CK 2001
May 20-22, Seoul, Korea

Numerical Analysis of the Instantaneous Motions of Panel-and-Hinge Frameworks and its Application to Computer Vision

LLUÍS ROS AND FEDERICO THOMAS

Institut de Robòtica i Informàtica Industrial (UPC-CSIC)
Gran Capità 2-4, 08034 Barcelona, Spain
email: {llros, fthomas}@iri.upc.es

Abstract: One of the main aims of this paper is to draw the reader's attention to some problems and results from Structural Geometry which can be easily reformulated in terms familiar to the researches in Computational Kinematics, thus establishing connecting threads between both disciplines. In particular, we herein study the problem of finding the spatial interpretation of polyhedral projections using the correspondence of this problem with that of analysing the instantaneous kinematics of panel-and-hinge frameworks. A robust algorithm to solve this problem is presented, based on the computation of the singular value decomposition of the rigidity matrix associated with such frameworks.

1 Introduction

Consider the pictures of impossible objects as those in Figure 1. How can we tell that these line drawings are impossible projections of a polyhedral scene? On the other hand, if we are given a correct projection (Figure 1, bottom-right), how can we reconstruct all possible 3D scenes it represents? Emulating the humans' hability to mentally interpret line drawing projections of 3D scenes has been one of the goals of Computer Vision and Artificial Intelligence since the early seventies. Although many ways to approach these problems have been proposed so far, we here show a kinematic formulation that yields several advantages over the best solutions provided so far [6]. More precisely, we take a *line drawing*—a line diagram made of straight line segments (the edges) and points were two or more segments meet (the vertices)—and focus on the following two problems:

- *Realizability*: decide whether the drawing is the correct projection of some 3-dimensional scene of polyhedral objects.
- *Reconstruction*: if the drawing is correct, obtain all its possible reconstructions, that is, all polyhedral scenes that project onto it. Since a correct drawing can be generated from an infinite number of scenes, we will ask for a parameterization of all these possibilities.

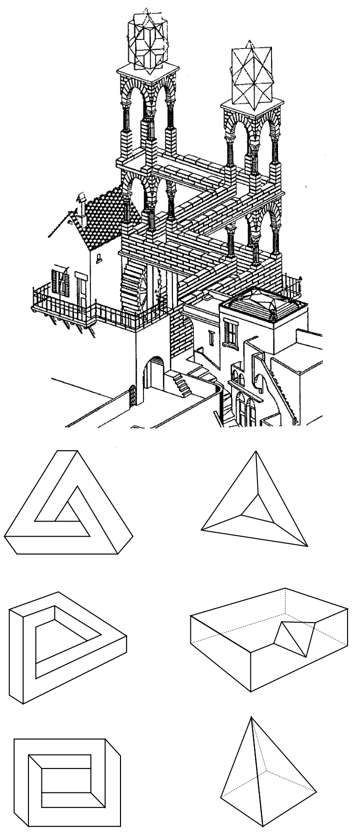

Figure 1: Top: The impossible waterfall by Escher. Bottom-left: incorrect line drawings. Bottom-right: correct line drawings.

A kinematic formulation allows solving these two problems in a unified way. There is a bijective mapping between the reconstructions of a line drawing and the instantaneous motions of an associated articulated mechanism: a panel-and-hinge framework with a panel for every face of the drawing and a hinge for every edge. This was discovered by W. Whiteley, who also used it to prove the converse of a theorem by J. C. Maxwell on the force patterns that keep a bar-and-joint structure in equilibrium [11]. We will see that this mapping permits a concise and numerically feasible solution for the realizability problem; namely, a drawing will be a correct projection if and only if the kernel of a related matrix contains a vector with non-zero components. Despite its importance, we believe that the correspondence between reconstructions and instantaneous motions has never been exploited by the Machine Vision community investigating the problem, probably because it came to light in the context of Structural Geometry [9, 10] and Rigidity Theory [3], a usually unnoticed source of results for this community. We develop further on this mapping and see that it allows us to define a linear parameterization of all reconstructions of a correct drawing, thus solving the reconstruction problem in a simple way.

The paper is organized as follows. Section 2 introduces some necessary background and assumptions. Section 3 studies the instantaneous kinematics of panel-and-hinge frameworks to later exploit its correspondence with the realizability of line drawings. Section 4 explains this correspondence and uses it to devise an algorithm that solves the realizability and reconstruction problems. This algorithm is applied to an example in Section 5 and how the test can be implemented in floating-point-arithmetic is detailed in Section 6. Advantages over previously existing approaches and points for further research are finally highlighted in Section 7.

2 Background

We assume a line drawing is depicting the projection of a *polyhedral surface* (or *polysurface* for short), a piecewise linear and continuous 2-manifold, made up with planar polygons, glued in pairs along their edges. To simplify, we will deal with projections of *a single* polysurface onto the XY plane. This is not too restrictive since [5] explains how to extend the results to line drawings depicting more complicated scenes, with several objects and possible occlusions between them. The *vertices*, *edges* and *faces* of a line drawing directly correspond to their spatial counterparts on the polysurface.

We say that a line drawing is *correct*, or *reconstructable*, if there exists a polysurface that projects onto it. Such a polysurface is called a *reconstruction* or *lifting* of the line drawing. A lifting is called *sharp* if adjacent faces have non-coplanar planes.

We also assume that a drawing $\mathcal{D}$ is given along with its *incidence structure*. The incidence structure tells the combinatorial structure of the spatial reconstructions of $\mathcal{D}$ —basically, which vertices will be incident to which faces. Formally, it is a triple $I = (V, F, R)$, where V is the set of vertices of $\mathcal{D}$ and F is the set of its faces. We put a face in F for every subset of vertices that must be kept coplanar in the reconstruction. $R \subseteq V \times F$ is the *incidence set*: there is an *incidence pair* (v, f) in R if vertex v must lie on face f in 3-space. The incidence structure can be computed by applying the method in [6, page 45], after a labeling its edges using standard techniques like those in [4].

Vector quantities relative to the instantaneous kinematics of rigid bodies will be herein represented and manipulated using Grassmann-Cayley algebra, a formalism that reflects the projective nature of instantaneous kinematic properties. See [9, 7] for an introduction to this algebra and its application to Kinematics. The basic objects of this algebra are the so-called extensors. An *n-extensor* is an expression of the form $\mathbf{p}_1 \vee \mathbf{p}_2 \vee \cdots \vee \mathbf{p}_n$, with $n \leq 3$, where the $\mathbf{p}_i$'s are points of projective 3-space, each corresponding to a point p_i in Euclidean 3-space (possibly at infinity). Assuming that $\underline{\mathbf{P}}$ is the $n \times 4$ matrix whose rows are the homogeneous coordinates of the points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, then $\mathbf{p}_1 \vee \mathbf{p}_2 \vee \cdots \vee \mathbf{p}_n$ is defined as the $\binom{4}{n}$-tuple of all $n \times n$ minors of $\underline{\mathbf{P}}$, listed in a pre-specified order. One can easily see that the 2-extensor of two points $\mathbf{p_1} \vee \mathbf{p_2}$ is a 6-tuple of Plücker coordinates for the line through p_1 and p_2. Actually, this 6-tuple uniquely defines an oriented line segment of fixed length on this line, also termed a *line-bound vector* in some texts. Analogously, the 3-extensor of three points $\mathbf{p_1} \vee \mathbf{p_2} \vee \mathbf{p_3}$ is a 4-tuple of Plücker coordinates for the plane through p_1, p_2 and p_3, defining any patch on this plane, with a fixed area and orientation.

Now, we can use these definitions to represent the instantaneous motions of a rigid body undergoing a rotation with angular velocity ω about its instantaneous rotation axis. We call a *rotor* the 2-extensor $\mathbf{S} = \mathbf{q} \vee \mathbf{r}$, where $\mathbf{q}$ and $\mathbf{r}$ correspond to two points, q and r, located on the instantaneous rotation axis such that $||q - r||$ equals the angular velocity ω. Note that the rotor's elements are the Plücker coordinates of the instantaneous rotation axis, and the line-bound vector it represents has a length equal to ω. The angular velocity can be made explicit on a rotor by choosing two points q' and r' separated by a unitary distance, and writing $\mathbf{S} = \omega \cdot \mathbf{q}' \vee \mathbf{r}'$ instead of $\mathbf{S} = \mathbf{q} \vee \mathbf{r}$.

Moreover, we represent the linear velocity $v_p = (v_x, v_y, v_z)$ that this rotation induces on a point $p = (p_x, p_y, p_z)$ of the body, with the 3-extensor $\mathbf{S} \vee \mathbf{p} = \mathbf{q} \vee \mathbf{r} \vee \mathbf{p}$. It can be shown that the first three elements of the 4-tuple $\mathbf{S} \vee \mathbf{p}$ are v_x, v_y and v_z, and that the fourth is the dot product $v_p \cdot p$, using a proper order of the minors involved in $\mathbf{S} \vee \mathbf{p}$ [11].

3 Panel-and-Hinge Frameworks

A *panel-and-hinge framework* is a pair $\mathcal{F}^{ph} = (P, H)$, where P is a collection of planar polygonal *panels* $\{P_1, P_2, \ldots, P_n\}$, all rigid, and H is a collection of *hinges* $\{\ldots, \mathbf{H}_{i,j}, \ldots\}$, each articulating a pair of panels. $\mathbf{H}_{i,j}$ denotes the hinge articulating P_i and P_j. We represent a hinge by a 2-extensor of its supporting line, that is, $\mathbf{H}_{i,j} = \mathbf{a} \vee \mathbf{b}$, where a and b are two points on the rotation axis of the hinge. It will be advantageous to redundantly represent each hinge and we will include both $\mathbf{H}_{i,j} = \mathbf{a} \vee \mathbf{b}$ and $\mathbf{H}_{j,i} = \mathbf{b} \vee \mathbf{a}$ in H. Note that $\mathbf{H}_{i,j} = -\mathbf{H}_{j,i}$.

A *path* between two panels of $\mathcal{F}^{ph}$, say P_{i_1} and P_{i_n}, is an alternate sequence of panels and hinges $\{P_{i_1}, \mathbf{H}_{i_1,i_2}, P_{i_2}, \mathbf{H}_{i_2,i_3}, \ldots, P_{i_{n-1}}, \mathbf{H}_{i_{n-1},i_n}, P_{i_n}\}$, such that no panel is repeated, and every panel P_{i_k} is adjacent to the next one $P_{i_{k+1}}$ through the hinge $\mathbf{H}_{i_k,i_{k+1}}$. A *cycle* of $\mathcal{F}^{ph}$ is a closed path that begins and ends in the same panel, i.e., $P_{i_1} = P_{i_n}$ (Figure 2).

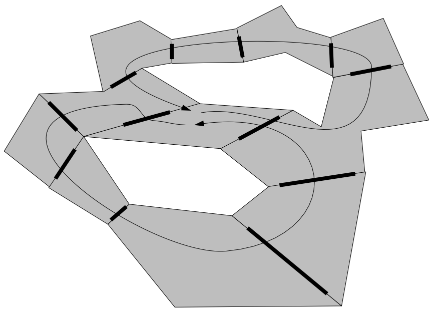

Figure 2: A panel-and-hinge framework with two cycles. Hinges are indicated as thick black segments.

As in any mechanism, a usual question is to ask what are the possible instantaneous motions undergone by the panels of $\mathcal{F}^{ph}$ or, more precisely, which are the possible rotors $\mathbf{S}_{i,0}$ that every panel P_i can have, with respect to an absolute reference frame, here labelled as frame 0.

Assume P_j is instantaneously rotating with reference to P_i with angular velocity $\omega_{i,j}$. Also, let $\mathbf{S}_{i,0}$ and $\mathbf{S}_{j,0}$ be the rotors of P_i and P_j, respectively, with reference to the absolute frame. Then, the absolute velocity of a point p on P_i can be expressed as the velocity of p relative to panel P_j plus the velocity of p when rigidly linked to P_j, i.e.,

$$\mathbf{S}_{i,0} \vee \mathbf{p} = \omega_{i,j} \cdot \mathbf{H}_{i,j} \vee \mathbf{p} + \mathbf{S}_{j,0} \vee \mathbf{p},$$

which, using the distributy law for extensors [7], yields to

$$\mathbf{S}_{i,0} - \mathbf{S}_{j,0} = \omega_{i,j} \cdot \mathbf{H}_{i,j}. \tag{1}$$

This means that the three rotors involved in a mechanism of two hinged panels cannot be in an arbitrary position, since they must satisfy this linear dependence. One can see that this holds whenever the lines of the three rotors are coplanar and copunctual. This result motivates the following definitions.

An *instantaneous motion* for $\mathcal{F}^{ph}$ is an assignment of a rotor $\mathbf{S}_{i,0}$ to each panel $P_i \in P$ such that for each $\mathbf{H}_{i,j} \in H$, $\mathbf{S}_{i,0} - \mathbf{S}_{j,0} = \omega_{i,j}\mathbf{H}_{i,j}$ for some choice of scalars $\omega_{i,j}$. Such an assignment induces velocities on the points of the panels that are actually satisfying the kinematic constraints imposed by the hinges. In other words, the fact that $\mathbf{S}_{i,0} - \mathbf{S}_{j,0} = \omega_{i,j}\mathbf{H}_{i,j}$ guarantees that the induced velocities on any point p of the hinge $\mathbf{H}_{i,j}$ will be the same either if we compute them as $\mathbf{S}_{i,0} \vee \mathbf{p}$, or as $\mathbf{S}_{j,0} \vee \mathbf{p}$. We note that, since

$$\omega_{j,i}\mathbf{H}_{j,i} = \mathbf{S}_{j,0} - \mathbf{S}_{i,0} = -\omega_{i,j}\mathbf{H}_{i,j} = \omega_{i,j}\mathbf{H}_{j,i},$$

then $\omega_{i,j} = \omega_{j,i}$ and there is a single scalar assigned to each hinge.

A *motion assignment* for $\mathcal{F}^{ph}$ is an assignment of a scalar $\omega_{i,j}$ to each hinge $\mathbf{H}_{i,j} \in H$, such that $\omega_{i,j} = \omega_{j,i}$ and $\sum_{\mathbf{H}_{i,j} \in C} \omega_{i,j} \cdot \mathbf{H}_{i,j} = 0$ for every cycle C of panels and hinges in $\mathcal{F}^{ph}$. If every hinge is a 2-extensor of two points on the axis, separated by a unitary distance, then $\omega_{i,j}$ can be interpreted as the angular velocity between P_i and P_j, and the following theorem guarantees that a motion assignment is equivalent to an instantaneous motion for $\mathcal{F}^{ph}$.

Theorem 3.1. *For a panel-and-hinge framework $\mathcal{F}^{ph} = (P, H)$ with a selected panel P_0 designated as the absolute reference frame, there exists a one-to-one correspondence between instantaneous motions relative to this panel and motion assignments.*

Proof. If we are given an instantaneous motion, then the scalars $\omega_{i,j}$ satisfying the equations $\mathbf{S}_{i,0} - \mathbf{S}_{j,0} = \omega_{i,j}\mathbf{H}_{i,j}$ already define a motion assignment. Certainly, if we consider any cycle C of panels and hinges, write down the equations $\mathbf{S}_{i,0} - \mathbf{S}_{j,0} = \omega_{i,j}\mathbf{H}_{i,j}$ for all hinges in C, and sum them all, we conclude that $\sum \omega_{i,j} \cdot \mathbf{H}_{i,j} = 0$. This gives one half of the correspondence.

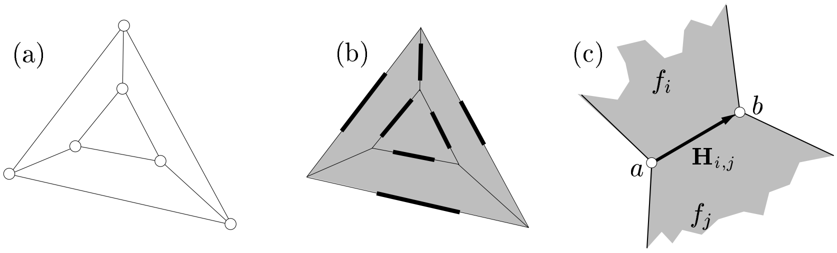

Figure 3: A line drawing (a) and its associated framework (b). (c) Orientation of the hinges.

If we are given a motion assignment, we can define a corresponding instantaneous motion as follows. We let $\mathbf{S}_{0,0} = 0$, as the panel P_0 is not moving with respect to itself. Then, to compute $\mathbf{S}_{i,0}$ for any other panel P_i we select an arbitrary path T of panels and hinges from P_0 to P_i and let:

$$\mathbf{S}_{i,0} = -\sum_{\mathbf{H}_{i,j} \in T} \omega_{i,j}\mathbf{H}_{i,j}, \tag{2}$$

where the sum is along all hinges of T. We note that the value of $\mathbf{S}_{i,0}$ is independent from the chosen path T, because any two different paths, say T_1 and T_2, from P_0 to P_i will form a closed cycle C on which the sum is zero. That is, taking into account that a hinge $\mathbf{H}_{i,j}$ in T_2 corresponds to the hinge $\mathbf{H}_{j,i}$ of C we can write

$$0 = \sum_{\mathbf{H}_{i,j} \in C} \omega_{i,j}\mathbf{H}_{i,j} = \sum_{\mathbf{H}_{i,j} \in T_1} \omega_{i,j}\mathbf{H}_{i,j} - \sum_{\mathbf{H}_{i,j} \in T_2} \omega_{i,j}\mathbf{H}_{i,j},$$

and hence,

$$\sum_{\mathbf{H}_{i,j} \in T_1} \omega_{i,j}\mathbf{H}_{i,j} = \sum_{\mathbf{H}_{i,j} \in T_2} \omega_{i,j}\mathbf{H}_{i,j},$$

which completes the other half of the correspondence. □

4 A Kinematic Test of Realizability

If $\mathcal{D}$ is a line drawing with incidence structure $S = (V, F, I)$, we can associate $\mathcal{D}$ with a panel-and-hinge framework $\mathcal{F}^{ph}(\mathcal{D})$ by putting a panel for each face $f \in F$, and a hinge articulating two panels if their associated faces are adjacent in $\mathcal{D}$. Figure 3 shows a projected truncated tetrahedron (a) and its associated framework (b). It will be helpful to give a common orientation to the hinges as follows. Assuming that the incidence structure has the topology of an orientable surface, we can orient the faces in F, designating an *outer* and *inner* side for each of them. Let the outer side be the one facing an observer placed at the center of projection. Then, for the hinge $\mathbf{H}_{i,j} = a \vee b$, we select a and b so that the vector $b - a$ turned 90 degrees clockwise points from face f_i to face f_j. See Figure 3c.

The following theorem reveals one sense of the anounced relationship between the liftings of $\mathcal{D}$ and the motion assignments of $\mathcal{F}^{ph}(\mathcal{D})$.

Theorem 4.1 (Whiteley 1982). *If $\mathcal{D}$ is the correct projection of a polysurface $\mathcal{P}$ without vertical faces, then $\mathcal{F}^{ph}(\mathcal{D})$ has a motion assignment that assigns a non-null angular velocity $\omega_{i,j}$ to every pair of adjacent panels whose corresponding faces of $\mathcal{P}$ lie on different planes.*

Proof. Let π denote the plane where $\mathcal{D}$ lies. An instantaneous motion can be found as follows. First, we take every panel P_i of $\mathcal{F}^{ph}(\mathcal{D})$ and assign a velocity vector to every point p of P_i: the vector from p to the lifted position of p on the polysurface $\mathcal{P}$ in 3-space (see Figure 4). Since all these velocities are orthogonal to π, the instantaneous rotation axis of P_i must be a straight line on π. Moreover, let β be the plane of the lifted face of P_i. Actually, the velocities of the points on P_i, as defined, are all proportional to the distance between p and the line r of intersection of π with β. Thus, r is the instantaneous rotation axis of P_i. Let $\mathbf{S}_{i,0}$ be a 2-extensor representing r. According to Equation 1, any hinge $\mathbf{H}_{i,j}$ between two adjacent panels P_j and P_i, satisfies $\mathbf{S}_{i,0} - \mathbf{S}_{j,0} = \omega_{i,j}\mathbf{H}_{i,j}$ for some scalar $\omega_{i,j}$, since the three lines of $\mathbf{S}_{i,0}$, $\mathbf{S}_{j,0}$ and $\mathbf{H}_{i,j}$ are concurrent to a point. This means that, as defined, the assigned rotors $\mathbf{S}_{i,0}$ are an instantaneous motion of $\mathcal{F}^{ph}(\mathcal{D})$. By Theorem 3.1 the scalars $\omega_{i,j}$ directly give a motion assignment.

If f_i and f_j are two adjacent faces of $\mathcal{P}$, and β_i, β_j are their respective planes, we observe that:

- If $\beta_i = \beta_j$, then the lines of $\mathbf{S}_{i,0}$ and $\mathbf{S}_{j,0}$ coincide, meaning that the only scalar $\omega_{i,j}$ that satisfies Equation 1 is zero.
- If $\beta_i \neq \beta_j$, then the lines of $\mathbf{S}_{i,0}$ and $\mathbf{S}_{j,0}$ are different and $\mathbf{S}_{i,0} - \mathbf{S}_{j,0}$ is non-null in Equation 1, meaning that $\omega_{i,j} \neq 0$,

which proves the theorem. □

The converse of this theorem is also true:

Theorem 4.2 (Whiteley 1982). *For a line drawing $\mathcal{D}$ with the incidence structure of a polysurface, and its associated panel-and-hinge framework $\mathcal{F}^{ph}(\mathcal{D})$, if $\mathcal{F}^{ph}(\mathcal{D})$ has a motion assignment, then $\mathcal{D}$ is a correct projection of a polysurface $\mathcal{P}$, with different planes for adjacent faces on each edge where $\omega_{i,j} \neq 0$.*

Proof. We verify the theorem by constructing a lifted polysurface $\mathcal{P}$ of $\mathcal{D}$. If $\mathcal{F}^{ph}(\mathcal{D})$ has a motion assignment, then there are scalars $\omega_{i,j}$ assigned to the hinges of $\mathcal{F}^{ph}(\mathcal{D})$ such that $\sum \omega_{i,j} \cdot \mathbf{H}_{i,j} = 0$ for every cycle of panels and hinges in $\mathcal{F}^{ph}(\mathcal{D})$. Thus, by Theorem 3.1 we can find an instantaneous motion of $\mathcal{F}^{ph}(\mathcal{D})$ that assigns a rotor $\mathbf{S}_{i,0}$ to every panel P_i. These axes will all lie on the plane π of the drawing, since their 2-extensors are all linear combinations of the 2-extensors $\mathbf{H}_{i,j}$ of the hinges. Hence, every vertex of the framework is instantaneously moving with a velocity vector that is orthogonal to π. The tips of these vectors are now taken as the liftings of the corresponding vertices of $\mathcal{D}$. As defined, for each face of $\mathcal{D}$ its lifted vertices will all be coplanar, since their height is proportional to the distance between the vertex and the corresponding instantaneous rotation axis. Also, along a hinge $\mathbf{H}_{i,j}$ articulating two panels P_i and P_j the velocity vectors induced by $\mathbf{S}_{i,0}$ and $\mathbf{S}_{j,0}$ will be exactly the same, meaning that the lifted faces for P_i and P_j coincide along their common edge. We conclude that the tips of the velocity vectors provide a lifting of $\mathcal{D}$.

The defined lifting will be sharp if every two adjacent panels, say P_i and P_j, receive non-coincident planes. Since $\mathbf{S}_{i,0} - \mathbf{S}_{j,0} = \omega_{i,j}\mathbf{H}_{i,j}$, this happens when $\omega_{i,j} \neq 0$. □

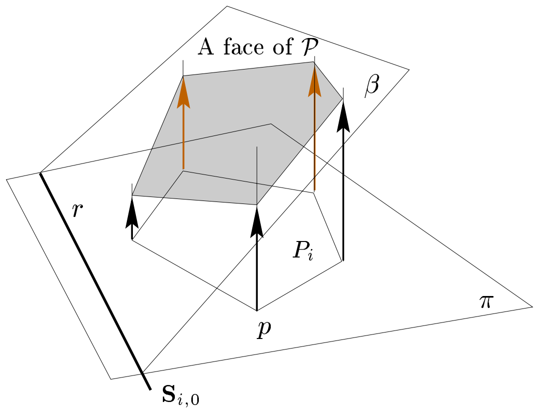

Figure 4: A motion assignment on $\mathcal{F}^{ph}(\mathcal{D})$.

Now, we can gather all scalars $\omega_{i,j}$ for all edges of $\mathcal{D}$ into a tuple $\omega = \{\ldots, \omega_{i,j}, \ldots\} \in \mathbb{R}^e$, where e is the number of edges of the drawing. If we collect all vector equations $\sum \omega_{i,j} \cdot \mathbf{H}_{i,j} = 0$ for all cycles of $\mathcal{F}^{ph}(\mathcal{D})$, we can express them in matrix form as

$$\underline{\mathbf{R}}_{\mathcal{D}} \cdot \omega = 0, \tag{3}$$

where $\underline{\mathbf{R}}_{\mathcal{D}}$ is called the *rigidity matrix* and contains as many rows as cycles in $\mathcal{F}^{ph}(\mathcal{D})$, and as many columns as there are hinges in $\mathcal{F}^{ph}(\mathcal{D})$. Let $Ker(\underline{\mathbf{R}}_{\mathcal{D}})$ denote the kernel of $\underline{\mathbf{R}}_{\mathcal{D}}$. With these definitions and Theorems 4.1 and 4.2 we finally have a set of necessary and sufficient conditions for $\mathcal{D}$ to be realizable.

Theorem 4.3 (Realizability of $\mathcal{D}$). *A line drawing $\mathcal{D}$ with the incidence structure of a polysurface has a sharp lifting if, and only if, there is some vector $\omega \in Ker(\underline{\mathbf{R}}_{\mathcal{D}})$ all of whose coordinates are different from zero.*

Proof. ($\Rightarrow$) If $\mathcal{D}$ is realizable, there exists a sharp lifting $\mathcal{P}$ with all pairs of adjacent faces lying on different planes. By Theorem 4.1 $\mathcal{P}$ defines a motion assignment with non-null scalars $\omega_{i,j}$ on all hinges of $\mathcal{F}^{ph}(\mathcal{D})$, and these scalars are the required non-null coordinates of a vector ω of $Ker(\underline{\mathbf{R}}_{\mathcal{D}})$.

($\Leftarrow$) If there is a vector $\omega \in Ker(\underline{\mathbf{R}}_{\mathcal{D}})$ with all coordinates $\omega_{i,j}$ different from zero, by Theorem 4.2 every pair of adjacent faces will lie on different planes of the spatial lifting. □

This theorem is of central importance, as it directly solves the realizability problem in an efficient way. Indeed, one can check that there exists a vector $\omega \in Ker(\underline{\mathbf{R}}_{\mathcal{D}})$ with all its coordinates different from zero by first computing a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ of $Ker(\underline{\mathbf{R}}_{\mathcal{D}})$, arranging these vectors as columns of a matrix and then verifying that there are no null rows in it. An important corollary of the previous results which directly solves the reconstruction problem, is the following.

Corollary 4.1 (Reconstruction of $\mathcal{D}$). *There is a one-to-one correspondence between vectors $\omega \in Ker(\underline{\mathbf{R}}_{\mathcal{D}})$ and the liftings of $\mathcal{D}$.*

Proof. By Theorem 3.1 there is a one-to-one correspondence between a vector ω of motion assignments and instantaneous motions $\mathbf{S}_{i,0}$ of $\mathcal{F}^{ph}(\mathcal{D})$. Different rotors $\mathbf{S}_{i,0}$ induce different velocity vectors on the panel P_i and hence different liftings of its vertices. □

Consequently, we need only to sweep $Ker(\underline{\mathbf{R}}_{\mathcal{D}})$ to generate all liftings of $\mathcal{D}$: from a motion assignment we compute its instantaneous motion, which gives a rotor $\mathbf{S}_{i,0}$ for each panel P_i. The extensor $\mathbf{S}_{i,0} \vee \mathbf{p}$ gives the velocity of a point p of this panel. This velocity will be orthogonal to the plane of the drawing, and its Z-coordinate will provide the height of p in the lifting. If the starting motion assignment is written as a linear combination of a basis of $Ker(\underline{\mathbf{R}}_{\mathcal{D}})$, then this process provides a linear parameterization of all liftings of $\mathcal{D}$.

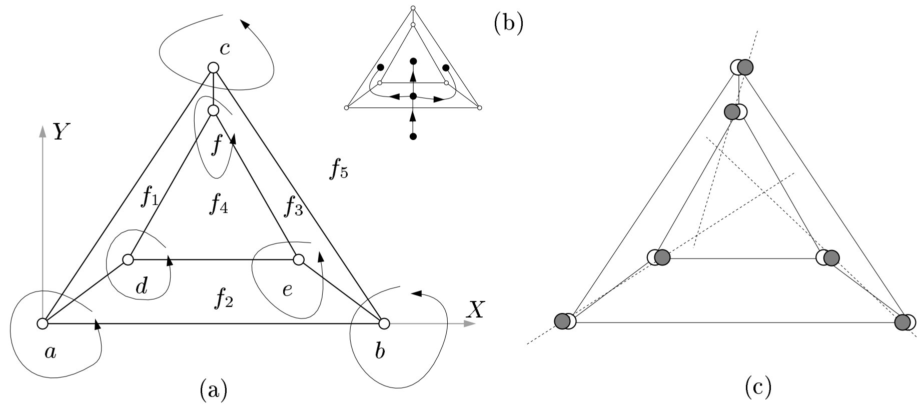

Figure 5: Testing a truncated tetrahedron. (a) Cycles involved. (b) Paths to compute the instantaneous motions. (c) A perturbation that slightly moves all vertices takes the drawing out of this realizable configuration. The proximity to a correct drawing, though, can be detected with the singular value decomposition of $\underline{\mathbf{R}}_{\mathcal{D}}$.

5 Example

Let us follow the process above on an example, the drawing of a truncated tetrahedron in Figure 5. The coordinates of its vertices are $\mathbf{a} = (0,\ 0,\ 0,\ 1)$, $\mathbf{b} = (16,\ 0,\ 0,\ 1)$, $\mathbf{c} = (8,\ 12,\ 0,\ 1)$, $\mathbf{d} = (4,\ 3,\ 0,\ 1)$, $\mathbf{e} = (12,\ 3,\ 0,\ 1)$ and $\mathbf{f} = (8,\ 10,\ 0,\ 1)$, and the 2-extensors of the hinges $\mathbf{H}_{i,j}$ between each pair of faces f_i and f_j are:

$$\mathbf{H}_{1,2} = \mathbf{a} \vee \mathbf{d} = \begin{pmatrix} -4 \\ -3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{H}_{2,3} = \mathbf{b} \vee \mathbf{e} = \begin{pmatrix} 4 \\ -3 \\ 0 \\ 0 \\ 0 \\ 48 \end{pmatrix}, \quad \mathbf{H}_{3,1} = \mathbf{c} \vee \mathbf{f} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ -16 \end{pmatrix},$$

$$\mathbf{H}_{1,4} = \mathbf{d} \vee \mathbf{f} = \begin{pmatrix} -4 \\ -7 \\ 0 \\ 0 \\ 0 \\ 16 \end{pmatrix}, \quad \mathbf{H}_{2,4} = \mathbf{e} \vee \mathbf{d} = \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 24 \end{pmatrix}, \quad \mathbf{H}_{3,4} = \mathbf{f} \vee \mathbf{e} = \begin{pmatrix} -4 \\ 7 \\ 0 \\ 0 \\ 0 \\ -96 \end{pmatrix},$$

$$\mathbf{H}_{1,5} = \mathbf{c} \vee \mathbf{a} = \begin{pmatrix} 8 \\ 12 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{H}_{2,5} = \mathbf{a} \vee \mathbf{b} = \begin{pmatrix} -16 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{H}_{3,5} = \mathbf{b} \vee \mathbf{c} = \begin{pmatrix} 8 \\ -12 \\ 0 \\ 0 \\ 0 \\ 192 \end{pmatrix}.$$

Before going on, we note that it is not necessary to put an equation in $\underline{\mathbf{R}}_{\mathcal{D}} \cdot \omega = 0$, for all cycles of panels of $\mathcal{F}^{ph}(\mathcal{D})$. Actually, if we regard $\mathcal{F}^{ph}(\mathcal{D})$ as a graph G where the vertex and edge set are the panels and hinges of $\mathcal{F}^{ph}(\mathcal{D})$, respectively, it suffices to put a cycle equation for every independent cycle of G. It is a well-known result of Graph Theory that the set of all cycles of a graph has the structure of a vector space. Then, a set of *independent cycles* is just defined as a basis of this vector space [1]. Since the truncated terahedron is a spherical polyhedron, it is easy to see that we only need one equation $\sum \omega_{i,j} \cdot \mathbf{H}_{i,j} = 0$ for every cycle of panels and hinges around every vertex, given that the equations of other cycles are linearly dependent on these.

Taking the cycles shown in Figure 5a, the rigidity matrix will have the following block form, where each zero entry is a column sextuple of zeroes:

$$\underline{\mathbf{R}}_{\mathcal{D}} = \begin{pmatrix} -\mathbf{H}_{1,2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_{1,5} & -\mathbf{H}_{2,5} & \mathbf{0} \\ \mathbf{0} & -\mathbf{H}_{2,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{H}_{2,5} & -\mathbf{H}_{3,5} \\ \mathbf{0} & \mathbf{0} & -\mathbf{H}_{3,1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{H}_{1,5} & \mathbf{0} & \mathbf{H}_{3,5} \\ \mathbf{H}_{1,2} & \mathbf{0} & \mathbf{0} & -\mathbf{H}_{1,4} & \mathbf{H}_{2,4} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{2,3} & \mathbf{0} & \mathbf{0} & -\mathbf{H}_{2,4} & \mathbf{H}_{3,4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H}_{3,1} & \mathbf{H}_{1,4} & \mathbf{0} & -\mathbf{H}_{3,4} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

It can be checked that the rank of this matrix is 8, and that a basis of its 1-dimensional kernel is the vector:

$$\omega = (-8, -8, -24, \frac{-24}{7}, \frac{-16}{7}, \frac{-24}{7}, 2, 1, 2). \tag{4}$$

Clearly, no component of ω is null and, consequently, $\mathcal{D}$ has a sharp lifting. If μ is a free parameter, the motion assignments have the form

$$\begin{aligned} \omega &= (\omega_{1,2}, \omega_{2,3}, \omega_{3,1}, \omega_{1,4}, \omega_{2,4}, \omega_{3,4}, \omega_{1,5}, \omega_{2,5}, \omega_{3,5}) \\ &= \mu\,(-8, -8, -24, \frac{-24}{7}, \frac{-16}{7}, \frac{-24}{7}, 2, 1, 2). \end{aligned}$$

According to the proof of Theorem 3.1, we compute the rotor of a panel P_i by choosing a path from a reference panel P_k to P_i and then applying Equation 2. Choosing the paths in Figure 5b, where the background panel is taken as the reference frame, we get:

$$\begin{aligned} \mathbf{S}_{2,5} &= \omega_{2,5}\,\mathbf{H}_{2,5}, \\ \mathbf{S}_{1,5} &= \omega_{2,5}\,\mathbf{H}_{2,5} + \omega_{1,2}\,\mathbf{H}_{1,2}, \\ \mathbf{S}_{3,5} &= \omega_{2,5}\,\mathbf{H}_{2,5} - \omega_{2,3}\,\mathbf{H}_{2,3}, \\ \mathbf{S}_{4,5} &= \omega_{2,5}\,\mathbf{H}_{2,5} - \omega_{2,4}\,\mathbf{H}_{2,4}. \end{aligned}$$

That is,

$$\mathbf{S}_{2,5} = \begin{pmatrix} -16\,\mu \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{S}_{1,5} = \begin{pmatrix} 16\,\mu \\ 24\,\mu \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{S}_{3,5} = \begin{pmatrix} 16\,\mu \\ -24\,\mu \\ 0 \\ 0 \\ 0 \\ 384 \end{pmatrix}, \ \mathbf{S}_{4,5} = \begin{pmatrix} \frac{16}{7}\,\mu \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{384}{7}\,\mu \end{pmatrix}.$$

Then, the velocity of each vertex v of $\mathcal{D}$ is computed from the rotor of a panel P_l incident with v as $\mathbf{S}_{l,5} \vee \mathbf{v}$. The third coordinate of $\mathbf{S}_{l,5} \vee \mathbf{v}$ gives the Z coordinate of this vertex on the lifted polysurface: $Z_a = 0$, $Z_b = 0$, $Z_c = 0$, $Z_d = 48$, $Z_e = 48$, $Z_f = 32$.

Finally, we can check that every quadrilateral face lifts coplanarly to 3-space verifying that the determinant of its four lifted vertices is identically zero. Also, every edge must have non coplanar incident planes and the determinant of the two end-vertices of the edge and two other points, each on an adjacent face, must not be identically null. For example, for face $abcd$ and edge fc:

$$det_{abcd} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 4 & 3 & 48\,\mu & 1 \\ 8 & 10 & 32\,\mu & 1 \\ 8 & 12 & 0 & 1 \end{vmatrix} = 0, \qquad det_{fcab} = \begin{vmatrix} 8 & 10 & 32\,\mu & 1 \\ 8 & 12 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 16 & 0 & 0 & 1 \end{vmatrix} = 6144\,\mu.$$

6 Testing Realizability in Floating-Point Arithmetic

Consider the projection of a truncated tetrahedron in Figure 5a. Note that it is only correct when the supporting lines of edges ad, cf, be meet at a common point. However, any small perturbation in the vertex locations destroy this concurrence, as shown in Figure 5c. In practice, this means that it is difficult to implement a realizability test on a computer using floating-point arithmetic, because small round-off errors will produce such perturbations, and there is low probability that a correct drawing would be properly classified. However, the fact that it suffices to analyse the kernel of a matrix to determine a drawing's correctness, allows an easy way around this problem, using the singular value decomposition of $\underline{\mathbf{R}}_{\mathcal{D}}$. We briefly introduce the concept and follow its application with an example. Additional material on the singular value decomposition can be found in [2, Chapter 9].

The singular value decomposition (or SVD for short) provides a powerful technique for dealing with sets of equations or matrices that are either singular or else numerically very close to singular. SVD methods are based on the following theorem of linear algebra.

Theorem 6.1. *Any $m \times n$ matrix $\underline{\mathbf{A}}$ whose number of rows m is greater than or equal to its number of columns n, can be written as the product of an $m \times n$ column-orthogonal matrix $\underline{\mathbf{U}}$, an $n \times n$ diagonal matrix $\underline{\mathbf{W}}$ with positive or zero elements $w_1, \ldots, w_n$, the* singular values*, and the transpose of an $n \times n$ orthogonal matrix $\underline{\mathbf{V}}$,*

$$\underline{\mathbf{A}}_{m\times n} = \underline{\mathbf{U}}_{m\times n}\, \underline{\mathbf{W}}_{n\times n}\, \underline{\mathbf{V}}^t_{n\times n},$$

such that $\underline{\mathbf{U}}$ and $\underline{\mathbf{V}}$ have orthonormal columns. That is, if $\underline{\mathbf{I}}$ is the identity matrix, $\underline{\mathbf{U}}^t\ \underline{\mathbf{U}} = \underline{\mathbf{I}}$ and $\underline{\mathbf{V}}^t\ \underline{\mathbf{V}} = \underline{\mathbf{I}}$.

The SVD can also be carried out when $m < n$. In this case, the singular values $w_{m+1}, \ldots, w_n$, are all zero, and the corresponding columns of $\underline{\mathbf{U}}$ are also zero.

The SVD explicitly constructs orthonormal bases for the kernel and the image space of a matrix. Specifically, the columns of $\underline{\mathbf{U}}$ whose same-numbered elements w_j are nonzero are an orthonormal basis spanning the image space. The columns of $\underline{\mathbf{V}}$ whose same-numbered elements w_j are zero are an orthonormal basis for the kernel. Hence, the rank of the matrix is equal to the number of non-zero singular values in $\underline{\mathbf{W}}$. These properties can be used to tell whether the rigidity matrix $\underline{\mathbf{R}}_{\mathcal{D}}$ is singular or close to singular, indicating the proximity of the vertices of $\mathcal{D}$ to those of a realizable drawing, say $\mathcal{D}^*$. The number of close-to-zero singular values will indicate the dimension of $Ker(\underline{\mathbf{R}}_{\mathcal{D}^*})$, and the columns of $\underline{\mathbf{V}}$ corresponding to these small singular values will be a reasonable approximation of a basis of $Ker(\underline{\mathbf{R}}_{\mathcal{D}^*})$. Let us see this with an example.

Suppose we perturb the X-coordinates of the correct line drawing in Figure 5a, as shown in Figure 5c, letting the vertices as follows: $\mathbf{a} = (-0.01,\ 0,\ 0,\ 1)$, $\mathbf{b} = (15.99,\ 0,\ 0,\ 1)$,

$\mathbf{c} = (8.01,\ 12,\ 0,\ 1)$, $\mathbf{d} = (4.01,\ 3,\ 0,\ 1)$, $\mathbf{e} = (12.01,\ 3,\ 0,\ 1)$, $\mathbf{f} = (7.99,\ 10,\ 0,\ 1)$. We get an incorrect drawing with the hinges:

$$\mathbf{H}_{1,2} = \mathbf{a} \vee \mathbf{d} = \begin{pmatrix} -4.02 \\ -3 \\ 0 \\ 0 \\ 0 \\ -0.03 \end{pmatrix}, \quad \mathbf{H}_{2,3} = \mathbf{b} \vee \mathbf{e} = \begin{pmatrix} 3.98 \\ -3 \\ 0 \\ 0 \\ 0 \\ 47.97 \end{pmatrix}, \quad \mathbf{H}_{3,1} = \mathbf{c} \vee \mathbf{f} = \begin{pmatrix} 0.02 \\ 2 \\ 0 \\ 0 \\ 0 \\ -15.78 \end{pmatrix},$$

$$\mathbf{H}_{1,4} = \mathbf{d} \vee \mathbf{f} = \begin{pmatrix} -3.98 \\ -7 \\ 0 \\ 0 \\ 0 \\ 16.13 \end{pmatrix}, \quad \mathbf{H}_{2,4} = \mathbf{e} \vee \mathbf{d} = \begin{pmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \\ 24 \end{pmatrix}, \quad \mathbf{H}_{3,4} = \mathbf{f} \vee \mathbf{e} = \begin{pmatrix} -4.02 \\ 7 \\ 0 \\ 0 \\ 0 \\ -96.13 \end{pmatrix},$$

$$\mathbf{H}_{1,5} = \mathbf{c} \vee \mathbf{a} = \begin{pmatrix} 8.02 \\ 12 \\ 0 \\ 0 \\ 0 \\ 0.12 \end{pmatrix}, \quad \mathbf{H}_{2,5} = \mathbf{a} \vee \mathbf{b} = \begin{pmatrix} -16 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{H}_{3,5} = \mathbf{b} \vee \mathbf{c} = \begin{pmatrix} 7.98 \\ -12 \\ 0 \\ 0 \\ 0 \\ 191.88 \end{pmatrix},$$

which yield a rigidity matrix whose SVD is $\underline{\mathbf{R}}_{\mathcal{D}} = \underline{\mathbf{U}}_{18\times 9}\ \underline{\mathbf{W}}_{9\times 9}\ \underline{\mathbf{V}}^{t}_{9\times 9}$, with the singular values:

$$\begin{aligned} \underline{\mathbf{W}}(1,1) &= 274.5898, & \underline{\mathbf{W}}(4,4) &= 31.4712, & \underline{\mathbf{W}}(7,7) &= 12.7619, \\ \underline{\mathbf{W}}(2,2) &= 142.9735, & \underline{\mathbf{W}}(5,5) &= 24.8156, & \underline{\mathbf{W}}(8,8) &= 4.3301, \\ \underline{\mathbf{W}}(3,3) &= 52.6653, & \underline{\mathbf{W}}(6,6) &= 18.1010, & \underline{\mathbf{W}}(9,9) &= 0.03207. \end{aligned}$$

We observe that $\underline{\mathbf{W}}(9,9)$ is quite small, revealing that $\underline{\mathbf{R}}_{\mathcal{D}}$ is close to singular and near a configuration with a 1-dimensional kernel. The ninth column of $\underline{\mathbf{V}}$ is a reasonable approximation of a vector spanning this kernel:

$$(-0.2948, -0.2903, -0.8821, -0.1265, -0.0852, -0.1235, 0.07428, 0.03642, 0.07258).$$

Since no component of this vector is close to zero, compared to the rest, we conclude that the input drawing, though incorrect, is near a configuration with sharp liftings. The reader can check that this vector and that of Equation 4 are almost aligned.

7 Conclusions

We have presented an algorithm to decide the correctness of a line drawing which relies on a result by W. Whiteley in [11]. Namely, that instantaneous motions of its associated panel-and-hinge framework are in one-to-one correspondence with the spatial reconstructions. Then, telling whether a drawing is correct reduces to the study of the instantaneous motion space of these frameworks. We have further exploited this correspondence leading to an algorithm whose main advantages over the classical solution by K. Sugihara [6] are the following:

1. One can decide the correctness of a drawing by simply computing a vector basis of the kernel of a matrix, whose entries are just Plücker coordinates of the edges. This offers a simpler alternative to Sugihara's approach based on Linear Programming.

2. As opposed to Sugihara's, this method is straightforwardly implementable in floating point arithmetic, as it can be made robust to the super-strictness problems induced by numerical round-off errors by simply using the singular value decomposition to find a basis of the aforementioned kernel.

3. Deciding whether an edge is convex or concave in the reconstruction can be done in a constructive fashion, rather than by solving a constraint satisfaction problem, as in Sugihara's approach. The only pre-requisite to this end is that the incidence structure be known or, equivalently, that the input line drawing has its boundary edges identified.

Grassmann-Cayley algebra has become useful to study the instantaneous motions of panel-and-hinge frameworks. Actually, this algebra proves to be an exellent tool for representing and solving kinetostatic problems [8]. Moreover, it has an intrinsic duality that would enable to readily obtain the forces that can be transmitted between adjancent panels, given the feasible velocities between them. This would permit to see the realizability problem as one of forces in equilibrium in panel-and-hinge structures. J. C. Maxwell and W. Whiteley already showed that a drawing is correct if and only if we can assign a force to every bar in a given planar bar-and-joint framework so that the sum of all forces from bars incident to each vertex is zero. It remains to check whether both formulations in terms of forces are equivalent.

References

[1] BIGGS, N. *Algebraic Graph Theory.* Cambridge Tracts in Mathematics. Cambridge University Press, 1974.

[2] FORSYTHE, G. E., MALCOLM, M. A., AND MOLER, C. B. *Computer Methods for Mathematical Computations.* Prentice-Hall, Englewood Cliffs, New Jersey, 1977.

[3] GRAVER, J., SERVATIUS, B., AND SERVATIUS, H. *Combinatorial Rigidity*, vol. 2 of *Graduate Studies in Mathematics.* American Mathematical Society, 1993.

[4] PARODI, P., LANCEWICKI, R., VIJH, A., AND TSOTSOS, J. K. Empirically-derived estimates of the complexity of labelling line drawings of polyhedral scenes. *Artificial Intelligence 105*, 1-2 (1998), 47–75.

[5] ROS, L. *A Kinematic-Geometric Approach to Spatial Interpretation of Line Drawings.* PhD thesis, Polytechnic University of Catalonia, May 2000. Available at `http://www-iri.upc.es/people/ros`.

[6] SUGIHARA, K. *Machine Interpretation of Line Drawings.* The MIT Press, 1986.

[7] WHITE, N. L. Grassmann-Cayley algebra and robotics. *Journal of Intelligent and Robotic Systems 11* (1994), 91–107.

[8] WHITE, N. L. Geometric applications of the Grassmann-Cayley algebra. In *Handbook of Discrete and Computational Geometry.* CRC Press, 1997.

[9] WHITELEY, W. Introduction to Structural Geometry I: Infinitesimal motions and infinitesimal rigidity. Preprint. Champlain Regional College. 900 Riverside Drive, St. Lambert, Quebec, April 1977.

[10] WHITELEY, W. Introduction to Structural Geometry II: Statics and stresses. Preprint. Champlain Regional College. 900 Riverside Drive, St. Lambert, Quebec, February 1978.

[11] WHITELEY, W. Motions and stresses of projected polyhedra. *Structural Topology 7* (1982), 13–38.