Tessellation README

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Metropolis Hastings and Gibbs Derivation

First, following the steps in AdaptSPEC, we rewrite the Whittle Likelihood as follows using the notation change that $y_n(\omega_j) = \log(I_n(\omega_j))$:

$$p(x|f) = (2\pi)^{n/2} \prod_{j=0}^{n-1} \exp\{-\frac{1}{2}[\log f(\omega_j) + I_n(\omega_j)/f(\omega_j)]\}$$

$$= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\log(f(\omega_j)) + \frac{\exp\{\log(I_n(\omega_j))\}}{\exp\{\log(f(\omega_j))\}}\right]\right\}$$

Then we use the equation given in the paper: $\log(f_j) = g(\omega_j) = \alpha_0 + h(\omega_j) = \alpha_0 + X_j^T\beta$ in order to replace the unknown spectral density with α_0's and β's

$$= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\log(f(\omega_j)) + \frac{\exp\{y_n(\omega_j)\}}{\exp\{\log(f(\omega_j))\}}\right]\right\}$$

$$\mathcal{L}(\alpha_0, \beta, \tau^2|y) = (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\alpha_0 + X_j^T\beta + \frac{\exp\{y_j\}}{\exp\{\alpha_0 + X_j^T\beta\}}\right]\right\}$$

$$= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\alpha_0 + X_j^T\beta + \exp\{y_j - \alpha_0 - X_j^T\beta\}\right]\right\}$$

$$= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}[\alpha_0 n + 1_n X\beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T\beta\}]\right\}$$

Priors

Following closely* the priors outlined in AdaptSPEC, we give the parameters the following priors:

$$\pi(\alpha) \sim N(0, \sigma_\alpha^2)$$
$$\pi(\beta) \sim N(0, \tau^2, D_K)$$
$$\pi(\tau^2|\lambda) \sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0}{\lambda}\right)$$
$$\pi(\lambda) \sim IG\left(\frac{1}{2}, \frac{1}{\eta_0^2}\right)$$

Such that D_K is a positive definite diagonal matrix of dimension K.

*Instead of giving τ^2 a Uniform prior as in AdaptSPEC we have given it a half-t prior. Following the work laid out in Wand et.al. we are able to write the half-t prior as an Inverse-Gamma PDF with the parameters as written above.

Posterior Distribution

Now, we go through the derivation of calculating the full posterior distribution of the Whittle Likelihood:

$$\pi(\beta, \tau^2, \alpha_0, \lambda | y) \propto \mathcal{L}(\beta, \tau^2, \alpha_0, \lambda | y) \cdot \pi(\beta) \cdot \pi(\tau^2 | \lambda) \cdot \pi(\lambda) \cdot \pi(\alpha_0)$$

In order to be precise and organized we will first take each prior given and rewrite the prior as it will be used in the posterior distribution.

First, $\pi(\alpha_0) \sim N(0, \sigma_\alpha^2)$:

$$\pi(\alpha_0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\alpha_0^2}{2\sigma_\alpha^2}\right\}$$

$$\propto \exp\left\{-\frac{\alpha_0^2}{2\sigma_\alpha^2}\right\}$$

Second, $\pi(\beta) \sim N(0, \tau^2 D_k)$:

$$\pi(\beta) = \frac{1}{\sqrt{(2\pi)^K |\tau^2 D_K|}} \exp\left\{-\frac{1}{2}\beta^T (\tau^2 D_K)^{-1} \beta\right\}$$

$$\propto (\tau^2)^{-K/2} \cdot \exp\left\{-\frac{1}{2}\beta^T (\tau^2 D_K)^{-1} \beta\right\}$$

Third, $\pi(\tau^2 | \lambda) \sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0}{\lambda}\right)$:

$$\pi(\tau^2 | \lambda) = \frac{\left(\frac{\nu_0}{\lambda}\right)^{\nu_0/2}}{\Gamma\left(\frac{\nu_0}{2}\right)} \cdot \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{\nu_0}{\tau^2 \lambda}\right\}$$

$$\propto \frac{1}{\lambda^{\nu_0/2}} \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{\nu_0}{\tau^2 \lambda}\right\}$$

Fourth, $\pi(\lambda) \sim IG\left(\frac{1}{2}, \frac{1}{\eta_0^2}\right)$:

$$\pi(\lambda) = \frac{\left(\frac{1}{\eta_0^2}\right)^{1/2}}{\Gamma(1/2)} \left(\frac{1}{\lambda}\right)^{3/2} \exp\left\{-\frac{1}{\lambda\eta_0^2}\right\}$$

$$\propto \left(\frac{1}{\lambda}\right)^{3/2} \exp\left\{-\frac{1}{\lambda\eta_0^2}\right\}$$

Now, putting this together we have:

$$\pi(\beta, \tau^2, \alpha_0, \lambda | y) \propto (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}[\alpha_0 n + 1_n X\beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\}]\right\} \cdot \exp\left\{-\frac{\alpha_0^2}{2\sigma_\alpha^2}\right\}$$

$$\cdot (\tau^2)^{-K/2} \cdot \exp\left\{-\frac{1}{2}\beta^T (\tau^2 D_K)^{-1} \beta\right\} \cdot \frac{1}{\lambda^{\nu_0/2}} \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{\nu_0}{\tau^2 \lambda}\right\} \cdot \left(\frac{1}{\lambda}\right)^{3/2} \exp\left\{-\frac{1}{\lambda\eta_0^2}\right\}$$

Conditional Posterior Distributions

Conditional Posterior Distribution of β and α_0

$$\pi(\beta, \alpha_0 | y) \propto \exp\left\{-\frac{1}{2}\left[\alpha_0 n + \frac{\alpha_0^2}{\sigma_\alpha^2} + 1_n X\beta + \frac{1}{\tau^2}\beta^T (D_K)^{-1}\beta + \sum_{j=0}^{n-1} \exp\left\{y_j - \alpha_0 - X_j^T\beta\right\}\right]\right\}$$

Conditional Posterior Distribution of τ^2

$$\pi(\tau^2 | y) \propto (\tau^2)^{-K/2} \cdot \exp\left\{-\frac{1}{2}\beta^T (\tau^2 D_K)^{-1}\beta\right\} \cdot \left(\frac{1}{\tau^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda}\right\}$$

$$\propto (\tau^2)^{-\frac{K-\nu_0}{2}-1} \cdot \exp\left\{-\frac{1}{\tau^2}\left[\frac{1}{2}\beta^T (D_K)^{-1}\beta + \frac{\nu_0}{\lambda}\right]\right\}$$

Hence, $\pi(\tau^2 | y) \sim IG\left(\frac{K+\nu_0}{2}, \frac{1}{2}\beta^T (D_K)^{-1}\beta + \frac{\nu_0}{\lambda}\right)$

Conditional Posterior Distribution of λ

$$\pi(\lambda | y) \propto \frac{1}{\lambda^{\nu_0/2}} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda}\right\} \cdot \left(\frac{1}{\lambda}\right)^{3/2} \cdot \exp\left\{-\frac{1}{\lambda\eta_0^2}\right\}$$

$$\propto \frac{1}{\lambda^{\nu_0/2}\lambda^{3/2}} \cdot \exp\left\{-\frac{\nu_0}{\tau^2\lambda} - \frac{1}{\lambda\eta_0^2}\right\}$$

$$\propto \lambda^{-\left(\frac{\nu_0+1}{2}\right)-1} \cdot \exp\left\{-\frac{1}{\lambda}\left[\frac{\nu_0}{\tau^2} + \frac{1}{\eta_0^2}\right]\right\}$$

Hence, $\pi(\lambda | y) \sim IG\left(\frac{\nu_0+1}{2}, \frac{\nu_0}{\tau^2} + \frac{1}{\eta_0^2}\right)$

Gradient and Hessian of $\pi(\alpha_0, \beta | y)$

Gradient of $\pi(\alpha_0, \beta | y)$

$$\bigtriangledown \log \pi(\alpha_0, \beta | y) = \bigtriangledown \log f(\cdot) = \begin{bmatrix} \frac{\partial}{\partial\alpha_0} \log f(\alpha_0, \beta) \\ \bigtriangledown_\beta \log f(\alpha_0, \beta) \end{bmatrix}$$

$$\frac{\partial}{\partial\alpha_0} \log f(\alpha_0, \beta) = \frac{\partial}{\partial\alpha_0}\left[-\frac{1}{2}\left[\alpha_0 n + \frac{\alpha_0^2}{\sigma_{\alpha_0}^2} + 1_n X\beta + \frac{1}{\tau^2}\beta^T (D_K)^{-1}\beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T\beta\}\right]\right]$$

$$= -\frac{1}{2}\left[n + \frac{2\alpha_0}{\sigma_{\alpha_0}^2} + \sum_{j=0}^{n-1} -\exp\{y_j - \alpha_0 - X_j^T\beta\}\right]$$

$$= -\frac{n}{2} - \frac{\alpha_0}{\sigma_\alpha^2} + \frac{1}{2}\sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T\beta\}$$

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$$\nabla_\beta \log f(\alpha_0, \beta) = \frac{\partial}{\partial \beta} \left[-\frac{1}{2} \left[1_n X \beta + \frac{1}{\tau^2} \beta^T (D_K)^{-1} \beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\} \right] \right]$$

$$= \frac{\partial}{\partial \beta} \left[-\frac{1}{2} 1_n X \beta \right] + \frac{\partial}{\partial \beta} \left[-\frac{1}{2\tau^2} \beta^T (D_K)^{-1} \beta \right] + \frac{\partial}{\partial \beta} \left[-\frac{1}{2} \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\} \right]$$

$$= -\frac{1}{2} \left[X^T 1_n^T + 2 D_K^{-1} \beta + \sum_{j=0}^{n-1} -X_j \exp\{y_j - \alpha_0 - X_j^T \beta\} \right]$$

Therefore, the gradient of the conditional posterior of β and α_0 conditioned on y is :

$$\nabla \log \pi(\alpha_0, \beta | y) = \nabla \log f(\cdot) = \begin{bmatrix} -\frac{n}{2} - \frac{\alpha_0}{\sigma_\alpha^2} + \frac{1}{2} \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\} \\ -\frac{1}{2} \left[X^T 1_n^T + 2 D_K^{-1} \beta + \sum_{j=0}^{n-1} -X_j \exp\{y_j - \alpha_0 - X_j^T \beta\} \right] \end{bmatrix}$$