1. Principles of Classical Mechanics: A Lagrangian Approach

This document explores the transition from Newtonian mechanics to the more sophisticated Lagrangian formulation. We will cover the Principle of Least Action, the Euler-Lagrange equations, and their applications to complex systems like the spherical pendulum and coupled oscillators.

1.1 The Principle of Least Action

In Newtonian mechanics, we typically analyze forces $\vec{F} = m\vec{a}$. However, for systems with constraints, Hamilton's Principle provides a more elegant framework. We define the **Lagrangian** L as the difference between kinetic energy T and potential energy V:

$$L = T - V$$

The action $\mathcal{S}$ is defined as the integral of the Lagrangian over time:

$$\mathcal{S} = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt$$

where q_i are the generalized coordinates. The principle states that the actual path taken by the system is the one that makes the action stationary ($\delta\mathcal{S} = 0$).

1.1.1 Derivation of Euler-Lagrange Equations

To find the path that minimizes the action, we apply the calculus of variations. For a single coordinate q, the variation of the action is:

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) dt = 0$$

By applying integration by parts to the second term and assuming the endpoints are fixed ($\delta q(t_1) = \delta q(t_2) = 0$), we arrive at the fundamental **Euler-Lagrange Equation**:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad (1.1)$$

> ***Exercise 1.1*** *Consider a free particle in 3D space with mass m. The potential $V = 0$. Write the Lagrangian in Cartesian coordinates and show that the Euler-Lagrange equations recover Newton's First Law: $m\ddot{x} = 0, m\ddot{y} = 0, m\ddot{z} = 0$.*

1.2 Central Force Motion and Orbitals

One of the most powerful applications of Lagrangian mechanics is the reduction of the two-body problem to a one-body problem using reduced mass μ.

1.2.1 The Two-Body Lagrangian

Given two masses m_1 and m_2 interacting via a central potential $V(r)$, the Lagrangian in polar coordinates (r, θ) is:

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

where the reduced mass is defined as:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

1.2.2 Conservation Laws

From equation (1.1), we can identify conserved quantities (Noether's Theorem). If the Lagrangian does not depend on a specific coordinate q_i (a "cyclic" coordinate), then the conjugate momentum p_i is conserved:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \ell \quad \text{(Angular Momentum)}$$

The radial equation of motion then becomes:

$$\mu \ddot{r} = -\frac{dV}{dr} + \frac{\ell^2}{\mu r^3} = -\frac{dV_{eff}}{dr}$$

where V_{eff} is the effective potential:

$$V_{eff}(r) = V(r) + \frac{\ell^2}{2\mu r^2} \quad (1.2)$$

Potential Type	Force Law	Expected Orbit
Gravitational	$-GmM/r$	Elliptic/Hyperbolic
Harmonic	$\frac{1}{2}kr^2$	Isotropic Ellipse
Centrifugal	$\ell^2/2\mu r^2$	Repulsive

1.3 Small Oscillations and Coupled Systems

For a system near a stable equilibrium point q_0, we can approximate the potential using a Taylor expansion:

$$V(q) \approx V(q_0) + V'(q_0)(q - q_0) + \frac{1}{2}V''(q_0)(q - q_0)^2$$

Since $V'(q_0) = 0$ at equilibrium, the system behaves like a simple harmonic oscillator.

1.3.1 The Double Pendulum

The double pendulum is a classic example of a system that exhibits chaotic behavior at high energies but behaves linearly at low amplitudes. The Lagrangian for lengths l_1, l_2 and masses m_1, m_2 is:

$$L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

> ***Exercise 1.2*** *Linearize the double pendulum equations for small θ_1, θ_2. Show that the normal mode frequencies ω satisfy the secular equation:* $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ *where $\mathbf{M}$ is the mass matrix and $\mathbf{K}$ is the stiffness matrix.*

1.4 Hamiltonian Mechanics and Phase Space

The Hamiltonian H is the Legendre transform of the Lagrangian, shifting the focus from $(q, \dot{q})$ to (q, p).

$$H(q, p, t) = \sum p_i \dot{q}_i - L$$

The equations of motion are now first-order differential equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (1.3)$$

Comparison Table

Feature	Lagrangian Mechanics	Hamiltonian Mechanics
Variables	$q, \dot{q}$ (Configuration Space)	q, p (Phase Space)
Order	n Second-order ODEs	$2n$ First-order ODEs
Foundation	Energy Extremization	Flow in Phase Space
Geometry	Tangent Bundle	Cotangent Bundle

> ***Remark 1.1*** *In many physical systems, H represents the total energy $T + V$. This is true if the transformation to generalized coordinates is time-independent and the potential is velocity-independent.*

1.4.1 Poisson Brackets

The evolution of any observable $A(q, p)$ can be expressed using Poisson Brackets:

$$\{A, H\} = \sum \left(\frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$$

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}$$

This formulation is the direct precursor to the Heisenberg picture in quantum mechanics, where $\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}]$.

1.5 Dissipative Systems (Extensions)

In the presence of non-conservative forces like friction, we introduce the **Rayleigh Dissipation Function** $\mathcal{R}$:

$$\mathcal{R} = \frac{1}{2} \sum_{i,j} c_{ij} \dot{q}_i \dot{q}_j$$

The modified Euler-Lagrange equation becomes:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \frac{\partial \mathcal{R}}{\partial \dot{q}_i} = 0$$

This allows for the modeling of damped harmonic oscillators and fluid resistance within the same variational framework.