Forward Euler Error Bound

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With this theorem I can find an upper bound for the forward Euler method given the first-order ODE, your chosen step-size, the smallest Lipschitz value, and the time-interval you are estimating the ODE on. I will provide a comprehensive proof of this theorem.

1 Theorem:

[Forward Euler Error Bound] Consider the IVP

$$y'(t) = f(t, y(t)), \quad t \in [t_0, T], \quad y(t_0) = y_0$$

with the assumptions:

1. $f(t, y)$ is continuous on $[t_0, T] \times \mathbb{R}$.
2. $f(t, y)$ satisfies a Lipschitz condition in y with constant L.
3. $y \in C^2([t_0, T])$.

Then, the global error of the Forward Euler method satisfies:

$$|y(t_n) - y_n| \leq e^{(T-t_0)L}|e_0| + \frac{e^{(T-t_0)L} - 1}{L} \cdot \frac{h}{2}\|y''\|_\infty$$

for $t_n \in [t_0, T]$.

2 Proof:

Lemma:
Let $t \in \mathbb{R}$ where $t \geq -1$ then $1+t \leq e^t$, and $0 \leq (1+t)^m \leq e^{mt}$ for some $m \in \mathbb{Z}$.
Proof of Lemma: For any $t \in \mathbb{R}$,
$e^t = 1 + t + \frac{e^{-\xi t^2}}{2} \geq 1 + t$ for some $\xi \in (0, t)$

Proof of Theorem:
Choose $N(h)$ such that $t_{N(h)} \leq T$ and $t_{N(h)+1} > T$, define e_n as $e_{n+1} = y(t_{n+1}) - y_h(t_{n+1})$.

$e_{n+1} = e_n + h(f(t_n, y(t_n)) - f(t_n, y_h(t_n))) + T_{n+1}$ where T_{n+1} is the local truncation error at $t = t_{n+1}$ which is $\frac{h^2}{2}y''(\xi_n), \xi_n \in [t_n, t_{n+1}]$. We know this from expanding $y(t_{n+1})$ about t_n to the second derivative term using Taylor's theorem and subtracting $y_h(t_n)$.

This would imply $|T_{n+1}| \leq \frac{h^2}{2}\|y''\|_\infty$.

$|f(t_n, y(t_n)) - f(t_n, y_h(t_n))| \leq L|y(t_n) - y_h(t_n)| = L|e_n|$
(We know this by the Lipschitz condition)

We then have that

$$|e_{n+1}| \leq (1 + hL)|e_n| + \frac{h^2}{2}\|y''\|_\infty \leq (1 + hL)^n|e_n| + (1 + (1 + hL) + (1 + hL)^2 + ... + (1 + hL)^{n-1})\frac{h^2}{2}\|y''\|_\infty \tag{1}$$

Recall the convergence of a geometric series: $\sum_{n=1}^{n} r^n = \frac{r^n - 1}{r-1}$ This then gives: $(1 + hL)^n|e_n| + \frac{(1+hL)^n - 1}{hL}\frac{h^2}{2}\|y''\|_\infty$. By Lemma: $(1 + hL)^n \leq e^{n(hL)}$ for any integer $n \in [1, N]$. By definition $n \cdot h \leq T - t_0$ so we have

$$|e_n| \leq e^{(T-t_0)L}|e_0| + \frac{e^{(T-t_0)L} - 1}{L}\frac{h^2}{2}\|y''\|_\infty \forall n \in [1, N - 1] \tag{2}$$