BDF2

Consider the polynomial interpolation of $Y(t)$ at the points t_{n-1}, t_n, t_{n+1}. The backward differentiation method approximates $Y(t)$ using that interpolant and then differentiates it to find an approximation for $f(t_{n+1}, y(t_{n+1}))$. This derives the second-order backward differentiation method. An identical derrivation will give you the backward differentiation method for order 3 through 6, by just interpolating at one more point each time.

$Y(t) \approx P(t)$

$$
\begin{aligned}
&= Y(t_{n-1})\frac{(t-t_n)(t-t_{n+1})}{(t_{n-1}-t_n)(t_{n-1}-t_{n+1})} + Y(t_n)\frac{(t-t_{n-1})(t-t_{n+1})}{(t_n-t_{n-1})(t_n-t_{n+1})} \\
&\quad + Y(t_{n+1})\frac{(t-t_{n-1})(t-t_n)}{(t_{n+1}-t_{n-1})(t_{n+1}-t_n)} \\
&= \frac{1}{h^2}\left(\frac{1}{2}Y(t_{n-1})(t-t_n)(t-t_{n+1}) - Y(t_n)(t-t_{n-1})(t-t_{n+1}) \right. \\
&\quad \left. +\frac{1}{2}Y(t_{n+1})(t-t_{n-1})(t-t_n)\right) \\
\Rightarrow P'(t) &= \frac{1}{h^2}\left[\frac{1}{2}Y(t_{n-1})(t-t_n+t-t_{n+1}) - Y(t_n)(t-t_{n-1}+t-t_{n+1}) \right. \\
&\quad \left. +\frac{1}{2}Y(t_{n+1})(t-t_{n-1}+t-t_n)\right] \\
\Rightarrow P'(t_{n+1}) &= \frac{1}{h^2}\left[\frac{1}{2}Y(t_{n-1})(h) - Y(t_n)(2h) + \frac{1}{2}Y(t_{n+1})(3h)\right] \approx f(t_{n+1}, y(t_{n+1}))
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow -\frac{3}{2h}Y(t_{n+1}) &= \frac{1}{2h}Y(t_{n-1}) - \frac{2}{h}Y(t_n) - f(t_{n+1}, y_{n+1}) \\
\Rightarrow Y(t_{n+1}) &= -\frac{1}{3}Y(t_{n-1}) + \frac{4}{3}Y(t_n) + \frac{2h}{3}f(t_{n+1}, y_{n+1})
\end{aligned}
$$

This yields the BDF2 method:

$$
y_{n+1} = -\frac{1}{3}y_{n-1} + \frac{4}{3}y_n + \frac{2h}{3}f(t_{n+1}, y_{n+1})
$$