Supplementary Material to:LiDAR-Aided Visual-Inertial Localization with Semantic Maps

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This is the supplementary material of our work, including the computation of the Jacobian matrices of measurement models and the derivation of general ICP's covariance estimation for point-to-point, point-to-line, and point-to-plane correspondence. In our work, we used the point-to-line, and point-to-plane correspondence.

1 Visual Measurement Module

1.1 Visual Keypoints Observation Model

$$\begin{aligned}\mathbf{0} &= \mathbf{h}_{d_i}\left(\mathbf{x}_k, \mathbf{n}_{d_i}\right) = {}^{P}\mathbf{q}_{d_i} - {}^{P}\mathbf{p}_{d_i} - \mathbf{n}_{d_i} \\ &\simeq \mathbf{h}_{d_i}\left(\hat{\mathbf{x}}_k, \mathbf{0}\right) + \mathbf{H}_{d_i}\delta\mathbf{x}_k + \mathbf{J}_{d_i}\mathbf{n}_{d_i} = \mathbf{z}_{d_i} + \mathbf{H}_{d_i}\delta\mathbf{x}_k + \mathbf{J}_{d_i}\mathbf{n}_{d_i}\end{aligned} \tag{1}$$

$\mathbf{H}_{d_i}$ is the Jacobian matrix of $\mathbf{h}_{d_i}\left(\mathbf{x}_k, \mathbf{n}_{d_i}\right)$ with respect to $\delta\mathbf{x}_k$:

$$\mathbf{H}_{d_i} = \left.\frac{\partial \mathbf{h}_{d_i}\left(\hat{\mathbf{x}}_k \boxplus \delta\mathbf{x}_k, \mathbf{n}_{d_i}\right)}{\partial \delta\mathbf{x}_k}\right|_{\delta\mathbf{x}_k=\mathbf{0}} = \left.\frac{\mathbf{h}_{d_i}\left(\mathbf{x}_k, \mathbf{n}_{d_i}\right)}{{}^{P}\mathbf{p}_{d_i}} \frac{{}^{P}\mathbf{p}_{d_i}}{{}^{C}\mathbf{p}_{d_i}} \frac{{}^{C}\mathbf{p}_{d_i}}{\delta\mathbf{x}_k}\right|_{\delta\mathbf{x}_k=\mathbf{0}} = \mathbf{J}_1 * \mathbf{J}_2 * \mathbf{J}_3 \tag{2}$$

where:

$$\mathbf{J}_1 = \frac{\mathbf{h}_{d_i}\left(\mathbf{x}_k, \mathbf{n}_{d_i}\right)}{{}^{P}\mathbf{p}_{d_i}} = -\mathbf{I} \tag{3}$$

$$\mathbf{J}_2 = \frac{{}^{P}\mathbf{p}_{d_i}}{{}^{C}\mathbf{p}_{d_i}} = \begin{bmatrix} f_x\frac{1}{z} & 0 & -f_x\frac{x}{z^2} \\ 0 & f_y\frac{1}{z} & -f_y\frac{y}{z^2} \end{bmatrix} \tag{4}$$

$$\mathbf{J}_3 = \frac{{}^{C}\mathbf{p}_{d_i}}{\delta\mathbf{x}_k} = \begin{bmatrix} -{}^{C}\mathbf{R}_B{}^{G}\mathbf{R}_{B_k}^{T} & 0_{3\times3} & {}^{B}\mathbf{R}_C^{\top}\left[{}^{G}\mathbf{R}_{B_k}^{\top}\left({}^{G}\mathbf{p}_{d_i} - {}^{G}\mathbf{p}_{B_k}\right)\right]\times & 0_{3\times3} \end{bmatrix} \tag{5}$$

Here, ${}^{C}\mathbf{p}_{d_i} = \left[x^T, y^T, z^T\right]^T$. f_x and f_y are camera focal lengths.

$\mathbf{J}_{d_i}$ is the Jacobian matrix of $\mathbf{h}_{d_i}\left(\mathbf{x}_k, \mathbf{n}_{d_i}\right)$ with respect to $\mathbf{n}_{d_i}$ and:

$$\mathbf{J}_{d_i} = \frac{\mathbf{h}_{d_i}\left(\mathbf{x}_k, \mathbf{n}_{d_i}\right)}{\mathbf{n}_{d_i}} = -\mathbf{I} \tag{6}$$

1.2 Visual Distance Observation Model

$$\begin{aligned} \mathbf{0} = \mathbf{h}_{s_j}\left(\mathbf{x}_k, \mathbf{n}_{s_j}\right) &= \mathcal{D}_k\left({}^{P}\mathbf{p}_{s_j}\right) - \mathbf{n}_{s_j} \\ &\simeq \mathbf{h}_{s_j}\left(\hat{\mathbf{x}}_k, \mathbf{0}\right) + \mathbf{H}_{s_j}\delta\mathbf{x}_k + \mathbf{J}_{s_j}\mathbf{n}_{s_j} = \mathbf{z}_{s_j} + \mathbf{H}_{s_j}\delta\mathbf{x}_k + \mathbf{J}_{s_j}\mathbf{n}_{s_j} \end{aligned} \tag{7}$$

$\mathbf{H}_{s_i}$ is the Jacobian matrix of $\mathbf{h}_{s_i}\left(\mathbf{x}_k, \mathbf{n}_{s_i}\right)$ with respect to $\delta\mathbf{x}_k$:

$$\mathbf{H}_{s_j} = \left.\frac{\partial \mathbf{h}_{s_j}\left(\hat{\mathbf{x}}_k \boxplus \delta\mathbf{x}_k, \mathbf{n}_{s_j}\right)}{\partial \delta\mathbf{x}_k}\right|_{\delta\mathbf{x}_k=\mathbf{0}} = \left.\frac{\mathbf{h}_{s_i}\left(\mathbf{x}_k, \mathbf{n}_{s_i}\right)}{{}^{P}\mathbf{p}_{s_j}} \frac{{}^{P}\mathbf{p}_{s_j}}{{}^{C}\mathbf{p}_{s_j}} \frac{{}^{C}\mathbf{p}_{s_j}}{\delta\mathbf{x}_k}\right|_{\delta\mathbf{x}_k=\mathbf{0}} = \mathbf{J}_1 * \mathbf{J}_2 * \mathbf{J}_3 \tag{8}$$

where $\mathbf{J}_2$ and $\mathbf{J}_3$ are the same as the counterpart in Sec. 1.1(only need to substitute ${}^{G}\mathbf{p}_{d_i}$ for ${}^{G}\mathbf{p}_{s_j}$) and:

$$\mathbf{J}_1 = \begin{bmatrix} \frac{\partial \mathcal{D}_k}{\partial u} & \frac{\partial \mathcal{D}_k}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\mathcal{D}_k\left(u+1, v\right) - \mathcal{D}_k\left(u-1, v\right)}{2} & \frac{\mathcal{D}_k\left(u, v+1\right) - \mathcal{D}_k\left(u, v-1\right)}{2} \end{bmatrix} \tag{9}$$

Here, $\mathcal{D}_k\left(u, v\right)$ denotes the look-up table distance from the k-th DT images for the pixel $\left[u^T, v^T\right]^T$.

$\mathbf{J}_{s_j}$ is the Jacobian matrix of $\mathbf{h}_{s_j}\left(\mathbf{x}_k, \mathbf{n}_{s_j}\right)$ with respect to $\mathbf{n}_{s_j}$ and:

$$\mathbf{J}_{s_j} = \left.\frac{\mathbf{h}_{s_j}\left(\mathbf{x}_k, \mathbf{n}_{s_j}\right)}{\mathbf{n}_{s_j}}\right|_{\mathbf{n}_{s_j}=\mathbf{0}} = -\mathbf{I} \tag{10}$$

2 Pose Measurement Module

2.1 Pose Measurement Model for the position

$$\begin{aligned} \mathbf{0} = \mathbf{h}_p\left(\mathbf{x}_k, \boldsymbol{\nu}_k\right) &= {}^{G}\mathbf{p}_{B_k} - \Psi\left({}^{G}\widetilde{\mathbf{p}}_{B_k}, \boldsymbol{\nu}_k\right) \\ &\simeq \mathbf{h}_p\left(\hat{\mathbf{x}}_k, \mathbf{0}\right) + \mathbf{H}_p\delta\mathbf{x}_k + \mathbf{J}_p\boldsymbol{\nu}_k = \mathbf{z}_p + \mathbf{H}_p\delta\mathbf{x}_k + \mathbf{J}_p\boldsymbol{\nu}_k \end{aligned} \tag{11}$$

$\mathbf{H}_p$ is the Jacobian matrix of $\mathbf{h}_p\left(\mathbf{x}_k, \boldsymbol{\nu}_k\right)$ with respect to $\delta\mathbf{x}_k$:

$$\mathbf{H}_p = \left.\frac{\partial \mathbf{h}_p}{\partial \delta\mathbf{x}_k}\right|_{\delta\mathbf{x}_k=\mathbf{0}} = \begin{bmatrix} \frac{\partial \mathbf{h}_p}{\partial \delta\mathbf{p}_k} & \frac{\partial \mathbf{h}_p}{\partial \delta\mathbf{v}_k} & \frac{\partial \mathbf{h}_p}{\partial \delta\boldsymbol{\theta}_k} & \frac{\partial \mathbf{h}_p}{\partial \delta\mathbf{b}_{\mathbf{a}}} & \frac{\partial \mathbf{h}_p}{\partial \delta\mathbf{b}_\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{12}$$

$\mathbf{J}_p$ is the Jacobian matrix of $\mathbf{h}_p\left(\mathbf{x}_k, \boldsymbol{\nu}_k\right)$ with respect to $\boldsymbol{\nu}_k$:

$$\mathbf{J}_p = \left.\frac{\partial \mathbf{h}_p}{\partial \boldsymbol{\nu}_k}\right|_{\boldsymbol{\nu}_k=\mathbf{0}} = -\left.\frac{\partial \Psi\left({}^{G}\widetilde{\mathbf{p}}_{B_k}, \boldsymbol{\nu}_k\right)}{\partial \boldsymbol{\nu}_k}\right|_{\boldsymbol{\nu}_k=\mathbf{0}} = -\begin{bmatrix} \frac{\partial \Psi}{\partial \delta\boldsymbol{\theta}_{k-1}} & \frac{\partial \Psi}{\partial \delta\mathbf{p}_{k-1}} & \frac{\partial \Psi}{\partial \delta\widetilde{\boldsymbol{\theta}}_{k-1}} & \frac{\partial \Psi}{\partial \delta\widetilde{\mathbf{p}}_{k-1}} \end{bmatrix} \tag{13}$$

$$\frac{\partial \Psi}{\partial \delta \boldsymbol{\theta}_{k-1}} = -{}^{G}\overline{\mathbf{R}}_{B_{k-1}} \left({}^{B}\mathbf{R}_{L} \left({}^{L_{k-1}}\hat{\mathbf{R}}_{L_k} {}^{L}\mathbf{p}_{B} + {}^{L_{k-1}}\hat{\mathbf{p}}_{L_k} \right) + {}^{L}\hat{\mathbf{p}}_{B} \right) - {}^{G}\overline{\mathbf{p}}_{B_{k-1}} \tag{14}$$

$$\frac{\partial \Psi}{\partial \delta \mathbf{p}_{k-1}} = -\mathbf{I} \tag{15}$$

$$\frac{\partial \Psi}{\partial \delta \widetilde{\boldsymbol{\theta}}_{k-1}} = -{}^{G}\overline{\mathbf{R}}_{B_{k-1}} {}^{B}\overline{\mathbf{R}}_{L} {}^{L_{k-1}}\hat{\mathbf{R}}_{L_k} \left[{}^{L}\mathbf{p}_{B} \right] \times \tag{16}$$

$$\frac{\partial \Psi}{\partial \delta \widetilde{\mathbf{p}}_{k-1}} = {}^{G}\overline{\mathbf{R}}_{B_{k-1}} {}^{B}\mathbf{R}_{L} \tag{17}$$

2.2 Pose Measurement Model for the rotation

$$\begin{aligned} \mathbf{0} &= \mathbf{h}_r\left(\mathbf{x}_k, \boldsymbol{\nu}_k\right) = \log\left(\Phi({}^{G}\widetilde{\mathbf{R}}_{B_k}, \boldsymbol{\nu}_k)^{-1} \cdot {}^{G}\mathbf{R}_{B_k}\right)^{\vee} \\ &\simeq \mathbf{h}_r(\hat{\mathbf{x}}_k, \mathbf{0}) + \mathbf{H}_r \delta \mathbf{x}_k + \mathbf{J}_r \boldsymbol{\nu}_k = \mathbf{z}_r + \mathbf{H}_r \delta \mathbf{x}_k + \mathbf{J}_r \boldsymbol{\nu}_k \end{aligned} \tag{18}$$

$\mathbf{H}_r$ is the Jacobian matrix of $\mathbf{h}_r\left(\mathbf{x}_k, \boldsymbol{\nu}_k\right)$ with respect to $\delta \mathbf{x}_k$:

$$\mathbf{H}_r = \left.\frac{\partial \mathbf{h}_r}{\partial \delta \mathbf{x}_k}\right|_{\delta \mathbf{x}_k = \mathbf{0}} = \begin{bmatrix} \frac{\partial \mathbf{h}_r}{\partial \delta \mathbf{p}_k} & \frac{\partial \mathbf{h}_r}{\partial \delta \mathbf{v}_k} & \frac{\partial \mathbf{h}_r}{\partial \delta \boldsymbol{\theta}_k} & \frac{\partial \mathbf{h}_r}{\partial \delta \mathbf{b}_{\mathbf{a}}} & \frac{\partial \mathbf{h}_r}{\partial \delta \mathbf{b}_{\omega}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \frac{\partial \mathbf{h}_r}{\partial \delta \boldsymbol{\theta}_k} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{19}$$

$$\frac{\partial \mathbf{h}_r}{\partial \delta \boldsymbol{\theta}_k} = \mathbf{A}\left(\log\left(\Phi({}^{G}\widetilde{\mathbf{R}}_{B_k}, \boldsymbol{\nu}_k)^{-1} \cdot {}^{G}\mathbf{R}_{B_k}\right)^{\vee} \right) \tag{20}$$

Here, for $\mathbf{n} \in \mathfrak{so}(3)$ and $\mathbf{n} = \theta \mathbf{a}\,(||\mathbf{a}|| = 1)$:

$$\mathbf{A}\left(\mathbf{n}\right) = \frac{\theta}{2}\cot\frac{\theta}{2}\mathbf{I} + \left(1 - \frac{\theta}{2}\cot\frac{\theta}{2}\right)\mathbf{a}\mathbf{a}^{T} + \frac{\theta}{2}\left[\mathbf{a}\right]\times \tag{21}$$

$\mathbf{J}_r$ is the Jacobian matrix of $\mathbf{h}_r\left(\mathbf{x}_k, \boldsymbol{\nu}_k\right)$ with respect to $\boldsymbol{\nu}_k$:

$$\mathbf{J}_r = \left.\frac{\partial \mathbf{h}_r}{\partial \boldsymbol{\nu}_k}\right|_{\boldsymbol{\nu}_k = \mathbf{0}} = \begin{bmatrix} \frac{\partial \mathbf{h}_r}{\partial \delta \boldsymbol{\theta}_{k-1}} & \mathbf{0} & \frac{\partial \mathbf{h}_r}{\partial \delta \widetilde{\boldsymbol{\theta}}_{k-1}} & \mathbf{0} \end{bmatrix} \tag{22}$$

$$\frac{\partial \mathbf{h}_r}{\partial \delta \boldsymbol{\theta}_{k-1}} = -\mathbf{A}\left(\log\left({}^{B}\mathbf{R}_{L} {}^{L_{k-1}}\hat{\mathbf{R}}_{L_k}^{-1} {}^{B}\mathbf{R}_{L}^{-1} {}^{G}\overline{\mathbf{R}}_{B_{k-1}} \right)^{\vee} \right) {}^{G}\overline{\mathbf{R}}_{B_{k-1}} \tag{23}$$

$$\frac{\partial \mathbf{h}_r}{\partial \delta \widetilde{\boldsymbol{\theta}}_{k-1}} = -\mathbf{A}\left(\log\left({}^{B}\mathbf{R}_{L} {}^{L_{k-1}}\hat{\mathbf{R}}_{L_k}^{-1} {}^{B}\mathbf{R}_{L}^{-1} {}^{G}\overline{\mathbf{R}}_{B_{k-1}}^{-1} \right)^{\vee} \right) {}^{G}\overline{\mathbf{R}}_{B_{k-1}} {}^{B}\mathbf{R}_{L} {}^{L_{k-1}}\hat{\mathbf{R}}_{L_k} \tag{24}$$

3 Covariance for Lidar Odometry

Denote J as the objective function[1]; G_i as a residual term introduced by measurement z_i (in this problem, it is a point or point pair); x is the state variable to be optimized (in this problem, it is 6 DoF lidar pose).

$$J = \sum_i J_i = \sum_i \|G_i\|^2 = \sum_i G_i^T G_i \tag{25}$$

According to Eq.26, we can get covariance about x. Also, suppose Cov(z) is a diagonal matrix whose diagonal entries are σ^2. Then we have:

$$\mathrm{Cov(x)} \approx \sigma^2 \left(\frac{\partial^2 J}{\partial \mathrm{x}^2}\right)^{-1} \left(\frac{\partial^2 J}{\partial \mathrm{z} \partial \mathrm{x}}\right) \left(\frac{\partial^2 J}{\partial \mathrm{z} \partial \mathrm{x}}\right)^T \left(\frac{\partial^2 J}{\partial \mathrm{x}^2}\right)^{-1} \tag{26}$$

$$\frac{\partial J_i}{\partial x} = \frac{\partial G_i^T G_i}{\partial x} = 2 G_i^T \frac{\partial G_i}{\partial x} \tag{27}$$

We derived the closed-form ICP covariance for point-to-point, point-to-line, and point-to-plane correspondence. In our paper, we only used the point-to-line and point-to-plane correspondence.

3.1 Point-Point Correspondence:

Q_i, P_i are point correspondences from two sets of point clouds. $x = (R, t)$ is relative rotation and translation; $z = (P_i, Q_i)$.

$$G_i = G(Q_i, P_i) = R \cdot Q_i + t - P_i \tag{28}$$

$$\frac{\partial J_i}{\partial x} = -2 \left[(t^T - P_i^T) R \lfloor Q_i \rfloor_\times, -Q_i^T R^T - (t^T - P_i^T) \right] \tag{29}$$

And

$$\left(\frac{\partial J_i}{\partial x}\right)^T = \begin{pmatrix} \left(\frac{\partial J_i}{\partial R}\right)^T \\ \left(\frac{\partial J_i}{\partial t}\right)^T \end{pmatrix} = -2 \begin{pmatrix} -\lfloor Q_i \rfloor_\times R^T (t - P_i) \\ -R Q_i - (t - P_i) \end{pmatrix} \tag{30}$$

So

$$\frac{\partial}{\partial z} \left(\frac{\partial J_i}{\partial x}\right)^T = 2 \begin{pmatrix} -\lfloor Q_i \rfloor_\times R^T & -\left\lfloor R^T (t - P_i) \right\rfloor_\times \\ -I_3 & R \end{pmatrix} \tag{31}$$

$$\left(\frac{\partial^2 J}{\partial \mathrm{x}^2}\right) = 2\begin{pmatrix} \lfloor Q_i \rfloor_\times \left\lfloor R^T(t-P_i)\right\rfloor_\times & \lfloor Q_i \rfloor_\times R^T \\ -R\lfloor Q_i \rfloor_\times & I_3 \end{pmatrix} \tag{32}$$

3.2 Point-Plane Correspondence:

P_i, Q_i are point-plane correspondences from two sets of point clouds. $x = (R, t)$ is relative rotation and translation; $z = (P_i, Q_i, n)$.

$$G_i = G(Q_i, P_i, n) = n^T(R \cdot Q_i + t - P_i) \tag{33}$$

Also, we fix the observation about P_i, n, which from our lidar map.

$$\frac{\partial J_i}{\partial x} = 2G_i^T \frac{\partial G_i}{\partial x} = 2\left(RQ_i + t - P_i\right)^T nn^T \left[-R\lfloor Q_i \rfloor_\times, I_3\right] \tag{34}$$

And

$$\left(\frac{\partial J_i}{\partial x}\right)^T = 2\begin{pmatrix} \lfloor Q_i \rfloor_\times R^T nn^T\left(RQ_i + t - P_i\right) \\ nn^T\left(RQ_i + t - P_i\right) \end{pmatrix} \tag{35}$$

In where $\left(\frac{\partial J_i}{\partial x}\right)^T \in R_{6\times 1}$

$$\frac{\partial}{\partial z}\left(\frac{\partial J_i}{\partial x}\right)^T = 2\begin{pmatrix} \alpha - \left\lfloor R^T nn^T\left(t - P_i\right)\right\rfloor_\times \\ nn^T R \end{pmatrix} \tag{36}$$

In where $\alpha = \frac{\partial\left(-\left\lfloor R^T n\right\rfloor_\times Q_i n^T RQ_i\right)}{\partial Q_i} = -\left\lfloor R^T n\right\rfloor_\times \left(\left(n^T RQ_i\right) I_3 + Q_i n^T R\right)$.

$$\frac{\partial}{\partial x}\left(\frac{\partial J_i}{\partial x}\right)^T = 2\begin{pmatrix} \lfloor Q_i \rfloor_\times \left\lfloor R^T nn^T RQ_i\right\rfloor_\times & \lfloor Q_i \rfloor_\times R^T nn^T \\ -nn^T R\lfloor Q_i \rfloor_\times & nn^T \end{pmatrix} \tag{37}$$

3.3 Point-Line/Edge Correspondence

3D lines can be represented implicitly by so-called Plucker lines. A Plucker line P_{i1}, P_{i2} is described by two point in the line. This line representation allows to conveniently determine the distance of a 3D point Q_i to the line:

$$G_i = G(P_{i1}, P_{i2}, Q_i) = \lfloor P_{i1} - P_{i2} \rfloor_\times \left(RQ_i + t - P_{i1}\right). \tag{38}$$

we calculate the partial derivative:

$$\frac{\partial G_i}{\partial x} = \lfloor P_{i1} - P_{i2} \rfloor_\times \left[-R \lfloor Q_i \rfloor_\times, I_3 \right] \tag{39}$$

We set $d = P_{i1} - P_{i2}$,

$$\frac{\partial J_i}{\partial x} = 2G_i^T \frac{\partial G_i}{\partial x} = 2(\lfloor d \rfloor_\times (RQ_i + t - P_{i1}))^T \lfloor d \rfloor_\times \left[-R \lfloor Q_i \rfloor_\times, I_3 \right] \tag{40}$$

And

$$\left(\frac{\partial J_i}{\partial x} \right)^T = 2 \begin{pmatrix} -\lfloor Q_i \rfloor_\times R^T \lfloor d \rfloor_\times \lfloor d \rfloor_\times (RQ_i + t - P_{i1}) \\ -\lfloor d \rfloor_\times \lfloor d \rfloor_\times (RQ_i + t - P_{i1}) \end{pmatrix} \tag{41}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial J_i}{\partial x} \right)^T = 2 \begin{pmatrix} A & -\lfloor Q_i \rfloor_\times R^T (I_3 - dd^T) \\ (I_3 - dd^T) R \lfloor Q_i \rfloor_\times & -(I_3 - dd^T) \end{pmatrix} \tag{42}$$

in where $A = \lfloor Q_i \rfloor_\times (R^T (I_3 - dd^T) R \lfloor Q_i \rfloor_\times - \left\lfloor R^T (I_3 - dd^T)(RQ_i + t - P_{i1}) \right\rfloor_\times)$, and $\lfloor d \rfloor_\times \lfloor d \rfloor_\times = (I_3 - dd^T)$

$$\frac{\partial}{\partial z} \left(\frac{\partial J_i}{\partial x} \right)^T = 2 \begin{pmatrix} -\left\lfloor R^T d \right\rfloor_\times (Q_i d^T R + d^T R Q_i) \\ (I_3 - dd^T) R \lfloor Q_i \rfloor_\times R \end{pmatrix} \tag{43}$$

4 Appendix 1

$$\begin{gathered} \lfloor a \rfloor_\times^T = -\lfloor a \rfloor_\times \\ \tfrac{\partial n n^T b}{\partial n^T} = n^T b I_3 + n b^T \\ \tfrac{\partial A n n^T b}{\partial n^T} = A n^T b + A n b^T \\ (\lfloor a \rfloor_\times \lfloor b \rfloor_\times)^T = \lfloor b \rfloor_\times \lfloor a \rfloor_\times \neq \lfloor a \rfloor_\times \lfloor b \rfloor_\times \\ \lfloor a \rfloor_\times a = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \end{gathered} \tag{44}$$

$$\begin{aligned} \frac{\partial J_i}{\partial x} &= 2 \left(RQ_i + t - P_i \right)^T \left(-R \lfloor Q_i \rfloor_\times, I_3 \right) \\ &= 2 \left(Q_i^T R^T + t^T - P_i^T \right) \left(-R \lfloor Q_i \rfloor_\times, I_3 \right) \\ &= -2 \left[Q_i^T \lfloor Q_i \rfloor_\times + (t^T - P_i^T) R \lfloor Q_i \rfloor_\times, Q_i^T R^T + (t^T - P_i^T) \right] \\ &= -2 \left[(t^T - P_i^T) R \lfloor Q_i \rfloor_\times, Q_i^T R^T + (t^T - P_i^T) \right] \end{aligned} \tag{45}$$

$J = \sum J_i$, $\frac{\partial J}{\partial x} = \sum \frac{\partial J_i}{\partial x}$, $\frac{\partial^2 J}{\partial^2 x} = \sum \frac{\partial^2 J_i}{\partial^2 x}$.

References

[1] P. S. Manoj, B. Liu, R. Yan, and W. Lin, "A closed-form estimate of 3d icp covariance," in *IAPR International Conference on Machine Vision Applications*, 2015.

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