $\sum_i p_i = 1$. Consumption takes place only after that the realization of i is observed. The agent invests his endowment in a portfolio of assets that will be liquidated at the end of the period to finance consumption. We assume that financial markets are complete. It implies that for each state i, there exists an associated state price (per unit of probability) $\pi_i \geq 0$. In other words, the agent must pay $p_i \pi_i$ at the beginning of the period to increase his consumption by one unit in state i. Vector $(C_1, ..., C_N)$ is the state-contingent consumption plan of the agent, and equation (2) is the budget constraint of the investor. This vector can also be seen as a portfolio of Arrow-Debreu securities. The objective function in (1) is the ex ante expected utility of the investor who selects this portfolio. This application has been intensively used in the theory of finance during the last three decades.

In this presentation, we assumed that the agent had no income ex-post. Introducing state-contingent incomes is not a problem when markets are complete. Indeed, X can also be seen as the ex ante market value of these state-contingent incomes, i.e., $X = \sum_i p_i \pi_i Y_i$ where Y_i is the household's income in state i. This is the main feature of markets completeness, as agents can transfer their individual risks to the market.

The last application of program (1) is the lifetime consumption-saving problem under certainty. Consider a household who lives for N periods, from period $i = 1$ to period $i = N$. There is no uncertainty about the net present value of his incomes. The household's net discounted wealth at the beginning of period $i = 1$ equals X. Vector $(C_1, ..., C_N)$ represents the time-dependent consumption plan, with C_i measuring consumption in period i. The objective function is to maximize the discounted utility of consumption over the lifetime of the household. Parameter p_i in (1) is the discount factor associated to period i. It is often assumed that $p_i = \beta^i$, to escape problems of time-consistency in decision making. People can finance their consumption in period i by purchasing in period 1 zero-coupon bonds maturing in period i. The gross rate of return of such a bond is denoted $(p_i \pi_i)^{-1}$. Constraint (2) is the household's lifetime budget constraint. Program (1) has been a cornerstone of the literature on the Permanent Income Hypothesis in macroeconomics.

In spite of their technical equivalence, these two problems are different in nature. For example, the concavity of the utility function represents risk aversion in the portfolio problem, whereas it implies aversion to consumption fluctuations in the consumption-saving problem. It means at the same time that the agent is willing to perfectly insure risks if insurance prices are fair ($\pi_i = 1$ for all i), and that he is willing to smooth consumption over time if the return on bonds equals the rate of impatience. This aversion to fluctuations of consumption across time or states is measured by the Arrow-Pratt index of absolute aversion, which is defined by $A(C) = -u''(C)/u'(C)$. It is more convenient in general to use an index of relative aversion, $\rho(C) = CA(C) = -Cu''(C)/u'(C)$. In the consumption-saving problem, ρ is the inverse of the well-known elasticity of intertemporal substitution.

These dual interpretations of the theoretical model suggest two ways to estimate ρ. Viewing it as a degree of risk aversion, one can estimate it by answering to the following question: What is the share of one's wealth that one is ready to pay to escape the risk of gaining or losing a share α of it with equal probability? Let x be this (certainty equivalent) share of wealth. Suppose that the agent has constant relative risk aversion (CRRA), which implies that

$$u(C) = \frac{C^{1-\rho}}{1-\rho}. \tag{4}$$

Normalizing wealth to unity,[1] it implies that x is the solution of the following equation:

$$0.5\frac{(1-\alpha)^{1-\rho}}{1-\rho} + 0.5\frac{(1+\alpha)^{1-\rho}}{1-\rho} = \frac{(1-x)^{1-\rho}}{1-\rho}. \tag{5}$$

Table 1 relates x to ρ, when $\alpha = 10\%$ or $\alpha = 30\%$.

RRA	$\alpha = 10\%$	$\alpha = 30\%$
$\rho = 0.5$	0.3%	2.3%
$\rho = 1$	0.5%	4.6%
$\rho = 4$	2.0%	16.0%
$\rho = 10$	4.4%	24.4%
$\rho = 40$	8.4%	28.7%

Table 1: Relative certainty equivalent loss x associated to the risk of gaining or losing a share α of wealth, with constant relative risk aversion ρ.

If we focus on the risk of gaining or losing 10% of one's wealth, we would consider an answer $x = 0.5\%, ..., 2\%$ as a sensible answer to the question. It implies that it is reasonable to believe that relative risk aversion is somewhere between 1 and 4. Saying it differently, a relative risk aversion superior to 10 seems foolish, as it implies very high relative risk premiums. Notice in particular that a relative risk aversion of 40 implies that one would be ready to pay as much as 8.4% to escape the risk of gaining or losing 10% of one's wealth!

The other approach is to see ρ as the degree of relative aversion to consumption fluctuations. Consider an agent who consumes income $1 - \alpha$ in

[1] With constant relative risk aversion, the certainty equivalent share x is independent of initial wealth.

each even year, and $1+\alpha$ in each odd year. He is offered to smooth his consumption by paying a premium x on his average income. What is the critical value of x which makes the agent indifferent between the two consumption plans? Under CRRA, the relationship between x, ρ and α is governed by exactly the same formula (5).[2] Looking at Table 1, an interval $[1,4]$ for the degree of relative aversion to consumption fluctuations over time seems to be reasonable. Barsky, Juster, Kimball and Shapiro (1997) used experimental data from the Health and Retirement Study in the U.S. to measure risk aversion and aversion to consumption fluctuations for people above 50 of age. They reported values of ρ that are slightly larger than those suggested here.

The beauty and the simplicity of the above model come from the additivity of the objective function with respect to states of the world or time, depending upon which application we have in mind. All nice properties that we will report in the remaining of this chapter will be derived from this additive hypothesis. In the case of risk, we know that this hypothesis can be derived from a more fundamental axiom, i.e., the independence axiom. A similar axiom can be built to derive an additive property for preferences with respect to time. We also know that the independence axiom has a weak predictive power in some specific risky situations, mainly those involving low probabilities. A branch of the economics of uncertainty provides more general (non-additive) decision criteria to deal with these specific cases. A similar step has been made in the economics of time, with non-time-additive models. Those involving habit formations seem to be quite promising, but will not be examined in this paper (see for example Constantinides (1990)).

Another problem arises when we try to mix risk and time. When future consumption levels are uncertain, the objective function is usually defined as the *discounted expected* lifetime utility. This implies that the utility function represents at the same time the attitude towards risk and the attitude towards time of the decision maker. But imagine an agent that does not give the same x when answering to the two questions mentioned above. This agent may not have the discounted expected lifetime utility as an objective function. Kreps and Porteus (1978) and Selden (1978) suggested a model that would disentangle the degree of risk aversion from the degree of aversion to

[2] To be exact, one of the two terms in the left-hand side of equation (5) should be multiplied by rate of impatience, depending upon whether we start with an even year or an odd year.

consumption fluctuations. In this model, the additivity across states in each period is preserved, together with the additivity across period in each state. But the additivity in the full space (risk, time) is not preserved, contrary to what we have with the discounted expected utility model. Again, we will not cover this generalization in this chapter. Applications of Kreps-Porteus preferences in macroeconomics are explored by Weil (1990).

3 The standard static portfolio problem

After this discussion about the basic decision model (1), it is now time to explore the main features of its solution. We do this with the risk application in mind. We consider an agent who lives for one period and who has to invest his endowment X in a portfolio that will be liquidated at the end of the period to finance his final consumption. Of course, this model is completely unrealistic because it is static, and because we assume market completeness. This model is considered as a benchmark, and will be extended to include dynamic strategies and markets incompleteness in subsequent sections.

First-order condition (3) holding in each possible state, it must also hold in expectation. This condition is written as

$$Eu'(\widetilde{C}) = \xi E\widetilde{\pi} = \frac{\xi}{R}, \tag{6}$$

where $E\widetilde{z} = \sum_i p_i z_i$ is the expectation operator and $R = [E\widetilde{\pi}]^{-1}$ is the gross risk free rate. Indeed, an asset guaranteeing one unit of consumption with certainty at the end of the period costs $\sum_i p_i \pi_i = E\widetilde{\pi}$ at the beginning of the period. Multiplying both sides of equality (3) by c_i, taking the expectation and using conditions (2) and (6) also yields

$$E\widetilde{C}u'(\widetilde{C}) = RXEu'(\widetilde{C}) \tag{7}$$

Let us assume that a Two-fund Separation result holds here, i.e., that it is optimal for the agent to limit his choice to the allocation of his endowment between two specific funds of assets. The first fund is risk free with gross return R. The second fund is the set of all risky assets, or "stocks", of the economy. Its gross return over the risk free rate is denoted $\widetilde{R}_s$. Let w denote the share of the initial endowment that is invested in the stock fund. The final consumption in this case equals $\widetilde{C} = X(R + w(\widetilde{R}_s - R))$. The optimal share w of wealth invested in stocks is then obtained by rewriting condition (7) as

$$E\widetilde{R}_s u'(X(R + w(\widetilde{R}_s - R))) = 0. \tag{8}$$

What do we know about the properties of the optimal share of wealth invested risky assets as a function of the parameters of the problem? Without entering into the details of the proofs which can be found for example in Eeckhoudt and Gollier (1995), we know the following three properties of the optimal w :

- If $E\widetilde{R}_s > R$, it is optimal to invest in the stock fund $(w > 0)$;
- Consider two agents with an identical initial endowment X, but respectively with utility function u and v. Individual u invests less in the risky fund than individual v, at all levels of X and for any distribution of $\widetilde{R}_s - R$, if and only if u is more risk-averse than v in the sense of Arrow-Pratt, i.e., if $-u''(C)/u'(C)$ is larger than $-v''(C)/v'(C)$ for all C.
- An increase in initial wealth X always increases the optimal share invested in the risky fund, for any distribution of $\widetilde{R}_s - R$, if and only if the index of relative risk aversion $\rho(C)$ is decreasing in C.

These are clear-cut properties. The third one in particular is an hypothesis on preferences that can be tested with data about household portfolios. The first result is contradicted by the observation that a large proportion of the population does not hold any risky asset. This may be due to the existence of a participation cost to financial markets. Notice also that there is an important literature on the effect of a change in the distribution of excess returns on the demand for stocks. Contrary to the intuition, it is not true in general that a first-order stochastically dominated shift in distribution of returns, or also a Rothschild-Stiglitz increase in risk in returns, does not necessarily reduce the demand for stocks. This literature did not provide any testable property of optimal portfolio strategies and will therefore not be covered here.[3]

[3] Gollier (1995) obtained the necessary and sufficient condition on the change of distribution for a reduction in the demand for stocks by all risk-averse investors.

We can also estimate the optimal w by calibrating the model. Assuming the lognormality of the distribution of $\widetilde{R}_s$ together with the constancy of relative risk aversion ρ, it can be shown that the solution of equation (8) is

$$w = R\frac{E\widetilde{R}_s - R}{\sigma_s^2}\frac{1}{\rho}, \tag{9}$$

where σ_s^2 is the variance of $\widetilde{R}_s$. Condition (9) can also be obtained without lognormality and CRRA, but by assuming that the portfolio risk is small. In that case, a first-order Taylor expansion of $u'(X(R + w(R_s - R)))$ around RX in equation (8) directly yields (9) as an approximation.

Historically, the equity premium $E\widetilde{R}_s - R$ has been around 6% per year over the century in U.S. markets.[4] The standard deviation of yearly U.S. stock returns over the same period equals 16%. The real risk free rate $R - 1$ averaged at 1% per year. Combining this information with formula (9) yields an optimal relative share of wealth invested in risky assets that equals 220% and 55% respectively for a relative risk aversion of 1 and 4. This very large shares with respect to observed portfolio compositions by U.S. households is commonly referred to as the Equity Premium Puzzle, as introduced by Mehra and Prescott (1985). A more standard way to present this puzzle is as follows: in order to explain actual portfolio composition in the U.S., one needs to assume a degree of relative risk aversion around 40 for the representative agent. From our discussion about the level of ρ, this is highly implausible. Kocherlakota (1996) provides a survey about the potential ways to solve the puzzle.

Equations (8) and (9) have been obtained under the assumption that all households invest in the same two funds. In particular, the structure of the portfolio of stocks is fixed. Since Wilson (1968), we know that this is the case at equilibrium only for utility functions exhibiting harmonic absolute risk aversion (HARA). Absolute risk aversion is harmonic if its inverse is linear, i.e., if $[A(C)]^{-1} = -u'(C)/u''(C)$ is linear in C. This is the case for logarithmic, power and exponential utility functions. When utility functions are not HARA, some agents will prefer to overinvest in some specific risky assets that they find underrepresented in the stock fund. For example, some agents may engage in portfolio insurance schemes by purchasing call options on stocks. It implies that C would not anymore be linear in the overall return

[4] These summary statistics are from Kocherlakota (1996).

$\widetilde{R}_s$ of the economy, contrary to the specification $C(R_s) = X(R+w(R_s-R))$.[5] The simplicity of the Two-Fund Separation hypothesis is to fully describe the portfolio by a single variable, w. A complete description of optimal portfolios when the Two-fund separation result does not hold requires us to solve the system of equation (2), (3). The effect of a change in the risk attitude of the investor is similar to what we stated in the two-fund case, but this requires to be more cautious about what we mean by a more risky portfolio.

4 The optimal static portfolio composition with background risk

Up to now, we assumed that the only source of risk faced by the household is the portfolio risk. This is far from realistic, since most of the observed volatility of households' earnings comes from variations in labour incomes. Typically, risks related to human capital cannot be traded on Wall Street. And financial intermediaries like insurers are not willing to underwrite such risks mainly because of moral hazard. Unemployment insurance is ineffective in most countries. This is particularly true for risks of long term unemployment, which is one of the central concerns of middle class households in Europe. Thus, it appears to be important to adapt the basic portfolio problem to include an uninsurable background risk. The question is how does the presence of a risk on human capital affect the demand for stocks.

We will assume that the risk on human capital is independent of the portfolio risk. This is clearly unrealistic. Most shocks to the economy affect in the same direction the marginal productivity of labour (wages) and the marginal productivity of capital (portfolio returns). It implies that human capital is usually positively correlated with assets value. But allowing for statistical dependence is not a problem. In particular, a positive correlation makes human capital substitute for stocks. Therefore, an increase in the correlation will reduce the demand for stocks. It remains to treat the independence case.

The intuition is strong to suggest that independent risks are substitutes. We mean by this that the presence of one risk reduces the demand for other independent risks. This would mean that background risks have a tempering

[5] Leland (1980) determines the characteristics of agents who should purchase call option on the aggregate risk in the economy.

effect on the demand for stocks. Households that are subject to a larger (mean-preserving) uncertainty about their future labour incomes should be more conservative on their portfolio.

Our theoretical model to treat this question is quite simple. Consider an agent with utility u on his final consumption. He is endowed with capital X at the beginning of the period, which can be invested in Arrow-Debreu securities as in the previous section. What is new here is that the final wealth of the agent equals the sum of the value of his portfolio and his labour income. Because the risk free part of this income has been included, discounted, in X, we assume that this added has a zero mean. This background risk is denoted $\widetilde{\varepsilon}$ and is independent of the state of the world i. The portfolio problem is now written as

$$\max_{C_1,\ldots,C_N} \quad \sum_{i=1}^{N} p_i Eu(C_i + \widetilde{\varepsilon}), \tag{10}$$

subject to the unchanged budget constraint (2).

Let us define indirect utility function v, with $v(Z) = Eu(Z + \widetilde{\varepsilon})$ for all Z. In consequence, the above problem can be rewritten as

$$\max_{C_1,\ldots,C_N} \quad \sum_{i=1}^{N} p_i v(C_i), \tag{11}$$

subject to (2). We conclude that the introduction of an independent background risk is equivalent to the transformation of the original utility function u into the indirect utility function v. This change in the attitude towards portfolio risks can be signed if and only if the degree of concavity of these two functions can be ranked in the sense of Arrow-Pratt. More specifically, the intuition that the background risk affects negatively the demand for stocks would be sustained by the theory if v is more concave than u. Technically, this means that

$$E\widetilde{\varepsilon} = 0 \quad \Longrightarrow \quad \frac{-Eu''(C + \widetilde{\varepsilon})}{Eu'(C + \widetilde{\varepsilon})} \geq \frac{-u''(C)}{u'(C)} \tag{12}$$

for all C and $\widetilde{\varepsilon}$. This property does not hold in general. All utility functions which satisfy property (12) are said to be "risk vulnerable", to follow a terminology introduced by Gollier and Pratt (1996). Using Taylor approximations, it is easy to prove that this property holds for small risks if and only if

$$A''(C) \geq 2A'(C)A(C), \tag{13}$$

where $A(C) = -u''(C)/u'(C)$ is absolute risk aversion. A simple sufficient condition for risk vulnerability is that A be decreasing and convex. Because the proof of this result is simple, we reproduce it here. The left-hand condition in (12) can be rewritten as

$$EA(C+\tilde{\varepsilon})u'(C+\tilde{\varepsilon}) \geq A(C)Eu'(C+\tilde{\varepsilon}) \tag{14}$$

But a decreasing and convex function A implies that

$$EA(C+\tilde{\varepsilon})u'(C+\tilde{\varepsilon}) \geq [EA(C+\tilde{\varepsilon})]\,[Eu'(C+\tilde{\varepsilon})] \geq A(C)Eu'(C+\tilde{\varepsilon}). \tag{15}$$

This is what we had to prove. The first inequality in (15) comes from the fact that both A and u' are decreasing in ε. The second inequality is an application of Jensen's inequality, together with $A'' \geq 0$ and $E\tilde{\varepsilon} = 0$.

Observe that the classically used power utility function (4) has a decreasing and convex absolute risk aversion, since $A(C) = \rho/C$. Our conclusion is that the introduction of a background risk in the calibration of household portfolios will imply a rebalancement towards the risk free asset.

5 The optimal dynamic portfolio composition with complete markets

One of the main deficiencies of the standard portfolio problem that we examined in section 3 comes from the fact that it is static. Technically, it is descriptive of the situation of an agent who invests his wealth in prospect of his retirement that will take place in one year. In this section, we will examine the optimal portfolio composition for people who have more periods to go before retirement. In other words, we will examine the relationship between portfolio risk and time horizon.

In the formal literature, the horizon-riskiness issue has received the greatest attention addressing portfolios appropriate to age. Samuelson (1989) and several others have asked: "As you grow older and your investment horizon shortens, should you cut down your exposure to lucrative but risky equities?" Conventional wisdom answers affirmatively, stating that long-horizon investors can tolerate more risk because they have more time to recoup transient losses. This dictum has not received the imprimatur of science, however. As Samuelson (1963, 1989) in particular points out, this "time-diversification" argument relies on a fallacious interpretation of the Law of Large Numbers: repeating an investment pattern over many periods does not cause risk to wash out in the long run. In the next section, we examine an alternative concept of time diversification.

To address this question, we consider the problem of an agent who has to manage a portfolio over time in order to maximize the expected utility of his consumption at retirement. In this section, we abstract ourself from the consumption-saving problem by assuming that this portfolio is specific for retirement and that it cannot be used for consumption before retirement. Also, we normalize the risk free rate to zero. This implies that a young and an old investor with the same discounted wealth today can secure the same level of consumption at retirement by investing in the riskless asset. If the risk free rate would be positive, the younger investor would implicitly be wealthier.

We can now understand why the age of the investor has an ambiguous effect on the optimal portfolio composition. Contrary to the old investor, the young agent has an option to invest in stocks in the subsequent periods. This option has a positive value, which makes the younger agent implicitly wealthier, at least on average. Under decreasing absolute risk aversion (DARA), that makes him less risk-averse. This wealth effect affects positively the share of wealth invested in stocks. But taking a risk has not the same comparative statics effect than getting its expected net payoff for sure, as stressed in section 4. The fact that the younger agent will take a portfolio risk in the future plays the role of a background risk with respect to his portfolio choice problem when he is young. This risk effect goes the opposite direction than the wealth effect, under risk vulnerability. All this is made more complex by the dynamic aspect of the problem, in the sense that the portfolio risk that the young will take in the future can be made contingent to the accumulated portfolio value in previous periods.

It is standard to solve this kind of dynamic problem by using backward induction. Let T denote the number of periods before retirement. Because we assumed that the only decision maker's concern is the welfare of the investor at the time of retirement, the objective is to maximize $Eu(\widetilde{C}_T)$. Assuming complete markets against all risks occurring during the last period, the problem of the decision maker is exactly the same as problem (1), (2). Denoting X_{T-1} for the wealth that has been accumulated at the beginning of the last period, this problem is rewritten as

$$V_{T-1}(X_{T-1}) = \max_{C_{1T},\dots,C_{NT}} \quad \sum_{i=1}^{N} p_i u(C_{iT}) \tag{16}$$

$$s.t. \quad \sum_{i=1}^{N} p_i \pi_i C_{iT} = X_{T-1} \tag{17}$$

where C_{iT} is the demand for the Arrow-Debreu security associated to state i. Because we assume that there is no serial correlation in asset returns and that the random walk of returns is stationary, we did not indexed state prices π_i and probabilities p_i by T. In short, financial risks are assumed to be the same at each period. Again, with a negative serial correlation in assets returns, investing in stocks in the future may serve as a partial insurance for portfolio risks taken today. This would provide an additional incentive for young investors to raise the share of their wealth invested in stocks. We do not consider this possibility here.

Equation (16) introduced the (Bellman) value function V_{T-1} into the picture. $V_{T-1}(X_{T-1})$ is the maximum expected utility of consumption at retirement that can be obtained when the household accumulated a capital X_{T-1} at the end of period $T-1$. Then, the portfolio problem at the beginning of period $T-1$ when the accumulated capital at that date is X_{T-2} can be written as

$$V_{T-2}(X_{T-2}) = \max_{X_{1T-1},\dots,X_{NT-1}} \quad \sum_{i=1}^{N} p_i V_{T-1}(X_{iT-1}) \tag{18}$$

$$s.t. \quad \sum_{i=1}^{N} p_i \pi_i X_{iT-1} = X_{T-2} \tag{19}$$

where X_{iT-1} is at the same time the demand at $T-1$ for the Arrow-Debreu security associated to state i and the accumulated capital at the end of period $T-1$ if state i occurs. The optimal portfolio strategy in period $T-1$ is to find a portfolio which maximizes the expected value V_{T-1} (which is itself the maximal expected utility of final consumption) of the accumulated wealth at the end of the period. This will generate a dynamic portfolio strategy that is optimal, in the sense that it will maximize the expected utility of final consumption. Pursuing this method by backward induction, we obtain the full description of the optimal nonmyopic portfolio strategy, which is given by the set of functions $\{X_{it}(X_{t-1}) \mid i = 1, ..., N; \quad t = 1, ..., T\}$, with $X_{iT}(X) = C_{iT}(X)$.

The question is to determine the impact of index t on the optimal portfolio composition $(X_{1t}(X), ..., X_{Nt}(X))$ for a given wealth X accumulated at the beginning of the period. To illustrate, let us limit the analysis to the comparison between the optimal portfolio composition in periods $T-1$ and T, assuming the same $X = X_{T-2} = X_{T-1}$. We see that the only difference between these two decision problems (16) and (18) comes from the replacement of the original utility function u on retirement consumption by the value function V_{T-1} on the accumulated wealth at the end of the period. We know that this is the case if and only if the degree of concavity of V_{T-1} is comparable to the degree of concavity of u in the sense of Arrow-Pratt. Thus, we need to evaluate the degree of concavity of V_{T-1}. This is done as follows. At the last period, the optimal portfolio composition is characterized by $C_{iT}(X)$ that is the solution of

$$u'(C_{iT}(X)) = \xi(X)\pi_i \tag{20}$$

subject to the budget constraint $\sum_i p_i \pi_i C_{iT}(X) = X$. Fully differentiating this condition with respect to X and eliminating π_i yields

$$C'_{iT}(X) = \frac{-\xi'(X)}{\xi(X)} \tau(C_{iT}(X)), \tag{21}$$

where $\tau(C) = -u'(C)/u''(C) = [A(C)]^{-1}$ is the degree of absolute risk tolerance of the agent. From the budget constraint, we have that $\sum_i p_i \pi_i C'_{iT}(X) = 1$. Using (21), it implies that

$$\frac{-\xi'(X)}{\xi(X)} = \left[\sum_i p_i \pi_i \tau(C_{iT}(X))\right]^{-1}. \tag{22}$$

On the other side, we know from the standard Lagrangian method that $V'_{T-1}(X) = \xi(X)$. It implies that

$$\frac{-V'_{T-1}(X)}{V''_{T-1}(X)} = \sum_i p_i \pi_i \tau(C_{iT}(X)). \tag{23}$$

The left-hand side of this equation is the weighted expectation of the absolute risk tolerance evaluated at the random final wealth. It is an expectation under the assumption that $\sum_i p_i \pi_i = 1$, which means that the risk free rate is zero. Condition (23) means that the relative risk tolerance of the value function used to measure the optimal attitude toward portfolio risk at $T-1$ is a weighted average of ex post absolute risk tolerance. This result has first been obtained by Wilson (1968) in the context of static risk sharing. Suppose that function τ be convex. By the Jensen's inequality, we then obtain that

$$\frac{-V'_{T-1}(X)}{V''_{T-1}(X)} \geq \tau \left(\sum_i p_i \pi_i C_{iT}(X) \right) = \tau(X) = \frac{-u'(X)}{u''(X)}. \tag{24}$$

Because this is true for any X, V_{T-1} is less concave than u in the sense of Arrow-Pratt. We conclude that *the convexity of absolute risk tolerance is necessary and sufficient for younger people to take more portfolio risk,* ceteris paribus. By symmetry, the concavity of absolute risk tolerance is necessary and sufficient for younger people to take less portfolio risk. This result has first been obtained by Gollier and Zeckhauser (1998) who extended this result to the more difficult case of incomplete markets.

The limit case is when absolute risk tolerance is linear, which corresponds to the set of HARA utility functions. For HARA preferences, the age of the investor has no effect on the optimal composition of her portfolio. Because this set of functions contains the only one for which a complete analytical solution for the optimal portfolio strategies can be obtained, it is not a surprise that most economist specialized in this field recommend a age-independent portfolio strategy. It must be stressed however that there is no strong argument in favor of HARA functions, except for their simplicity. Whether absolute risk tolerance is concave or convex, or neither of the two, remains an open question for empirical investigation. Guiso, Jappelli and Terlizesse (1995) observed a bell curve for the relationship between age and the share of wealth invested in stocks, suggesting that τ is neither concave nor convex.

We did not take into account of an important phenomenon in this analysis. Namely, younger people usually face a larger background risk on their human capital. As we grow older, the uncertainty on one's human capital is revealed by labour markets. From our discussion in section 4, it is likely to imply a tempering effect on the optimal demand for stocks.

6 Self-insurance and time diversification

Up to now, we assumed that households have no control on what they put in and out of the fund of assets at each period. In other words, we assumed that the capital is accumulated for a single objective, which is consumption at old age. Obviously, this is not a reasonable assumption. Most households also use savings for a precautionary motive. They increase their saving in case of an unexpected transitory increase in income, and they reduce their saving effort, or even they become borrowers, in case of an adverse transitory shock on their incomes. Reciprocally, agents can decide to reduce their saving effort if their have been lucky on their portfolio. This means that agents can smooth shocks on their accumulated wealth by increasing or reducing their consumption over several periods. As we will see, consumption smoothing plays the role of self-insurance. That implies in turn more risk taking.

The simplest model that we could imagine for self-insurance over time is a model in which agents live for N periods $t = 1, ..., N$. They have a cash-on-hand K prior to period 1. At each period, they receive a non-capital income Y. They consume C_t from it in period t. The remaining is saved in a risk free asset whose gross return is R. Agents are allowed to take risk prior to period 1, but they are prohibited to do so afterwards. This simplified assumption is made here because we want to isolate the self-insurance effect of time. We will come back to this point later on.

Let $\widetilde{\varepsilon}$ denote the net payoff of the risk taken prior to period 1. In period 1, the discounted wealth X will be the sum of the cash-on-hand, the discounted value of future incomes and the actual payoff of the lottery: $X = K + \sum_t R^{-t} Y + \varepsilon$. Given X, agents select the consumption plan that maximizes their discounted lifetime utility $V(X)$, which takes the following form:

$$V(X) = \max_{C_1,\dots,C_N} \quad \sum_{t=1}^{N} \beta^t u(C_t) \tag{25}$$

$$s.t. \quad \sum_{t=1}^{N} R^{-t} C_t = X, \tag{26}$$

where β is the discount factor on utility. It yields an optimal consumption plan $C_t(X)$ which is a function of the discounted wealth of the agent. To solve this problem, we can use the fact that it is equivalent to program (1) with $p_t = \beta^t$ and $\pi_t = (R\beta)^{-t}$. From equation (23), we directly infer that

$$\frac{-V'(X)}{V''(X)} = \sum_{t=1}^{N} R^{-t} \tau(C_t(X)). \tag{27}$$

Condition (27) characterizes the degree of tolerance towards the risk $\widetilde{\varepsilon}$ taken prior to period 1. The simplest case is obtained when $\beta = R$. In that case, we know that it is optimal to smooth consumption perfectly. It yields

$$\frac{-V'(X)}{V''(X)} = \left[\sum_{t=1}^{N} R^{-t}\right] \tau\left(\frac{X}{\sum_{t=1}^{N} R^{-t}}\right) = \left[\sum_{t=1}^{N} R^{-t}\right] \tau\left(Y + \frac{K+\varepsilon}{\sum_{t=1}^{N} R^{-t}}\right) \tag{28}$$

when the risk free rate is small. Now, compare *two agents whose wealth levels per period ($X/\sum R^{-t}$) are the same*. But one of the two agents has one period to go ($N = 1$), whereas the other agent has $N > 1$ periods to go. Equation (28) tells us that the agent with time horizon N will be $\sum R^{-t} \approx N$ times more risk tolerant than the agent with only one period to go. Thus, we conclude that there is a strong time diversification effect which takes place in this model. The intuition is quite simple. Each dollar of loss or gain will be equally split into $1/N$ of a dollar reduction or increase in the consumption in each period. This is an efficient risk-sharing scheme of the different future selves representing the household over time. This ability to diversify the single risk over time induces the agent to be more willing to purchase it.

Our ceteris paribus assumption above was that agents have the same wealth level per period. This means that a reduction in the time horizon does not affect the feasible consumption level per period. This is the case for example when the agent has no cash-on-hand in period 1 in such a way that wealth X comes solely from discounting future incomes. We can alternatively compare two agents with different time horizons, but with identical discounted wealth X. In this case, the agent with a longer time horizon will consume less at each period. Under DARA, that will induce more risk aversion, and it is not clear which of the time diversification effect and this wealth effect will dominate. The limit case is when absolute risk tolerance is homogenous of degree 1 with respect to consumption. This is the case under CRRA, since we have $\tau(C) = C/\gamma$. Equation (28) is rewritten as $-V'(X)/V''(X) = X/\gamma$ in that case. This is independent of the time horizon. Under CRRA, the wealth effect just compensates the time diversification effect: two CRRA agents with the same aggregate wealth but facing different time horizons will have the same attitude towards the single risk $\widetilde{\varepsilon}$. When absolute risk tolerance is subhomogeneous., the agent will be more tolerant to a single risk if it is resolved earlier in his life.

Remember that equation (28) holds only under the assumption that a constant consumption over the lifetime is optimal, i.e., when $R\beta = 1$. When R and β^{-1} are not equal, we must use equation (27) with the optimal consumption plan ($C_1, ..., C_N$). If absolute risk tolerance is convex, Jensen's inequality applied to this equation implies that

$$\frac{-V'(X)}{V''(X)} \geq \left[\sum_{t=1}^{N} R^{-t}\right] \tau \left(\frac{\sum_{t=1}^{N} R^{-t} C_t}{\sum_{t=1}^{N} R^{-t}}\right) = \left[\sum_{t=1}^{N} R^{-t}\right] \tau \left(\frac{X}{\sum_{t=1}^{N} R^{-t}}\right). \quad (29)$$

This is equation (28), except that the equality has been replaced by an inequality. Thus, under convex absolute risk tolerance, the time diversification effect is even stronger than explained above. Technically, this result has been obtained by following the same procedure than the one for the complementarity of repeated risks in section 5. Here, the optimal fluctuations of consumption over time, rather than across states, are complementary to risk taking if τ is convex.

The model presented in this section is not realistic because of the assumption that there is a single risk in the lifetime of the investor. We now combine the different effects of time horizon on the optimal instantaneous portfolio:

1. The *complementarity effect* of repeated risks over time: the option to take risk in the future raises the willingness to take risk today under convex absolute risk tolerance;

2. The *time diversification effect*: the opportunity to smooth shocks on capital by small variations of consumption over long horizons raises the willingness to take risk;

3. The *wealth effect*: for a given discounted wealth, a longer horizon means less consumption at each period, which reduces the willingness to take risk under DARA.

We consider the following model. Investors can consume, save and take risk at each period from $t = 1$ to $t = T$. At each period, there is some uncertainty about which state of the world $i = 1, ..., N$ will prevail at the end of the period. Within each period t, the agent begins with the selection of a portfolio. After observing the state of the world and the value X_t of the portfolio, the agent decides how much to consume (C_t) and how much to save for the next period (S_t). We can decompose the above-mentioned effects by using backward induction again. This is done in the following steps:

- In period T, the agent selects his optimal portfolio of Arrow-Debreu securities by solving program (16), where X_{T-1} is replaced by S_{T-1}. The transformation from u to V_{T-1} describes the complementarity (or substituability) effect of repeated risk.

- At the end of period $T-1$, after observing the state of the world and the associated value X_{T-1} of the portfolio, the agent solves his consumption-saving problem which is written as

$$\widehat{V}_{T-1}(X_{T-1}) = \max_{C} \quad u(C) + \beta V_{T-1}(R(X_{T-1} - C)). \qquad (30)$$

  This operation describes the time diversification effect and the wealth effect of time horizon.

- The agent determines the optimal composition of his portfolio at the beginning of period $T-1$ by solving program (18), but with function V_{T-1} being replaced by function $\widehat{V}_{T-1}$ defined by (30) to take into account of the possibility to smooth consumption over time.

Going back to the original question of how does time horizon affect the optimal structure of households' portfolios, we must compare the degree of concavity of $\widehat{V}_{T-1}$ with respect to the degree of concavity of u. If $\widehat{V}_{T-1}$ is less concave than u, it would imply that, ceteris paribus, one selects a riskier portfolio in period $T-1$ than in period T. The CRRA function is an instructive benchmark. Because CRRA is a special case of HARA, we know from section 5 that $V_{T-1}(.) \equiv hu(.)$: the option to invest in risky assets in period T does not affect the degree of concavity of the value function. It implies that program (30) is a special case of program (25) with β being replaced by βh. Because u is CRRA, we also know that the time diversification effect is completely offset by the wealth effect, so that $\widehat{V}_{T-1}(.) \equiv \widehat{h}u(.)$: $\widehat{V}_{T-1}$ as the same concavity than u. We conclude that myopia is optimal under

CRRA. The optimal dynamic portfolio strategy is obtained by doing as if each period would be the last one before retirement. There is no effect of time horizon on the optimal portfolio, ceteris paribus. This result was already in Merton (1969) and Samuelson (1969). Mossin (1969) showed that CRRA is also necessary for optimal myopia.

It is easy to combine our other results. For example, a larger time horizon induces riskier portfolios if absolute risk tolerance is convex and subhomogeneous. On the contrary, a larger time horizon induces more conservative portfolios if absolute risk tolerance is concave and superhomogeneous.

7 The liquidity constraint

In the previous sections, we assumed that markets are frictionless. There are no transaction costs to exchange assets and no cap on the risk that can be taken on financial markets. Moreover, we assumed that the borrowing rate is equal to the lending rate on the credit market. Whereas the effect of the introduction of market imperfections on the optimal portfolio is easy to examine in a static model, the analysis becomes complex in a dynamic framework. In this section, we will focus on the effect of a liquidity constraint on the optimal portfolio. A liquidity constraint is an extreme form of inefficiency on the credit market. It forces households to maintain a sufficient level of cash-on-hand in each period. In other words, it prohibits households to borrow too much. A weaker version of the liquidity constraint is when the borrowing rate is larger than the lending rate.

The effect of a liquidity constraint on the optimal portfolio is easy to understand. Remember that it is optimal for households to smooth adverse shocks on their portfolio return by reducing consumption over a long period of time. This is done by reducing saving immediately after the adverse shock to finance short term consumption. This is the mechanism behind consumption smoothing. If the shock is large enough, it can be the case that some households become short of cash, i.e. that they become net borrowers for a short period of time. Under the liquidity constraint, this is not allowed. Thus, these households will be forced to drastically reduce their short term consumption. They will not be able to time diversify their portfolio risk anymore. This will raise their degree of aversion towards portfolio risk.

The simplest model to describe this phenomenon is similar to the one presented in section 6. We assume that households live for N periods, and that they start with a cash-on-hand K. The only possibility to take risk is at the beginning of period $t = 1$. Households have a permanent income Y. They must maintain a positive cash-on-hand permanently. We also assume that $R\beta = 1$. We showed above how this simplest model can be extended by the introduction of repeated risk taking and $R\beta \neq 1$.

Let ε denote the payoff of the risk taken at the beginning of period 1. Thus, when households determine their optimal consumption plan , they are endowed with a cash-on-hand $K + \varepsilon$. Because perfect consumption smoothing is optimal in the absence of the liquidity constraint, households would consume

$$C = Y + \frac{K + \varepsilon}{\sum_{t=1}^{N} R^{-t}} \tag{31}$$

at each period. It yields a degree of risk aversion to risk $\widetilde{\varepsilon}$ ex ante that is given by equation (28). How is the degree of aversion to risk $\widetilde{\varepsilon}$ affected by the constraint that the cash-on-hand must be positive? In the unconstrained case, the cash-on-hand at the end of the first period would be

$$K + \varepsilon + Y - C = (K + \varepsilon)\frac{-1 + \sum_{t=1}^{N} R^{-t}}{\sum_{t=1}^{N} R^{-t}}. \tag{32}$$

If ε is larger than K, the constraint will not be binding and the local measure of concavity of the value function V , measured at $X = K + \varepsilon + Y \sum_t R^{-t}$, will be as in equation (28). Of course, the difference is for the local measure of the value function at lower wealth levels, where the liquidity constraint is binding. Indeed, when $\varepsilon < -K$, households would like to finance short term consumption by borrowing money, but they are not allowed to do so. They are forced to swallow the drop in their wealth in one shot by reducing their immediate consumption accordingly. They will consume Y afterwards. For those values of ε, we obtain that

$$\frac{-V'(X)}{V''(X)} = \tau(Y + K + \varepsilon). \tag{33}$$

Under DARA, we have that

$$\tau(Y + K + \varepsilon) \leq \tau(Y + \frac{K + \varepsilon}{\sum_{t=1}^{N} R^{-t}}) \leq \left[\sum_{t=1}^{N} R^{-t}\right] \tau(Y + \frac{K + \varepsilon}{\sum_{t=1}^{N} R^{-t}}). \tag{34}$$

This means that the local degree of concavity of the value function is smaller with a binding liquidity constraint than without it. Moreover, it is reduced by a factor at least equal to $\sum_{t=1}^{N} R^{-t} \approx N$. There is no more time diversification.

The implication of this analysis is that households that are more likely to be liquidity constrained will adopt a more conservative portfolio strategy.

8 Conclusion

The theory of dynamic portfolio management remains a fascinating field of research. The similarity between the static portfolio problem under uncertainty and the dynamic consumption problem under certainty is helpful to understand how risk and time interact with each other to generate a dynamically optimal portfolio strategy. We have shown in particular that the concavity or convexity of the absolute risk tolerance with respect to consumption is important to characterize the effect of time on risk taking. This is because the absolute tolerance to portfolio risk in the static portfolio problem is a weighted average of ex post absolute risk tolerances on consumption. Similarly, the absolute risk tolerance on wealth in the dynamic consumption problem under certainty is equal to the discounted value of future absolute risk tolerances on consumption. Thus, absolute risk tolerances can be summed, contrary to absolute risk aversions.

This very basic result is at the origin of our understanding of how portfolio risks occurring at different points in time interact with each other. We also showed how the familiar notion of time diversification can be justified on a theoretical ground. We presented several testable hypotheses using data on household portfolios:

1. Wealthier people own more risky assets (under decreasing absolute risk aversion);

2. Wealthier people invest a larger share of their wealth in risky assets (under decreasing relative risk aversion);

3. Households with a riskier human capital invest less in risky assets (under risk vulnerability);

4. Households who can invest longer in risky assets will invest more in them (under convex absolute risk tolerance);
5. Households who are more likely to be liquidity constraint in the future will invest less in risky assets (under decreasing absolute risk aversion).

Empirical evidences related to these hypotheses are presented in various chapters of this book.

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