Edge crossings in random arrangements

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Joint research of

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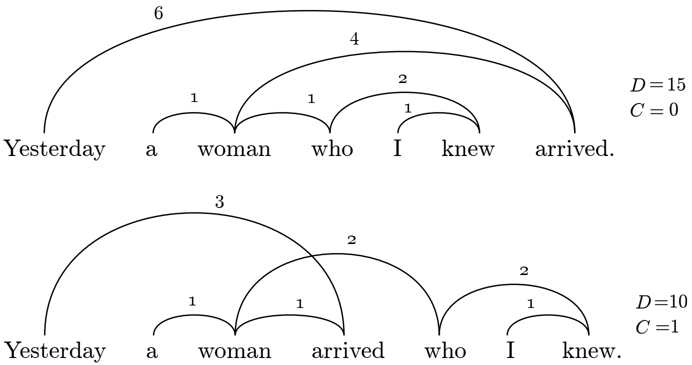
- Alemany-Puig, Ferrer-i-Cancho. Edge crossings in random arrangements. *Journal of Statistical Mechanics*. In press.
- Alemany-Puig, Mora, Ferrer-i-Cancho. Reappraising the distribution of the number of crossings of graphs on a sphere. In prep.
- Alemany-Puig, Ferrer-i-Cancho. Fast calculation of the variance of edge crossings in random linear arrangements. In prep.

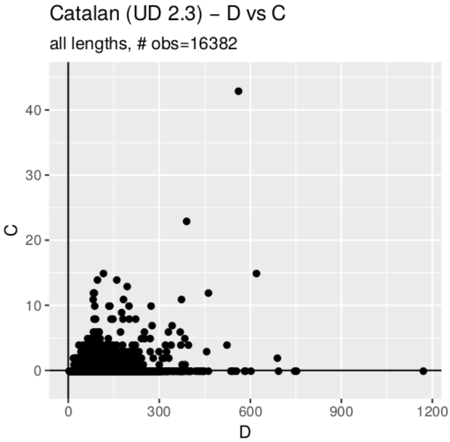
Motivation

- Quantitative Linguistics: D vs C.



D_{min} is the solution to the minimum linear arrangement problem.

- Quantitative Linguistics: D vs C.



$$D_z = \frac{D - \mathbb{E}_{rla}[D]}{\sqrt{\mathbb{V}_{rla}[D]}}$$

$$C_z = \frac{C - \mathbb{E}_{rla}[C]}{\sqrt{\color{red}{\mathbb{V}_{rla}[C]}}}$$

- Computation of $\mathbb{V}_{rla}[C]$ in [Alemany-Puig and Ferrer-i-Cancho, 2019]

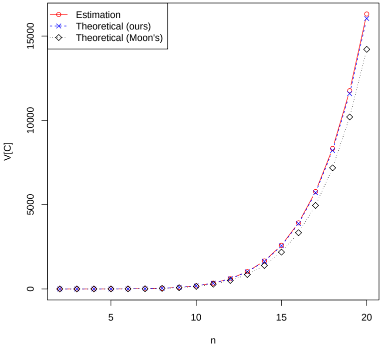
Motivation

- Generalisation to other embeddings - From $\mathbb{V}_{rla}[C]$ to $\mathbb{V}_{*}[C]$

$$\mathbb{V}[C] = \mathbb{E}[(C - \mathbb{E}[C])^2]$$

In [Alemany-Puig and Ferrer-i-Cancho, 2020]

- Correction of previous work [Moon, 1965].



In [Alemany-Puig et al., 2020]

Concepts and notation

$\mathbb{E}_*[C], \mathbb{V}_*[C]$: expectation/variance of C in an **arbitrary layout *** such that

1. Only independent edges can cross,
   edges $\{s,t\} = st \in E, \{u,v\} = uv \in E$ are independent iff $\{s,t\} \cap \{u,v\} = \emptyset$. Then, $\{st, uv\} \in Q$.
2. two edges can cross only once, and
3. our notion of crossings is edge-pairwise defined.
   (e edges crossing at the same point incur in $\binom{e}{2}$ crossings)

Examples of layouts

- $*$: arbitrary layout, $\mathbb{E}_*[C]$, $\mathbb{V}_*[C]$.
- *rla*: random linear arrangement, $\mathbb{E}_{rla}[C]$, $\mathbb{V}_{rla}[C]$.
- *rsa*: random spherical arrangement, $\mathbb{E}_{rsa}[C]$, $\mathbb{V}_{rsa}[C]$.
- *rap*: random arrangement on the plane, $\mathbb{E}_{rap}[C]$, $\mathbb{V}_{rap}[C]$.

Given $\{st, uv\} \in Q$, let $\alpha(st, uv) = 1$ iff edges st and uv cross.

$$C = \sum_{\{st,uv\}\in Q} \alpha(st, uv),$$
$$\mathbb{E}_*\left[C\right] = \sum_{\{st,uv\}\in Q} \mathbb{E}_*\left[\alpha(st, uv)\right]$$
$$= |Q|\delta_*$$

δ_*: probability that two independent edges cross in $*$.
Example: $\delta_{rla} = 1/3$, $\delta_{rsa} = 1/8$ [Moon, 1965].

$\mathbb{V}_*[C]$, the variance of C in any layout
Variance

$$\mathbb{V}_*[C] = \mathbb{E}_*\left[(C - \mathbb{E}_*[C])^2\right]$$

$\mathbb{V}_*[C]$, the variance of C in any layout
Variance

$$\mathbb{V}_*[C] = \mathbb{E}_*\left[(C - \mathbb{E}_*[C])^2\right]$$
$$= \mathbb{E}_*\left[\left(\sum_{\{st,uv\}\in Q} \alpha(st,uv) - |Q|\delta_*\right)^2\right]$$

$\mathbb{V}_*[C]$, the variance of C in any layout

Variance

$$\begin{aligned}
\mathbb{V}_*[C] &= \mathbb{E}_*\left[(C - \mathbb{E}_*[C])^2\right] \\
&= \mathbb{E}_*\left[\left(\sum_{\{st,uv\}\in Q} \alpha(st,uv) - |Q|\delta_*\right)^2\right] \\
&= \sum_{\{st,uv\}\in Q} \sum_{\{wx,yz\}\in Q} \mathbb{E}_*\left[\alpha(st,uv)\alpha(wx,yz) - \delta_*^2\right]
\end{aligned}$$

$\mathbb{V}_*[C]$, the variance of C in any layout
Variance

$$
\begin{aligned}
\mathbb{V}_*[C] &= \mathbb{E}_*\left[(C - \mathbb{E}_*[C])^2\right] \\
&= \mathbb{E}_*\left[\left(\sum_{\{st,uv\}\in Q} \alpha(st,uv) - |Q|\delta_*\right)^2\right] \\
&= \sum_{\{st,uv\}\in Q}\ \sum_{\{wx,yz\}\in Q} \mathbb{E}_*\left[\alpha(st,uv)\alpha(wx,yz) - \delta_*^2\right] \\
&= \sum_{\omega\in\Omega} f_\omega \mathbb{E}_*[\gamma_\omega],
\end{aligned}
$$

$$\mathbb{V}_*[C] = \sum_{\{st,uv\}\in Q} \sum_{\{wx,yz\}\in Q} \mathbb{E}_*\left[\alpha(st,uv)\alpha(wx,yz) - \delta_*^2\right]$$
$$= \sum_{\omega\in\Omega} f_\omega \mathbb{E}_*[\gamma_\omega]$$

The variance is a weighted sum of

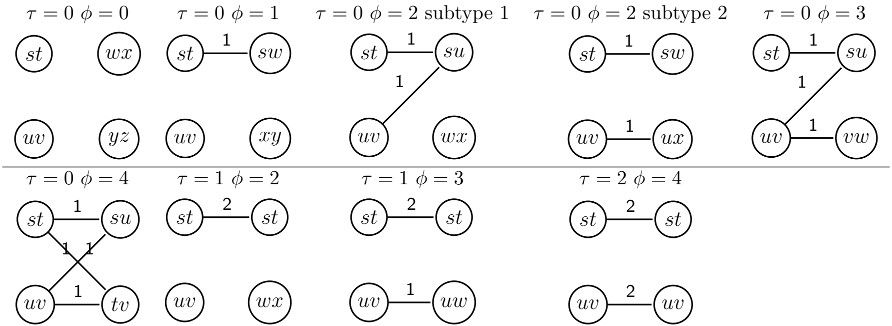
- Graph-dependent terms: f_ω, the amount of products $\alpha(st,uv)\alpha(wx,yz) \longrightarrow 9$ types

$$\Omega = \{00, 01, 021, 022, 03, 04, 12, 13, 24\}$$

- Layout-dependent terms:

$$\mathbb{E}_*[\gamma_\omega] = \mathbb{E}_*[\alpha(st,uv)\alpha(wx,yz)] - \delta_*^2$$
$$= p_{*,\omega} - \delta_*^2$$

Classifying $\alpha(st, uv)\alpha(wx, yz)$



s, t, u, v, w, x, y, z represent different vertices.

Classifying $\alpha(st, uv)\alpha(wx, yz)$

$\omega \in \Omega$	$(\{e_1, e_2\}, \{e_3, e_4\})$	τ	ϕ	$p_{rla,\,\omega}$	$\mathbb{E}_{rla}[\gamma_\omega]$
00	$(\{st, uv\}, \{wx, yz\})$	0	0	1/9	0
24	$(\{st, uv\}, \{st, uv\})$	2	4	1/3	2/9
13	$(\{st, uv\}, \{st, uw\})$	1	3	1/6	1/18
12	$(\{st, uv\}, \{st, wx\})$	1	2	2/15	1/45
04	$(\{st, uv\}, \{su, tv\})$	0	4	0	-1/9
03	$(\{st, uv\}, \{su, vw\})$	0	3	1/12	-1/36
021	$(\{st, uv\}, \{su, wx\})$	0	2	1/10	-1/90
022	$(\{st, uv\}, \{sw, ux\})$	0	2	7/60	1/180
01	$(\{st, uv\}, \{sw, xy\})$	0	1	1/9	0

where

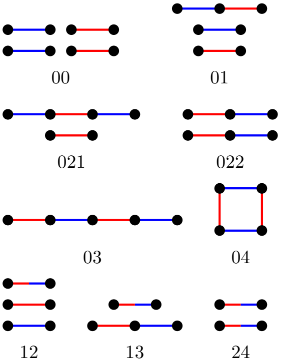
$$p_{*,\,\omega} = \mathbb{P}r\left[\alpha(e_1, e_2)\alpha(e_3, e_4) = 1\right]$$

with $\{e_1, e_2\}, \{e_3, e_4\} \in Q$.

Classifying $\alpha(st, uv)\alpha(wx, yz)$: subgraphs

$\mathbb{P}r\left[\alpha(e_1, e_2)\alpha(e_3, e_4) = 1\right]$

$$f_\omega = a_\omega n_G(F_\omega)$$



$\omega \in \Omega$	a_ω	F_ω
00	6	$\mathcal{L}_2 \oplus \mathcal{L}_2 \oplus \mathcal{L}_2 \oplus \mathcal{L}_2$
24	1	$\mathcal{L}_2 \oplus \mathcal{L}_2$
13	2	$\mathcal{L}_3 \oplus \mathcal{L}_2$
12	6	$\mathcal{L}_2 \oplus \mathcal{L}_2 \oplus \mathcal{L}_2$
04	2	$\mathcal{C}_4$
03	2	$\mathcal{L}_5$
021	2	$\mathcal{L}_4 \oplus \mathcal{L}_2$
022	4	$\mathcal{L}_3 \oplus \mathcal{L}_3$
01	4	$\mathcal{L}_3 \oplus \mathcal{L}_2 \oplus \mathcal{L}_2$

$\mathbb{V}_*[C]$, the variance of C in any layout

Probabilities of the types of products

$p_{*,\,\omega} = \mathbb{P}[\alpha(st, uv)\alpha(wx, yz) = 1]$, for $\alpha(st, uv)\alpha(wx, yz)$ of type ω

$$
\begin{aligned}
p_{*,\,00} &= \delta_*^2, & \mathbb{E}_*[\gamma_{00}] &= 0,\\
p_{*,\,01} &= \delta_*^2, & \mathbb{E}_*[\gamma_{01}] &= 0,\\
p_{*,\,24} &= \delta_*, & \mathbb{E}_*[\gamma_{24}] &= \delta_*(1-\delta_*),
\end{aligned}
$$

$$
\begin{aligned}
p_{rla,\,04} &= 0, & \mathbb{E}_{rla}[\gamma_{04}] &= -\delta_{rla}^2,\\
p_{rap,\,04} &= 0, & \mathbb{E}_{rap}[\gamma_{04}] &= -\delta_{rap}^2,\\
p_{rsa,\,04} &= 0, & \mathbb{E}_{rsa}[\gamma_{04}] &= -\delta_{rsa}^2.
\end{aligned}
$$

$\mathbb{V}_*[C]$, the variance of C in any layout

Variance in types of graphs with known structure

$$\mathbb{V}_*[C] = \sum_{\omega \in \Omega} f_\omega \mathbb{E}_*[\gamma_\omega]$$

Complete graphs
Complete bipartite graphs
Cycle graphs
Path graphs
...

$$\mathbb{V}_* [C] = \sum_{\omega \in \Omega} f_\omega \mathbb{E}_* [\gamma_\omega]$$

Complete graphs

$$\begin{aligned}\mathbb{V}_* [C(\mathcal{K}_n)] = 3\binom{n}{4}((n-4)(n-5)(&\mathbb{E}_* [\gamma_{12}] + 4(\mathbb{E}_* [\gamma_{021}] + \mathbb{E}_* [\gamma_{022}]))\\ &+ 4(n-4)(\mathbb{E}_* [\gamma_{13}] + 2\mathbb{E}_* [\gamma_{03}])\\ &+ 2\mathbb{E}_* [\gamma_{04}] + \mathbb{E}_* [\gamma_{24}])\\ \mathbb{V}_{rla} [C(\mathcal{K}_n)] = 0 &\end{aligned}$$

Complete bipartite graphs
Cycle graphs
Path graphs
...

$$\mathbb{V}_*[C] = \sum_{\omega\in\Omega} f_\omega \mathbb{E}_*[\gamma_\omega]$$

Complete graphs

Complete bipartite graphs

$$\begin{aligned}
\mathbb{V}_*[C(\mathcal{K}_{n_1,n_2})] = \quad & 2(\mathbb{E}_*[\gamma_{24}] + \mathbb{E}_*[\gamma_{04}])\binom{n_1}{2}\binom{n_2}{2} \\
& + 12(\mathbb{E}_*[\gamma_{03}] + \mathbb{E}_*[\gamma_{13}])\left[\binom{n_1}{3}\binom{n_2}{2} + \binom{n_1}{2}\binom{n_2}{3}\right] \\
& + 36(\mathbb{E}_*[\gamma_{12}] + \mathbb{E}_*[\gamma_{022}] + 2\mathbb{E}_*[\gamma_{021}])\binom{n_1}{3}\binom{n_2}{3} \\
& + 24\mathbb{E}_*[\gamma_{022}]\left[\binom{n_1}{2}\binom{n_2}{4} + \binom{n_1}{4}\binom{n_2}{2}\right]
\end{aligned}$$

$$\mathbb{V}_{rla}[C(\mathcal{K}_{n_1,n_2})] = \frac{1}{90}\binom{n_1}{2}\binom{n_2}{2}((n_1+n_2)^2 + n_1 + n_2)$$

Cycle graphs

Path graphs

$\mathbb{V}_*[C]$, the variance of C in any layout

Variance in types of graphs with known structure

$$\mathbb{V}_*[C] = \sum_{\omega\in\Omega} f_\omega \mathbb{E}_*[\gamma_\omega]$$

Complete graphs

Complete bipartite graphs

Cycle graphs

$$\begin{aligned}\mathbb{V}_*[C(\mathcal{C}_n)] &= \frac{1}{2}n(4(n-4)\mathbb{E}_*[\gamma_{13}] + (n-3)\mathbb{E}_*[\gamma_{24}] + 4\mathbb{E}_*[\gamma_{03}] \\ &\quad + (n-5)[2(n-4)\mathbb{E}_*[\gamma_{12}] + 4(\mathbb{E}_*[\gamma_{021}] + \mathbb{E}_*[\gamma_{022}])]) \\ \mathbb{V}_{rla}[C(\mathcal{C}_n)] &= \frac{1}{45}n^3 + \frac{1}{90}n^2 - \frac{1}{3}n\end{aligned}$$

Path graphs

...

$$\mathbb{V}_* [C] = \sum_{\omega \in \Omega} f_\omega \mathbb{E}_* [\gamma_\omega]$$

Complete graphs
Complete bipartite graphs
Cycle graphs
Path graphs

$$\begin{aligned}\mathbb{V}_* [C(\mathcal{L}_n)] = \frac{1}{2}(&4(n-3)(n-4)\mathbb{E}_* [\gamma_{13}] + (n-2)(n-3)\mathbb{E}_* [\gamma_{24}] + 4(n-4)\mathbb{E}_* [\gamma_{03}] \\ &+ (n-4)(n-5)(2(n-3)\mathbb{E}_* [\gamma_{12}] + 4(\mathbb{E}_* [\gamma_{021}] + \mathbb{E}_* [\gamma_{022}])))\\ \mathbb{V}_{rla} [C(\mathcal{L}_n)] = \frac{1}{45}&n^3 - \frac{1}{18}n^2 - \frac{11}{45}n + \frac{2}{3}\end{aligned}$$

...

	Cost	
Algorithm	Time	Space
Naive algorithm	$O(m^4)$	$O(1)$
Fast algorithm (general graphs)	$O(k_{max} n\langle k^2\rangle)$	$O(n)$

where

$$k_{max}: \text{ max degree}, \quad n\langle k^2\rangle = \sum_{u\in V} k_u^2, \quad O(k_{max} n\langle k^2\rangle) = o(nm^2)$$

Computation of $\mathbb{V}_*[C]$

$$
\begin{aligned}
f_{24} &= q \\
f_{13} &= K - 4q - 2n_G(\mathcal{L}_4) \\
f_{12} &= 2[(m+2)q + n_G(\mathcal{L}_4) - K] \\
f_{04} &= 2n_G(\mathcal{C}_4) \\
f_{03} &= \Lambda_1 - 2n_G(\mathcal{L}_4) - 8n_G(\mathcal{C}_4) - 2n_G(Z) \\
f_{021} &= 2q + (m+5)n_G(\mathcal{L}_4) + 8n_G(\mathcal{C}_4) + 3n_G(Z) + \Phi_1 \\
&\qquad - n_G(Y) - \Lambda_1 - \Lambda_2 - K \\
f_{022} &= 4q + 5n_G(\mathcal{L}_4) + 2n_G(Z) + 4n_G(\mathcal{C}_4) + \Phi_2 - \Lambda_2 - 2K - n_G(\mathcal{L}_5)
\end{aligned}
$$

Z: paw graph, $Y = \mathcal{C}_3 \oplus \mathcal{L}_2$

Computation of $\mathbb{V}_*[C]$

$$
\begin{aligned}
\mathbb{V}_*[C] = \quad & q(\mathbb{E}_*[\gamma_{24}] - 4\mathbb{E}_*[\gamma_{13}] + 2(m+2)\mathbb{E}_*[\gamma_{12}] + 2\mathbb{E}_*[\gamma_{021}] + 4\mathbb{E}_*[\gamma_{022}]) \\
& + K(\mathbb{E}_*[\gamma_{13}] - 2\mathbb{E}_*[\gamma_{12}] - \mathbb{E}_*[\gamma_{021}] - 2\mathbb{E}_*[\gamma_{022}]) \\
& + n_G(\mathcal{L}_4)(-2\mathbb{E}_*[\gamma_{13}] + 2\mathbb{E}_*[\gamma_{12}] - 2\mathbb{E}_*[\gamma_{03}] + (m+5)\mathbb{E}_*[\gamma_{021}] + 5\mathbb{E}_*[\gamma_{022}]) \\
& + n_G(\mathcal{C}_4)(2\mathbb{E}_*[\gamma_{04}] - 8\mathbb{E}_*[\gamma_{03}] + 8\mathbb{E}_*[\gamma_{021}] + 4\mathbb{E}_*[\gamma_{022}]) \\
& + n_G(Z)(-2\mathbb{E}_*[\gamma_{03}] + 3\mathbb{E}_*[\gamma_{021}] + 2\mathbb{E}_*[\gamma_{022}]) \\
& - n_G(\mathcal{L}_5)\mathbb{E}_*[\gamma_{022}] - n_G(Y)\mathbb{E}_*[\gamma_{021}] \\
& + \Lambda_1(\mathbb{E}_*[\gamma_{03}] - \mathbb{E}_*[\gamma_{021}]) \\
& - \Lambda_2(\mathbb{E}_*[\gamma_{021}] + \mathbb{E}_*[\gamma_{022}]) \\
& + \Phi_1\mathbb{E}_*[\gamma_{021}] + \Phi_2\mathbb{E}_*[\gamma_{022}]
\end{aligned}
$$

Z: paw graph, $Y = \mathcal{C}_3 \oplus \mathcal{L}_2$

$$q = \sum_{\{st,uv\}\in Q} 1 = |Q| = \frac{1}{2}\left[m(m+1) - \sum_{u\in V} k_u^2\right]$$

$$K = \sum_{\{st,uv\}\in Q} (k_s + k_t + k_u + k_v)$$

$$\Phi_1 = \sum_{\{st,uv\}\in Q} (k_s k_t + k_u k_v)$$

$$\Phi_2 = \sum_{\{st,uv\}\in Q} (k_s + k_t)(k_u + k_v)$$

$$\Lambda_1 = \sum_{\{st,uv\}\in Q} (a_{su}(k_t + k_v) + a_{sv}(k_t + k_u) + a_{tu}(k_s + k_v) + a_{tv}(k_s + k_u))$$

$$\Lambda_2 = \sum_{\{st,uv\}\in Q} (a_{su} + a_{sv} + a_{tu} + a_{tv})(k_s + k_t + k_u + k_v)$$

	Cost	
Algorithm	Time	Space
Naive algorithm	$O\left(m^4\right)$	$O\left(1\right)$
Fast algorithm (general graphs)	$O\left(k_{max} n\langle k^2\rangle\right)$	$O\left(n\right)$
Faster algorithm (general graphs)	$O\left(k_{max} g\right)$	$O\left(n+g\right)$
Forests	$O\left(n\right)$	$O\left(n\right)$

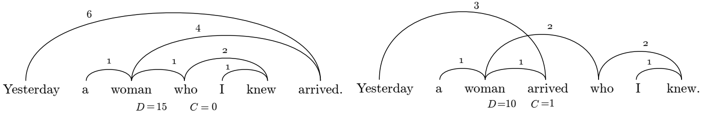
where

$$k_{max} : \text{max degree}, \quad n\langle k^2\rangle = \sum_{u\in V} k_u^2, \quad O\left(k_{max} n\langle k^2\rangle\right) = o\left(nm^2\right),$$

$$g = O\left(m + n_G(\mathcal{L}_3) - n_G(\mathcal{C}_3)\right)$$

Applications

Finding a relationship between D and C



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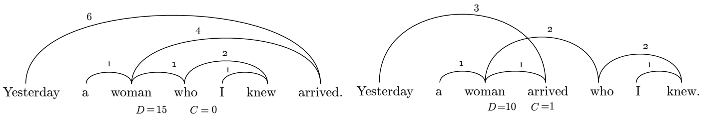
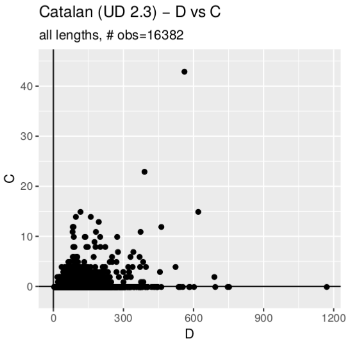
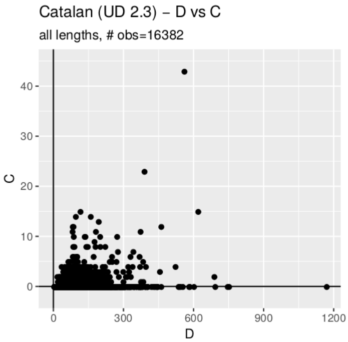




Applications

Finding a relationship between D_z and C_z

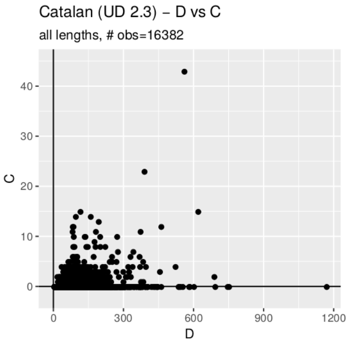
$$D_z = \frac{D - \mathbb{E}_{rla}[D]}{\sqrt{\mathbb{V}_{rla}[D]}}, \qquad C_z = \frac{C - \mathbb{E}_{rla}[C]}{\sqrt{\mathbb{V}_{rla}[C]}}$$

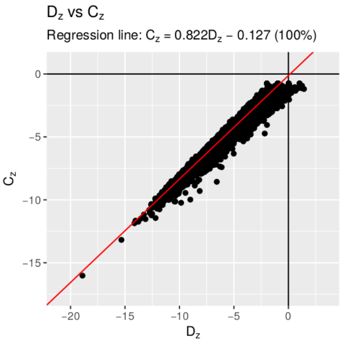


Applications

Finding a relationship between D_z and C_z

$$D_z = \frac{D - \mathbb{E}_{rla}[D]}{\sqrt{\mathbb{V}_{rla}[D]}}, \qquad C_z = \frac{C - \mathbb{E}_{rla}[C]}{\sqrt{\mathbb{V}_{rla}[C]}}$$





Applications

Revising [Moon, 1965]

$$\mathbb{V}_{rsa}\left[C\right] = \sum_{\omega\in\Omega} f_\omega \mathbb{E}_{rsa}\left[\gamma_\omega\right]$$

$$p_{rsa,\,00} = \left(\int_0^\pi \frac{\alpha}{4\pi} f(\alpha) d\alpha\right)^2 = \frac{1}{64}$$

$$p_{rsa,\,12} = \int_0^\pi \left(\frac{\alpha}{4\pi}\right)^2 f(\alpha) d\alpha = \frac{\pi^2 - 8}{64\pi^2}$$

$$p_{rsa,\,021} = 2 \iiint_0^\pi \frac{a - b - c + \pi}{4\pi} \frac{\beta}{4\pi} f(\alpha, \beta, c)\, d\alpha\, d\beta\, dc \approx 0.0126651,$$

$$p_{rsa,\,022} = 4 \int \cdots \int_0^\pi \left(\frac{b - a - c + \pi}{4\pi}\right) \left(\frac{b' - a' - c' + \pi}{4\pi}\right) f(\alpha, \beta, c) d\alpha d\beta d\beta' dc dc'$$
$$\approx 0.01856605$$

Applications

Revising [Moon, 1965]

$$\mathbb{V}_{rsa}[C] = \sum_{\omega\in\Omega} f_\omega \mathbb{E}_{rsa}[\gamma_\omega]$$

$$\mathbb{E}_{rsa}[\gamma_{00}] = 0, \qquad \mathbb{E}_{rsa}[\gamma_{01}] = 0,$$

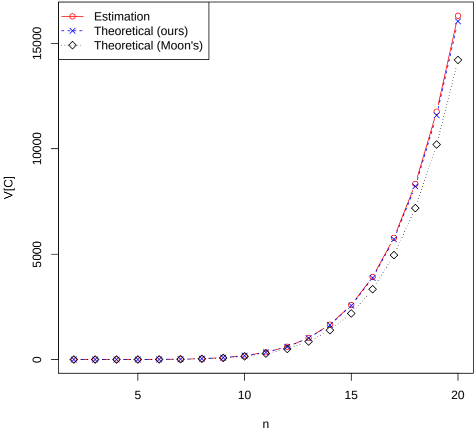
$$\mathbb{E}_{rsa}[\gamma_{24}] = \frac{7}{64}, \qquad \mathbb{E}_{rsa}[\gamma_{04}] = -\frac{1}{64},$$

$$\mathbb{E}_{rsa}[\gamma_{12}] = \frac{\pi^2 - 8}{64\pi^2}, \qquad \mathbb{E}_{rsa}[\gamma_{13}] \approx 0.031265,$$

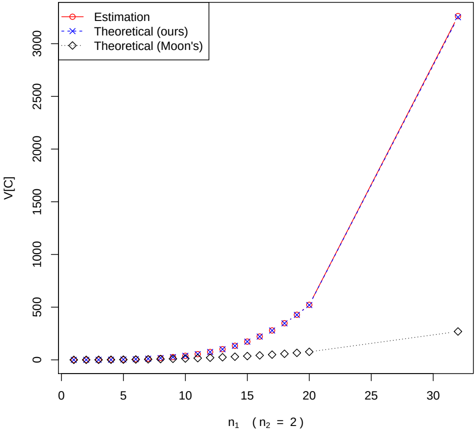
$$\mathbb{E}_{rsa}[\gamma_{03}] \approx -0.0052083334, \qquad \mathbb{E}_{rsa}[\gamma_{021}] \approx -0.0029598521,$$

$$\mathbb{E}_{rsa}[\gamma_{022}] \approx 0.00294105.$$

Complete graphs $\mathcal{K}_n$



Complete bipartite graphs $\mathcal{K}_{n_1,n_2}$ (n_2 fixed).



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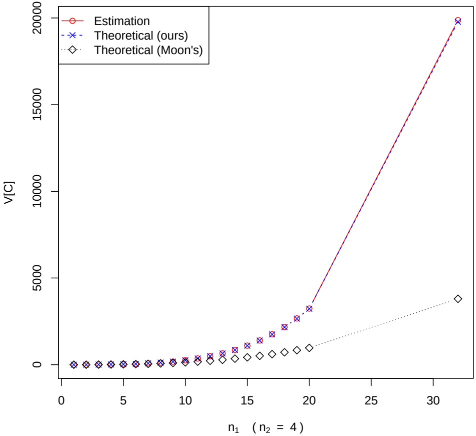

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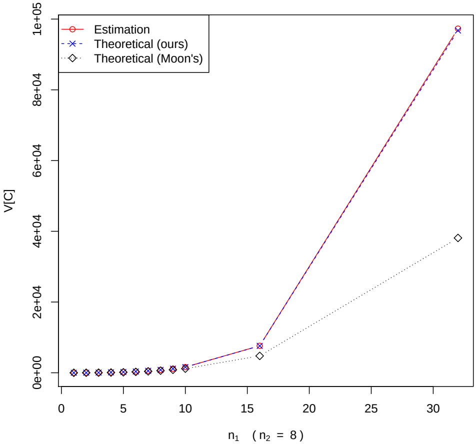
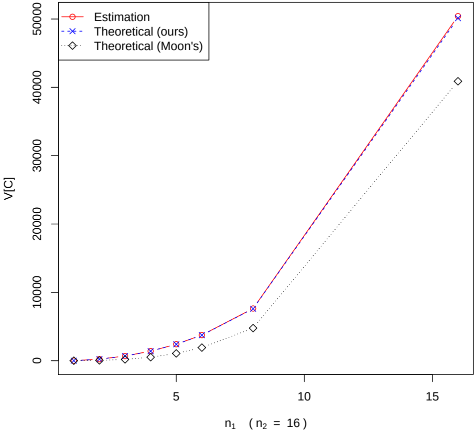


Complete bipartite graphs $\mathcal{K}_{n_1,n_2}$ (n_2 fixed).



Applications

Revising [Moon, 1965]

"in most cases the two variables appearing are independent and hence the expectation of their product equals the product of their individual expectations, i.e., zero" [Moon, 1965]

$$\mathbb{E}_{rsa}[\gamma_{021}] = 0, \qquad \mathbb{E}_{rsa}[\gamma_{03}] = 0,$$

$$\mathbb{E}_{rsa}[\gamma_{022}] = 0, \qquad \mathbb{E}_{rsa}[\gamma_{13}] = \frac{\pi^2 - 8}{64\pi^2} \approx 0.0029598,$$

Our results:

$$\mathbb{E}_{rsa}[\gamma_{021}] \approx -0.0029598521, \qquad \mathbb{E}_{rsa}[\gamma_{03}] \approx -0.0052083334,$$

$$\mathbb{E}_{rsa}[\gamma_{022}] \approx 0.00294105, \qquad \mathbb{E}_{rsa}[\gamma_{13}] \approx 0.0156249999.$$

Conclusions

- Three papers: [Alemany-Puig and Ferrer-i-Cancho, 2020], [Alemany-Puig et al., 2020], [Alemany-Puig and Ferrer-i-Cancho, 2019]
- Algorithms are templates that solve a subgraph counting.
- We have revised Moon's work. Apply this theory to other works?
- Extend this research to road networks.
- New linguistic law?
- Future research: calculate $\mathbb{E}_* [\gamma_\omega]$ in other layouts (calculate δ_*, and $\mathbb{E}_* [\gamma_\omega]$ for $\omega \in \{021, 022, 03, 12, 13\}$)

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The end

Questions?