

# Tessellation README

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## Metropolis Hastings and Gibbs Derivation

First, following the steps in AdaptSPEC, we rewrite the Whittle Likelihood as follows using the notation change that  $y_n(\omega_j) = \log(I_n(\omega_j))$ :

$$\begin{aligned} p(x|f) &= (2\pi)^{n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}[\log f(\omega_j) + I_n(\omega_j)/f(\omega_j)]\right\} \\ &= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\log(f(\omega_j)) + \frac{\exp\{\log(I_n(\omega_j))\}}{\exp\{\log(f(\omega_j))\}}\right]\right\} \end{aligned}$$

Then we use the equation given in the paper:  $\log(f_j) = g(\omega_j) = \alpha_0 + h(\omega_j) = \alpha_0 + X_j^T \beta$  in order to replace the unknown spectral density with  $\alpha_0$ 's and  $\beta$ 's

$$\begin{aligned} &= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\log(f(\omega_j)) + \frac{\exp\{y_n(\omega_j)\}}{\exp\{\log(f(\omega_j))\}}\right]\right\} \\ \mathcal{L}(\alpha_0, \beta, \tau^2|y) &= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\alpha_0 + X_j^T \beta + \frac{\exp\{y_j\}}{\exp\{\alpha_0 + X_j^T \beta\}}\right]\right\} \\ &= (2\pi)^{-n/2} \prod_{j=0}^{n-1} \exp\left\{-\frac{1}{2}\left[\alpha_0 + X_j^T \beta + \exp\{y_j - \alpha_0 - X_j^T \beta\}\right]\right\} \\ &= (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}[\alpha_0 n + 1_n X \beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\}]\right\} \end{aligned}$$

## Priors

Following closely\* the priors outlined in AdaptSPEC, we give the parameters the following priors:

$$\begin{aligned} \pi(\alpha) &\sim N(0, \sigma_\alpha^2) \\ \pi(\beta) &\sim N(0, \tau^2, D_K) \\ \pi(\tau^2|\lambda) &\sim IG\left(\frac{\nu_0}{2}, \frac{\nu_0}{\lambda}\right) \\ \pi(\lambda) &\sim IG\left(\frac{1}{2}, \frac{1}{\eta_0^2}\right) \end{aligned}$$

Such that  $D_K$  is a positive definite diagonal matrix of dimension  $K$ .

\*Instead of giving  $\tau^2$  a Uniform prior as in AdaptSPEC we have given it a half-t prior. Following the work laid out in Wand et.al. we are able to write the half-t prior as an Inverse-Gamma PDF with the parameters as written above.

### Posterior Distribution

Now, we go through the derivation of calculating the full posterior distribution of the Whittle Likelihood:

$$\pi(\beta, \tau^2, \alpha_0, \lambda | y) \propto \mathcal{L}(\beta, \tau^2, \alpha_0, \lambda | y) \cdot \pi(\beta) \cdot \pi(\tau^2 | \lambda) \cdot \pi(\lambda) \cdot \pi(\alpha_0)$$

In order to be precise and organized we will first take each prior given and rewrite the prior as it will be used in the posterior distribution.

First,  $\pi(\alpha_0) \sim N(0, \sigma_\alpha^2)$ :

$$\begin{aligned} \pi(\alpha_0) &= \frac{1}{\sqrt{2\pi\sigma_\alpha^2}} \exp \left\{ -\frac{\alpha_0^2}{2\sigma_\alpha^2} \right\} \\ &\propto \exp \left\{ -\frac{\alpha_0^2}{2\sigma_\alpha^2} \right\} \end{aligned}$$

Second,  $\pi(\beta) \sim N(0, \tau^2 D_K)$ :

$$\begin{aligned} \pi(\beta) &= \frac{1}{\sqrt{(2\pi)^K |\tau^2 D_K|}} \exp \left\{ -\frac{1}{2} \beta^T (\tau^2 D_K)^{-1} \beta \right\} \\ &\propto (\tau^2)^{-K/2} \cdot \exp \left\{ -\frac{1}{2} \beta^T (\tau^2 D_K)^{-1} \beta \right\} \end{aligned}$$

Third,  $\pi(\tau^2 | \lambda) \sim IG \left( \frac{\nu_0}{2}, \frac{\nu_0}{\lambda} \right)$ :

$$\begin{aligned} \pi(\tau^2 | \lambda) &= \frac{\left( \frac{\nu_0}{\lambda} \right)^{\nu_0/2}}{\Gamma \left( \frac{\nu_0}{2} \right)} \cdot \left( \frac{1}{\tau^2} \right)^{\frac{\nu_0}{2}+1} \cdot \exp \left\{ -\frac{\nu_0}{\tau^2 \lambda} \right\} \\ &\propto \frac{1}{\lambda^{\nu_0/2}} \left( \frac{1}{\tau^2} \right)^{\frac{\nu_0}{2}+1} \cdot \exp \left\{ -\frac{\nu_0}{\tau^2 \lambda} \right\} \end{aligned}$$

Fourth,  $\pi(\lambda) \sim IG \left( \frac{1}{2}, \frac{1}{\eta_0^2} \right)$ :

$$\begin{aligned} \pi(\lambda) &= \frac{\left( \frac{1}{\eta_0^2} \right)^{1/2}}{\Gamma(1/2)} \left( \frac{1}{\lambda} \right)^{3/2} \exp \left\{ -\frac{1}{\lambda \eta_0^2} \right\} \\ &\propto \left( \frac{1}{\lambda} \right)^{3/2} \exp \left\{ -\frac{1}{\lambda \eta_0^2} \right\} \end{aligned}$$

Now, putting this together we have:

$$\begin{aligned} \pi(\beta, \tau^2, \alpha_0, \lambda | y) &\propto (2\pi)^{-n/2} \exp \left\{ -\frac{1}{2} [\alpha_0 n + 1_n X \beta + \sum_{j=0}^{n-1} \exp\{y_j - \alpha_0 - X_j^T \beta\}] \right\} \cdot \exp \left\{ -\frac{\alpha_0^2}{2\sigma_\alpha^2} \right\} \\ &\quad \cdot (\tau^2)^{-K/2} \cdot \exp \left\{ -\frac{1}{2} \beta^T (\tau^2 D_K)^{-1} \beta \right\} \cdot \frac{1}{\lambda^{\nu_0/2}} \left( \frac{1}{\tau^2} \right)^{\frac{\nu_0}{2}+1} \cdot \exp \left\{ -\frac{\nu_0}{\tau^2 \lambda} \right\} \cdot \left( \frac{1}{\lambda} \right)^{3/2} \exp \left\{ -\frac{1}{\lambda \eta_0^2} \right\} \end{aligned}$$

## Conditional Posterior Distributions

### Conditional Posterior Distribution of $\beta$ and $\alpha_0$

$$\pi(\beta, \alpha_0 | y) \propto \exp \left\{ -\frac{1}{2} \left[ \alpha_0 n + \frac{\alpha_0^2}{\sigma_\alpha^2} + 1_n X \beta + \frac{1}{\tau^2} \beta^T (D_K)^{-1} \beta + \sum_{j=0}^{n-1} \exp \{ y_j - \alpha_0 - X_j^T \beta \} \right] \right\}$$

### Conditional Posterior Distribution of $\tau^2$

$$\begin{aligned} \pi(\tau^2 | y) &\propto (\tau^2)^{-K/2} \cdot \exp \left\{ -\frac{1}{2} \beta^T (\tau^2 D_K)^{-1} \beta \right\} \cdot \left( \frac{1}{\tau^2} \right)^{\frac{\nu_0}{2} + 1} \cdot \exp \left\{ -\frac{\nu_0}{\tau^2 \lambda} \right\} \\ &\propto (\tau^2)^{-\frac{K+\nu_0}{2} - 1} \cdot \exp \left\{ -\frac{1}{\tau^2} \left[ \frac{1}{2} \beta^T (D_K)^{-1} \beta + \frac{\nu_0}{\lambda} \right] \right\} \end{aligned}$$

Hence,  $\pi(\tau^2 | y) \sim IG \left( \frac{K+\nu_0}{2}, \frac{1}{2} \beta^T (D_K)^{-1} \beta + \frac{\nu_0}{\lambda} \right)$

### Conditional Posterior Distribution of $\lambda$

$$\begin{aligned} \pi(\lambda | y) &\propto \frac{1}{\lambda^{\nu_0/2}} \cdot \exp \left\{ -\frac{\nu_0}{\tau^2 \lambda} \right\} \cdot \left( \frac{1}{\lambda} \right)^{3/2} \cdot \exp \left\{ -\frac{1}{\lambda \eta_0^2} \right\} \\ &\propto \frac{1}{\lambda^{\nu_0/2} \lambda^{3/2}} \cdot \exp \left\{ -\frac{\nu_0}{\tau^2 \lambda} - \frac{1}{\lambda \eta_0^2} \right\} \\ &\propto \lambda^{-(\frac{\nu_0+1}{2})-1} \cdot \exp \left\{ -\frac{1}{\lambda} \left[ \frac{\nu_0}{\tau^2} + \frac{1}{\eta_0^2} \right] \right\} \end{aligned}$$

Hence,  $\pi(\lambda | y) \sim IG \left( \frac{\nu_0+1}{2}, \frac{\nu_0}{\tau^2} + \frac{1}{\eta_0^2} \right)$

## Gradient and Hessian of $\pi(\alpha_0, \beta | y)$

### Gradient of $\pi(\alpha_0, \beta | y)$

$$\nabla \log \pi(\alpha_0, \beta | y) = \nabla \log f(\cdot) = \begin{bmatrix} \frac{\partial}{\partial \alpha_0} \log f(\alpha_0, \beta) \\ \nabla_\beta \log f(\alpha_0, \beta) \end{bmatrix}$$

$$\begin{aligned} \frac{\partial}{\partial \alpha_0} \log f(\alpha_0, \beta) &= \frac{\partial}{\partial \alpha_0} \left[ -\frac{1}{2} \left[ \alpha_0 n + \frac{\alpha_0^2}{\sigma_{\alpha_0}^2} + 1_n X \beta + \frac{1}{\tau^2} \beta^T (D_K)^{-1} \beta + \sum_{j=0}^{n-1} \exp \{ y_j - \alpha_0 - X_j^T \beta \} \right] \right] \\ &= \end{aligned}$$