Tessellation README

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Metropolis Hastings and Gibbs Derivation

First, following the steps in AdaptSPEC, we rewrite the Whittle Likelihood as follows using the notation change that $y_n(\omega_j) = \log(I_n(\omega_j))$:

$$
\begin{aligned}
p(x|f) &= (2\pi)^{n/2} \prod_{j=0}^{n-1} \exp\{-\frac{1}{2}[\log f(\omega_j) + I_n(\omega_j)/f(\omega_j)]\} \\
&= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\log(f(\omega_j)) + \frac{\exp\{\log(I_n(\omega_j))\}}{\exp\{\log(f(\omega_j))\}}\right]\right\}
\end{aligned}
$$

Then we use the equation given in the paper: $\log(f_j) = g(\omega_j) = \alpha_0 + h(\omega_j) = \alpha_0 + X_j^T\beta$ in order to replace the unknown spectral density with α_0's and β's

$$
\begin{aligned}
&= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\log(f(\omega_j)) + \frac{\exp\{y_n(\omega_j)\}}{\exp\{\log(f(\omega_j))\}}\right]\right\} \\
\mathcal{L}(\alpha_0, \beta, \tau^2|y) &= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\alpha_0 + X_j^T\beta + \frac{\exp\{y_j\}}{\exp\{\alpha_0 + X_j^T\beta\}}\right]\right\} \\
&= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\alpha_0 + X_j^T\beta + \exp\{y_j - \alpha_0 - X_j^T\beta\}\right]\right\} \\
&= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}[\alpha_0 n + 1_n X\beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T\beta\}]\right\}
\end{aligned}
$$

Priors

Following closely* the priors outlined in AdaptSPEC, we give the parameters the following priors:

$$
\begin{aligned}
\pi(\alpha) &\sim N(0, \sigma_\alpha^2) \\
\pi(\beta) &\sim N(0, \tau^2, D_K) \\
\pi(\tau^2|\lambda) &\sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0}{\lambda}\right) \\
\pi(\lambda) &\sim IG\left(\frac{1}{2}, \frac{1}{\eta_0^2}\right)
\end{aligned}
$$

Such that D_K is a positive definite diagonal matrix of dimension K.

*Instead of giving τ^2 a Uniform prior as in AdaptSPEC we have given it a half-t prior. Following the work laid out in Wand et.al. we are able to write the half-t prior as an Inverse-Gamma PDF with the parameters as written above.

Posterior Distribution

Now, we go through the derivation of calculating the full posterior distribution of the Whittle Likelihood:

$$\pi(\beta, \tau^2, \alpha_0, \lambda | y) \propto \mathcal{L}(\beta, \tau^2, \alpha_0, \lambda | y) \cdot \pi(\beta) \cdot \pi(\tau^2|\lambda) \cdot \pi(\lambda) \cdot \pi(\alpha_0)$$

In order to be precise and organized we will first take each prior given and rewrite the prior as it will be used in the posterior distribution.

First, $\pi(\alpha_0) \sim N(0, \sigma_\alpha^2)$:

$$\begin{aligned}\pi(\alpha_0) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\alpha_0^2}{2\sigma_\alpha^2}\right\} \\ &\propto \exp\left\{-\frac{\alpha_0^2}{2\sigma_\alpha^2}\right\}\end{aligned}$$

Second, $\pi(\beta) \sim N(0, \tau^2 D_k)$:

$$\begin{aligned}\pi(\beta) &= \frac{1}{\sqrt{(2\pi)^K |\tau^2 D_K|}} \exp\left\{-\frac{1}{2}\beta^T(\tau^2 D_K)^{-1}\beta\right\} \\ &\propto (\tau^2)^{-K/2} \cdot \exp\left\{-\frac{1}{2}\beta^T(\tau^2 D_K)^{-1}\beta\right\}\end{aligned}$$

Third, $\pi(\tau^2|\lambda) \sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0}{\lambda}\right)$:

$$\begin{aligned}\pi(\tau^2|\lambda) &= \frac{\left(\frac{\nu_0}{\lambda}\right)^{\nu_0/2}}{\Gamma\left(\frac{\nu_0}{2}\right)} \cdot \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda}\right\} \\ &\propto \frac{1}{\lambda^{\nu_0/2}} \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda}\right\}\end{aligned}$$

Fourth, $\pi(\lambda) \sim IG\left(\frac{1}{2}, \frac{1}{\eta_0^2}\right)$:

$$\begin{aligned}\pi(\lambda) &= \frac{\left(\frac{1}{\eta_0^2}\right)^{1/2}}{\Gamma(1/2)} \left(\frac{1}{\lambda}\right)^{3/2} \exp\left\{-\frac{1}{\lambda\eta_0^2}\right\} \\ &\propto \left(\frac{1}{\lambda}\right)^{3/2} \exp\left\{-\frac{1}{\lambda\eta_0^2}\right\}\end{aligned}$$

Now, putting this together we have:

$$\begin{aligned}\pi(\beta, \tau^2, \alpha_0, \lambda|y) \propto (2\pi)^{-n/2} \exp&\left\{-\frac{1}{2}[\alpha_0 n + 1_n X\beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T\beta\}]\right\} \cdot \exp\left\{-\frac{\alpha_0^2}{2\sigma_\alpha^2}\right\} \\ &\cdot (\tau^2)^{-K/2} \cdot \exp\left\{-\frac{1}{2}\beta^T(\tau^2 D_K)^{-1}\beta\right\} \cdot \frac{1}{\lambda^{\nu_0/2}} \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda}\right\} \cdot \left(\frac{1}{\lambda}\right)^{3/2} \exp\left\{-\frac{1}{\lambda\eta_0^2}\right\}\end{aligned}$$

Conditional Posterior Distributions

Conditional Posterior Distribution of β and α_0

$$\pi(\beta, \alpha_0|y) \propto \exp\left\{-\frac{1}{2}\left[\alpha_0 n + \frac{\alpha_0^2}{\sigma_\alpha^2} + 1_n X\beta + \frac{1}{\tau^2}\beta^T (D_K)^{-1}\beta + \sum_{j=0}^{n-1} \exp\left\{y_j - \alpha_0 - X_j^T\beta\right\}\right]\right\}$$

Conditional Posterior Distribution of τ^2

$$\begin{aligned}\pi(\tau^2|y) &\propto (\tau^2)^{-K/2} \cdot \exp\left\{-\frac{1}{2}\beta^T(\tau^2 D_K)^{-1}\beta\right\} \cdot \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda}\right\} \\ &\propto (\tau^2)^{-\frac{K-\nu_0}{2}-1} \cdot \exp\left\{-\frac{1}{\tau^2}\left[\frac{1}{2}\beta^T(D_K)^{-1}\beta + \frac{\nu_0}{\lambda}\right]\right\}\end{aligned}$$

Hence, $\pi(\tau^2|y) \sim IG\left(\frac{K+\nu_0}{2}\ ,\ \frac{1}{2}\beta^T(D_K)^{-1}\beta + \frac{\nu_0}{\lambda}\right)$

Conditional Posterior Distribution of λ

$$\begin{aligned}\pi(\lambda|y) &\propto \frac{1}{\lambda^{\nu_0/2}} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda}\right\} \cdot \left(\frac{1}{\lambda}\right)^{3/2} \cdot \exp\left\{-\frac{1}{\lambda\eta_0^2}\right\} \\ &\propto \frac{1}{\lambda^{\nu_0/2}\lambda^{3/2}} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda} - \frac{1}{\lambda\eta_0^2}\right\} \\ &\propto \lambda^{-\left(\frac{\nu_0+1}{2}\right)-1} \cdot \exp\left\{-\frac{1}{\lambda}\left[\frac{\nu_0}{\tau^2} + \frac{1}{\eta_0^2}\right]\right\}\end{aligned}$$

Hence, $\pi(\lambda|y) \sim IG\left(\frac{\nu_0+1}{2}\ ,\ \frac{\nu_0}{\tau^2} + \frac{1}{\eta_0^2}\right)$

Gradient and Hessian of $\pi(\alpha_0, \beta|y)$

Gradient of $\pi(\alpha_0, \beta|y)$

$$\triangledown \log \pi(\alpha_0, \beta|y) = \triangledown \log f(\cdot) = \begin{bmatrix}\frac{\partial}{\partial \alpha_0}\log f(\alpha_0,\beta) \\ \triangledown_\beta \log f(\alpha_0,\beta)\end{bmatrix}$$

$$\begin{aligned}\frac{\partial}{\partial\alpha_0}\log f(\alpha_0,\beta) &= \frac{\partial}{\partial\alpha_0}\left[-\frac{1}{2}\left[\alpha_0 n + \frac{\alpha_0^2}{\sigma_{\alpha_0}^2} + 1_n X\beta + \frac{1}{\tau^2}\beta^T(D_K)^{-1}\beta + \sum_{j=0}^{n-1}\exp\{y_j - \alpha_0 - X_j^T\beta\}\right]\right] \\ &= -\frac{1}{2}\left[n + \frac{2\alpha_0}{\sigma_{\alpha_0}^2} + \sum_{j=0}^{n-1} -\exp\{y_j - \alpha_0 - X_j^T\beta\}\right] \\ &= -\frac{n}{2} - \frac{\alpha_0}{\sigma_\alpha^2} + \frac{1}{2}\sum_{j=0}^{n-1}\exp\{y_j - \alpha_0 - X_j^T\beta\}\end{aligned}$$

$$
\begin{aligned}
\bigtriangledown_\beta \log f(\alpha_0,\beta) &= \frac{\partial}{\partial\beta}\left[-\frac{1}{2}\left[1_n X\beta + \frac{1}{\tau^2}\beta^T(D_K)^{-1}\beta + \sum_{j=0}^{n-1}\exp\{y_j-\alpha_0-X_j^T\beta\}\right]\right] \\
&= \frac{\partial}{\partial\beta}\left[-\frac{1}{2}1_n X\beta\right] + \frac{\partial}{\partial\beta}\left[-\frac{1}{2\tau^2}\beta^T(D_K)^{-1}\beta\right] + \frac{\partial}{\partial\beta}\left[-\frac{1}{2}\sum_{j=0}^{n-1}\exp\{y_j-\alpha_0-X_j^T\beta\}\right] \\
&= -\frac{1}{2}\left[X^T 1_n^T + 2D_K^{-1}\beta + \sum_{j=0}^{n-1} -X_j\exp\{y_j-\alpha_0-X_j^T\beta\}\right]
\end{aligned}
$$

Therefore, the gradient of the conditional posterior of β and α_0 conditioned on y is :

$$
\bigtriangledown \log\pi(\alpha_0,\beta|y) = \bigtriangledown \log f(\cdot) = \begin{bmatrix} -\frac{n}{2} - \frac{\alpha_0}{\sigma_\alpha^2} + \frac{1}{2}\sum_{j=0}^{n-1}\exp\{y_j-\alpha_0-X_j^T\beta\} \\ -\frac{1}{2}\left[X^T 1_n^T + 2D_K^{-1}\beta + \sum_{j=0}^{n-1} -X_j\exp\{y_j-\alpha_0-X_j^T\beta\}\right] \end{bmatrix}
$$

Hessian of $\pi(\beta,\alpha_0|y)$

$$
\bigtriangledown^2 \log f(\cdot) = \begin{bmatrix} \frac{\partial^2}{\partial\alpha_0^2} & \left(\frac{\partial^2}{\partial\alpha_0\partial\beta}\right)^T \\ \frac{\partial^2}{\partial\beta\partial\alpha_0} & \bigtriangledown_\beta^2 f \end{bmatrix}
$$

$$
\begin{aligned}
\frac{\partial^2 \log f}{\partial\alpha_0^2} &= \frac{\partial}{\partial\alpha_0}\left[-\frac{n}{2} - \frac{\alpha_0}{\sigma_\alpha^2} + \frac{1}{2}\sum_{j=0}^{n-1}\exp\left\{y_j-\alpha_0-X_j^T\beta\right\}\right] \\
&= -\frac{1}{\sigma_\alpha^2} + \frac{1}{2}\sum_{j=0}^{n-1} -\exp\{y_j-\alpha_0-X_j^T\beta\} \\
&= -\frac{1}{\sigma_\alpha^2} - \frac{1}{2}\sum_{j=0}^{n-1}\exp\{y_j-\alpha_0-X_j^T\beta\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \log f}{\partial\beta\partial\alpha_0} &= \frac{\partial}{\partial\alpha}\left[-\frac{1}{2}X^T 1_n^T - \frac{D_K^{-1}}{\tau^2}\beta + \frac{1}{2}\sum_{j=0}^{n-1}\exp\{y_j-\alpha_0-X_j^T\beta\}\right] \\
&= -\frac{1}{2}\sum_{j=0}^{n-1} X_j\exp\{y_j-\alpha_0-X_j^T\beta\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \log f}{\partial\alpha_0\partial\beta} &= \frac{\partial}{\partial\beta}\left[-\frac{n}{2} - \frac{\alpha_0}{\sigma_{\alpha_0}^2} + \frac{1}{2}\sum_{j=0}^{n-1}\exp\{y_j-\alpha_0-X_j^T\beta\}\right] \\
&= -\frac{1}{2}\sum_{j=0}^{n-1} X_j\exp\{y_j-\alpha_0-X_j^T\beta\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 \log f}{\partial^2 \beta} &= \frac{\partial}{\partial \beta}\left[-\frac{1}{2}\left[X^T 1_n^T + \frac{2}{\tau^2} D_K^{-1}\beta + \sum_{j=0}^{n-1} -X_j \exp\{y_j - \alpha_0 - X_j^T \beta\}\right]\right] \\
&= -\frac{D_K^{-1}}{\tau^2} + -\frac{1}{2}\sum_{j=0}^{n-1} X_j X_j^T \exp\{y_j - \alpha_0 - X_j^T \beta\}
\end{aligned}
$$

Therefore, the Hessian is:

$$
\bigtriangledown^2 \log f(\cdot) = \begin{bmatrix} -\frac{1}{\sigma_\alpha^2} - \frac{1}{2}\sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\} & \left(-\frac{1}{2}\sum_{j=0}^{n-1} X_j \exp\{y_j - \alpha_0 - X_j^T \beta\}\right)^T \\ -\frac{1}{2}\sum_{j=0}^{n-1} X_j \exp\{y_j - \alpha_0 - X_j^T \beta\} & -\frac{D_K^{-1}}{\tau^2} + -\frac{1}{2}\sum_{j=0}^{n-1} X_j X_j^T \exp\{y_j - \alpha_0 - X_j^T \beta\} \end{bmatrix}
$$