Tessellation README

2023-01-23

Metropolis Hastings Derivation

First, following the steps in AdaptSPEC, we rewrite the Whittle Likelihood as follows using the notation change that $y_n(\omega_j) = \log(I_n(\omega_j))$:

$$
\begin{aligned}
p(x|f) &= (2\pi)^{n/2} \prod_{j=0}^{n-1} \exp\{-\frac{1}{2}[\log f(\omega_j) + I_n(\omega_j)/f(\omega_j)]\} \\
&= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\log(f(\omega_j)) + \frac{\exp\{\log(I_n(\omega_j))\}}{\exp\{\log(f(\omega_j))\}}\right]\right\}
\end{aligned}
$$

Then we use the equation given in the paper: $\log(f_j) = g(\omega_j) = \alpha_0 + h(\omega_j) = \alpha_0 + X_j^T\beta$ in order to replace the unknown spectral density with α_0's and β's

$$
\begin{aligned}
&= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\log(f(\omega_j)) + \frac{\exp\{y_n(\omega_j)\}}{\exp\{\log(f(\omega_j))\}}\right]\right\} \\
\mathcal{L}(\alpha_0, \beta, \tau^2|y) &= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\alpha_0 + X_j^T\beta + \frac{\exp\{y_j\}}{\exp\{\alpha_0 + X_j^T\beta\}}\right]\right\} \\
&= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\alpha_0 + X_j^T\beta + \exp\{y_j - \alpha_0 - X_j^T\beta\}\right]\right\} \\
&= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}[\alpha_0 n + 1_n X\beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T\beta\}]\right\}
\end{aligned}
$$

Priors

Following closely* the priors outlined in AdaptSPEC, we give the parameters the following priors:

$$
\begin{aligned}
\pi(\alpha) &\sim N(0, \sigma_\alpha^2) \\
\pi(\beta) &\sim N(0, \tau^2, D_K) \\
\pi(\tau^2|\lambda) &\sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0}{\lambda}\right) \\
\pi(\lambda) &\sim IG\left(\frac{1}{2}, \frac{1}{\eta_0^2}\right)
\end{aligned}
$$

Such that D_K is a positive definite diagonal matrix of dimension K.

*Instead of giving τ^2 a Uniform prior as in AdaptSPEC we have given it a half-t prior. Following the work laid out in Wand et.al. we are able to write the half-t prior as an Inverse-Gamma PDF with the parameters as written above.

Posterior Distribution of the Whittle Likelihood

Now, we go through the derivation of calculating the full posterior distribution of the Whittle Likelihood:

$$\pi(\beta, \tau^2, \alpha_0, \lambda | y) \propto \mathcal{L}(\beta, \tau^2, \alpha_0, \lambda | y) \cdot \pi(\beta) \cdot \pi(\tau^2 | \lambda) \cdot \pi(\lambda) \cdot \pi(\alpha_0)$$

In order to be precise and organized we will first take each prior given and rewrite the prior as it will be used in the posterior distribution.

First, $\pi(\alpha_0) \sim N(0, \sigma_\alpha^2)$:

$$\begin{aligned} \pi(\alpha_0) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{\alpha_0^2}{2\sigma_\alpha^2} \right\} \\ &\propto \exp\left\{ -\frac{\alpha_0^2}{2\sigma_\alpha^2} \right\} \end{aligned}$$

Second, $\pi(\beta) \sim N(0, \tau^2 D_k)$:

$$\begin{aligned} \pi(\beta) &= \frac{1}{\sqrt{(2\pi)^K |\tau^2 D_K|}} \exp\left\{ -\frac{1}{2} \beta^T (\tau^2 D_K)^{-1} \beta \right\} \\ &\propto (\tau^2)^{-K/2} \cdot \exp\left\{ -\frac{1}{2} \beta^T (\tau^2 D_K)^{-1} \beta \right\} \end{aligned}$$

Third, $\pi(\tau^2 | \lambda) \sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0}{\lambda}\right)$:

$$\begin{aligned} \pi(\tau^2 | \lambda) &= \frac{\left(\frac{\nu_0}{\lambda}\right)^{\nu_0/2}}{\Gamma\left(\frac{\nu_0}{2}\right)} \cdot \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{ -\frac{\nu_0}{\tau^2 \lambda} \right\} \\ &\propto \frac{1}{\lambda^{\nu_0/2}} \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{ -\frac{\nu_0}{\tau^2 \lambda} \right\} \end{aligned}$$

Fourth, $\pi(\lambda) \sim IG\left(\frac{1}{2}, \frac{1}{\eta_0^2}\right)$:

$$\begin{aligned} \pi(\lambda) &= \frac{\left(\frac{1}{\eta_0^2}\right)^{1/2}}{\Gamma(1/2)} \left(\frac{1}{\lambda}\right)^{3/2} \exp\left\{ -\frac{1}{\lambda \eta_0^2} \right\} \\ &\propto \left(\frac{1}{\lambda}\right)^{3/2} \exp\left\{ -\frac{1}{\lambda \eta_0^2} \right\} \end{aligned}$$