Proposed Notational Conventions

This document proposes some notational rules drawn primarily from the recent finance text by Campbell, Lo, and MacKinlay, though with some important deviations.

1 General Rules

- The variable i will be used as a subscript to indicate asset i out of a list of up to N assets
- R with no subscripts will be taken to be the constant, riskless gross rate of return ('gross' meaning including return of principal) and r will signify net rate of return, where gross and net returns are related by $R = 1 + r$.
- The timing convention is that a variable is dated t if it is known by the end of period t. Thus $R_{i,t}$ indicates the return on asset i held from the end of period $t-1$ into period t.
- $w_{i,t}$ indicates the portfolio weight given to asset i for the holding period from the end of period t into period $t+1$.
- Use boldface for vectors and matrices, and regular face for scalars. Thus $w_{i,t}$ is a scalar and $\mathbf{w}_t$ is the vector of the values $w_{i,t}$ for all i.
- $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean μ and standard deviation σ.
- The mean expectation of a variable is designated by a bar over the variable. Thus the statement that $r_i \sim \mathcal{N}(\overline{r}_i, \sigma^2_{r_i})$ indicates that r_i is distributed normally with mean value $\overline{r}$.
- $\Pr(.)$ denotes the probability of an event
- $E_t[X_s]$ is the expectation given all information known as of time t of the value of variable X as of time s. The unconditional expectation of a variable whose value varies with time is given by $E[X_t]$.
- A $\sim$ over a variable in an expectations formula indicates that the variable's value is stochastic viewed from the perspective of the period with respect to which the expectation is being taken. Thus if r_i is distributed lognormally as specified above, we can write $E_t[\tilde{r}_{i,t+1}] = \overline{r}_i$.

2 Parameter Definitions

β	–	Time discount factor between periods
ρ	–	Coefficient of Relative Risk Aversion
X_t	–	Gross resources ('cash-on-hand') available for spending in period t
S_t	–	'Savings' left over at the end of period t after consumption
C_t	–	Consumption in period t
V_t	–	Value function
u_t	–	Utility function

B_t	–	Utility from wealth/bequest function
W_t	–	Proportion of portfolio in risky assets
R_t	–	Total portfolio-weighted return between $t-1$ and t
Y_t	–	Noncapital income in period t
P_t	–	Permanent income in period t
ϵ_t	–	Transitory shock to noncapital income in period t
η_t	–	Shock to permanent income in period t
d_t	–	Probability of dying at the end of period t
p_z	–	Probability that the variable z is equal to zero

3 A Specific Problem

Using these conventions and a few others I will solve the notation problem recursively from the last period of life.

Consider the optimization problem in the last period of life. The consumer obtains utility $u(C_T)$ from consumption and $B(S_T)$ from any wealth S_T that is unconsumed and thus is left behind as a bequest. The value function gives the utility obtained from choosing last-period consumption C_T optimally:

$$\begin{array}{rcl} V_T(X_T) & = & \max_{\{C_T\}} u(C_T) + B(S_T) \\ & \text{such that} & \\ S_T & = & X_T - C_T \\ S_T & \geq & 0 \end{array}$$

In the second-to-last period of life, the consumer must choose not only the level of consumption but also how to allocate any savings S_{T-1} not consumed between the riskless asset and N different potential risky asset categories. Designate the proportion (or weight) of S_{T-1} put into each of these possible assets $w_{i,T-1}$, and define $W_{T-1} = (1 - \sum_{i=1}^{N} w_{i,T-1})$ as the total portfolio share in all risky assets implying that the portfolio share in the riskless asset will be (1-W_{T-1}). Define the gross return on asset category i between period $T-1$ and period T as $R_{i,T}$ as per the guidelines, and define the portfolio-weighted average rate of return on savings between periods $T-1$ and T as

$$R_T = R(1 - W) + \sum_{i=1}^{N} w_{i,T-1} R_{i,T}. \tag{1}$$

Designating the total noncapital income that the household receives in the last period of life Y_T, total cash-on-hand in period T will be given by

$$X_T = R_T S_{T-1} + Y_T. \tag{2}$$

In addition, assume that there is some probability d_T that the consumer will die after performing this period's consumption but before beginning next period. Finally, assume that the consumer is not allowed to die in debt and therefore cannot borrow. The consumer's problem under these circumstances is

$$\begin{aligned}
V_{T-1}(X_{T-1}) &= \max_{\{C_{T-1},\mathbf{w}_{T-1}\}} u(C_{T-1}) + (1-d_{T-1})\beta E_{T-1} V_T(X_T) + d_{T-1}B(S_{T-1}) \\
&\text{such that} \\
X_T &= R_T S_{T-1} + Y_T \\
S_{T-1} &= X_{T-1} - C_{T-1} \\
S_{T-1} &\geq 0
\end{aligned}$$

or, substituting the constraints into the problem, we have:

$$\begin{aligned}
&V_{T-1}(X_{T-1}) = \\
&\max_{\{C_{T-1},\mathbf{w}_{T-1}\}} \quad u(C_{T-1}) + (1-d_{T-1})\beta E_{T-1}\left[V_T(\tilde{R}_T[X_{T-1}-C_{T-1}] + \tilde{Y}_T)\right] + d_{T-1}B(X_{T-1}-C_{T-1}) \\
&\text{such that} \quad C_{T-1} \leq X_{T-1}
\end{aligned}$$

An analogous equation will hold for all earlier periods of life, so that we can state the general maximization problem as

$$\begin{aligned}
&V_t(X_t) = \\
&\max_{\{C_t,\mathbf{w}_t\}} \quad u(C_t) + (1-d_t)\beta E_t\left[V_{t+1}(\tilde{R}_{t+1}[X_t - C_t] + \tilde{Y}_{t+1})\right] + d_t B(X_t - C_t) \\
&\text{such that} \quad C_t \leq X_t
\end{aligned}$$

In the process of solving maximization problems numerically, it is often useful to define a function which yields the expected value associated with pursuing every possible choice for the control variables. For this problem, the only way in which C_t and X_t affect V_{t+1} is through their effects on S_t. We can now define a function

$$\Omega_t(S_t, w_{1,t}, \ldots, w_{N,t}) = (1-d_t)\beta E_t\left[V_{t+1}(\tilde{R}_{t+1}S_t + \tilde{Y}_{t+1})\right] + d_t B(S_t)$$

which returns the expected value from pursuing any possible choice of S_t and portfolio share configuration, and the maximization problem can be rewritten somewhat more simply as:

$$\begin{aligned}
V_t(X_t) &= \max_{\{C_t,\mathbf{w}_t\}} u(C_t) + \beta\Omega_t(S_t, \mathbf{w}_t) \\
&\text{such that} \\
S_t &= X_t - C_t \\
S_t &\geq 0.
\end{aligned}$$

Written in this way, the problem is formidably difficult to solve because it involves simultaneous nonlinear maximization with respect to $N+1$ choice variables, C_t and N portfolio shares (any $N-1$ shares will define the Nth share via the constraint that the sum of shares equals one). One way of solving problems of this kind numerically is to define a series of functions

$$\begin{aligned}
\Omega_{N,t}(S_t, w_{2,t}, \ldots, w_{N-1,t}) &= \max_{\{w_{N,t}\}} \Omega_t(S_t, w_{2,t}, \ldots, w_{N,t}) \\
\Omega_{N-1,t}(S_t, w_{2,t}, \ldots, w_{N-2,t}) &= \max_{\{w_{N-1,t}\}} \Omega_{N,t}(S_t, w_{2,t}, \ldots, w_{N,t})
\end{aligned}$$

which, given a fixed choice for S_t and all of the portfolio shares up to a given share, finds the optimal value for that portfolio share given optimal choice of the remainder of portfolio shares. The logic of this process is exactly equivalent to the logic behind the traditional recursive solution to dynamic optimization problems, and the process leads eventually to a function which yields the optimal value of any value of savings S_t given optimal choice of portfolio shares, $\Omega_{*,t}(S_t)$ where the * is used to indicate that optimal choice of portfolio shares is happening in the background. The problem now can be written

$$\begin{aligned} V_t(X_t) &= \max_{\{C_t\}} u(C_t) + \Omega_{*,t}(X_t - C_t) \\ &\text{such that} \\ C_t &\leq X_t \end{aligned}$$

Problems of this kind are usually solved using the first order conditions. Define the derivative of a function $f(x, y, \ldots)$ with respect to its arguments as $f^x, f^y, \ldots$. There will be one first order condition with respect to the portfolio share of each possible investment, possibly along with short sales constraints that require $0 \leq w_{i,t} \leq 1$ for all i. Then there will be N first order conditions with respect to the portfolio shares, one for each of the series of equations listed above, and each of these will take the form:

$$0 = \beta E_t \left[(\tilde{R}_{i,t} - R) S_t (1 - d_t) V^X_{t+1}(\tilde{X}_{t+1}) \right].$$

The first order condition for this problem with respect to consumption can be written:

$$0 = u^c(C_{t+1}) + (1 - d_t)\beta E_t \left[\tilde{R}_{t+1} V^X_{t+1}(\tilde{X}_{T+1}) \right] + d_t B^S_t(X_t - C_t).$$

This equation will define the optimal level of consumption for any given choice of portfolio shares $\mathbf{w}_t$, unless the C_{t+1} which satisfies this equation is greater than X_t, in which case the no-dying-in-debt constraint will bind and the consumer will spend $C_t = X_t$.

A prototypical specification for the income process is as follows. Realized income in period t is given by permanent income in period t times a multiplicative shock ϵ_t:

$$Y_t = P_t \epsilon_t \tag{3}$$

Often we assume that there is some positive probability that $\epsilon_t = 0$; write this probability as $p_\epsilon = \Pr(\epsilon_t = 0)$. Often we assume that if the realization of ϵ_t is nonzero then ϵ_t is distributed lognormally with a mean such that $E_{t-1}[\tilde{\epsilon}_t] = 1$. This implies that $E_{t-1}[\tilde{\epsilon}_t | \epsilon_t > 0] = 1/(1 - p_\epsilon)$. Using the fact that for a lognormally distributed variable z

$$\log E[z] = E[\log z] + \frac{1}{2}\text{var}[\log z]$$

we have that

$$\log E[\epsilon | \epsilon > 0] = E[\log \epsilon | \epsilon > 0] + \text{var}_t[\log \epsilon | \epsilon > 0]$$

Using the approximation that $\log[1/(1 - p_\epsilon)] \approx p_\epsilon$ if p_ϵ is small, this gives us that

$$\begin{aligned} p_\epsilon &\approx E[\log \epsilon | \epsilon > 0] + \text{var}_t[\log \epsilon | \epsilon > 0] \\ E[\log \epsilon | \epsilon > 0] &\approx p_\epsilon - \sigma^2_\epsilon / 2 \end{aligned}$$

and thus our assumption is that

$$\log \epsilon_t \sim \mathcal{N}(p_\epsilon - \sigma_\epsilon^2/2, \sigma_\epsilon^2) \tag{4}$$

Similarly, but more simply, we often assume that permanent income is growing at some rate G_t from period to period but is hit by a multiplicative shock η_t:

$$P_{t+1} = G_{t+1} P_t \eta_{t+1}$$

where usually the assumption is that η_t is lognormally distributed such that $E_{t-1}[\eta_t] = 1$ implying that the distribution for η is $\eta \sim \mathcal{N}(-\sigma_\eta^2/2, \sigma_\eta^2)$.