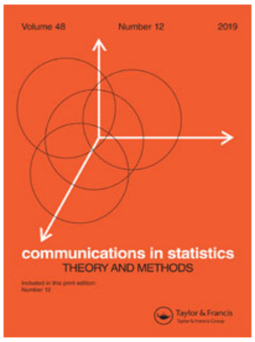

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Selected singular-generalized-hyperbolic distributions with applications to order statistics and reliability

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ABSTRACT
In this paper, the truncated version of the selected multivariate generalized-hyperbolic distributions is introduced. Considering special truncations, the joint distribution of the consecutive order statistics from the multivariate generalized-hyperbolic (GH) distribution is derived. It is shown that this joint distribution can be expressed as mixtures of the truncated selected-GH distributions. All of these truncated distributions are expressed as the selected singular-GH distributions. These results are used to obtain some expressions for the reliability measures such as mean residual life time, mean inactivity time and regression mean residual life for *k*-out-of-*n* systems.

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1. Introduction

Relationship between the order statistics and skewed distributions was first considered by Loperfido (2002), that was independent rediscovering of the result in Roberts (1966), and later generalized by Azzalini and Capitanio (2003). Other relationships between the order statistics and generalizations of the normal distribution have been studied by several authors (see, e.g., Crocetta and Loperfido 2005; Arellano-Valle and Genton 2007, 2008; Jamalizadeh and Balakrishnan 2009 and Arellano-Valle et al. 2014). Moreover, Jamalizadeh and Balakrishnan (2009) have obtained the exact distributions of order statistics from trivariate normal and Student-t distributions based on generalized skew-normal and skew-t distributions. Also, Arellano-Valle and Genton (2007) and Jamalizadeh and Balakrishnan (2010) have derived the distributions of L-statistics from the multivariate normal and elliptical distributions. Madadi, Khalilpoor, and Jamalizadeh (2015) have used the joint and the conditional distributions of order statistics from a trivariate normal distribution to obtain some reliability measures for systems with three dependent components with normal lifetimes. However, little is known about the joint distribution of order statistics from the normal mean-variance mixture distributions.

In this paper, the selected distribution from the multivariate generalized-hyperbolic (GH) distribution and the truncated selected-GH (TSGH) distribution are introduced. Considering some special truncation sets, these TSGH distributions are expressed as the selected distribution from the singular-GH distributions, it is called the selected singular-GH distributions.

Furthermore, the exact joint distribution of consecutive order statistics from the GH random vector is derived as a mixture of TSGH distributions. Then, using the properties of

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TSGH and this mixture representation, some reliability measures, such as mean residual life time (MRL) function, mean inactivity time (MIT) function and regression mean residual life (mrl) function for k-out-of-n systems, are derived.

This paper is organized as follows: Section 2 contains a review of GH and selected-GH distributions. Section 3 introduces the TSGH distributions with special truncations and some conditional distributions. In Section 4, the joint and the conditional distributions of order statistics from the multivariate GH distribution are derived and the exchangeable case is discussed in detail. Some reliability measures such as MRL, mrl and MIT, for a system with dependent components and log-generalized-hyperbolic life times, are computed in Section 5. Section 6 gives some concluding remarks.

2. Preliminaries

In this section, the class of multivariate GH and the multivariate selected-GH distributions are briefly reviewed, and some of their basic properties are discussed.

2.1. Multivariate GH distribution

Blæsild (1981) introduced the class of GH distributions as the mean-variance mixtures of normal distributions. Recently, the GH distributions have been used by many authors to model financial data (e.g., Rydberg 1999; Cont and Tankov 2004). A useful representation of the GH distribution can be given by using the generalized-inverse-gaussian distribution.

A positive random variable W follows a generalized-inverse-gaussian (GIG) distribution, denoted by $W \sim GIG(\lambda, \chi, \psi)$, if its probability density function (pdf) is given by

$$c(\lambda, \chi, \psi)\ w^{\lambda-1} \mathrm{e}^{-\frac{1}{2}(\chi w^{-1}+\psi w)}, \quad w > 0$$

where

$$c(\lambda, \chi, \psi) = \frac{\left(\frac{\psi}{\chi}\right)^{\frac{\lambda}{2}}}{2K_\lambda\left(\sqrt{\chi\psi}\right)} \tag{1}$$

and $K_\lambda(x)$ is a modified Bessel function of the third kind with index λ and the parameters satisfy

$$\begin{cases} \chi > 0, \psi \geq 0; & \lambda < 0 \\ \chi > 0, \psi > 0; & \lambda = 0 \\ \chi \geq 0, \psi > 0; & \lambda > 0 \end{cases}$$

The random vector $\mathbf{X}$ has the p-variate GH distribution with parameters $\boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\tau}, \boldsymbol{\Sigma}, \lambda, \chi, \psi)$, denoted by $\mathbf{X} \sim GH_p(\boldsymbol{\theta})$, if

$$\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{\tau} W + \sqrt{W}\mathbf{Z} \tag{2}$$

where $\boldsymbol{\mu} \in \mathbb{R}^p$,$\boldsymbol{\tau} \in \mathbb{R}^p$, $\mathbf{Z} \sim N_p(\mathbf{0}, \boldsymbol{\Sigma})$ and $W \sim GIG(\lambda, \chi, \psi)$. In this case, it is easy to see that the cumulative distribution function (cdf) of $\mathbf{X}$, for $\mathbf{x} \in \mathbb{R}^p$, is

$$F_{GH_p}(\mathbf{x}; \boldsymbol{\theta}) = c(\lambda, \chi, \psi) \int_0^\infty w^{\lambda-1} \mathrm{e}^{-\frac{1}{2}(\chi w^{-1}+\psi w)} \Phi_p\left((\mathbf{x} - \boldsymbol{\mu}) w^{-\frac{1}{2}} - \boldsymbol{\tau} w^{\frac{1}{2}}; \boldsymbol{\Sigma}\right) dw \tag{3}$$

The pdf of Equation (2) in the non singular case (Σ has rank p) is (see McNeil, Frey and Embrechts 2015)

$$f_{GH_p}(\mathbf{x};\boldsymbol{\theta}) = c\frac{K_{\lambda-\frac{p}{2}}\left(\sqrt{(\chi+q(\mathbf{x}))\left(\psi+\boldsymbol{\tau}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\tau}\right)}\right)}{\sqrt{(\chi+q(\mathbf{x}))\left(\psi+\boldsymbol{\tau}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\tau}\right)}^{\frac{p}{2}-\lambda}} \exp\left(\boldsymbol{\tau}^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right) \tag{4}$$

where $\mathbf{x} \in \mathbb{R}^p$, $q(\mathbf{x}) = (\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ and the normalizing constant is

$$c = \frac{\sqrt{\chi\psi}^{-\lambda}\psi^{\lambda}\left(\psi+\boldsymbol{\tau}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\tau}\right)^{\frac{p}{2}-\lambda}}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}K_{\lambda}(\sqrt{\chi\psi})}$$

when $|.|$ denotes the determinant.

The following remark states some special cases of the GH distribution.

Remark 2.1. When $\lambda = -0.5$, GH becomes the Normal Inverse Gaussian (NIG) distribution. The limiting case of GH with $\lambda > 0$ and $\chi = 0$ yields the variance gamma (VG) distribution. The limiting case of GH with $\lambda < 0$ and $\psi = 0$ leads to the skewed t distribution. If $\lambda = (p+1)/2$, we get the p-dimensional hyperbolic distribution. Moreover, if in Equation (3), we set $\boldsymbol{\tau} = \mathbf{0}$, then we have a symmetric GH distribution which is an elliptical distribution.

To present the marginal and conditional distributions of the GH distribution, let $\mathbf{X} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^n$ be jointly distributed as

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} \sim GH_{m+n}\left(\begin{pmatrix}\mu_{\mathbf{X}}\\ \mu_{\mathbf{Y}}\end{pmatrix}, \begin{pmatrix}\tau_{\mathbf{X}}\\ \tau_{\mathbf{Y}}\end{pmatrix}, \begin{pmatrix}\Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}}\\ & \Sigma_{\mathbf{YY}}\end{pmatrix}, \lambda, \chi, \psi\right) \tag{5}$$

Then, we have the following lemma that dates back to Blæsild (1981), which shows that a GH distribution is closed under marginalization, conditioning and linear transformations.

Lemma 2.2. *If the random vectors* $\mathbf{X}$ *and* $\mathbf{Y}$ *are distributed as in Equation (5), then*

(i) $\mathbf{X} \sim GH_m(\boldsymbol{\mu}_X, \boldsymbol{\tau}_{\mathbf{X}}, \boldsymbol{\Sigma}_{\mathbf{XX}}, \lambda, \chi, \psi)$;

(ii) *For* $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{Y}|(\mathbf{X}=\mathbf{x}) \sim GH_n\left(\boldsymbol{\mu}_{\mathbf{Y.X}}, \boldsymbol{\tau}_{\mathbf{Y.X}}, \boldsymbol{\Sigma}_{\mathbf{YY.X}}, \lambda_{\mathbf{Y.X}}, \chi_{\mathbf{Y.X}}, \psi_{\mathbf{Y.X}}\right)$;

(iii) *If* $\mathbf{B} \in \mathbb{R}^{k\times m}$ *and* $\mathbf{b} \in \mathbb{R}^k$, *then* $\mathbf{B}X+\mathbf{b} \sim GH_k\left(\mathbf{B}\boldsymbol{\mu}_{\mathbf{X}}+\mathbf{b}, \mathbf{B}\boldsymbol{\tau}_{\mathbf{X}}, \mathbf{B}\boldsymbol{\Sigma}_{\mathbf{XX}}\mathbf{B}^T, \lambda, \chi, \psi\right)$, *where*

$$\begin{aligned}
&\boldsymbol{\mu}_{\mathbf{Y.X}} = \boldsymbol{\mu}_{\mathbf{Y}} + \boldsymbol{\Sigma}_{\mathbf{XY}}^T\boldsymbol{\Sigma}_{\mathbf{XX}}^{-1}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}}), \qquad && \boldsymbol{\tau}_{\mathbf{Y.X}} = \boldsymbol{\tau}_{\mathbf{Y}} - \boldsymbol{\Sigma}_{\mathbf{XY}}^T\boldsymbol{\Sigma}_{\mathbf{XX}}^{-1}\boldsymbol{\tau}_{\mathbf{X}}\\
&\boldsymbol{\Sigma}_{\mathbf{YY.X}} = \boldsymbol{\Sigma}_{\mathbf{YY}} - \boldsymbol{\Sigma}_{\mathbf{XY}}^T\boldsymbol{\Sigma}_{\mathbf{XX}}^{-1}\boldsymbol{\Sigma}_{\mathbf{XY}}, && \lambda_{\mathbf{Y.X}} = \lambda - \frac{m}{2}\\
&\chi_{\mathbf{Y.X}} = \chi + (\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}})^T\boldsymbol{\Sigma}_{XX}^{-1}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{X}}), && \psi_{\mathbf{Y.X}} = \psi + \boldsymbol{\tau}_{\mathbf{X}}^T\boldsymbol{\Sigma}_{\mathbf{XX}}^{-1}\boldsymbol{\tau}_{\mathbf{X}}
\end{aligned}$$

2.2. Multivariate selection distribution

Let $\mathbf{X} \in \mathbb{R}^m$ and $\mathbf{Y} \in \mathbb{R}^n$ be two random vectors and $\mathbf{C}$ be a measurable subset of $\mathbb{R}^m$. Arellano-Valle, Branco and Genton (2006) defined the selection distribution as the conditional distribution of $\mathbf{Y}$, given $\mathbf{X} \in \mathbf{C}$. Specifically, an n-dimensional random vector $\mathbf{U}$ is said to have a multivariate selection distribution with parameters depending on the characteristics of $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{C}$, if $\mathbf{U} \overset{d}{=} \mathbf{Y}|(\mathbf{X} \in \mathbf{C})$. If $\mathbf{Y}$ has the pdf $f_{\mathbf{Y}}(.)$, then the random vector $\mathbf{U}$ has the pdf of the form

$$f_{\mathbf{U}}(\mathbf{u}) = f_{\mathbf{Y}}(\mathbf{u})\frac{P(\mathbf{X}\in\mathbf{C}|\mathbf{Y}=\mathbf{u})}{P(\mathbf{X}\in\mathbf{C})}$$

Let $\mathbf{C} = \{\mathbf{x} \in \mathbb{R}^m \,|\mathbf{x} > \mathbf{0}\}$ and $\mathbf{X}$ and $\mathbf{Y}$ are jointly normally distributed as

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \sim N_{m+n}\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathbf{X}} \\ \boldsymbol{\mu}_{\mathbf{Y}} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{\mathbf{XX}} & \boldsymbol{\Sigma}_{\mathbf{XY}} \\ & \boldsymbol{\Sigma}_{\mathbf{YY}} \end{pmatrix}\right) \tag{6}$$

Then, we have

$$\mathbf{U} \stackrel{d}{=} \mathbf{Y}\,|(\mathbf{X} > \mathbf{0}) \tag{7}$$

and this stochastic representation yields the unified skew-normal (SUN) distribution, denoted by $\mathbf{U} \sim SUN_{n,m}(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\Sigma}_{\mathbf{YY}}, \boldsymbol{\Sigma}_{\mathbf{XX}}, \boldsymbol{\Sigma}_{\mathbf{XY}})$. The pdf of Equation (7) is given by (see Arellano-Valle and Azzalini 2006)

$$\phi_{SUN_{n,m}}(\mathbf{u}; \boldsymbol{\theta}) = \phi_n(\mathbf{u}; \boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\Sigma}_{\mathbf{YY}}) \frac{\Phi_m\left(\boldsymbol{\mu}_{\mathbf{X}} + \boldsymbol{\Sigma}_{\mathbf{XY}}\boldsymbol{\Sigma}_{\mathbf{YY}}^{-1}(\mathbf{u}-\boldsymbol{\mu}_{\mathbf{Y}}); \boldsymbol{\Sigma}_{\mathbf{XX.Y}}\right)}{\Phi_m(\boldsymbol{\mu}_{\mathbf{X}}; \boldsymbol{\Sigma}_{\mathbf{XX}})}, \quad \mathbf{u} \in \mathbb{R}^n \tag{8}$$

where $\Sigma_{\mathbf{XX.Y}} = \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{YY}}^{-1}\Sigma_{\mathbf{YX}}$, $\phi_n(\cdot; \boldsymbol{\mu}_{\mathbf{Y}}, \Sigma_{\mathbf{YY}})$ denotes the pdf of $N_n(\boldsymbol{\mu}_{\mathbf{Y}}, \Sigma_{\mathbf{YY}})$, and $\Phi_m(\cdot; \Sigma_{\mathbf{XX.Y}})$ and $\Phi_m(\cdot; \Sigma_{\mathbf{XX}})$ denote the cdfs of $N_m(\mathbf{0}, \Sigma_{\mathbf{XX.Y}})$ and $N_m(\mathbf{0}, \Sigma_{\mathbf{XX}})$, respectively.

Now, suppose $n = 1$ in Equation (7), i.e., $U \stackrel{d}{=} Y|\,(\mathbf{X} > 0)$, where

$$\begin{pmatrix} \mathbf{X} \\ Y \end{pmatrix} \sim N_{m+1}\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathbf{X}} \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \Sigma_{\mathbf{XX}} & \boldsymbol{\sigma}_{\mathbf{X}Y} \\ & \sigma_{YY} \end{pmatrix}\right)$$

Then, $U \sim SUN_{1,m}(\mu_Y, \boldsymbol{\mu}_{\mathbf{X}}, \sigma_{YY}, \Sigma_{\mathbf{XX}}, \boldsymbol{\sigma}_{\mathbf{X}Y})$ and (see Jamalizadeh and Balakrishnan 2010)

$$E(U) = \mu_Y + \frac{\sum_{i=1}^m \frac{\sigma_{\mathbf{X}Y,i}}{\sqrt{\sigma_{XX,ii}}}\phi\left(\frac{\mu_{\mathbf{X},i}}{\sqrt{\sigma_{\mathbf{XX},ii}}}\right)\Phi_{m-1}\left(\boldsymbol{\mu}_{\mathbf{X},-i} - \frac{\mu_{X,i}}{\sigma_{XX,ii}}\boldsymbol{\sigma}_{XX,-ii}; \Sigma_{\mathbf{XX},-i|i}\right)}{\Phi_m(\boldsymbol{\mu}_{\mathbf{X}}; \Sigma_{\mathbf{XX}})} \tag{9}$$

where $\boldsymbol{\mu}_{\mathbf{X}}$, $\boldsymbol{\sigma}_{\mathbf{X}Y}$ and $\Sigma_{\mathbf{XX}}$ are partitioned, for $i = 1,\ldots, m$, as

$$\boldsymbol{\mu}_{\mathbf{X}} = \begin{pmatrix} \mu_{\mathbf{X},i} \\ \boldsymbol{\mu}_{\mathbf{X},-i} \end{pmatrix}, \quad \boldsymbol{\sigma}_{\mathbf{X}Y} = \begin{pmatrix} \sigma_{\mathbf{X}Y,i} \\ \boldsymbol{\sigma}_{\mathbf{X}Y,-i} \end{pmatrix}, \quad \Sigma_{\mathbf{XX}} = \begin{pmatrix} \sigma_{XX,ii} & \boldsymbol{\sigma}_{\mathbf{X}X,-ii}^T \\ \boldsymbol{\sigma}_{\mathbf{X}X,-ii} & \Sigma_{\mathbf{XX},-i-i} \end{pmatrix} \tag{10}$$

respectively, and $\Sigma_{\mathbf{XX},-i|i} = \Sigma_{\mathbf{XX},-i-i} - \frac{\boldsymbol{\sigma}_{\mathbf{X}X,-ii}\boldsymbol{\sigma}_{\mathbf{X}X,-ii}^T}{\sigma_{XX,ii}}$.

Note that, if $(\mathbf{X}^T, \mathbf{Y}^T)^T$ in Equation (7) is a random vector following a singular normal distribution, then the random vector $\mathbf{U}$ has a singular SUN distribution, denoted by $\mathbf{U} \sim SSUN_{n,m}(\boldsymbol{\theta})$. Arellano-Valle and Azzalini (2006) showed that, under the following singularity conditions,

$$Rank(\Sigma_{\mathbf{XX}}) = m,\, Rank(\Sigma_{\mathbf{YY}}) = n,\, Rank\left(\begin{pmatrix} \Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}} \\ & \Sigma_{\mathbf{YY}} \end{pmatrix}\right) < m + n \tag{11}$$

the random vector $\mathbf{U}$ in Equation (7) has the pdf quite similar to Equation (8). Since $|\Sigma_{\mathbf{XX.Y}}| = 0$, $\Phi_m(\mathbf{z}; \Sigma_{\mathbf{XX.Y}})$ must be computed via

$$\Phi_m(\mathbf{z}; \Sigma_{\mathbf{XX.Y}}) = (2\pi)^{-\frac{r}{2}} \int_{\{\mathbf{u}:\mathbf{Cu}\le\mathbf{z}\}} \exp\left(-\tfrac{1}{2}\mathbf{u}^T\mathbf{u}\right) d\mathbf{u} \tag{12}$$

where $r = Rank(\Sigma_{\mathbf{XX.Y}}) < m$ and $\mathbf{CC}^T = \Sigma_{\mathbf{XX.Y}}$.

2.3. Multivariate selected-GH distribution

Let $\mathbf{X}$ and $\mathbf{Y}$ be the random vectors of dimensions m and n, respectively, with a joint distribution as in Equation (5). Then, the n-dimensional random vector $\mathbf{U}$ is said to have the multivariate selected-GH (SGH) distribution with the parameters $\boldsymbol{\theta}=(\boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\tau}_{\mathbf{Y}}, \boldsymbol{\tau}_{\mathbf{X}}, \Sigma_{\mathbf{YY}}, \Sigma_{\mathbf{XX}},$

$\Sigma_{\mathbf{XY}}$,λ, χ, ψ), denoted by $\mathbf{U} \sim SGH_{n,m}(\boldsymbol{\theta})$, if

$$\mathbf{U} \overset{d}{=} \boldsymbol{Y} | (\mathbf{X} > \mathbf{0}) \tag{13}$$

The corresponding pdf of $\mathbf{U}$, for $\mathbf{x} \in \mathbb{R}^n$, is

$$f_{SGH_{n,m}}(\mathbf{x}; \boldsymbol{\theta}) = f_{GH_n}(\mathbf{x}; \boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\tau}_{\mathbf{Y}}, \Sigma_{\mathbf{YY}}, \lambda, \chi, \psi) \times \frac{F_{GH_m}(\boldsymbol{\mu}_{\mathbf{X.Y}}; -\boldsymbol{\tau}_{\mathbf{X.Y}}, \Sigma_{\mathbf{XX.Y}}, \lambda_{\mathbf{X.Y}}, \chi_{\mathbf{X.Y}}, \psi_{\mathbf{X.Y}})}{F_{GH_m}(\boldsymbol{\mu}_{\mathbf{X}}; -\boldsymbol{\tau}_{\mathbf{X}}, \Sigma_{\mathbf{XX}}, \lambda, \chi, \psi)} \tag{14}$$

where $F_{GH_m}(\cdot; \boldsymbol{\tau}, \Sigma, \lambda, \chi, \psi)$ denotes the cdf of $GH_m(\mathbf{0}, \boldsymbol{\tau}, \Sigma, \lambda, \chi, \psi)$ and $\boldsymbol{\mu}_{\mathbf{X.Y}}, \boldsymbol{\tau}_{\mathbf{X.Y}}$, $\Sigma_{\mathbf{XX.Y}}$, $\lambda_{\mathbf{X.Y}}$, $\chi_{\mathbf{X.Y}}$ and $\psi_{\mathbf{X.Y}}$ are given in Lemma 2.2.

3. Selected singular-GH distribution

At the beginning of this section, the selected singular-GH (SSGH) distribution is defined.

Definition 3.1. Let $\mathbf{X}$ and $\mathbf{Y}$ be jointly distributed as in Equation (5) with the singularity conditions in Equation (11). Then, the random vector $\mathbf{U} = \mathbf{Y} | (\mathbf{X} > \mathbf{0})$ has SSGH distribution with a set of parameters $\boldsymbol{\theta} = (\boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\tau}_{\mathbf{Y}}, \boldsymbol{\tau}_{\mathbf{X}}, \Sigma_{\mathbf{YY}}, \Sigma_{\mathbf{XX}}, \Sigma_{\mathbf{XY}}, \lambda, \chi, \psi)$, denoted by $\mathbf{U} \sim SSGH_{n,m}(\boldsymbol{\theta})$

It is noteworthy that the density of the SSGH distribution is also given by Equation (14), irrespective to the fact that $|\Sigma_{\mathbf{XX.Y}}| = 0$. The only difference is in the computation of $F_{GH_m}(.;.)$ where $\Phi_m(.; \Sigma_{\mathbf{XX.Y}})$ will be computed according to Equation (12).

The focus now turns to the univariate SSGH distributions that corresponds to the multivariate SSGH distribution when $n = 1$ whenever the value of $m = 1, 2, \ldots$.

Lemma 3.2. *Let $W \sim GIG(\lambda, \chi, \psi)$. Then, for any $a_1, a_2 \in \mathbb{R}$, $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^k$, a $k \times k$ covariance matrix Γ and $r > 0$,*

$$E\left(W^r \phi\left(\frac{a_1}{\sqrt{W}} - a_2\sqrt{W}\right) \Phi_k\left(\frac{1}{\sqrt{W}}\mathbf{b}_1 - \sqrt{W}\mathbf{b}_2; \Gamma\right)\right) = \frac{c(\lambda, \chi, \psi) \exp(a_1 a_2) F_{GH_k}(\mathbf{b}_1; \mathbf{b}_2, \Gamma, \lambda + r, a_1^2 + \chi, a_2^2 + \psi)}{c(\lambda + r, a_1^2 + \chi, a_2^2 + \psi)}$$

where $\phi(\cdot)$ is the pdf of the standard normal distribution.

Proof. See Jamalizadeh et al. (2017). □

Lemma 3.3. *Suppose $U \sim SSGH_{1,m}(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\mu_Y, \boldsymbol{\mu}_{\mathbf{X}}, \tau_Y, \boldsymbol{\tau}_{\mathbf{X}}, \sigma_{YY}, \Sigma_{\mathbf{XX}}, \boldsymbol{\sigma}_{\mathbf{X}Y}, \lambda, \chi, \psi)$, and consider the partitions for $\boldsymbol{\mu}_{\mathbf{X}}$, $\Sigma_{\mathbf{XX}}$, $\boldsymbol{\sigma}_{\mathbf{X}Y}$ as in Equation (10) and $\boldsymbol{\tau}_{\mathbf{X}} = \left(\tau_{X,i}, \boldsymbol{\tau}_{\mathbf{X},-i}^T\right)^T$. Then,*

$$E_{\boldsymbol{\theta}}(U) = \mu_Y + \frac{c(\lambda, \chi, \psi)}{F_{GH_m}(\mu_{\mathbf{X}}; -\tau_{\mathbf{X}}, \Sigma_{\mathbf{XX}}, \lambda, \chi, \psi)} \times \left\{ \frac{\tau_Y\, F_{GH_m}(\mu_{\mathbf{X}}; -\tau_{\mathbf{X}}, \Sigma_{\mathbf{XX}}, \lambda + 1, \chi, \psi)}{c(\lambda + 1, \chi, \psi)} + \sum_{i=1}^{m} \frac{\sigma_{\mathbf{X}Y,i}\, F_{GH_{m-1}}(\mathbf{c}_i; -\mathbf{d}_i, \Sigma_{\mathbf{XX},-i|i}, \lambda + 1/2, \chi + a_i^2, \psi + b_i^2) e^{-a_i b_i}}{\sqrt{2\pi\sigma_{XX,ii}}\; c(\lambda + 1/2, \chi + a_i^2, \psi + b_i^2)} \right\}$$

where $a_i = \frac{\mu_{X,i}}{\sqrt{\sigma_{XX,ii}}}$, $b_i = \frac{\tau_{X,i}}{\sqrt{\sigma_{XX,ii}}}$, $\mathbf{c}_i = \boldsymbol{\mu}_{\mathbf{X},-i} - \frac{\mu_{X,i}}{\sigma_{XX,ii}} \boldsymbol{\sigma}_{XX,-ii}$, $\mathbf{d}_i = \boldsymbol{\tau}_{\mathbf{X},-i} - \frac{\tau_{X,i}}{\sigma_{XX,ii}} \boldsymbol{\sigma}_{XX,-ii}$ and $c(\lambda, \chi, \psi)$ is given by Equation (1).

Proof. Using representations in Equations (2) and (13), we see

$$U|\left(W=w\right)\sim SSUN_{1,m}\left(\mu_Y+w\tau_Y,\boldsymbol{\mu}_{\mathbf{X}}+w\boldsymbol{\tau}_{\mathbf{X}},w\sigma_{YY},w\Sigma_{\mathbf{XX}},w\sigma_{\mathbf{X}Y}\right)$$

Then, applying double expectation formula and Equation (9), we have

$$\begin{aligned} E_{\boldsymbol{\theta}}\left(U\right) &= E\left\{E\left(U|W\right)\right\} \\ &= \mu_Y+\frac{1}{F_{GH_m}\left(\boldsymbol{\mu}_{\mathbf{X}};-\boldsymbol{\tau}_{\mathbf{X}},\Sigma_{\mathbf{XX}},\lambda,\chi,\psi\right)}\times\left\{\tau_Y E\left(W\Phi_m\left(W^{-\frac{1}{2}}\boldsymbol{\mu}_{\mathbf{X}}+W^{\frac{1}{2}}\boldsymbol{\tau}_{\mathbf{X}};\Sigma_{\mathbf{xx}}\right)\right)\right. \\ &\quad \left.+\sum_{i=1}^{m}\frac{\sigma_{\mathbf{X}Y,i}}{\sqrt{\sigma_{XX,ii}}}E\left(W^{\frac{1}{2}}\phi\left(W^{-\frac{1}{2}}a_i+W^{\frac{1}{2}}b_i\right)\Phi_{m-1}\left(W^{-1/2}\mathbf{c}_i+W^{1/2}\mathbf{d}_i;\Sigma_{\mathbf{XX},-i|i}\right)\right)\right\} \end{aligned}$$

The proof can now be completed using Lemma 3.2. □

In the following lemma, it is shown that the family of SSGH distributions is closed under marginalization and conditioning to given values of some components. But first, let $\mathbf{U}_1$ and $\mathbf{U}_2$ be two random vectors of dimensions n_1 and $n-n_1$, respectively, such that

$$\mathbf{U}=\left(\mathbf{U}_1^T,\mathbf{U}_2^T\right)^T\sim SSGH_{n,m}\left(\boldsymbol{\mu}_{\mathbf{Y}},\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\tau}_{\mathbf{Y}},\boldsymbol{\tau}_{\mathbf{X}},\Sigma_{\mathbf{YY}},\Sigma_{\mathbf{XX}},\Sigma_{\mathbf{XY}},\lambda,\chi,\psi\right) \tag{15}$$

Corresponding to $\mathbf{U}_1$ and $\mathbf{U}_2$, consider the following partitions of $\boldsymbol{\mu}_{\mathbf{Y}}$, $\boldsymbol{\tau}_{\mathbf{Y}}$, $\Sigma_{\mathbf{YY}}$ and $\Sigma_{\mathbf{XY}}$ with the dimensions matching suitably:

$$\mathbf{Y}=\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix},\boldsymbol{\mu}_{\mathbf{Y}}=\begin{pmatrix}\boldsymbol{\mu}_{\mathbf{Y}_1}\\ \boldsymbol{\mu}_{\mathbf{Y}_2}\end{pmatrix},\boldsymbol{\tau}_{\mathbf{Y}}=\begin{pmatrix}\boldsymbol{\tau}_{\mathbf{Y}_1}\\ \boldsymbol{\tau}_{\mathbf{Y}_2}\end{pmatrix},\Sigma_{\mathbf{YY}}=\begin{pmatrix}\Sigma_{\mathbf{Y}_1\mathbf{Y}_1} & \Sigma_{\mathbf{Y}_1\mathbf{Y}_2}\\ \Sigma_{\mathbf{Y}_2\mathbf{Y}_1} & \Sigma_{\mathbf{Y}_2\mathbf{Y}_2}\end{pmatrix},\Sigma_{\mathbf{XY}}=\begin{pmatrix}\Sigma_{\mathbf{XY}_1}\\ \Sigma_{\mathbf{XY}_2}\end{pmatrix}$$

Lemma 3.4. *Let $\mathbf{U}_1$ and $\mathbf{U}_2$ be distributed as in Equation (15). Then, we have*

(i) $\mathbf{U}_1\sim SSGH_{n_1,m}\left(\boldsymbol{\mu}_{\mathbf{Y}_1},\boldsymbol{\mu}_{\mathbf{X}},\boldsymbol{\tau}_{\mathbf{Y}_1},\boldsymbol{\tau}_{\mathbf{X}},\Sigma_{\mathbf{Y}_1\mathbf{Y}_1},\Sigma_{\mathbf{XX}},\Sigma_{\mathbf{XY}_1},\lambda,\chi,\psi\right)$;

(ii) *For* $\mathbf{x}\in\mathbb{R}^{n_1}$, $\mathbf{U}_2|\left(\mathbf{U}_1=\mathbf{x}\right)\sim SSGH_{n-n_1,m}\left(\boldsymbol{\theta}_{2.1}\right)$,
where $\boldsymbol{\theta}_{2.1}=\left(\boldsymbol{\mu}_{\mathbf{Y},2.1},\boldsymbol{\mu}_{\mathbf{X},2.1},\boldsymbol{\tau}_{\mathbf{Y},2.1},\boldsymbol{\tau}_{\mathbf{X},2.1},\Sigma_{\mathbf{YY},22.1},\Sigma_{\mathbf{XX},2.1},\Sigma_{\mathbf{XY},2.1},\lambda_{2.1},\chi_{2.1},\psi_{2.1}\right)$, *with*

$$\begin{array}{ll} \boldsymbol{\mu}_{\mathbf{Y},2.1}=\boldsymbol{\mu}_{\mathbf{Y}_2}+\Sigma_{\mathbf{Y}_2\mathbf{Y}_1}\Sigma_{\mathbf{Y}_1\mathbf{Y}_1}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{Y}_1}\right) & \boldsymbol{\mu}_{\mathbf{X},2.1}=\boldsymbol{\mu}_{\mathbf{X}}+\Sigma_{\mathbf{XY}_1}\Sigma_{\mathbf{Y}_1\mathbf{Y}_1}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{Y}_1}\right) \\ \boldsymbol{\tau}_{\mathbf{Y},2.1}=\boldsymbol{\tau}_{\mathbf{Y}_2}-\Sigma_{\mathbf{Y}_2\mathbf{Y}_1}\Sigma_{\mathbf{Y}_1\mathbf{Y}_1}^{-1}\boldsymbol{\tau}_{\mathbf{Y}_1} & \boldsymbol{\tau}_{\mathbf{X},2.1}=\boldsymbol{\tau}_{\mathbf{X}}-\Sigma_{\mathbf{XY}_1}\Sigma_{\mathbf{Y}_1\mathbf{Y}_1}^{-1}\boldsymbol{\tau}_{\mathbf{Y}_1} \\ \Sigma_{\mathbf{YY},22.1}=\Sigma_{\mathbf{Y}_2\mathbf{Y}_2}-\Sigma_{\mathbf{Y}_2\mathbf{Y}_1}\Sigma_{\mathbf{Y}_1\mathbf{Y}_1}^{-1}\Sigma_{\mathbf{Y}_1\mathbf{Y}_2} & \Sigma_{\mathbf{XX},2.1}=\Sigma_{\mathbf{XX}}-\Sigma_{\mathbf{XY}_1}\Sigma_{\mathbf{Y}_1\mathbf{Y}_1}^{-1}\Sigma_{\mathbf{XY}_1}^T \\ \Sigma_{\mathbf{XY},2.1}=\Sigma_{\mathbf{XY}_2}-\Sigma_{\mathbf{XY}_1}\boldsymbol{\Sigma}_{\mathbf{Y}_1\mathbf{Y}_1}^{-1}\boldsymbol{\Sigma}_{\mathbf{Y}_1\mathbf{Y}_2} & \lambda_{2.1}=\lambda-\frac{n_1}{2} \\ \chi_{2.1}=\chi+\left(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{Y}_1}\right)^T\Sigma_{\mathbf{Y}_1\mathbf{Y}_1}^{-1}\left(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{Y}_1}\right) & \psi_{2.1}=\psi+\boldsymbol{\tau}_{\mathbf{Y}_1}^T\Sigma_{\mathbf{Y}_1\mathbf{Y}_1}^{-1}\boldsymbol{\tau}_{\mathbf{Y}_1} \end{array}$$

Proof. Using the stochastic representation in Equation (13), we get

$$\begin{pmatrix}\mathbf{U}_1\\ \mathbf{U}_2\end{pmatrix}\stackrel{d}{=}\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix}\Bigg|\left(\mathbf{X}>\mathbf{0}\right)$$

Then, using the fact $\mathbf{U}_1\stackrel{d}{=}\mathbf{Y}_1|\left(\mathbf{X}>\mathbf{0}\right)$ and Definition 3.1, the case (i) can be obtained. For (ii), it can be shown $\mathbf{U}_2|\left(\mathbf{U}_1=\mathbf{x}\right)\stackrel{d}{=}\mathbf{Y}^*|\left(\mathbf{X}^*>\mathbf{0}\right)$, where $\binom{\mathbf{X}^*}{\mathbf{Y}^*}\stackrel{d}{=}\binom{\mathbf{X}}{\mathbf{Y}_2}\Big|\left(\mathbf{Y}_1=\mathbf{x}\right)$ and so, the required result is obtained from Lemma 2.2 and Definition 3.1. □

Using Lemma 3.2, it can be shown that the MGF of $\mathbf{U}\sim SSGH_{n,m}\left(\boldsymbol{\theta}\right)$ is given by

$$\begin{aligned} M_{SSGH_{n,m}}\left(\mathbf{t}\right) &= \frac{c(\lambda,\chi,\psi)}{c(\lambda,\chi,\psi-2\mathbf{t}^T\boldsymbol{\tau}_{\mathbf{Y}}-\mathbf{t}^T\Sigma_{\mathbf{YY}}\mathbf{t})}e^{\mathbf{t}^T\boldsymbol{\mu}_{\mathbf{Y}}} \\ &\quad\times\frac{F_{GH_m}(\boldsymbol{\mu}_{\mathbf{X}};-\boldsymbol{\tau}_{\mathbf{X}}-\Sigma_{\mathbf{XY}}\mathbf{t},\Sigma_{\mathbf{XX}},\lambda,\chi,\psi-2\mathbf{t}^T\boldsymbol{\tau}_{\mathbf{Y}}-\mathbf{t}^T\Sigma_{\mathbf{YY}}\mathbf{t})}{F_{GH_m}(\boldsymbol{\mu}_{\mathbf{X}};-\boldsymbol{\tau}_{\mathbf{X}},\Sigma_{\mathbf{XX}},\lambda,\chi,\psi)} \end{aligned} \tag{16}$$

for $\psi - 2\mathbf{t}^T\boldsymbol{\tau}_{\mathbf{Y}} - \mathbf{t}^T\Sigma_{\mathbf{YY}}\mathbf{t} > 0$, where $c(\lambda, \chi, \psi)$ is defined in Equation (1).

3.1. Truncated selected-GH distribution

Let $\mathbf{U} = (U_1, \ldots, U_n)^T$ be the selected-GH random vector in Equation (13) and $\mathbf{C} \subset \mathbb{R}^n$. Then, the random vector $\mathbf{U}^{\mathbf{C}}$ has truncated selected-GH (TSGH) distribution by the truncation set $\mathbf{C}$, denoted by $\mathbf{U}^{\mathbf{C}} \sim TSGH_{n,m}(\boldsymbol{\theta}, \mathbf{C})$, if

$$\begin{aligned}\mathbf{U}^{\mathbf{C}} &\stackrel{d}{=} \mathbf{U} \mid (\mathbf{U} \in \mathbf{C}) \\ &\stackrel{d}{=} \mathbf{Y} \mid (\mathbf{X} > \mathbf{0}, \mathbf{Y} \in \mathbf{C})\end{aligned} \tag{17}$$

where $\left(\mathbf{X}^T, \mathbf{Y}^T\right)^T$ has the GH distribution in Equation (5). The pdf of $\mathbf{U}^{\mathbf{C}}$ is

$$\begin{aligned}f_{TSGH_{n,m}}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{C}) = &f_{GH_n}(\mathbf{x}; \boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\tau}_{\mathbf{Y}}, \Sigma_{\mathbf{YY}}, \lambda, \chi, \psi) \\ &\times \frac{F_{GH_m}(\boldsymbol{\mu}_{\mathbf{X.Y}}; -\boldsymbol{\tau}_{\mathbf{X.Y}}, \Sigma_{\mathbf{XX.Y}}, \lambda_{\mathbf{X.Y}}, \chi_{\mathbf{X.Y}}, \psi_{\mathbf{X.Y}})}{P(\mathbf{X} > \mathbf{0}, \mathbf{Y} \in \mathbf{C})}, \quad \mathbf{x} \in \mathbf{C}\end{aligned} \tag{18}$$

In the sequel, consider the following subsets of $\mathbb{R}^n$, for $n = 2, 3, \ldots$,

$$\begin{aligned}\mathbf{C}_n &= \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{D}_n\mathbf{y} > \mathbf{0}\} \\ \mathbf{A}_n^t &= \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{D}_n\mathbf{y} > \mathbf{0},\ y_n < t\} \\ \mathbf{B}_n^t &= \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{D}_n\mathbf{y} > \mathbf{0},\ y_1 > t\}\end{aligned} \tag{19}$$

where $\mathbf{D}_n \in \mathbb{R}^{(n-1)\times n}$ is a difference matrix such that $\mathbf{D}_n\mathbf{Y} = (Y_2 - Y_1, Y_3 - Y_2, \ldots, Y_n - Y_{n-1})^T$, i.e., the i-th row of $\mathbf{D}_n$ is $\mathbf{e}_{i+1,n}^T - \mathbf{e}_{i,n}^T$, $i = 1, \ldots, n-1$, where $\mathbf{e}_{1,n}, \ldots, \mathbf{e}_{n,n}$ are the n-dimensional unit basis vectors.

Invoking Equation (18), the pdf of $\mathbf{U}^{\mathbf{C}_n}$ is

$$\begin{aligned}f_{TSGH_{n,m}}(\mathbf{x}; \boldsymbol{\theta}, \mathbf{C}_n) = &f_{GH_n}(\mathbf{x}; \boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\tau}_{\mathbf{Y}}, \Sigma_{\mathbf{YY}}, \lambda, \chi, \psi) \\ &\times \frac{F_{GH_m}(\boldsymbol{\mu}_{\mathbf{X.Y}}; -\boldsymbol{\tau}_{\mathbf{X.Y}}, \Sigma_{\mathbf{XX.Y}}, \lambda_{\mathbf{X.Y}}, \chi_{\mathbf{X.Y}}, \psi_{\mathbf{X.Y}})}{F_{GH_{m+n-1}}(\boldsymbol{\mu}_{\mathbf{c}}; -\boldsymbol{\tau}_{\mathbf{c}}, \Sigma_{\mathbf{c}}, \lambda, \chi, \psi)}, \quad \mathbf{x} \in \mathbf{C}_n\end{aligned}$$

where

$$\boldsymbol{\mu}_{\mathbf{C}} = \begin{pmatrix}\boldsymbol{\mu}_{\mathbf{X}} \\ \mathbf{D}_n\boldsymbol{\mu}_{\mathbf{Y}}\end{pmatrix}, \boldsymbol{\tau}_{\mathbf{C}} = \begin{pmatrix}\boldsymbol{\tau}_{\mathbf{X}} \\ \mathbf{D}_n\boldsymbol{\tau}_{\mathbf{Y}}\end{pmatrix}, \ \Sigma_{\mathbf{C}} = \begin{pmatrix}\Sigma_{\mathbf{XX}} & \Sigma_{\mathbf{XY}}\mathbf{D}_n^T \\ & \mathbf{D}_n\Sigma_{\mathbf{YY}}\mathbf{D}_n^T\end{pmatrix} \tag{20}$$

In the following proposition, the distribution of TSGH random vectors $\mathbf{U}^{\mathbf{C}_n}$, $\mathbf{U}^{\mathbf{A}_n^t}$ and $\mathbf{U}^{\mathbf{B}_n^t}$ are expressed as the SSGH distributions.

Proposition 3.5. *We have*

(i) $TSGH_{n,m}(\boldsymbol{\theta}, \mathbf{C}_n) \equiv SSGH_{n,m+n-1}\left(\boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\mu}_{\mathbf{C}}, \boldsymbol{\tau}_{\mathbf{Y}}, \boldsymbol{\tau}_{\mathbf{C}}, \Sigma_{\mathbf{YY}}, \Sigma_{\mathbf{C}}, \Delta_{\mathbf{C}}, \lambda, \chi, \psi\right)$;

(ii) $TSGH_{n,m}\left(\boldsymbol{\theta}, \mathbf{A}_n^t\right) \equiv SSGH_{n,m+n}(\boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\mu}_{\mathbf{A}}, \boldsymbol{\tau}_{\mathbf{Y}}, \boldsymbol{\tau}_{\mathbf{A}}, \Sigma_{\mathbf{YY}}, \Sigma_{\mathbf{A}}, \Delta_{\mathbf{A}}, \lambda, \chi, \psi)$;

(iii) $TSGH_{n,m}\left(\boldsymbol{\theta}, \mathbf{B}_n^t\right) \equiv SSGH_{n,m+n}(\boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\mu}_{\mathbf{B}}, \boldsymbol{\tau}_{\mathbf{Y}}, \boldsymbol{\tau}_{\mathbf{B}}, \Sigma_{\mathbf{YY}}, \Sigma_{\mathbf{B}}, \Delta_{\mathbf{B}}, \lambda, \chi, \psi)$,

where $\boldsymbol{\mu}_{\mathbf{C}}$, $\boldsymbol{\tau}_{\mathbf{C}}$ *and* $\Sigma_{\mathbf{C}}$ *are defined in (20) and*

$$\boldsymbol{\mu}_{\mathbf{A}} = \begin{pmatrix}\boldsymbol{\mu}_{\mathbf{C}} \\ t - \mu_{Y\,n}\end{pmatrix}, \ \boldsymbol{\mu}_{\mathbf{B}} = \begin{pmatrix}\boldsymbol{\mu}_{\mathbf{C}} \\ \mu_{Y,1} - t\end{pmatrix}, \qquad \boldsymbol{\tau}_{\mathbf{A}} = \begin{pmatrix}\boldsymbol{\tau}_{\mathbf{C}} \\ -\tau_{Y,n}\end{pmatrix}$$

$$\boldsymbol{\tau}_{\mathbf{B}} = \begin{pmatrix}\boldsymbol{\tau}_{\mathbf{C}} \\ \tau_{Y,1}\end{pmatrix}, \quad \Sigma_{\mathbf{A}} = \begin{pmatrix}\Sigma_{\mathbf{C}} & -\Delta_{\mathbf{C},n} \\ & \sigma_{YY,nn}\end{pmatrix}, \quad \Sigma_{\mathbf{B}} = \begin{pmatrix}\Sigma_{\mathbf{C}} & \Delta_{\mathbf{C},1} \\ & \sigma_{YY,11}\end{pmatrix}$$

$$\Delta_{\mathbf{C}} = \begin{pmatrix}\Sigma_{\mathbf{XY}} \\ \mathbf{D}_n\Sigma_{\mathbf{YY}}\end{pmatrix}, \quad \Delta_{\mathbf{A}} = \begin{pmatrix}\Delta_{\mathbf{C}} \\ -\Sigma_{\mathbf{YY},n}^T\end{pmatrix}, \qquad \Delta_{\mathbf{B}} = \begin{pmatrix}\Delta_{\mathbf{C}} \\ \Sigma_{\mathbf{YY},1}^T\end{pmatrix}$$

when $\Delta_{\mathbf{C},i}$ *and* $\Sigma_{\mathbf{YY},i}$ *are i-th column of* $\Delta_{\mathbf{C}}$ *and* $\Sigma_{\mathbf{YY}}$, *respectively.*

Proof. From Equations (17) and (19), we see that

$$\mathbf{U}^{\mathbf{C}_n} \stackrel{d}{=} \mathbf{Y} \mid (\mathbf{X}_{\mathbf{C}} > \mathbf{0}), \quad \mathbf{U}^{\mathbf{A}_n^t} \stackrel{d}{=} \mathbf{Y} \mid (\mathbf{X}_{\mathbf{A}} > \mathbf{0}), \quad \mathbf{U}^{\mathbf{B}_n^t} \stackrel{d}{=} \mathbf{Y} \mid (\mathbf{X}_{\mathbf{B}} > \mathbf{0}) \tag{21}$$

where

$$\mathbf{X}_{\mathbf{C}} = \begin{pmatrix} \mathbf{X} \\ \mathbf{D}_n \mathbf{Y} \end{pmatrix}, \quad \mathbf{X}_{\mathbf{A}} = \begin{pmatrix} \mathbf{X}_{\mathbf{C}} \\ t - Y_n \end{pmatrix}, \quad \mathbf{X}_{\mathbf{B}} = \begin{pmatrix} \mathbf{X}_{\mathbf{C}} \\ Y_1 - t \end{pmatrix}$$

So

$$\begin{pmatrix} \mathbf{X}_{\mathbf{C}} \\ \mathbf{Y} \end{pmatrix} \sim GH_{m+n-1}\left(\begin{pmatrix} \boldsymbol{\mu}_{\mathbf{C}} \\ \boldsymbol{\mu}_{\mathbf{Y}} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\tau}_{\mathbf{C}} \\ \boldsymbol{\tau}_{\mathbf{Y}} \end{pmatrix}, \begin{pmatrix} \Sigma_{\mathbf{C}} & \Delta_{\mathbf{C}} \\ & \Sigma_{\mathbf{YY}} \end{pmatrix}, \lambda, \chi, \psi \right)$$

which $\Sigma_{\mathbf{C}}$ and $\Sigma_{\mathbf{YY}}$ is non singular, but $\begin{pmatrix} \Sigma_{\mathbf{C}} & \Delta_{\mathbf{C}} \\ & \Sigma_{\mathbf{YY}} \end{pmatrix}$ is singular. Therefore, in view of Definition 3.1, the case (i) is obtained. The other two cases can be derived in a similar manner. □

In the following proposition, some conditional distributions are presented for the random vector $\mathbf{U}^{\mathbf{C}_n}$, that will be useful for the subsequent discussions in the next section.

Proposition 3.6. *For $\mathbf{U}^{\mathbf{C}_n} = (W_1, \ldots, W_n)^T$ and $t \in \mathbb{R}$, we have*
(i) $\mathbf{W}_{-n} | (W_n = t) \sim TSGH_{n-1,m}\left(\boldsymbol{\theta}_n, \mathbf{A}_{n-1}^t\right)$;
(ii) $\mathbf{W}_{-1} | (W_1 = t) \sim TSGH_{n-1,m}\left(\boldsymbol{\theta}_1, \mathbf{B}_{n-1}^t\right)$,
where $\boldsymbol{\theta}_i = \left(\boldsymbol{\mu}_i^{(2)}, \boldsymbol{\mu}_i^{(1)}, \boldsymbol{\tau}_i^{(2)}, \boldsymbol{\tau}_i^{(1)}, \Sigma_i^{(22)}, \Sigma_i^{(11)}, \Sigma_i^{(12)}, \lambda - \frac{1}{2}, \chi_i, \psi_i\right)$, *for* $i = 1$, *and* $i = n$,

$$\begin{aligned}
\boldsymbol{\mu}_i^{(2)} &= \boldsymbol{\mu}_{\mathbf{Y},-i} + \boldsymbol{\sigma}_{\mathbf{YY},-ii}\, \sigma_{\mathbf{YY},ii}^{-1} \left(t - \mu_{\mathbf{Y},i}\right), \quad \boldsymbol{\mu}_i^{(1)} = \boldsymbol{\mu}_{\mathbf{X}} + \Sigma_{\mathbf{XY},i}\, \sigma_{\mathbf{YY},ii}^{-1} \left(t - \mu_{\mathbf{Y},i}\right) \\
\boldsymbol{\tau}_i^{(2)} &= \boldsymbol{\tau}_{\mathbf{Y},-i} - \boldsymbol{\sigma}_{\mathbf{YY},-ii}\, \sigma_{\mathbf{YY},ii}^{-1} \tau_{\mathbf{Y},i}, \quad \boldsymbol{\tau}_i^{(1)} = \boldsymbol{\tau}_{\mathbf{X}} - \Sigma_{\mathbf{XY},i}\, \sigma_{\mathbf{YY},ii}^{-1} \tau_{\mathbf{Y},i} \\
\Sigma_i^{(22)} &= \Sigma_{\mathbf{YY},-i-i} - \boldsymbol{\sigma}_{\mathbf{YY},-ii}\, \sigma_{\mathbf{YY},ii}^{-1}\, \boldsymbol{\sigma}_{\mathbf{YY},-ii}^T, \quad \Sigma_i^{(11)} = \Sigma_{\mathbf{XX}} - \Sigma_{\mathbf{XY},i}\, \sigma_{\mathbf{YY},ii}^{-1}\, \Sigma_{\mathbf{XY},i}^T \\
\Sigma_i^{(12)} &= \Sigma_{\mathbf{XY},-i} - \Sigma_{\mathbf{XY},i}\, \sigma_{\mathbf{YY},ii}^{-1}\, \boldsymbol{\sigma}_{\mathbf{YY},-ii}^T, \quad \chi_i = \chi + \frac{\left(t - \mu_{\mathbf{Y},i}\right)^2}{\sigma_{\mathbf{YY},ii}}, \quad \psi_i = \psi + \frac{\tau_{\mathbf{Y},i}^2}{\sigma_{\mathbf{YY},ii}}
\end{aligned}$$

$\Sigma_{\mathbf{XY},i}$ is i-th column of $\Sigma_{\mathbf{XY}}$ and $\Sigma_{\mathbf{XY},-i}$ is obtained from $\Sigma_{\mathbf{XY}}$ by deleting its i-th column.

Proof. Employing the representation in Equation (17), we have

$$\begin{aligned}
\mathbf{W}_{-n} \mid (W_n = t) &\stackrel{d}{=} \mathbf{Y}_{-n} | \left(\mathbf{X} > \mathbf{0}, Y_n = t,\ \mathbf{Y}_{-n} \in \mathbf{A}_{n-1}^t\right) \stackrel{d}{=} \mathbf{V}_{-n} | \left(\mathbf{V}_{-n} \in \mathbf{A}_{n-1}^t\right) \\
\mathbf{W}_{-1} \mid (W_1 = t) &\stackrel{d}{=} \mathbf{Y}_{-1} | \left(\mathbf{X} > \mathbf{0}, Y_1 = t,\ \mathbf{Y}_{-1} \in \mathbf{B}_{n-1}^t\right) \stackrel{d}{=} \mathbf{V}_{-1} | \left(\mathbf{V}_{-1} \in \mathbf{B}_{n-1}^t\right)
\end{aligned}$$

where $\mathbf{V}_{-i} \stackrel{d}{=} \mathbf{Y}_{-i} | (\mathbf{X} > \mathbf{0}, Y_i = t)$. Applying Lemma 2.2 and representation in Equation (13), we deduce $\mathbf{V}_{-i} \sim SGH_{n-1,m}(\boldsymbol{\theta}_i)$. Then, the results are achieved invoking Equation (17). □

Using Equation (16) and Propositions 3.5 and 3.6, the following expressions are derived for the MGFs of W_1 given $W_n = t$ and W_n given $W_1 = t$ as,

$$\begin{aligned}
M_{W_1|W_n=t}(s) &= \frac{c(\lambda - \frac{1}{2}, \chi_n, \psi_n) F_{GH_{n+m-1}}(\boldsymbol{\mu}_{\mathbf{A}}; \boldsymbol{\theta}_n(s))}{c(\lambda - \frac{1}{2}, \chi_n, \psi_n - 2s\tau_n - s^2\sigma_n) F_{GH_m}(\boldsymbol{\mu}_{\mathbf{A}}; \boldsymbol{\theta}_n(0))} e^{s\mu_n(t)}, \quad \psi_n > 2s\tau_n + s^2\sigma_n \\
M_{W_n|W_1=t}(s) &= \frac{c(\lambda - \frac{1}{2}, \chi_1, \psi_1) F_{GH_{n+m-1}}(\boldsymbol{\mu}_{\mathbf{B}}; \boldsymbol{\theta}_1(s))}{c(\lambda - \frac{1}{2}, \chi_1, \psi_1 - 2s\tau_1 - s^2\sigma_1) F_{GH_{n+m-1}}(\boldsymbol{\mu}_{\mathbf{B}}; \boldsymbol{\theta}_1(0))} e^{s\mu_1(t)}, \quad \psi_1 > 2s\tau_1 + s^2\sigma_1
\end{aligned} \tag{22}$$

where χ_i and ψ_i are defined in Proposition 3.6, $\mu_i(t) = \mu_{Y,n-i+1} + \frac{\sigma_{YY,n1}}{\sigma_{YY,ii}}\left(t - \mu_{Y,i}\right)$,

$$\tau_i = \tau_{Y,n-i+1} - \frac{\sigma_{YY,n1}}{\sigma_{YY,ii}}\tau_{Y,i}\,, \quad \sigma_i = \sigma_{YY,n-i+1\ n-i+1} - \frac{\sigma^2_{YY,n1}}{\sigma_{YY,ii}}$$

and

$$\begin{aligned}\boldsymbol{\theta}_n(s) &= \left(-\boldsymbol{\tau}_{\mathbf{A}} - s\Delta_{\mathbf{A},1}, \Sigma_{\mathbf{A}}, \lambda - \tfrac{1}{2}, \chi_n, \psi_n - 2s\tau_n - s^2\sigma_n\right)\\ \boldsymbol{\theta}_1(s) &= \left(-\boldsymbol{\tau}_{\mathbf{B}} - s\Delta_{\mathbf{B},n-1}, \Sigma_{\mathbf{B}}, \lambda - \tfrac{1}{2}, \chi_1, \psi_1 - 2s\tau_1 - s^2\sigma_1\right)\end{aligned}$$

where $\left(\boldsymbol{\mu}_{\mathbf{A}}, \boldsymbol{\tau}_{\mathbf{A}}, \Delta_{\mathbf{A},1}, \Sigma_{\mathbf{A}}\right)$ and $\left(\boldsymbol{\mu}_{\mathbf{B}}, \boldsymbol{\tau}_{\mathbf{B}}, \Delta_{\mathbf{B},n-1}, \Sigma_{\mathbf{B}}\right)$ are obtained from Proposition 3.5, respectively, substituting $\boldsymbol{\theta} = \boldsymbol{\theta}_n$ and $\boldsymbol{\theta} = \boldsymbol{\theta}_1$ in Proposition 3.6.

Also, the following corollary is obtained from Equations (17) and (19).

Corollary 3.7. *For* $\mathbf{U}^{\mathbf{C}_n} = (W_1, \ldots, W_n)^T$ *and* $t \in \mathbb{R}$,

(i) $\mathbf{U}^{\mathbf{C}_n} | (W_1 > t) \sim TSGH_{n,m}\left(\boldsymbol{\theta}, \mathbf{B}_n^t\right)$;

(ii) $\mathbf{U}^{\mathbf{C}_n} | (W_n < t) \sim TSGH_{n,m}\left(\boldsymbol{\theta}, \mathbf{A}_n^t\right)$.

Applying Corollary 3.7 and Proposition 3.5 and using Equation (16), explicit expressions for the MGFs of $W_n | (W_1 > t)$ and $W_1 | (W_n < t)$ can be easily derived as,

$$\begin{aligned}M_{W_n|(W_1>t)}(s) &= \frac{c(\lambda,\chi,\psi)}{c(\lambda,\chi,\psi - 2\tau_{Y,n}s - \sigma_{YY,nn}s^2)} e^{s\mu_{Y,n}}\\ &\quad\times \frac{F_{GH_{m+n}}\left(\boldsymbol{\mu}_{\mathbf{B}}; -\boldsymbol{\tau}_{\mathbf{B}} - s\ \Delta_{\mathbf{B},n}, \Sigma_{\mathbf{B}}, \lambda, \chi, \psi - 2\tau_{Y,n}s - \sigma_{YY,nn}s^2\right)}{F_{GH_{m+n}}\left(\boldsymbol{\mu}_{\mathbf{B}}; -\boldsymbol{\tau}_{\mathbf{B}}, \Sigma_{\mathbf{B}}, \lambda, \chi, \psi\right)}\end{aligned} \tag{23}$$

for $\psi > 2\tau_{Y,n}s + \sigma_{YY,nn}s^2$, and

$$\begin{aligned}M_{W_1|(W_n<t)}(s) &= \frac{c(\lambda,\chi,\psi)}{c(\lambda,\chi,\psi - 2s\ \tau_{Y,1} - \sigma_{YY,11}s^2)} e^{s\mu_{Y,1}}\\ &\quad\times \frac{F_{GH_{m+n}}\left(\boldsymbol{\mu}_{\mathbf{A}}; -\boldsymbol{\tau}_{\mathbf{A}} - s\ \Delta_{\mathbf{A},1}, \Sigma_{\mathbf{A}}, \lambda, \chi, \psi - 2\tau_{Y,1}s - \sigma_{YY,11}s^2\right)}{F_{GH_{m+n}}\left(\boldsymbol{\mu}_{\mathbf{A}}; -\boldsymbol{\tau}_{\mathbf{A}}, \Sigma_{\mathbf{A}}, \lambda, \chi, \psi\right)}\end{aligned} \tag{24}$$

for $\psi > 2\tau_{Y,1}s + \sigma_{YY,11}s^2$, and the other parameters are similar to those in Proposition 3.5.

4. The joint distribution of consecutive order statistics from the multivariate GH distribution

Let $\mathbf{X} = (X_1, \ldots, X_n)^T \sim GH_n(\boldsymbol{\mu}, \boldsymbol{\tau}, \Sigma, \lambda, \chi, \psi)$ and $\mathbf{X}_{(1,\ldots,n)} = \left(X_{(1)}, \ldots, X_{(n)}\right)^T$ be the vector of order statistic from $\mathbf{X}$ and $\mathbf{X}_{(r,\ldots,r+k)} = \left(X_{(r)}, \ldots, X_{(r+k)}\right)^T$, for positive integers r and k such that $r + k = 2, \ldots, n$. Define the index sets $\mathbf{I}^{(k)}$ and $\mathbf{J}_r$ as follows

$$\begin{aligned}\mathbf{I}^{(k)} &= \left\{\mathbf{i} = \left(i_1, \ldots, i_{k+1}\right); 1 \le i_d \ne i_l \le n\right\}\\ \mathbf{J}_r &= \left\{\mathbf{j}_r = \left(j_1 \ldots j_{r-1}\right); 1 \le j_1 < \cdots < j_{r-1} \le n - k - 1\right\}\end{aligned}$$

and $\mathbf{S}_{\mathbf{j}_r} = \operatorname{diag}\left(s_1, \ldots, s_{n-k-1}\right)$, where $s_i = 1$ if $i \in \mathbf{j}_r$, otherwise $s_i = -1$. In particular, we set $\mathbf{S}_{\mathbf{j}_n} = \mathbf{I}_{n-k-1}$ and $\mathbf{S}_{\mathbf{j}_1} = -\mathbf{I}_{n-k-1}$. Further, define the vectors

$$\mathbf{p}_{r,k} = \left(\mathbf{1}_{r-1}^T, \mathbf{0}_{n-k-r}^T\right)^T, \quad \mathbf{q}_{r,k} = \left(\mathbf{0}_{r-1}^T, \mathbf{1}_{n-k-r}^T\right)^T$$

In this section, a mixture representation for the distribution of $\mathbf{X}_{(r,\ldots,r+k)}$ in terms of TSGH distributions is presented. To this end, let $\mathbf{X}$ be partitioned as $\mathbf{X} = \left(\mathbf{X}_{\mathbf{i}}^T, \mathbf{X}_{-\mathbf{i}}^T\right)^T$ and introduce the corresponding partitions of $\boldsymbol{\mu}$, $\boldsymbol{\tau}$ and $\boldsymbol{\Sigma}$ as follows:

$$\boldsymbol{\mu} = \begin{pmatrix}\boldsymbol{\mu}_{\mathbf{i}}\\ \boldsymbol{\mu}_{-\mathbf{i}}\end{pmatrix}, \boldsymbol{\tau} = \begin{pmatrix}\boldsymbol{\tau}_{\mathbf{i}}\\ \boldsymbol{\tau}_{-\mathbf{i}}\end{pmatrix}, \Sigma = \begin{pmatrix}\Sigma_{\mathbf{ii}} & \Sigma_{-\mathbf{ii}}\\ & \Sigma_{-\mathbf{i}-\mathbf{i}}\end{pmatrix}$$

Proposition 4.1. *The cdf of* $\mathbf{X}_{(r,\ldots,r+k)}$ *is the mixture*

$$F_{\mathbf{X}_{(r,\ldots,r+k)}}(\mathbf{x}) = \sum_{\mathbf{i}\in\mathbf{I}^{(k)}}\sum_{\mathbf{j}\in\mathbf{J}_r} P_{\mathbf{ij}} F_{TSGH_{k+1,n-k-1}}\left(\mathbf{x};\boldsymbol{\theta}_{\mathbf{ij}},\mathbf{C}_{k+1}\right),\ \mathbf{x}\in\mathbb{R}^{k+1}$$

where $\boldsymbol{\theta}_{\mathbf{ij}} = \left(\boldsymbol{\mu}_{\mathbf{i}},\boldsymbol{\eta}_{\mathbf{ij}},\boldsymbol{\tau}_{\mathbf{i}},\boldsymbol{\gamma}_{\mathbf{ij}},\boldsymbol{\Sigma}_{\mathbf{ii}},\boldsymbol{\Gamma}_{\mathbf{ij}},\boldsymbol{\Delta}_{\mathbf{ij}},\lambda,\chi,\psi\right)$ *and*

$$\begin{aligned}
\boldsymbol{\eta}_{\mathbf{ij}} &= \mathbf{S}_{\mathbf{j}}\left\{\mathbf{A}_{r,k}\boldsymbol{\mu}_{i_1 i_{k+1}} - \boldsymbol{\mu}_{-\mathbf{i}}\right\},\boldsymbol{\gamma}_{\mathbf{ij}} = \mathbf{S}_{\mathbf{j}}\left\{\mathbf{A}_{r,k}\boldsymbol{\tau}_{i_1 i_{k+1}} - \boldsymbol{\tau}_{-\mathbf{i}}\right\}\\
\boldsymbol{\Delta}_{\mathbf{ij}} &= \mathbf{S}_{\mathbf{j}}\left\{\mathbf{A}_{r,k}\boldsymbol{\sigma}_{i_1 i_{k+1}\mathbf{i}} - \boldsymbol{\Sigma}_{-\mathbf{ii}}\right\}\\
\boldsymbol{\Gamma}_{\mathbf{ij}} &= \mathbf{S}_{\mathbf{j}}\left\{\mathbf{A}_{r,k}\boldsymbol{\sigma}_{i_1 i_{k+1}}\mathbf{A}_{r,k}^T + \boldsymbol{\Sigma}_{-\mathbf{i}-\mathbf{i}} - \mathbf{A}_{r,k}\boldsymbol{\sigma}_{-i_1 i_{k+1}\mathbf{i}} - \boldsymbol{\sigma}_{-i_1 i_{k+1}\mathbf{i}}^T\mathbf{A}_{r,k}^T\right\}\mathbf{S}_{\mathbf{j}}
\end{aligned}$$

where $\mathbf{A}_{r,k} = (\mathbf{p}_{r,k},\mathbf{q}_{r,k})$, $\boldsymbol{\sigma}_{i_1 i_{k+1}}$ *is* (i_1,i_{k+1})*-th rows and* (i_1,i_{k+1})*-th columns of* $\boldsymbol{\Sigma}_{\mathbf{ii}}$, $\boldsymbol{\sigma}_{i_1 i_{k+1}\mathbf{i}}$ *is* (i_1,i_{k+1})*-th rows of* $\boldsymbol{\Sigma}_{\mathbf{ii}}$, $\boldsymbol{\sigma}_{-i_1 i_{k+1}\mathbf{i}}$ *is* (i_1,i_{k+1})*-th rows of* $\boldsymbol{\Sigma}_{-\mathbf{ii}}$ *and* $P_{\mathbf{ij}} = F_{GH_{n-1}}\left(\boldsymbol{\eta}^*_{\mathbf{ij}};-\boldsymbol{\gamma}^*_{\mathbf{ij}},\boldsymbol{\Gamma}^*_{\mathbf{ij}},\lambda,\chi,\psi\right)$ *where*

$$\boldsymbol{\eta}^*_{\mathbf{ij}} = \begin{pmatrix}\boldsymbol{\eta}_{\mathbf{ij}}\\ \mathbf{D}_{k+1}\boldsymbol{\mu}_{\mathbf{i}}\end{pmatrix},\quad \boldsymbol{\gamma}^*_{\mathbf{ij}} = \begin{pmatrix}\boldsymbol{\gamma}_{\mathbf{ij}}\\ \mathbf{D}_{k+1}\boldsymbol{\tau}_{\mathbf{i}}\end{pmatrix},\quad \boldsymbol{\Gamma}^*_{\mathbf{ij}} = \begin{pmatrix}\boldsymbol{\Gamma}_{\mathbf{ij}} & \left(\mathbf{D}_{k+1}\boldsymbol{\Delta}_{\mathbf{ij}}\right)^T\\ & \mathbf{D}_{k+1}\boldsymbol{\Sigma}_{\mathbf{ii}}\mathbf{D}_{k+1}^T\end{pmatrix}$$

Proof. Let $\mathbf{B}_{\mathbf{ij}} = \{\max\{\mathbf{X}_{\mathbf{j}}\} < X_{i_1} < \cdots < X_{i_{k+1}} < \min\{\mathbf{X}_{-\mathbf{i},-\mathbf{j}}\}\}$. Then, we get

$$\begin{aligned}
F_{\mathbf{X}_{(r,\ldots,r+k)}}(\mathbf{x}) &= \sum_{\mathbf{i}\in\mathbf{I}^{(k)}} P\left(\mathbf{X}_{(r,\ldots,r+k)} = \mathbf{X}_{\mathbf{i}}\right)P\left(\mathbf{X}_{\mathbf{i}} \le \mathbf{x}|\mathbf{X}_{(r,\ldots,r+k)} = \mathbf{X}_{\mathbf{i}}\right)\\
&= \sum_{\mathbf{i}\in\mathbf{I}^{(k)}}\sum_{\mathbf{j}\in\mathbf{J}_r} P(\mathbf{B}_{\mathbf{ij}})P\left(\mathbf{X}_{\mathbf{i}} \le \mathbf{x}|\mathbf{B}_{\mathbf{ij}}\right)
\end{aligned}$$

Further, $\mathbf{B}_{\mathbf{ij}}$ can be written as $\mathbf{B}_{\mathbf{ij}} = \left\{\mathbf{U}_{\mathbf{ij}} > \mathbf{0},\ \mathbf{X}_{\mathbf{i}}\in\mathbf{C}_{k+1}\right\}$ where $\mathbf{U}_{\mathbf{ij}} = \mathbf{S}_{\mathbf{j}}\left\{\mathbf{A}_{r,k}\mathbf{X}_{i_1,i_{k+1}} - \mathbf{X}_{-\mathbf{i}}\right\}$. So, we have $P\left(\mathbf{B}_{\mathbf{ij}}\right) = P_{\mathbf{ij}}$ and

$$\begin{pmatrix}\mathbf{U}_{\mathbf{ij}}\\ \mathbf{X}_{\mathbf{i}}\end{pmatrix} \sim GH_n\left(\begin{pmatrix}\boldsymbol{\eta}_{\mathbf{ij}}\\ \boldsymbol{\mu}_{\mathbf{i}}\end{pmatrix},\begin{pmatrix}\boldsymbol{\gamma}_{\mathbf{ij}}\\ \boldsymbol{\tau}_{\mathbf{i}}\end{pmatrix},\begin{pmatrix}\boldsymbol{\Gamma}_{\mathbf{ij}} & \boldsymbol{\Delta}_{\mathbf{ij}}\\ & \boldsymbol{\Sigma}_{\mathbf{ii}}\end{pmatrix},\lambda,\chi,\psi\right)$$

and the result is obtained from Equation (17). □

4.1. Exchangeable GH case

Let $\mathbf{X} = (X_1,\ldots,X_n)^T \sim GH_n\left(\boldsymbol{\mu}_n,\boldsymbol{\tau}_n,\boldsymbol{\Sigma}_n,\lambda,\chi,\psi\right)$ and

$$\boldsymbol{\mu}_n = \mu\mathbf{1}_n,\boldsymbol{\tau}_n = \tau\mathbf{1}_n,\quad \boldsymbol{\Sigma}_n = \sigma^2\left\{(1-\rho)\,\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n^T\right\} \tag{25}$$

where $\mu,\tau\in\mathbb{R}$, $\sigma > 0$, $\frac{-1}{n-1} < \rho < 1$. Now, from the exchangeability of $\mathbf{X} = (X_1,\ldots,X_n)^T$ and applying Proposition 4.1, after some simple calculations, the following result can be obtained.

Corollary 4.2. *If* $\mathbf{X} \sim GH_n\left(\boldsymbol{\mu}_n,\boldsymbol{\tau}_n,\boldsymbol{\Sigma}_n,\lambda,\chi,\psi\right)$*, then we have*

$$\mathbf{X}_{(r,\ldots,r+k)} \sim TSGH_{k+1,n-k-1}\left(\boldsymbol{\theta}_{r,k},\mathbf{C}_{k+1}\right)$$

where $\boldsymbol{\theta}_{r,k} = \left(\boldsymbol{\mu}_{k+1},\mathbf{0}_{n-k-1},\boldsymbol{\tau}_{k+1},\mathbf{0}_{n-k-1},\boldsymbol{\Sigma}_{k+1},\boldsymbol{\Gamma}_{r,k},\boldsymbol{\Lambda}_{r,k},\lambda,\chi,\psi\right)$ *and*

$$\begin{aligned}
\boldsymbol{\Gamma}_{r,k} &= \sigma^2(1-\rho)\left(\mathbf{I}_{n-k-1} + \mathbf{p}_{r,k}\mathbf{p}_{r,k}^T + \mathbf{q}_{r,k}\mathbf{q}_{r,k}^T\right)\\
\boldsymbol{\Lambda}_{r,k} &= \sigma^2(1-\rho)\left(\mathbf{p}_{r,k},\mathbf{0}_{n-k-1\times k-1},-\mathbf{q}_{r,k}\right)
\end{aligned}$$

Using Corollary 4.2 and Equation (22), the MGF of $X_{(r)}$ given $\left\{X_{(r+k)}=t\right\}$ and the MGF of $X_{(r+k)}$ given $\left\{X_{(r)}=t\right\}$ are obtained, for $\psi+\frac{\tau^2}{\sigma^2}>2\tau\left(1-\rho\right)s+\sigma^2\left(1-\rho^2\right)s^2$, as follows:

$$
\begin{aligned}
M_{X_{(r)}|(X_{(r+k)}=t)}\left(s\right) &= a\left(s,t\right)\frac{F_{GH_{n-1}}\left((t-\mu)\boldsymbol{\mu}^{(1)}_{r,k};\boldsymbol{\theta}^{(1)}_{r,k}(s,t)\right)}{F_{GH_{n-1}}\left((t-\mu)\boldsymbol{\mu}^{(1)}_{r,k};\boldsymbol{\theta}^{(1)}_{r,k}(0,t)\right)}\\
M_{X_{(r+k)}|(X_{(r)}=t)}\left(s\right) &= a\left(s,t\right)\frac{F_{GH_{n-1}}\left((t-\mu)\boldsymbol{\mu}^{(2)}_{r,k};\boldsymbol{\theta}^{(2)}_{r,k}(s,t)\right)}{F_{GH_{n-1}}\left((t-\mu)\boldsymbol{\mu}^{(2)}_{r,k};\boldsymbol{\theta}^{(2)}_{r,k}(0,t)\right)}
\end{aligned}
\tag{26}
$$

where

$$
a\left(s,t\right)=\frac{c\left(\lambda-\frac{1}{2},\chi+\left(\frac{t-\mu}{\sigma}\right)^2,\psi+\left(\frac{\tau}{\sigma}\right)^2\right)e^{s(\mu+\rho(t-\mu))}}{c\left(\lambda-\frac{1}{2},\chi+\left(\frac{t-\mu}{\sigma}\right)^2,\psi+\left(\frac{\tau}{\sigma}\right)^2-2\frac{1-\rho}{1+\rho}\tau s\;-\sigma^2\frac{(1-\rho)(1+2\rho)}{1+\rho}s^2\right)}
\tag{27}
$$

and

$$
\begin{aligned}
\boldsymbol{\theta}^{(i)}_{r,k}\left(s,t\right)=&\left(\tau\boldsymbol{\mu}^{(i)}_{r,k}-s\boldsymbol{\delta}^{(i)}_{r,k},\Sigma^{(ii)}_{r,k},\lambda-\frac{1}{2},\chi+\left(\frac{t-\mu}{\sigma}\right)^2,\psi+\left(\frac{\tau}{\sigma}\right)^2-2\frac{1-\rho}{1+\rho}\tau s\right.\\
&\left.-\sigma^2\frac{(1-\rho)(1+2\rho)}{1+\rho}s^2\right)
\end{aligned}
\tag{28}
$$

with

$$
\begin{aligned}
\boldsymbol{\mu}^{(1)}_{r,k}&=\left(1-\rho\right)\left(-\mathbf{q}^T_{r,k},\mathbf{e}^T_{k,k}\right)^T \qquad \boldsymbol{\mu}^{(2)}_{r,k}=\left(1-\rho\right)\left(\mathbf{p}^T_{r,k},-\mathbf{e}^T_{k,k}\right)^T\\
\boldsymbol{\delta}^{(1)}_{r,k}&=\sigma^2\left(1-\rho\right)\left(\mathbf{p}^T_{r,k}+\rho\mathbf{q}^T_{r,k},\mathbf{e}^T_{1,k-1},\rho\right)^T\\
\boldsymbol{\delta}^{(2)}_{r,k}&=\sigma^2\left(1-\rho\right)\left(-\rho\mathbf{p}^T_{r,k}-\mathbf{q}^T_{r,k},\mathbf{e}^T_{k-1,k-1},\rho\right)^T\\
\Sigma^{(11)}_{r,k}&=\sigma^2\left(1-\rho\right)\begin{pmatrix}\mathbf{I}_{n-k-1}+\mathbf{p}_{r,k}\mathbf{p}^T_{r,k}+\rho\mathbf{q}_{r,k}\mathbf{q}^T_{r,k} & \left(\mathbf{p}_{r,k},\mathbf{0}_{n-k-1\times k-1}\right)\mathbf{D}^T_k & -\rho\mathbf{q}_{r,k}\\ & \mathbf{D}_k\mathbf{D}^T_k & -\mathbf{e}_{k-1,k-1}\\ & & 1+\rho\end{pmatrix}\\
\Sigma^{(22)}_{r,k}&=\sigma^2\left(1-\rho\right)\begin{pmatrix}\mathbf{I}_{n-k-1}+\rho\mathbf{p}_{r,k}\mathbf{p}^T_{r,k}+\mathbf{q}_{r,k}\mathbf{q}^T_{r,k} & \left(\mathbf{0}_{n-k-1\times k-1},-\mathbf{q}_{r,k}\right)\mathbf{D}^T_k & -\rho\mathbf{p}_{r,k}\\ & \mathbf{D}_k\mathbf{D}^T_k & -\mathbf{e}_{1,k-1}\\ & & 1+\rho\end{pmatrix}
\end{aligned}
$$

Also, applying the results of Corollary 4.2 and MGFs in Equations (23) and (24), the MGFs of $X_{(r+k)}$ given $X_{(r)}>t$ and $X_{(r)}$ given $X_{(r+k)}<t$, for $\psi>2\tau s+\sigma^2s^2$, are derived as follows:

$$
\begin{aligned}
M_{X_{(r+k)}|(X_{(r)}>t)}\left(s\right)=&\frac{c(\lambda,\chi,\psi)}{c(\lambda,\chi,\psi-2\tau s-\sigma^2s^2)}e^{s\mu}\\
&\times\frac{F_{GH_n}\left((\mu-t)\mathbf{e}_{n,n};-\tau\mathbf{e}_{n,n}-s\left(\boldsymbol{\gamma}^{(1)T}_{r,k},\rho\sigma^2\right)^T,\Psi^{(11)}_{r,k},\lambda,\chi,\psi-2\tau s-\sigma^2s^2\right)}{F_{GH_n}\left((\mu-t)\mathbf{e}_{n,n};-\tau\mathbf{e}_{n,n},\Psi^{(11)}_{r,k},\lambda,\chi,\psi\right)}
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
M_{X_{(r)}|(X_{(r+k)}<t)}\left(s\right)=&\frac{c(\lambda,\chi,\psi)}{c(\lambda,\chi,\psi-2\tau s-\sigma^2s^2)}e^{s\mu}\\
&\times\frac{F_{GH_n}\left((t-\mu)\mathbf{e}_{n,n};\tau\mathbf{e}_{n,n}-s\left(\boldsymbol{\gamma}^{(2)T}_{r,k},-\rho\sigma^2\right)^T,\Psi^{(22)}_{r,k},\lambda,\chi,\psi-2\tau s-\sigma^2s^2\right)}{F_{GH_n}\left((t-\mu)\mathbf{e}_{n,n};\tau\mathbf{e}_{n,n},\Psi^{(22)}_{r,k},\lambda,\chi,\psi\right)}
\end{aligned}
\tag{30}
$$

where

$$\boldsymbol{\gamma}_{r,k}^{(1)} = \sigma^2 (1-\rho) \left(-\mathbf{q}_{r,k}^T, \mathbf{e}_{k,k}^T\right)^T, \boldsymbol{\gamma}_{r,k}^{(2)} = \sigma^2 (1-\rho) \left(\mathbf{p}_{r,k}^T, -\mathbf{e}_{1,k}^T\right)^T \tag{31}$$

and

$$\Psi_{r,k}^{(11)} = \begin{pmatrix} \mathbf{V}_{r,k} & \boldsymbol{\gamma}_{r,k}^{(2)} \\ & \sigma^2 \end{pmatrix}, \Psi_{r,k}^{(22)} = \begin{pmatrix} \mathbf{V}_{r,k} & -\boldsymbol{\gamma}_{r,k}^{(1)} \\ & \sigma^2 \end{pmatrix}$$

$$\mathbf{V}_{r,k} = \begin{pmatrix} \boldsymbol{\Gamma}_{r,k} & \boldsymbol{\Delta}_{r,k}\mathbf{D}_{k+1}^T \\ & \sigma^2 (1-\rho) \mathbf{D}_{k+1}\mathbf{D}_{k+1}^T \end{pmatrix} \tag{32}$$

The other parameters are as in Corollary 4.2.

5. Reliability measures

Consider a system consisting of n components with life times T_i, for $i = 1, 2, \ldots, n$. The life times of a series, parallel and k-out-of-n system are $T_{(1)}$, $T_{(n)}$ and $T_{(n-k+1)}$, respectively, where $T_{(1)}, \ldots, T_{(n)}$ are the order statistics corresponding to T_i, s.

For $i > j$ and $i, j = 1, \ldots, n$,

$$MR_{i,j}(t) = E\left(T_{(i)} - t | T_{(j)} > t\right) \tag{33}$$

$$mr_{i,j}(t) = E\left(T_{(i)} - t | T_{(j)} = t\right) \tag{34}$$

are MRL functions at the system level. $MR_{i,j}(t)$ is the MRL of an i-out-of-n system when at least $i - j + 1$ of its components are still working at time t or, in other words, when the j-th failure has not occurred up to time t. In reliability theory, $mr_{i,j}(t)$ is known as the regression mean residual life. It is the average residual life time of an i-out-of-n system if the j-th failure occurs at time t.

For $i < j$ and $i, j = 1, \ldots, n$,

$$MI_{i,j}(t) = E\left(t - T_{(i)} | T_{(j)} < t\right) \tag{35}$$

are MIT functions. $MI_{i,j}(t)$ is the mean time clasped since $T_{(i)}$ given that $T_{(j)} < t$. Assuming T_i, s, $i = 1, \ldots, n$, are dependent and have a common distribution $Log - GH_1\left(\mu, \tau, \sigma^2, \lambda, \chi, \psi\right)$ (log-generalized-hyperbolic with parameters μ, τ, σ,λ, χ, ψ), it follows that $(T_1, T_2, \ldots, T_n)^T \overset{d}{=} \left(e^{X_1}, e^{X_2}, \ldots, e^{X_n}\right)^T$ where $(X_1, \ldots, X_n)^T \sim GH_n\left(\boldsymbol{\mu}_n, \boldsymbol{\tau}_n,\right.$ $\left.\boldsymbol{\Sigma}_n, \lambda, \chi, \psi\right)$ and $\boldsymbol{\mu}_n$, $\boldsymbol{\tau}_n$ and $\boldsymbol{\Sigma}_n$ are given by Equation (25). Using Equations (34) and (26),

$$mr_{i,j}(t) = a(1, \ln(t)) \frac{F_{GH_{n-1}}\left((\ln(t) - \mu) \boldsymbol{\mu}_{j,i-j}^{(2)}; \boldsymbol{\theta}_{j,i-j}^{(2)}(1, \ln(t))\right)}{F_{GH_{n-1}}\left((\ln(t) - \mu) \boldsymbol{\mu}_{j,i-j}^{(2)}; \boldsymbol{\theta}_{j,i-j}^{(2)}(0, \ln(t))\right)} - t$$

where $a(.,.)$ is defined in Equation (27) and the parameters $\boldsymbol{\mu}_{j,i-j}^{(2)}$ and $\boldsymbol{\theta}_{j,i-j}^{(2)}(.,.)$ are derived from Equation (28) substituting $r = j$ and $k = i - j$.

For $i > j$ and $i, j = 1, \ldots, n$, it can be concluded from Equations (33) and (29) that

$$MR_{i,j}(t) = -t + \frac{c(\lambda,\chi,\psi) e^{\mu}}{c(\lambda,\chi,\psi - 2\tau - \sigma^2)} \times \frac{F_{GH_n}\left((\mu - \ln(t))\mathbf{e}_{n,n}; -\tau\mathbf{e}_{n,n} - \left(\boldsymbol{\gamma}_{j,i-j}^{(1)T}, \rho\sigma^2\right)^T, \Psi_{j,i-j}^{(11)}, \lambda, \chi, \psi - 2\tau - \sigma^2\right)}{F_{GH_n}\left((\mu - \ln(t))\mathbf{e}_{n,n}; -\tau\mathbf{e}_{n,n}, \Psi_{j,i-j}^{(11)}, \lambda, \chi, \psi\right)}$$

where $\boldsymbol{\gamma}^{(1)}_{j,i-j}$ and $\Psi^{(11)}_{j,i-j}$ are derived from Equations (31) and (32) by setting $r = j$ and $k = i-j$. For $i < j$ and $i, j = 1, \ldots, n$, we have from Equations (35) and (30),

$$MI_{i,j}(t) = t - \frac{c(\lambda,\chi,\psi)e^{\mu}}{c(\lambda,\chi,\psi-2\tau-\sigma^2)} \times \frac{F_{GH_n}\left((\ln(t)-\mu)\mathbf{e}_{n,n};\tau\mathbf{e}_{n,n}-\left(\boldsymbol{\gamma}^{(2)T}_{i,j-i},-\rho\sigma^2\right)^T, \Psi^{(22)}_{i,j-i},\lambda,\chi,\psi-2\tau-\sigma^2\right)}{F_{GH_n}\left((\ln(t)-\mu)\mathbf{e}_{n,n};\tau\mathbf{e}_{n,n},\Psi^{(22)}_{i,j-i},\lambda,\chi,\psi\right)}$$

where $\boldsymbol{\gamma}^{(2)}_{i,j-i}$ and $\Psi^{(11)}_{i,j-i}$ are derived from Equations (31) and (32) by setting $r = i$ and $k = j-i$.

6. Concluding remarks

In this paper, the truncated selected-GH distribution and selected singular-GH distribution are studied, and the joint distribution of consecutive order statistics from the GH distribution are obtained in terms of TSGH distributions. Using these results, some reliability measures are derived. The best non linear predictor of order statistics under the square error loss can be determined. These results can be extended to all normal mean-variance distributions.

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