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Convolution-invariant subclasses of generalized hyperbolic distributions

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ABSTRACT
It is rigorously shown that the generalized Laplace distributions and the normal inverse Gaussian distributions are the only subclasses of the generalized hyperbolic distributions that are closed under convolution. The result is obtained by showing that the corresponding two classes of variance mixing distributions—gamma and inverse Gaussian—are the only convolution-invariant classes of the generalized inverse Gaussian distributions.

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1. Preliminaries

Parametric classes of infinitely divisible distributions that are closed under convolution play a key role in continuous time stochastic modeling. Thus for any parametric family, it is of interest to identify its convolution-invariant subclasses. Within the generalized hyperbolic laws (GH), invariance under convolution of the generalized asymmetric Laplace distributions (also known as the Bessel function or variance gamma distributions) and the normal inverse Gaussian (NIG) distributions is well known and frequently reported in the literature, (cf. Barndorff-Nielsen, 1978; Kotz et al., 2001; Bibby and Sørensen, 2003). To quote from Bibby and Sørensen (2003, p. 227): “However, in the case of the NIG and VG [variance gamma] distributions, the convolution properties … imply that the value of the Lévy process will be NIG-distributed, respectively VG-distributed, at all time points. This makes the NIG and VG Lévy processes more natural generalized hyperbolic Lévy processes than the other generalized hyperbolic Lévy processes.” Some authors also mention that within the GH distributions, there are no other convolution invariant families, see Hammerstein (2010) or Fajardo and Farias (2004) (the latter, in fact, reports only the NIG class). Despite these few mentions, we did not find any explicit and rigorous argument for the characterization of the convolution-invariant GH subfamilies. The intention of this note is to provide such a one.

A wide range of infinitely divisible distributions can be obtained by mixing normal distributions. Such mixtures are distributed according to a density that is represented as a weighted

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average of normal densities. In particular, a variance mixture of normal densities is given through

$$f(x) = \int_0^\infty \frac{1}{\sqrt{\gamma}} \phi(x/\sqrt{\gamma}) \mathrm{d}F(\gamma),$$

where ϕ is the standard normal density and $\mathrm{d}F(\gamma)$ an arbitrary probability distribution on $[0, \infty)$ that serves as weights with which the (normal) densities are mixed together. The variance mixture of normal distributions is equivalently given as the distribution of $X = \sqrt{\Gamma}Z$, where Γ is distributed according to F and independently of a standard normal variable Z.

A well-known extension of this idea of mixing of normal densities is given in the following definition.

Definition 1. A random variable X (and also the corresponding distribution) is called a normal variance–mean mixture with a non-negative mixing variable Γ, variance scale $\sigma > 0$, mean scale $\mu \in \mathbb{R}$ if

$$X = \sigma\sqrt{\Gamma}Z + \mu\Gamma, \tag{1}$$

where Z is a standard normal variable independent of Γ.

This note is dealing with particular classes of the normal variance–mean mixtures obtained by restricting the distribution of Γ to some parametric subclass. Among discussed, the most general is the class of generalized hyperbolic distributions with the generalized inverse Gaussian distributions as the corresponding class of mixing distributions. For the purpose of establishing the notation and terminology, let us recall formal definitions of the two.

1.1 *Generalized Inverse Gaussian (GIG)*

This class of mixing distributions is given by the density

$$f(x) = \frac{(\psi/\chi)^{\lambda/2}}{2K_\lambda(\sqrt{\psi\chi})} x^{\lambda-1} e^{-(\psi x + \chi/x)/2}, x > 0,$$

where the parameters satisfy

$$\begin{aligned} \psi > 0\,, \chi \geq 0\,, &\text{ if } \lambda > 0, \\ \psi > 0\,, \chi > 0\,, &\text{ if } \lambda = 0, \\ \psi \geq 0\,, \chi > 0\,, &\text{ if } \lambda < 0. \end{aligned}$$

The moment-generating function of a GIG distribution is given by

$$M(t) = \left(\frac{\psi}{\psi - 2t}\right)^{\lambda/2} \frac{K_\lambda\left(\sqrt{\chi(\psi - 2t)}\right)}{K_\lambda(\sqrt{\psi\chi})}, t < \psi/2. \tag{2}$$

Let us mention two special cases.

- The inverse Gaussian distribution (the first passage time by a Brownian motion of a fixed level) is a GIG with $\lambda = -1/2$.
- The gamma distribution is GIG with $\chi = 0$.

1.2 *Generalized Hyperbolic (GH)*

This class is obtained as normal variance–mean mixtures (1) with Γ having a GIG distribution. The two special cases of GIG mentioned above specify the two corresponding GH classes, namely, the generalized asymmetric Laplace (GAL) distributions with the gamma mixing and the normal inverse Gaussian (NIG) distributions with the inverse Gaussian mixing. For more detailed information, we refer to Eberlein and Keller (1993), for the generalized hyperbolic distributions, to Jørgensen (1982) for the generalized inverse Gaussian distributions, and to Kotz et al. (2001) for the generalized Laplace distributions.

2. Convolutions of normal variance–mean mixtures

Let us review some fundamental properties of the normal variance–mean mixtures. First, observe that the distributions of two variance–mean normal mixtures coincide if and only if their mixing distributions are the same, see also Hammerstein (2010) for some related results.

Proposition 1. *Let $X_1 = \sqrt{\Gamma_1}Z_1 + \mu\Gamma_1$ and $X_2 = \sqrt{\Gamma_2}Z_2 + \mu\Gamma_2$ be variance–mean normal mixtures. Then they have the same distribution if and only if Γ_1 and Γ_2 are also identically distributed.*

Proof. Let us assume that X_1 and X_2 are identically distributed and consider a complex solution $z = z(t, \mu) \in \mathbb{C}$ of $z^2/2 + \mu z - it = 0$. Then

$$\mathbb{E}\left(e^{zX_1}\right) = \mathbb{E}\left(e^{zX_2}\right), \tag{3}$$

whenever any of the sides is well defined.

However,

$$\begin{aligned}\mathbb{E}\left(e^{zX_1}\right) &= \mathbb{E}\left(\mathbb{E}\left(e^{z\cdot Z\sqrt{\gamma}+\mu\gamma z}|\gamma = \Gamma_1\right)\right)\\ &= \mathbb{E}\left(e^{\Gamma_1(z^2/2+\mu z)}\right)\\ &= \mathbb{E}\left(e^{it\Gamma_1}\right),\end{aligned}$$

which is the characteristic function of X_1. Since the same is true for X_2, it follows from (3) that both the characteristic functions are equal. □

Consider variance–mean mixtures $X_1 = \sqrt{\Gamma_1}Z_1 + \mu\Gamma_1$ and $X_2 = \sqrt{\Gamma_2}Z_2 + \mu\Gamma_2$, where (Γ_1, Z_1) and (Γ_2, Z_2) are independent. When conditioned on $\Gamma_1 = \gamma_2$ and $\Gamma_2 = \gamma_2$, the variable $X_1 + X_2$ can be represented as

$$\sqrt{\gamma_1}Z_1 + \sqrt{\gamma_2}Z_2 + \mu(\gamma_1 + \gamma_2) \stackrel{d}{=} \sqrt{\gamma_1 + \gamma_2}Z + \mu(\gamma_1 + \gamma_2),$$

where Z is a standard normal random variable and $\stackrel{d}{=}$ stands for the equality of distributions. Consequently, the unconditional distribution of $X_1 + X_2$ is the same as that of

$$\sqrt{\Gamma_1 + \Gamma_2}Z + \mu(\Gamma_1 + \Gamma_2),$$

i.e., the sum $X_1 + X_2$ is also a variance–mean mixture with the same scale μ and the mixing variable $\Gamma = \Gamma_1 + \Gamma_2$.

For $\mu \in \mathbb{R}$ and a subclass $\mathcal{G}$ of distributions on positive line, we denote by $\mathcal{M}(\mathcal{G}, \mu)$ the sub family of normal variance–mean mixtures of the form $X = \sqrt{\Gamma}Z + \mu\Gamma$, where the distribution of Γ is in $\mathcal{G}$. With this notation and from the properties shown above, we have the following immediate result.

Proposition 2. *For each $\mu \in \mathbb{R}$, $\mathcal{M}(\mathcal{G}, \mu)$ is closed under convolution if and only if $\mathcal{G}$ is closed under convolution.*

3. Convolution invariance within GH distributions

It follows from Proposition 2 that it is sufficient to investigate the closeness under convolution for the corresponding variance mixing distributions, i.e., the GIG distributions, which is done in the subsequent results.

Lemma 1. *Let Γ_1 and Γ_2 be two independent GIG distributed variables, with corresponding parameters $(\chi_1, \psi_1, \lambda_1)$ and $(\chi_2, \psi_2, \lambda_2)$, where χ_1 and χ_2 are greater than zero. For $\Gamma = \Gamma_1 + \Gamma_2$ to be again GIG, say, with parameters (χ, ψ, λ), it is necessary that*

$$\begin{aligned}\psi &= \min(\psi_1, \psi_2),\\ \chi &= \left(\sqrt{\chi_1} + \sqrt{\chi_2}\right)^2,\\ \lambda &= \lambda_1 + \lambda_2 + 1/2\end{aligned}$$

and, additionally,

$$\left(2^2\chi_1\chi_2\right)^{1/4}\psi^{\lambda/2}K_{\lambda_1}\left(\sqrt{\chi_1\psi_1}\right)K_{\lambda_2}\left(\sqrt{\chi_2\psi_2}\right) = \left(\pi^2\chi\right)^{1/4}\psi_1^{\lambda_1/2}\psi_2^{\lambda_2/2}K_\lambda\left(\sqrt{\chi\psi}\right). \quad (4)$$

Proof. The equality $\psi = \min(\psi_1, \psi_2)$ follows from the domain of the moment-generating function given in (2).

If Γ is GIG with parameters (χ, ψ, λ), then for $t < \psi/2$:

$$\begin{aligned}&\sqrt{\frac{\psi^\lambda}{\psi_1^{\lambda_1}\psi_2^{\lambda_2}}}\sqrt{\frac{(\psi_1 - 2t)^{\lambda_1}(\psi_2 - 2t)^{\lambda_2}}{(\psi - 2t)^\lambda}}\frac{K_{\lambda_1}(\sqrt{\chi_1\psi_1})K_{\lambda_2}(\sqrt{\chi_2\psi_2})}{K_\lambda(\sqrt{\chi\psi})}\\ &\quad = \frac{K_{\lambda_1}\left(\sqrt{\chi_1(\psi_1 - 2t)}\right)K_{\lambda_2}\left(\sqrt{\chi_2(\psi_2 - 2t)}\right)}{K_\lambda\left(\sqrt{\chi(\psi - 2t)}\right)}, \quad (5)\end{aligned}$$

or, equivalently,

$$\begin{aligned}&A\cdot\exp\left(\sqrt{\chi_1(\psi_1 - 2t)} + \sqrt{\chi_2(\psi_2 - 2t)} - \sqrt{\chi(\psi - 2t)}\right)\\ &\quad\times\frac{(\psi_1 - 2t)^{\lambda_1/2+1/4}(\psi_2 - 2t)^{\lambda_2/2+1/4}}{(\psi - 2t)^{\lambda/2+1/4}}\\ &\quad = \frac{F_{\lambda_1}\left(\sqrt{\chi_1(\psi_1 - 2t)}\right)F_{\lambda_2}\left(\sqrt{\chi_2(\psi_2 - 2t)}\right)}{F_\lambda\left(\sqrt{\chi(\psi - 2t)}\right)}, \quad (6)\end{aligned}$$

where $F_\nu(x) = \sqrt{2x/\pi}\,e^x K_\nu(x)$ and

$$A = \sqrt{(2/\pi)\sqrt{\chi_1\chi_2/\chi}\cdot\psi^\lambda\psi_1^{-\lambda_1}\psi_2^{-\lambda_2}}\cdot K_{\lambda_1}\left(\sqrt{\chi_1\psi_1}\right)K_{\lambda_2}\left(\sqrt{\chi_2\psi_2}\right)/K_\lambda\left(\sqrt{\chi\psi}\right).$$

Using the asymptotics

$$\lim_{x\to\infty}\sqrt{\frac{2x}{\pi}}e^x K_\nu(x) = 1$$

and letting t decreasing to negative infinity in (6), we observe that the right-hand side converges to one while the left-hand side converges either to zero or to infinity as long as $\sqrt{2\chi_1} + \sqrt{2\chi_2} \neq \sqrt{2\chi}$.

Consequently, we can now assume that $\sqrt{\chi_1} + \sqrt{\chi_2} = \sqrt{\chi}$. When $t \to -\infty$, the left-hand side of (6) converges to the same value as

$$A \cdot \frac{(\psi_1 - 2t)^{\lambda_1/2+1/4}(\psi_2 - 2t)^{\lambda_2/2+1/4}}{(\psi^2 - 2t)^{\lambda+1/4}},$$

i.e., to zero or infinity unless $\lambda_1 + \lambda_2 + 1/2 = \lambda$. Under this constraint by allowing negative t to decrease without bound in (6), one obtains the additional relation as given by (4). □

In the previous lemma, we have exploited the behavior of the moment-generating function at zero. In the next one, we examine its behavior at the upper boundary of its domain to derive further restrictions on the parameters.

Lemma 2. *Let Γ_1 and Γ_2 be two independent and identically distributed GIG variables, with parameters (χ, ψ, λ), where χ is greater than zero. For $\Gamma = \Gamma_1 + \Gamma_2$ to be again GIG, and thus, by Lemma 1, having parameters $(2^2\chi, \psi, 2\lambda + 1/2)$, it is necessary for λ to be less then $-1/4$ and, additionally,*

$$\sqrt{\pi} \cdot 2^{2\lambda+1/2} \cdot G_{-2\lambda-1/2}\left(2\sqrt{\chi(\psi - 2t)}\right) = G^2_{-\lambda}\left(\sqrt{\chi(\psi - 2t)}\right), \; t < \psi/2, \tag{7}$$

where $G_\nu(x) = x^\nu K_\nu(x)$.

Proof. We use Equation (4) and the relation $K_{-\nu}(x) = K_\nu(x)$, to re-write Equation (5) as

$$B \cdot (\psi - 2t)^{-1/4+|\lambda|-|\lambda+1/4|} = \frac{G^2_{|-\lambda|}\left(\sqrt{\chi(\psi - 2t)}\right)}{G_{|-2\lambda-1/2|}\left(2\sqrt{\chi(\psi - 2t)}\right)} \tag{8}$$

where $B = \sqrt{\pi} \cdot 2^{-|2\lambda+1/2|} \cdot \chi^{-1/4+|\lambda|-|\lambda+1/4|}$.

For a positive ν,

$$\lim_{x \to 0^+} G_\nu(x) = \Gamma(\nu)2^{\nu-1} \tag{9}$$

so if t converges from below to $\psi/2$ and λ is greater or equal to $-1/4$, then the left-hand side of (8) converges either to zero or is unbounded, while the right-hand side convergent to a non zero constant. □

We are ready for the main result that completely describes the convolution-invariant families within the GIG distributions.

Theorem 1. *Within the generalized inverse Gaussian distributions, there are only two subclasses that are closed under convolution: the gamma distributions and the inverse Gaussian distributions.*

Proof. Consider a subfamily of GIG distributions that is closed under convolution and let $(\chi_0, \psi_0, \lambda_0)$ be the parameters of a member of this subfamily.

Assume that $\chi_0 \neq 0$. If $\lambda_0 > -1/2$, then, by Lemma 2, the increasing sequence defined through the recurrence relation $\lambda_n = 2\lambda_{n-1} + 1/2$ is made of parameters of some members in the family. Since λ_n increases without bound the terms will be eventually positive, which is not permitted as shown in Lemma 2.

Now assume that $\lambda_0 < -1/2$, so that for sufficiently large n leads to $-\lambda_n - 1 > 0$. Differentiating both side of (7) with respect to $\sqrt{\chi(\psi - 2t)}$, and using the identity

$$[x^\nu K_\nu(x)]' = -x^{\nu-1}K_{\nu-1}(x), \; \nu > 0,$$

we obtain

$$\sqrt{\pi}2^{2\lambda_n+1/2}G_{-2\lambda_n-3/2}\left(2\sqrt{\chi_n(\psi_n-2t)}\right) \tag{10}$$
$$= G_{-\lambda_n}\left(\sqrt{\chi_n(\psi_n-2t)}\right)G_{-\lambda_n-1}\left(\sqrt{\chi_n(\psi_n-2t)}\right)$$

Let now consider the limits in the above when $t \to \psi_n/2$. Since $-\lambda_n - 1 > 0$, applying (9) yields

$$\sqrt{\pi}2^{2\lambda_n+1/2}\Gamma(-2\lambda_n-3/2)2^{-2\lambda_n-5/2} = \Gamma(-\lambda_n-1)2^{-\lambda_n-2}\Gamma(-\lambda_n)2^{-\lambda_n-1},$$

or, equivalently,

$$\sqrt{\pi}2^{2\lambda_n+1}\frac{-\lambda_n-1}{-2\lambda_n-3/2} = \frac{\Gamma^2(-\lambda_n)}{\Gamma(-2\lambda_n-1/2)}. \tag{11}$$

On the other hand, applying (9) in (7) yields

$$\sqrt{\pi}2^{2\lambda_n+1} = \frac{\Gamma^2(-\lambda_n)}{\Gamma(-2\lambda_n-1/2)},$$

which is only possible when $\lambda_n = -1/2$ contradicting that $\lambda_n < -1$.

We conclude that either $\lambda_0 = -1/2$, which corresponds to a member of the inverse Gaussian distributions that is closed under convolutions, or $\chi_0 = 0$, which corresponds to another convolution closed family, namely, that of the gamma distributions. □

An immediate consequence is the corresponding characterization for the GH distributions.

Corollary 1. *Within the class of the generalized hyperbolical distributions with the support over the entire real line, only two classes of distributions are closed under the convolution: the generalized Laplace distributions and the normal inverse Gaussian distributions.*

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