

# Tessellation README

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## Metropolis Hastings and Gibbs Derivation

First, following the steps in AdaptSPEC, we rewrite the Whittle Likelihood as follows using the notation change that  $y_n(\omega_j) = \log(I_n(\omega_j))$ :

$$\begin{aligned} p(x|f) &= (2\pi)^{n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}[\log f(\omega_j) + I_n(\omega_j)/f(\omega_j)]\right\} \\ &= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\log(f(\omega_j)) + \frac{\exp\{\log(I_n(\omega_j))\}}{\exp\{\log(f(\omega_j))\}}\right]\right\} \end{aligned}$$

Then we use the equation given in the paper:  $\log(f_j) = g(\omega_j) = \alpha_0 + h(\omega_j) = \alpha_0 + X_j^T \beta$  in order to replace the unknown spectral density with  $\alpha_0$ 's and  $\beta$ 's

$$\begin{aligned} &= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\log(f(\omega_j)) + \frac{\exp\{y_n(\omega_j)\}}{\exp\{\log(f(\omega_j))\}}\right]\right\} \\ \mathcal{L}(\alpha_0, \beta, \tau^2|y) &= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\alpha_0 + X_j^T \beta + \frac{\exp\{y_j\}}{\exp\{\alpha_0 + X_j^T \beta\}}\right]\right\} \\ &= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\alpha_0 + X_j^T \beta + \exp\{y_j - \alpha_0 - X_j^T \beta\}\right]\right\} \\ &= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}[\alpha_0 n + 1_n X \beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\}]\right\} \end{aligned}$$

## Priors

Following closely\* the priors outlined in AdaptSPEC, we give the parameters the following priors:

$$\begin{aligned} \pi(\alpha) &\sim N(0, \sigma_\alpha^2) \\ \pi(\beta) &\sim N(0, \tau^2, D_K) \\ \pi(\tau^2|\lambda) &\sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0}{\lambda}\right) \\ \pi(\lambda) &\sim IG\left(\frac{1}{2}, \frac{1}{\eta_0^2}\right) \end{aligned}$$

Such that  $D_K$  is a positive definite diagonal matrix of dimension  $K$ .

\*Instead of giving  $\tau^2$  a Uniform prior as in AdaptSPEC we have given it a half-t prior. Following the work laid out in Wand et.al. we are able to write the half-t prior as an Inverse-Gamma PDF with the parameters as written above.

### Posterior Distribution

Now, we go through the derivation of calculating the full posterior distribution of the Whittle Likelihood:

$$\pi(\beta, \tau^2, \alpha_0, \lambda|y) \propto \mathcal{L}(\beta, \tau^2, \alpha_0, \lambda|y) \cdot \pi(\beta) \cdot \pi(\tau^2|\lambda) \cdot \pi(\lambda) \cdot \pi(\alpha_0)$$

In order to be precise and organized we will first take each prior given and rewrite the prior as it will be used in the posterior distribution.

First,  $\pi(\alpha_0) \sim N(0, \sigma_\alpha^2)$ :

$$\begin{aligned} \pi(\alpha_0) &= \frac{1}{\sqrt{2\pi\sigma_\alpha^2}} \exp\left\{-\frac{\alpha_0^2}{2\sigma_\alpha^2}\right\} \\ &\propto \exp\left\{-\frac{\alpha_0^2}{2\sigma_\alpha^2}\right\} \end{aligned}$$

Second,  $\pi(\beta) \sim N(0, \tau^2 D_K)$ :

$$\begin{aligned} \pi(\beta) &= \frac{1}{\sqrt{(2\pi)^K |\tau^2 D_K|}} \exp\left\{-\frac{1}{2}\beta^T (\tau^2 D_K)^{-1} \beta\right\} \\ &\propto (\tau^2)^{-K/2} \cdot \exp\left\{-\frac{1}{2}\beta^T (\tau^2 D_K)^{-1} \beta\right\} \end{aligned}$$

Third,  $\pi(\tau^2|\lambda) \sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0}{\lambda}\right)$ :

$$\begin{aligned} \pi(\tau^2|\lambda) &= \frac{\left(\frac{\nu_0}{\lambda}\right)^{\nu_0/2}}{\Gamma\left(\frac{\nu_0}{2}\right)} \cdot \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda}\right\} \\ &\propto \frac{1}{\lambda^{\nu_0/2}} \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda}\right\} \end{aligned}$$

Fourth,  $\pi(\lambda) \sim IG\left(\frac{1}{2}, \frac{1}{\eta_0^2}\right)$ :

$$\begin{aligned} \pi(\lambda) &= \frac{\left(\frac{1}{\eta_0^2}\right)^{1/2}}{\Gamma(1/2)} \left(\frac{1}{\lambda}\right)^{3/2} \exp\left\{-\frac{1}{\lambda\eta_0^2}\right\} \\ &\propto \left(\frac{1}{\lambda}\right)^{3/2} \exp\left\{-\frac{1}{\lambda\eta_0^2}\right\} \end{aligned}$$

Now, putting this together we have:

$$\begin{aligned} \pi(\beta, \tau^2, \alpha_0, \lambda|y) &\propto (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}[\alpha_0 n + 1_n X \beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\}]\right\} \cdot \exp\left\{-\frac{\alpha_0^2}{2\sigma_\alpha^2}\right\} \\ &\quad \cdot (\tau^2)^{-K/2} \cdot \exp\left\{-\frac{1}{2}\beta^T (\tau^2 D_K)^{-1} \beta\right\} \cdot \frac{1}{\lambda^{\nu_0/2}} \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda}\right\} \cdot \left(\frac{1}{\lambda}\right)^{3/2} \exp\left\{-\frac{1}{\lambda\eta_0^2}\right\} \end{aligned}$$

## Conditional Posterior Distributions

### Conditional Posterior Distribution of $\beta$ and $\alpha_0$

$$\pi(\beta, \alpha_0 | y) \propto \exp \left\{ -\frac{1}{2} \left[ \alpha_0 n + \frac{\alpha_0^2}{\sigma_\alpha^2} + 1_n X \beta + \frac{1}{\tau^2} \beta^T (D_K)^{-1} \beta + \sum_{j=0}^{n-1} \exp \{ y_j - \alpha_0 - X_j^T \beta \} \right] \right\}$$

### Conditional Posterior Distribution of $\tau^2$

$$\begin{aligned} \pi(\tau^2 | y) &\propto (\tau^2)^{-K/2} \cdot \exp \left\{ -\frac{1}{2} \beta^T (\tau^2 D_K)^{-1} \beta \right\} \cdot \left( \frac{1}{\tau^2} \right)^{\frac{\nu_0}{2} + 1} \cdot \exp \left\{ -\frac{\nu_0}{\tau^2 \lambda} \right\} \\ &\propto (\tau^2)^{-\frac{K+\nu_0}{2} - 1} \cdot \exp \left\{ -\frac{1}{\tau^2} \left[ \frac{1}{2} \beta^T (D_K)^{-1} \beta + \frac{\nu_0}{\lambda} \right] \right\} \end{aligned}$$

Hence,  $\pi(\tau^2 | y) \sim IG \left( \frac{K+\nu_0}{2}, \frac{1}{2} \beta^T (D_K)^{-1} \beta + \frac{\nu_0}{\lambda} \right)$

### Conditional Posterior Distribution of $\lambda$

$$\begin{aligned} \pi(\lambda | y) &\propto \frac{1}{\lambda^{\nu_0/2}} \cdot \exp \left\{ -\frac{\nu_0}{\tau^2 \lambda} \right\} \cdot \left( \frac{1}{\lambda} \right)^{3/2} \cdot \exp \left\{ -\frac{1}{\lambda \eta_0^2} \right\} \\ &\propto \frac{1}{\lambda^{\nu_0/2} \lambda^{3/2}} \cdot \exp \left\{ -\frac{\nu_0}{\tau^2 \lambda} - \frac{1}{\lambda \eta_0^2} \right\} \\ &\propto \lambda^{-(\frac{\nu_0+1}{2})-1} \cdot \exp \left\{ -\frac{1}{\lambda} \left[ \frac{\nu_0}{\tau^2} + \frac{1}{\eta_0^2} \right] \right\} \end{aligned}$$

Hence,  $\pi(\lambda | y) \sim IG \left( \frac{\nu_0+1}{2}, \frac{\nu_0}{\tau^2} + \frac{1}{\eta_0^2} \right)$

## Gradient and Hessian of $\pi(\alpha_0, \beta | y)$

### Gradient of $\pi(\alpha_0, \beta | y)$

See (`gr_single.R` file)

$$\nabla \log \pi(\alpha_0, \beta | y) = \nabla \log f(\cdot) = \begin{bmatrix} \frac{\partial}{\partial \alpha_0} \log f(\alpha_0, \beta) \\ \nabla_\beta \log f(\alpha_0, \beta) \end{bmatrix}$$

$$\begin{aligned} \frac{\partial}{\partial \alpha_0} \log f(\alpha_0, \beta) &= \frac{\partial}{\partial \alpha_0} \left[ -\frac{1}{2} \left[ \alpha_0 n + \frac{\alpha_0^2}{\sigma_{\alpha_0}^2} + 1_n X \beta + \frac{1}{\tau^2} \beta^T (D_K)^{-1} \beta + \sum_{j=0}^{n-1} \exp \{ y_j - \alpha_0 - X_j^T \beta \} \right] \right] \\ &= -\frac{1}{2} \left[ n + \frac{2\alpha_0}{\sigma_{\alpha_0}^2} + \sum_{j=0}^{n-1} -\exp \{ y_j - \alpha_0 - X_j^T \beta \} \right] \\ &= -\frac{n}{2} - \frac{\alpha_0}{\sigma_{\alpha_0}^2} + \frac{1}{2} \sum_{j=0}^{n-1} \exp \{ y_j - \alpha_0 - X_j^T \beta \} \end{aligned}$$

$$\begin{aligned}
\nabla_{\beta} \log f(\alpha_0, \beta) &= \frac{\partial}{\partial \beta} \left[ -\frac{1}{2} \left[ 1_n X \beta + \frac{1}{\tau^2} \beta^T (D_K)^{-1} \beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\} \right] \right] \\
&= \frac{\partial}{\partial \beta} \left[ -\frac{1}{2} 1_n X \beta \right] + \frac{\partial}{\partial \beta} \left[ -\frac{1}{2\tau^2} \beta^T (D_K)^{-1} \beta \right] + \frac{\partial}{\partial \beta} \left[ -\frac{1}{2} \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\} \right] \\
&= -\frac{1}{2} \left[ X^T 1_n^T + 2D_K^{-1} \beta + \sum_{j=0}^{n-1} -X_j \exp\{y_j - \alpha_0 - X_j^T \beta\} \right]
\end{aligned}$$

Therefore, the gradient of the conditional posterior of  $\beta$  and  $\alpha_0$  conditioned on  $y$  is :

$$\nabla \log \pi(\alpha_0, \beta | y) = \nabla \log f(\cdot) = \begin{bmatrix} -\frac{n}{2} - \frac{\alpha_0}{\sigma_{\alpha}^2} + \frac{1}{2} \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\} \\ -\frac{1}{2} \left[ X^T 1_n^T + 2D_K^{-1} \beta + \sum_{j=0}^{n-1} -X_j \exp\{y_j - \alpha_0 - X_j^T \beta\} \right] \end{bmatrix}$$

**Hessian of  $\pi(\beta, \alpha_0 | y)$**

See (`he_single.R` file)

$$\nabla^2 \log f(\cdot) = \begin{bmatrix} \frac{\partial^2}{\partial \alpha_0^2} & \left( \frac{\partial^2}{\partial \alpha_0 \partial \beta} \right)^T \\ \frac{\partial^2}{\partial \beta \partial \alpha_0} & \nabla_{\beta}^2 f \end{bmatrix}$$

$$\begin{aligned}
\frac{\partial^2 \log f}{\partial \alpha_0^2} &= \frac{\partial}{\partial \alpha_0} \left[ -\frac{n}{2} - \frac{\alpha_0}{\sigma_{\alpha}^2} + \frac{1}{2} \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\} \right] \\
&= -\frac{1}{\sigma_{\alpha}^2} + \frac{1}{2} \sum_{j=0}^{n-1} -\exp\{y_j - \alpha_0 - X_j^T \beta\} \\
&= -\frac{1}{\sigma_{\alpha}^2} - \frac{1}{2} \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \log f}{\partial \beta \partial \alpha_0} &= \frac{\partial}{\partial \alpha} \left[ -\frac{1}{2} X^T 1_n^T - \frac{D_K^{-1}}{\tau^2} \beta + \frac{1}{2} \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\} \right] \\
&= -\frac{1}{2} \sum_{j=0}^{n-1} X_j \exp\{y_j - \alpha_0 - X_j^T \beta\}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \log f}{\partial \alpha_0 \partial \beta} &= \frac{\partial}{\partial \beta} \left[ -\frac{n}{2} - \frac{\alpha_0}{\sigma_{\alpha_0}^2} + \frac{1}{2} \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\} \right] \\
&= -\frac{1}{2} \sum_{j=0}^{n-1} X_j \exp\{y_j - \alpha_0 - X_j^T \beta\}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \log f}{\partial^2 \beta} &= \frac{\partial}{\partial \beta} \left[ -\frac{1}{2} \left[ X^T 1_n^T + \frac{2}{\tau^2} D_K^{-1} \beta + \sum_{j=0}^{n-1} -X_j \exp\{y_j - \alpha_0 - X_j^T \beta\} \right] \right] \\
&= -\frac{D_K^{-1}}{\tau^2} + -\frac{1}{2} \sum_{j=0}^{n-1} X_j X_j^T \exp\{y_j - \alpha_0 - X_j^T \beta\}
\end{aligned}$$

Therefore, the Hessian is:

$$\nabla^2 \log f(\cdot) = \begin{bmatrix} -\frac{1}{\sigma_\alpha^2} - \frac{1}{2} \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\} & \left( -\frac{1}{2} \sum_{j=0}^{n-1} X_j \exp\{y_j - \alpha_0 - X_j^T \beta\} \right)^T \\ -\frac{1}{2} \sum_{j=0}^{n-1} X_j \exp\{y_j - \alpha_0 - X_j^T \beta\} & -\frac{D_K^{-1}}{\tau^2} + -\frac{1}{2} \sum_{j=0}^{n-1} X_j X_j^T \exp\{y_j - \alpha_0 - X_j^T \beta\} \end{bmatrix}$$