Low order stabilized finite element for the fucll Biot formulation in Soil Mechanics at finite strain

Response to reviewers

We thank the reviewers for their constructive suggestions. We reproduce below their comments in blue type noting how we have addressed each of them.

Text changes to the paper to address the comments are written *in red italic type*. New version of tables and figures are not reproduced here to avoid clutter.

**Reviewing: 1**

Comments to the Author

The paper presents an interesting contribution to the stabilization of low order elements. It is clearly written, with suitable references.

The Biot equations described in the paper are limited to problems where material derivatives following water and soil are equal, which is not clear in many cases. Porosity is assumed to be constant too.

I would recommend to add a note explaining this limitations, and how the proposed method could be extended to cope with them.

Could a comparison between different stabilization techniques be included? Here there is a parameter which needs some insight to be estimated while other techniques do not require any additional parameter. I have found very much pertinent the cite of Hafez and Soliman.

The set of proposed examples do not include any where bending or localization is present. In some geotechnical engineering problems, there are structural elements such as walls or beams. Assuming them to be elastic, how is the accuracy of computations? It is important to note that this accuracy affects shear band computations too.

In the present work, only elastic problems are considered. The extension of the proposed formulation to elasto-plastic problems is a topic of current research.

The example presented in 4.3 does not include absorbing boundary conditions, hence the authors stop the computation before waves are reflected. Which approach could be used within the proposed method to implement absorbing BCs?

Leaving the consistent linearization and the estimation of the linearization parameter to 2 appendixes is a good idea, as the paper is more readable.

My conclusion is that the paper is excellent and should be published after the minor remarks stated above have been addresses by the authors.

**Reviewing: 2**

Comments to the Author

The authors in this work compare two stabilization schemes for a three-field (u-w-p) formulation of dynamic poromechanics. The first is based on the divergence of the momentum equation (DME) and the other on the polynomial pressure projection technique (PPP). While the work is interesting and the numerical results are quite thorough, I have reservations about the current presentation of the schemes and the discussion of stabilized methods. I hope the authors can address my specific comments below:

— The authors discuss both inf-sup (LBB) stability and volumetric locking, and at times use the terms interchangeably. This lack of clarity needs to be corrected, particularly as the paper is attempting to address both. Inf-sup stability addresses whether a block matrix containing algebraic constraints is well posed, so that it does not contain any singular modes. If the block matrix is denoted A, the question is whether there are any solutions q such that Aq=0. In the current context, these singular modes manifest themselves as fluid pressure oscillations, and the algebraic constraints stem from the undrained mixture behavior. On the other hand, volumetric locking addresses whether the mechanical stiffness sub-block (say, K) is overconstrained, such that the only admissible solution to the linear system Ku=b is u=0, regardless of b. This occurs for incompressible elasticity, when the solid itself (not the mixture) has a divergence constraint on its behavior. The term “locking” is adopted because the mesh is somehow locked in place (displacement u=0). It has nothing to do with fluid pressure oscillations, and can be observed for an uncoupled problem with no fluid at all. Describing spurious fluid pressure oscillations as “locking” is therefore inaccurate. The relationship between the two is that a common solution to the volumetric locking problem is to rewrite the momentum balance equation in a mixed form to more accurately approximate the divergence of the displacement field. This approach alters the definition of K so that it is no longer overconstrained. By adding a new unknown, however, one ends up with a block system with algebraic constraints that are then subject to inf-sup conditions. This system may have spurious modes of concern, but these are oscillations in the mean stress field, not the pore pressure field (as shown in Figure 11). I would ask the authors to edit their work and make sure these distinctions are apparent. Statements like “due to volumetric locking non-stabilized formulations show large-amplitude oscillations in the water pressure profile” are confusing, as a clear distinction needs to be made between incompressible solid behavior and undrained mixture behavior. There are quite a few references of this sort throughout the paper. I suggest the authors decide on a consistent terminology for the phenomena under investigation, and then re-read the paper to ensure this terminology is rigorously used.

We agree with the reviewer than we have not been careful enough with the vocabulary. We have revised the manuscript in depth in order to be consistent with the terminology.

Page 1, line 45

*The reduced u − p w formulation- is not applicable to high-frequency loading, such as that resulting from impact events [3]. On the other hand,it is used quite often, since many seismic applications fall within its remit and the formulation is simpler and,having less degrees of freedom per node, computationally less demanding. This widespread use has brought much attention to potential numerical problems that may arise during its solution and, particularly,* ***the development of stable formulations.***

***Unstable elements manifest themselves in*** *the numerical solution as over-stiffening of mechanical response and spurious high spatial variability* ***of the water pressure field****, eventually leading to instability, non-uniqueness and mesh-dependence of the solutions.* ***This pathology*** *appears when the problem conditions lead to incompressibility. For water-saturated soils this is approximately the case in undrained conditions. The system of discretized equations for the u−pw formulation, has then similar structure to that found when using a mixed u−p formulation of Solid Mechanics problems [4] . As the problem approaches zero permeability and given the quasi-incompressilibity of the soil and water constituents, it may become ill-posed from a mathematical point of view.*

***This numerical pathology*** *is due to an improper finite-dimensional space in the finite element discretization and may be identified using analytical tools such as the inf-sup condition [5]* ***or, alternatively, the patch-test****. More importantly, it can be also avoided or alleviated using a variety of numerical strategies. Typically, those strategies have been first applied in single-phase fluid or solid mechanics problems and later adapted to the coupled hydromechanical case.*

Page 2, line 8

***Stable formulations may be also achieved*** *with fractional step algorithms (split-operator methods); equations are solved in a staggered fashion and the time integration algorithm provides stabilization [4]. This approach requires semi-explicit integration steps that may significantly increase the computational burden.*

*Finally,* ***the spurious modes that appear due to the failure to satisfy the inf-sup condition*** *can be also addressed using simpler, single-step monolithic solvers. Two main strategies are still possible: either to use more complex, but stable, finite elements with different order interpolation of displacement and water pressure fields, or to apply stabilization procedures to originally unstable finite elements [10]. Equal-order elements are preferable for various reasons. Numerical implementation is simpler when the same shape functions are used for all variables [11]. Additionally, low order elements are preferred for its flexibility in adaptive remeshing codes such as the Particle Finite Element method, which is typically reliant on linear elements [12,13,14].*

Page 2, line 22

*Apart from undrained conditions, incompressibility may also arise in soil mechanics simply from the mechanical response of the solid phase. This happens, for instance, when failure is reached in Critical State soil models. As noted in the literature [19,18], this response may* ***~~(also)~~*** *lead to volumetric locking. Separate numerical techniques to treat this extra source of incompressibility are required, and they have been addressed in models using the reduced u−p w formulation. Sun et al [18,20] proposed a formulation with an assumed deformation gradient, that has the F-bar method as a limiting case. Monforte et al [21] examined three-field mixed formulations in which either the effective pressure or the Jacobian were added as supplementary nodal variables and stabilization techniques were employed in all scalar equations.*

*In contrast to the extensive work done with the reduced formulation,* ***the development of equal-order stable formulations*** *and the consequent need for stabilization has received much less attention for the full Biot case. Jeremić et al [22] employed a u − U − pw formulation where the pore pressure variable was added for stabilization purposes. They addressed elasto-plastic problems with a small strain formulation. They used elements with different interpolation order for pore pressure and other nodal variables and did not note any* ***spurious mode****, although the focus of their work was mostly on seismic applications.*

Page 2, line 52

*The focus of this work lies on the stabilization techniques that are incorporated in the solution to* ***avoid pore pressure spurious oscillations****. Two techniques, namely polynomial pressure projection (PPP) and divergence of momentum (DME),are implemented and tested. They both allow for a monolithic (i.e. non-staggered) solution of the field equations. Apart from that, and to mitigate the volumetric locking that may arise from the solid phase response a mixed stabilized formulation previously applied to the simplified u-pw formulation [21] is extended to the dynamic range.*

In page 11, line 21 has been rewritten as:

*On the contrary, the structure of the system matrix, Equation (46), allows to hypothesize that the u−w−pw formulation may suffer* ***spurious spatial oscillations of the water pressure field*** *irrespectively of the oedometric modulus and the time-step. The reason beneath this fact is that the system matrix of the u−w−pw formulation always has the same form that the Lagrangian problem (irrespectively of the value of the permeability); differently, using the u−pw formulation, the term K pw pw is significantly different from zero when the permeability or the time-step are large (i.e, when the time-step is larger than the critical time-step depicted in Equation (48)).*

In page 11, line 21 has been rewritten as:

*The use of a mixed formulation is not enough by itself to mitigate the* ***high amplitude oscillations in the effective mean stress field*** *if equal order interpolants are used for all the variables. Therefore a stabilization technique is also required here. For simplicity, PPP is also used, thus adding a stabilization term to the volume deformation balance equation. As such, the final form of the semi-discrete finite element equations of the u-theta-w-pw reads:*

In page 13, line 43 has been rewritten as:

*The oedometer problem has been frequently employed [9,17,34] to assess the accuracy of numerical consolidation models using the reduced u-pw formulation.* ***Unstable elements show large amplitude oscillations in the water pressure profile during the first steps.*** *This is also observed here. Figure 3 depicts the vertical profiles of normalized water pressure, pw/pmax, at t=0.1s. Figure 3(a) compares the results obtained with direct integration (i.e. without any stabilization technique) with those obtained using the PPP and DME methodologies. Direct integration results in the typical oscillatory profile of the pore pressure. Both stabilization techniques are able to eliminate the oscillations and give almost coincident results.*

In page 17, line 53 has been rewritten as:

*Significant pore pressures are generated even in the partially drained case, Figure 9 although conditions are far from incompressibility. Even if the beneficial effect of stabilization (using DME with $\tau\_0$=0.1) is clear in the spatial pore pressure pattern below the load,* ***spurious water pressure oscillations still plays here a relatively minor role****. This is visible in the comparative performance of different stabilization techniques, illustrated in Figure 10. The PPP solution was computed with Delta t=0.0025 s., a value fixed by the condition given in Eq. B42. For the DME solution, several time steps were tried.*

In page 18, line 51 has been rewritten as:

***The use of a stabilized mixed formulation to mitigate the solid-phase induced volumetric locking has significant effects in this partially drained case.*** *The distribution of the effective pressure, p', is depicted at time t=0.15 s. for both u-w-pw and u-theta-w-pw formulations. Locking-induced numerical pollution that appears when the p-wave is traveling without stabilization is eliminated with the proposed technique (see Figure~11). This volumetric locking arising from the effective response of the medium does not affect much motion at Point A in the free surface Figure 12, but it is very visible in the spurious mean stress oscillations that are recorded at point B (in Figure 12b observed at two consecutive elements just above and below B).*

*For the undrained conditions (k=10E-10 m/s) of this simulation* ***the unstabilized element is severely unstable.*** *As shown in Figure 13 large pore pressure oscillations are widespread across the domain, irrespective of location. The benefits of stabilization are highly significant in this case. Rayleigh-wave motion at A, Figure 14, shows smaller damping (for the same time step and time-integration algorithm) using DME. The unacceptable performance of the non-stabilized solution is much clearer on the pore pressure wave at B, whereas, again, PPP stabilization does show some minor high frequency oscillation.*

Finally, the second paragraph of the conclusions have been rewritten as:

*Several stabilization techniques, (polynomial pressure projection; divergence of momentum; mixed reformulation of field equations), are implemented* ***to avoid spurious water pressure oscillations and volumetric locking of the low order elements employed.*** *They all seem to perform well, as illustrated by several numerical examples for which reference solutions were available. The examples also illustrate that, with this formulation, numerical stabilization is required even in partly drained cases.*

— The patch test condition proposed in Equation (45) appears incorrect. In Equation (44), imagine the practical situation in which dynamic terms are negligible (or, alternatively, the timestep dt is taken extremely small). In this case, the blocks multiplied by alpha\_1 dissappear, and we arrive at the classical three-field Biot discretization of quasi-static poromechanics. There are a number of elements that are known to be unstable for this problem—for example, the Q1-Q1-Q1 element used in this work, the Q1-Q1-P0 (constant pressure) element, and the Q1-RT0-P0 (Raviart-Thomas) element. All of these elements would satisfy the condition n\_u+n\_w >= n\_p, yet all of them exhibit pressure oscillations under undrained conditions. Instead, a sufficient condition for stability of the three field problem is that it must be separately stable for the incompressible elasticity sub problem (u-p system) and the Darcy sub-problem (w-p system). That is, a better check is whether n\_u >= n\_p AND n\_w >= n\_p separately. A more rigorous study of three-field stability conditions can be found in [Hong and Kraus (2018). Parameter-robust stability of classical three-field formulation of Biot’s consolidation model, <https://arxiv.org/abs/1706.00724>].

We agree with the reviewer that we fail to identify the conditions in which the element passes the patch test. The patch test should be passed in the two limiting cases of the problem: (i) in the Darcy’s sub-problem and (ii) the incompressible elastic sub-problem.

*As such, in order to have a non-singular system and, consequently, a water pressure field free of spurious oscillations, the following conditions must be fulfilled:*

*n\_u >= n\_pw & n\_w > n\_pw*

*being n\_u, n\_w and n\_pw the number of degrees of freedom for displacement, water displacement and water pressure fields. In addition to the usual condition required in the for the u-p problem, n\_u > n\_p (see, for instance, Pastor et al (1999)), and additional condition arises from the Darcy's problem: n\_w > n\_p. These conditions, although necessary, are not sufficient and must be fulfilled in any assembly of elements of the mesh (Zienkiewicz et al; Pastor et al, 1999). A more rigorous study of the stability conditions of the three-field formulation may be found elsewhere (Hong Krauss, 2018).*

— The choice of finite element spaces in Equation (18) should be discussed a bit further. One major reason for the using the three-field (u-w-p) formulation is that one can arrive at a discrete form where no derivatives on the pressure field are required, and therefore a piecewise constant space for the pressures can be adopted. This then allows for element-wise mass conservation. See, for example, [Castelletto, White, Ferronato (2016) Scalable Algorithms for Three Field Mixed Finite Element Coupled Poromechanics. J. Comp. Phys, 2016]. Here, the divergence theorem is applied in such a way that the unstabilized formulation contains pressure gradients, and then the stabilizations also both require pressure gradients. Therefore, a linear pressure space is required. Please include some additional text on these details, as well as a precise definition of the finite element spaces used in the weak formulation.

We have included the definition of the spaces of the weak form and also the discrete spaces.

In the introduction (page 2, line 54) we have added a few words to first introduce the finite element spaces that will be used afterwards:

*This work presents a methodology to solve the full Biot poroelastic equations at large strains using a finite element mesh with low order elements. The motivation lies in extending the range of geotechnical applications of the Particle Finite Element Method (PFEM) to cover high frequency problems, such as impact loading. PFEM was originally developed to address fluid-structure interaction problems [12,28] and has been later adapted to study soil-structure interaction [14,29]. A cornerstone of PFEM is frequent re-meshing for which low order elements are very advantageous. The elements employed have equal order interpolation for all the nodal basic variables, chosen as the mixture displacement, water relative displacement and water pressure, a formulation with shorthand u − w − p w ,* ***resulting in a P1-P1-P1 element.***

After Equation (16), that describes the weak form, we have added:

*where, in Equations (14), (15) and (16), the solution fields belong to the spaces: u ∈ Su(Ω), w ∈ Sw (Ω) and pw ∈ Spw (Ω), that are defined as:*

*Equation (17)*

*where H1(Ω) is the space of square integrable functions whose gradients are also square integrable, L2(Ω) is the space of square integrable functions whereas H(div, Ω) is the space of square integrable vector functions whose divergence is also square integrable.*

Meanwhile, to better describe the discrete spaces, we have rewritten a paragraph before introducing the interpolants as:

After obtaining the weak form of the balance equations, Eqs. (14), (15) and (16), the semi-discrete equations of the hydromechanical formulation are obtained. In this work, linear and equal shape functions are used for all the field variables: solid displacement, water relative displacement and water pressure. In particular, for two-dimensional problems, three-nodded linear triangles are used that have 5 degrees of freedom at each node. Consequently, the continuous spaces of functions of the solution and test functions, Equation (2.2.1), are discretized into the discrete functional spaces:

Equation (19)

where T(Ω, h) = {u | u ∈ C0 (Ω), u|Ωe ∈ P1 (Ωe) ∀Ωe ∈ Th}, P1 (Ωe) is the space of linear polynomials in Ωe and Th is a partition of Ω made of non-overlapping elements Ωe .

Another paragraph has been introduced to stress that other continuous and discrete spaces of the solution may be employed and its use may be advantageous:

It is worth mentioning that applying the Divergence theorem to the first term of the fluid linear balance equation, Equation (16), the balance equations do not involve any gradient of the water pressure field (Ferronato et al 2010,Castelleto et al 2016). Therefore, it is no longer necessary to have a water pressure field whose gradients are square integrable. By doing so, displacements, Darcy's velocity and pore pressure may be discretized using linear triangles, lowest order Raviart-Thomas and piece-wise constant shape functions respectively, which allows for element-wise mass conservation (Ferronato et al 2010,Castelleto et al 2016). In the undrained limit, this element is unstable and exhibit pore pressure oscillations, that might be solved by employing a quadratic discretization for the skeleton displacement (Loftian & Sivaselvan, 2018).

— In Equation (49) and Appendix B, a new stabilization parameter value is proposed. Its effectiveness should be justified with a comparison to the other stabilization values that have already been proposed for the PPP scheme in the past. I suggest performing a mesh refinement study using the 1D soil column example, for which the authors already have an analytical solution. Please present the computed L2-error as a function of mesh refinement using the proposed stabilization value and previously suggested values. The DME scheme could also be included in this study. This should be done for a timestep less than the critical timestep (where stabilization is required) and one for greater than the critical timestep (where stabilization terms are redundant). See for example, a similar study in Figure 4 of Li and Wei (2018), Stabilized low order finite elements for strongly coupled poromechanical problems, Int. J. Number. Meth. Eng, DOI: 10.1002/nme.5815]. Those authors showed that the White and Borja tau=1/(2G) value seems to provide the best L2-approximation error, but it does not guarantee monotonicity of the solution. We emphasize that monotonicity is a very different property than stability, however. For example, the Taylor-Hood element is stable, but does not necessarily guarantee monotonic solutions. The best approximating solution in the space of admissible FE solutions is often non-monotonic. Other authors have proposed larger stabilization values that introduce additional numerical diffusion to remove undesirable overshoots, at the cost of a higher L2-approximation error. I believe the proposal here would fall in this class. Indeed, monotonicity is one of the requirements stated up front in the derivation of tau in Appendix B.

The proposed value for the stabilization factor should be understood as a rule of thumb to get an estimate of the correct order of magnitude and understand the dependency of the problem parameters (moduli, time-step, element size) on the stabilization factor. In fact, for the simplified displacement-water pressure formulation (neglecting inertial terms), Sun et al (2013) arrived at the same conclusion using a ratter complex rational.

The requirement of monotonicity is complex: in the annex we obtain the value of tau for a simplified formulation where only the water pressure is considered. By using the same rational (one dimensional oedometer), considering the simplified u-pw formulation (without inertial effects), using a completely implicit time marching scheme and discretizing both degrees of freedom with linear shape functions (P1-P1) a different estimate of tau is obtained: tau = 3/M – 12 \*Dt \*k /h^2 [In contraposition to that found for Terzaghy equation (only water pressure degrees of freedom): tau = 2/M – 12 \*Dt \*k /h^2]

In conclusion, it is our opinion that the requirement of monotonicity in the very simplified formulation employed in the estimation of the stabilization factor is not translated into a monotonic solution using the finite elements employed in this work since the stabilization factor depends, among others, on the degrees of freedom, finite element spaces and the dimension of the problem (1-D, 2-D, 3-D).

In order to show the correctness of the stabilization factor proposed herein, we have included, in Annex B, a table comparing the stabilization factor proposed herein with other proposals of the literature. Also, we have included a convergence analysis, where the effect of the element size is studied. The new text, figure and table now read:

*Table B1 presents some of the stabilization factors proposed in the literature for the PPP technique for u − p w elements using a completely implicit time marching scheme, assuming linear elasticity and neglecting dynamic effects. It is worth noting that the factor proposed here has a similar expression to the one found by Sun et al [18] using a more complex technique [61] ; in fact, Sun et al [18] include the term related to tanh to limit the overdiffusion. It is also remarkable the similitude with the stabilization factor proposed by Li & Wei [60] : they do not have the negative term and the inverse of the constrained modulus is multiplied by 3 instead of 2. The stabilization factor proposed by White and Borja [17] depends on the shear modulus instead of the constrained modulus.*

*In order to assess the stabilization technique, the one-dimensional oedometer is computed and the solution is compared with that of Terzaghy. The height is 1 m, the coefficient of consolidation is cv = 1 m2/s, whereas the constrained modulus is M = 100 kPa. The Poisson’s ratio has been fixed at ν = 1∕3; this way, the proposal of White and Borja [17] coincides with that of Sun et al [18] and the one derived in this work once the term k Δt/h2 tends to zero. However, it should be stressed that the proposal of White and Borja [17] would render different results by assuming different Poisson’s ratios.*

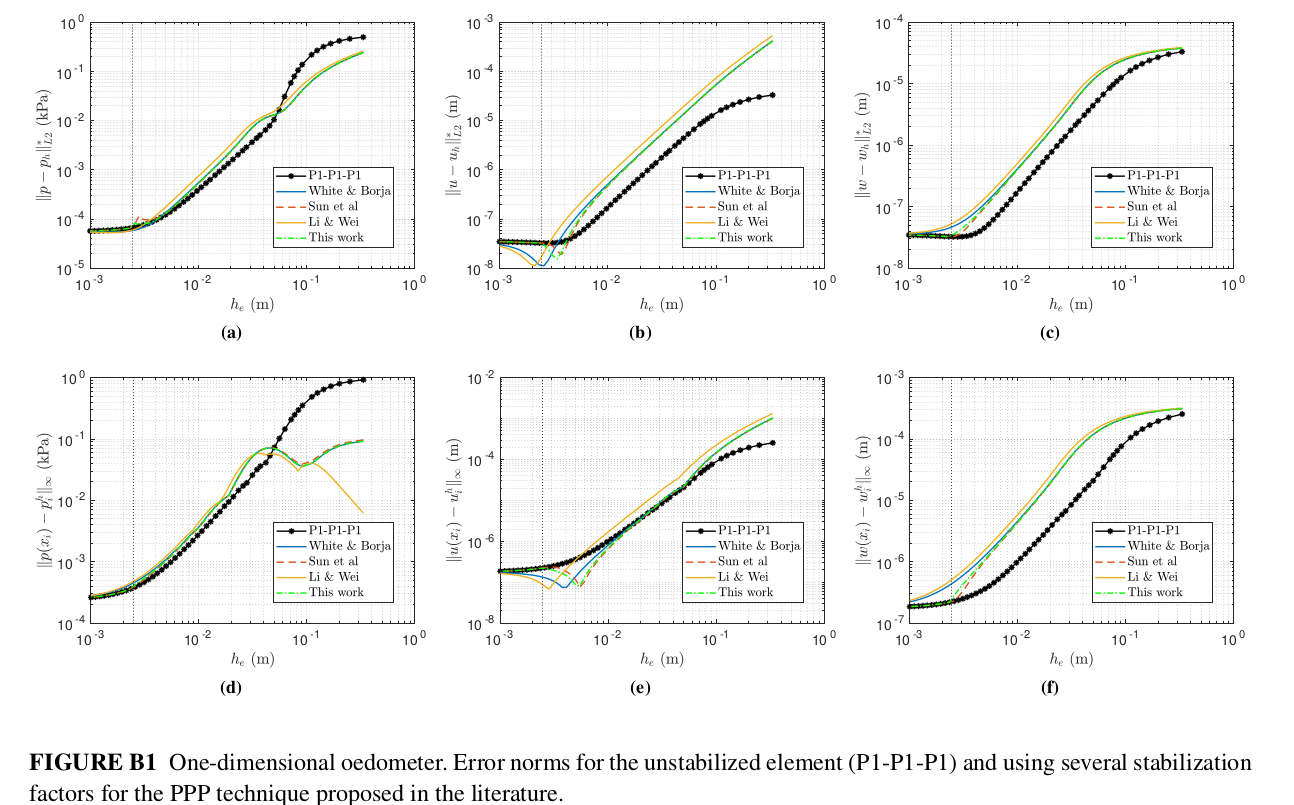
*Figure B1 the results of a convergence analysis, where the solution is computed using 500 time steps up to a final normalized time of T = 1E−3 . Two types of error norms are adopted in this study, namely, the L2 norm*

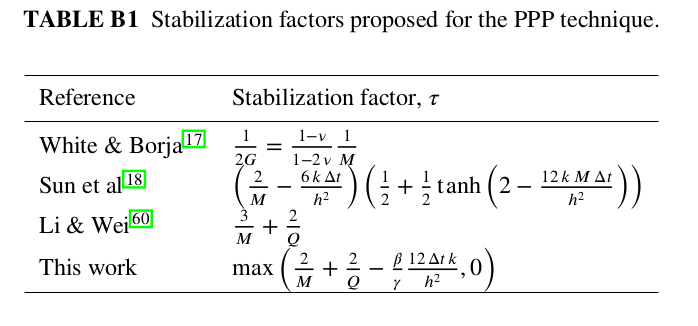
*Equation B43: definition of the L2 Norm*

*and the maximum nodal difference*

*Equation B44: definition of the maximum nodal difference norm.*

*As noted before, all the proposals for the value of the stabilization factor in the literature, see Table B1 and Equation (B42), are quite similar; therefore, the error norms for all variables show a similar trend irrespectively of the employed proposal. It should be noted that, with the exception of the water pressure nodal difference, the stabilization factor proposed by Li and Wei [60], give consistently higher error norms. Additionally, these figures also include -indicated by a discontinuous vertical line- the element size at which the proposed stabilization factor is zero (see Equation (B42)); as such, at the left of this line the solution is computed with a null stabilization factor, whereas on the right the stabilization factor is larger than zero. To deactivate the stabilization technique does not produce any undesirable effect.*





— The statement in Appendix B that the stabilization value 1/(2G) “may introduced excessive smoothing of the water pressure field” should be justified. Taking the limit of dt to zero in Equation (B42) one observes that the stabilization value 1/2G will be smaller than than the one proposed here, and therefore introduce \*less\* diffusion. Perhaps that authors more specifically mean that the White and Borja value is always applied regardless of the regime, whereas the proposed scheme lets it go to zero at late times when no longer needed? It should be noted though that at these late times, the scale of the numerical perturbation is small compared to the physical diffusion, and so the “excessive” smoothing is, in fact, quite negligible. The idea to de-activate the stabilization at late times is intriguing, however, and strikes me as a great idea.

The authors respectfully disagree with the reviewer. Taking dt to zero and assuming an Biot Modulus (*Q*) equal to infinity, the stabilization factor proposed in this work reduces to *tau = 2/M,* whereas the one proposed by White and Borja may be expressed as *tau = (1-nu)/M/(1-2nu).* For Poisson’s coefficients lower than 1/3, the stabilization factor proposed by White and Borja is lower than that derived in this work. Meanwhile, for Poisson’s ratios larger than 1/3 the proposed stabilization factor is lower than that of White and Borja. Not only that, the stabilizaton factor of White and Borja tends to *tau = Infinity,* for a Poisson’s ratio approaching 0.5.

These stabilization factors are assessed here in the solution of the one dimensional solution of an oedometer (H=1, M=1, k=1, increase of water pressure = 1) with 20 nodes, employing displacement-water pressure elements. The final time is T = 0.0001 (normalized time) and the solution is computed with only one time-step. The Figure reports the results in terms of the Poisson’s ratio.

The first thing to notice is that all the norms are independent of the Poisson’s ratio with the exception to that of White and Borja (1/2G). Not only that, the Linfinity norm is much larger than the applied load in cases with a large Poisson’s ratio.

It is our opinion that the dependency of the solution due to the Poisson’s ratio is an undesirable effect, since it is accepted that consolidation processes are governed by the coefficient of consolidation. Thus, by the way that the discrete finite element equations are written, the stabilization factor should depend on the constrained modulus, *M*, and not on the shear modulus, *G*.

In this exercise, the proposed stabilization factor is the one that renders the lowest values of the L2 norm and maximum nodal error on both: displacement and water pressure fields, with the exception of some cases using that proposed by White and Borja, specifically for Poisson’s ratios lower than 0.2.

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FIGURE: Comparison of stabilization factors in the one-dimensional oedometer test. On top: L2 norm divided by the square root of the area, On the bottom: maximum error at nodes. On the Left: water pressure. On the Right: displacement field.

As noted in the previous response, to address this fact in the manuscript, we have added:

*In order to assess the stabilization technique, the one-dimensional oedometer is computed and the solution is compared with that of Terzaghy. The height is 1 m, the coefficient of consolidation is cv = 1 m2/s, whereas the constrained modulus is M = 100 kPa.* ***The Poisson’s ratio has been fixed at ν = 1∕3; this way, the proposal of White and Borja [17] coincides with that of Sun et al [18] and the one derived in this work once the term k Δt/h2 tends to zero. However, it should be stressed that the proposal of White and Borja [17] would render different results by assuming different Poisson’s ratios.***