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On the property of multivariate generalized hyperbolic distribution and the Stein-type inequality

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ABSTRACT
This note consists of two parts . In the first part, we provide a pedagogic review on the multivariate generalized hyperbolic (MGH) distribution. We show that this probability family is close under margining, conditioning, and linear transforms; however, such property does *not* hold for its subclasses. In the second part, we obtain the Stein-type inequality in the context of MGH distribution. Moreover, we apply the Stein-type inequality to prove a lower bound for Var[$h(X)$]. Particularly, we present examples when X belongs to some well-known subclasses in MGH family.

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1. Introduction

Stein (1973) proved that

$$\mathrm{Cov}[h(X), Y] = \mathrm{Cov}[X, Y]\mathrm{E}[h^{'}(X)] \tag{1}$$

for a bivariate normal distributed random vector $(X, Y)^T$, where $h(x)$ is a differentiable function such that $E[|h'(X)|]$ exists. Equation (1) is referred as Stein's lemma or Stein identity in the literature. It has considerably widespread applications in various areas such as statistics, insurance, and finance. For instance, it can be used to construct the so-called Stein's unbiased risk estimate with superior admissibility (Johnstone 1986; Brown et al. 2006). In finance, Hamada and Valdez (2008) employed Stein's lemma to derive the Capital Asset Pricing Model. It also implies the well-known two-fund theorem in optimal portfolio theory. However, Diaconis and Zabell (1991) showed that (1) only holds for bivariate normal distribution. Hence, ongoing efforts have been devoted to the generalization of Stein identity. Landsman (2006) and Landsman and Neslehova (2008) proved a Stein-type identity for multivariate elliptical distribution, where $\mathrm{E}[h^{'}(X)]$ in the right-hand side of (1) is replaced by $\mathrm{E}[h^{'}(X^*)]$ and X^* is called the associated random variable. A more comprehensive study of their result can be found in Landsman, Vanduffel, and Yao (2013). Adcock (2007) derived a Stein-type identity for multivariate skew-normal distribution, where the skewness generates an additional term in the right-hand side of (1). Vanduffel and Yao (2017) generalize the Stein-type identity to the multivariate generalized hyperbolic (MGH) family.

In fact, the MGH distribution is extremely general and contains many well-known distributions as special cases such as the hyperbolic distribution, normal inverse Gaussian

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distribution, variance gamma distribution, Student-t distribution, and skew-Student t distribution. Thanks to such flexibility, Eberlein and Prause (1998) argued that this distribution family seems to be tailor-made to calibrate the returns of assets. The MGH distribution can be regarded as a multivariate normal variance–mean mixture where the subordinator follows the generalized inverse Gaussian (GIG) distribution. Such structure allows to model the skewness individually for the marginals and makes it possible to model the kurtosis and heavy tails. This is indeed adequate for many purposes in modeling financial assets' returns. For more details on the MGH family and its applications in finance, we refer to Chapter 3 in McNeil, Frey, and Embrechts (2005) and Prause (1999). In particular , Vanduffel and Yao (2017) employ the Stein-type identity to show that expected utility maximizers would invest in three funds (not in two) under the MGH family.

In the symmetric case of MGH distribution, Landsman, Vanduffel, and Yao (2015) obtain a Stein-type inequality and show that it naturally implies a lower bound for $\mathrm{Var}[h(X)]$, which is of interest in the literature. In particular, bounding $\mathrm{Var}[h(X)]$ is motivated from an isoperimetric problem, see Chernoff (1981). Cacoullos (1982) shows that

$$\mathrm{E}^2[h'(X)] \leq \mathrm{Var}[h(X)] \leq \mathrm{E}[(h'(X))^2]$$

if X is a standard normal distributed random variable. More general results can be found in Cacoullos and Papathanasiou (1997), Afendras, Papadatos and Papathanasiou (2011), Afendras and Papadatos (2014), and references therein. In this note, we extend the results of Landsman, Vanduffel, and Yao (2015) to general MGH family.

We make the following contributions in this note.

First, we provide a pedagogic review on the MGH family. Blœsild and Jensen (1981) showed that MGH family is close under margining, conditioning, and linear transform. McNeil, Frey, and Embrechts (2005) provide a detailed introduction on MGH family in Chapter 3; however, their proof for the close property under conditioning seems to be missing. To provide a complement, we reformulate these propositions with proofs. Particularly, we show that such tractable "closing" property does *NOT* hold for the subclasses of MGH family. To our knowledge, we are the first to point out this literally. *Second*, we generalize the Stein-type inequality to MGH family. *Third*, we use Stein-type inequality to develop lower bounds for $\mathrm{Var}[h(X)]$.

The rest of the paper is organized as follows. Section 2 provides the review study on MGH family. Section 3 presents the Stein-type identity and Stein-type inequality for MGH family. In Sec. 4, we derive lower bounds for $\mathrm{Var}[h(X)]$ in the context of MGH distribution. Section 5 concludes the paper.

2. Multivariate generalized hyperbolic family

2.1. MGH distribution as normal mean–variance mixture

A d-dimensional MGH distributed random vector $\mathbf{X}$ can be written as

$$\mathbf{X} = \mu + W\gamma + \sqrt{W}\mathbf{Z}, \mathbf{Z} \sim \mathbf{N}_d(\mathbf{0}, \mathbf{\Sigma}), \mu, \gamma \in \mathbb{R}^d \tag{2}$$

where W is a GIG distributed random variable and $\mathbf{Z}$ is d-dimensional multivariate normal distributed random vector with $\mathbf{0}$ mean and covariance matrix $\mathbf{\Sigma}$[1].

[1] In order to ensure the existence of the density function, we assume that $\mathbf{\Sigma}$ is positive definite.

Definition 1 (GIG distribution). A random variable W has a GIG distribution, written as $W \sim \mathrm{GIG}(\lambda, \chi, \psi)$, if its density is

$$f(w) = \frac{\chi^{-\lambda}(\sqrt{\chi\psi})^{\lambda}}{2K_{\lambda}(\sqrt{\chi\psi})} w^{\lambda-1} e^{-\frac{1}{2}(\chi w^{-1}+\psi w)}, \quad w > 0, \chi > 0, \psi > 0, \lambda \in \mathbb{R}$$

where K_{λ} is the modified Bessel function of the third kind.

Using notations

$$\tau_1 = \frac{\sqrt{\chi} K_{\lambda+1}(\sqrt{\chi\psi})}{\sqrt{\psi} K_{\lambda}(\sqrt{\chi\psi})}, \tau_2 = \frac{\chi K_{\lambda+2}(\sqrt{\chi\psi})}{\psi K_{\lambda}(\sqrt{\chi\psi})}$$

for the first two moments of W, we have

$$\mathrm{E}[W] = \tau_1, \mathrm{Var}[W] = \tau_2 - \tau_1^2$$

The domain of variation for the parameters of GIG distribution is

$$\begin{aligned} &\chi \geq 0, \quad \psi > 0, \quad \text{if } \lambda > 0 \\ &\chi > 0, \quad \psi > 0, \quad \text{if } \lambda = 0 \\ &\chi > 0, \quad \psi \geq 0, \quad \text{if } \lambda < 0 \end{aligned} \tag{3}$$

Definition 2 (MGH distribution). A random vector $\mathbf{X} \in \mathbb{R}^d$ is said to be MGH distributed if its density function is

$$f(\mathbf{x}) = a_d \frac{K_{\lambda-\frac{d}{2}}\left(\sqrt{(\chi + (\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu))(\psi + \gamma^T \mathbf{\Sigma}^{-1}\gamma)}\right)}{\left(\sqrt{(\chi + (\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu))(\psi + \gamma^T \mathbf{\Delta}^{-1}\gamma)}\right)^{\frac{d}{2}-\lambda}} e^{\gamma^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)}$$

$$a_d = \frac{\left(\sqrt{\psi}\right)^{\lambda} (\psi + \gamma^T \mathbf{\Sigma}^{-1}\gamma)^{\frac{d}{2}-\lambda}}{(2\pi)^{\frac{d}{2}} |\mathbf{\Sigma}|^{\frac{1}{2}} \left(\sqrt{\chi}\right)^{\lambda} K_{\lambda}\left(\sqrt{\chi\psi}\right)}$$

We denote it as $\mathbf{X} \sim \mathbf{MGH}_d(\lambda, \chi, \psi, \mu, \mathbf{\Sigma}, \gamma)$.

$\mathbf{X}$'s mean and covariance matrix are as follows**.**

$$\begin{aligned} \mathrm{E}[\mathbf{X}] &= \mathrm{E}[\mathrm{E}[\mathbf{X}|W]] = \mu + \tau_1\gamma \\ \mathrm{Cov}[\mathbf{X}] &= \mathrm{E}[\mathrm{Cov}[\mathbf{X}|W]] + \mathrm{Cov}[\mathrm{E}[\mathbf{X}|W]] = \tau_1\mathbf{\Sigma} + \left(\tau_2 - \tau_1^2\right)\gamma\gamma^T \end{aligned} \tag{4}$$

Proposition 3. *Let* $\mathbf{X} \sim \mathbf{MGH}_d(\lambda, \chi, \psi, \mu, \mathbf{\Sigma}, \gamma)$ *and* $(\mathbf{X}_1^T, \mathbf{X}_2^T)^T$ *be a partitioning of* $\mathbf{X}$ *with dimensions* d_1 *and* d_2*, respectively. Correspondingly, let* $(\mu_1^T, \mu_2^T)^T$ *and* $(\gamma_1^T, \gamma_2^T)^T$ *be the partitions of* μ *and* γ*, and let*

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}$$

where $\mathbf{\Sigma}_{11}$ *is* $d_1 \times d_1$ *Then the following statements hold:*

1. $\mathbf{X}_1 \sim \mathbf{MGH}_{d_1}(\lambda, \chi, \psi, \mu_1, \mathbf{\Sigma}_{11}, \gamma_1)$.
2. $\mathbf{Y} = \mathbf{B}\mathbf{X} + \mathbf{b}$ *is a regular affine transform*[2] *of* $\mathbf{X}$. $\mathbf{Y} \sim \mathbf{MGH}_{d_1}(\lambda, \chi, \psi, \mathbf{B}\mu + \mathbf{b}, \mathbf{B}\mathbf{\Sigma}\mathbf{B}^T, \mathbf{B}\gamma)$.
3. $\mathbf{X}_2|\mathbf{X}_1 = \mathbf{x}_1 \sim \mathbf{MGH}_{d_2}(\lambda - d_1/2, \chi + (\mathbf{x}_1 - \mu_1)^T \mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1 - \mu_1), \psi + \gamma_1^T \mathbf{\Sigma}_{11}^{-1}\gamma_1, \mu_2 + \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1 - \mu_1), \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}, \gamma_2 - \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\gamma_1)$.

[2] $\mathbf{B}$ is a $d_1 \times d$ full-ranked matrix. $\mathbf{b}$ is a d_1-dimensional vector.

Proof. Proofs for statement 1 and statement 2 can be found in Proposition 3.13 of McNeil, Frey, and Embrechts (2005). We prove statement 3.

Using Woodbury identity and the partition of the quadratic form corresponding to $\mathbf{\Sigma}^{-1}$, we have

$$\begin{aligned}
(\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu) &= (\mathbf{x}_1-\mu_1)^T\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1-\mu_1) + \left(\mathbf{x}_2-\mu_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1-\mu_1)\right)^T \\
&\quad \times\left(\mathbf{\Sigma}_{22}-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}\right)^{-1}\times\left(\mathbf{x}_2-\mu_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1-\mu_1)\right) \\
\gamma^T\mathbf{\Sigma}^{-1}\gamma &= \gamma_1^T\mathbf{\Sigma}_{11}^{-1}\gamma_1 + \left(\gamma_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\gamma_1\right)^T\times\left(\mathbf{\Sigma}_{22}-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}\right)^{-1} \\
&\quad \times\left(\gamma_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\gamma_1\right) \\
\gamma^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu) &= \gamma_1^T\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1-\mu_1) + \left(\gamma_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\gamma_1\right)^T \\
&\quad \times\left(\mathbf{\Sigma}_{22}-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}\right)^{-1}\times\left(\mathbf{x}_2-\mu_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1-\mu_1)\right)
\end{aligned}$$

Applying $f_{\mathbf{X}_2|\mathbf{X}_1=\mathbf{x}_1}(\mathbf{x}_2) = f(\mathbf{x}_1,\mathbf{x}_2)/f(\mathbf{x}_1)$, we can easily show that statement 3 holds. □

Remark 4 Blœsild and Jensen (1981). already showed that the MGH family is close under margining, conditioning, and linear transform in their α-δ-β-$\mathbf{\Delta}$ parameterization. However, in the χ-ψ-γ-$\mathbf{\Sigma}$ parameterization (McNeil, Frey, and Embrechts 2005), the proof for conditioning is missing. There is a transform between the two sets of parameterization

$$\gamma = \mathbf{\Delta}\beta, \chi{=}\delta^2,\ \psi = \alpha^2-\beta^T\mathbf{\Delta}\beta$$

However, to perform such transforms, the calculations in the proof of Proposition 3 are inevitable due to the partition, especially for the proof of conditioning. Hence, Proposition 3 contributes a complement proof to McNeil, Frey, and Embrechts (2005).

MGH family is extremely general and contains many frequently used multivariate distributions as special cases. In the sequel of this section, we present some well-known multivariate distribution that belongs to the MGH family.

2.2. Multivariate variance gamma distribution

Multivariate variance gamma (MVG) distribution is often used in modeling stocks' price processes, see Madan, Carr and Chang (1998) and Luciano and Schouten (2006). A d-dimensional random vector $\mathbf{X}$ follows the MVG distribution if its density function is of the form

$$f(\mathbf{x}) = a_d\frac{K_{\lambda-\frac{d}{2}}\left(\sqrt{(\psi+\gamma^T\mathbf{\Sigma}^{-1}\gamma)(\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)}\right)}{\left(\sqrt{(\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)/(\psi+\gamma^T\mathbf{\Sigma}^{-1}\gamma)}\right)^{-\lambda+\frac{d}{2}}}e^{\gamma^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)}$$

$$a_d = \frac{2\,(\psi/2)^{\lambda}}{(2\pi)^{\frac{d}{2}}\,|\mathbf{\Sigma}|^{\frac{1}{2}}\,\Gamma(\lambda)},\qquad \Gamma \text{ is gamma function}$$

and we denote $\mathbf{X}\sim\mathbf{MVG}_d(\lambda,\psi,\mu,\mathbf{\Sigma},\gamma)$. Likewise, MVG is also a mean–variance normal mixture where W follows gamma distribution, i.e., $W\sim\Gamma(\lambda,\psi/2)$. MVG distribution is close under margining and linear transform but its conditional distribution of MVG is again MGH distributed.

Proposition 5. *Let* $\mathbf{X} \sim \mathbf{MVG}_d(\lambda, \psi, \mu, \mathbf{\Sigma}, \gamma)$ *and* $(\mathbf{X}_1^T, \mathbf{X}_2^T)^T$ *be a partitioning of* $\mathbf{X}$ *with dimensions* d_1 *and* d_2*, respectively. Correspondingly, let* $(\mu_1^T, \mu_2^T)^T$ *and* $(\gamma_1^T, \gamma_2^T)^T$ *be the partitions of* μ *and* γ*, and let*

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}$$

where $\mathbf{\Sigma}_{11}$ *is* $d_1 \times d_1$*. Then the following statements hold:*

1. $\mathbf{X}_1 \sim \mathbf{MVG}_{d_1}(\lambda, \psi, \mu_1, \mathbf{\Sigma}_{11}, \gamma_1)$.
2. $\mathbf{Y} = \mathbf{BX} + \mathbf{b}$ *is a regular affine transform of* $\mathbf{X}$. $\mathbf{Y} \sim \mathbf{MVG}_{d_1}(\lambda, \psi, \mathbf{B}\mu + \mathbf{b}, \mathbf{B\Sigma B}^T, \mathbf{B}\gamma)$.
3. $\mathbf{X}_2|\mathbf{X}_1 = \mathbf{x}_1 \sim \mathbf{MGH}_{d_2}(\lambda - d_1/2, (\mathbf{x}_1 - \mu_1)^T \mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1 - \mu_1), \psi + \gamma_1^T \mathbf{\Sigma}_{11}^{-1}\gamma_1, \mu_2 + \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1 - \mu_1), \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}, \gamma_2 - \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\gamma_1)$.

2.3. Multivariate normal inverse Gaussian distribution

Normal inverse Gaussian (NIG) model is another frequently used subclass of MGH family in finance. Some studies have shown that the financial assets' returns can often be fitted extremely well by the NIG model (Barndorff-Nielsen 1997). Multivariate normal inverse Gaussian (MNIG) model is also used in engineering; see Øigård et al. (2005). We say a d-dimensional random vector $\mathbf{X}$ follows MNIG distribution if its density function is

$$f(\mathbf{x}) = \frac{\sqrt{\chi} K_{\frac{d+1}{2}}\left(\sqrt{(\chi + (\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu))(\psi + \gamma^T \mathbf{\Sigma}^{-1}\gamma)}\right)}{2^{\frac{d-1}{2}} |\mathbf{\Sigma}|^{\frac{1}{2}} \left(\pi \sqrt{(\chi + (\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu))/(\psi + \gamma^T \mathbf{\Sigma}^{-1}\gamma)}\right)^{\frac{d+1}{2}}} e^{\sqrt{\chi\psi} + \gamma^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)}$$

We denote it as $\mathbf{X} \sim \mathbf{MNIG}_d(\chi, \psi, \mu, \mathbf{\Sigma}, \gamma)$. Like MVG, MNIG is also close under margining and linear transform but its conditional distribution goes back to the MGH distribution.

Proposition 6. *Let* $\mathbf{X} \sim \mathbf{MNIG}_d(\chi, \psi, \mu, \mathbf{\Sigma}, \gamma)$ *and* $(\mathbf{X}_1^T, \mathbf{X}_2^T)^T$ *be a partitioning of* $\mathbf{X}$ *with dimensions* d_1 *and* d_2*, respectively. Correspondingly, let* $(\mu_1^T, \mu_2^T)^T$ *and* $(\gamma_1^T, \gamma_2^T)^T$ *be the partitions of* μ *and* γ*, and let*

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}$$

where $\mathbf{\Sigma}_{11}$ *is* $d_1 \times d_1$*. Then the following statements hold:*

1. $\mathbf{X}_1 \sim \mathbf{MNIG}_{d_1}(\chi, \psi, \mu_1, \mathbf{\Sigma}_{11}, \gamma_1)$.
2. $\mathbf{Y} = \mathbf{BX} + \mathbf{b}$ *is a regular affine transform of* $\mathbf{X}$. $\mathbf{Y} \sim \mathbf{MNIG}_{d_1}(\chi, \psi, \mathbf{B}\mu + \mathbf{b}, \mathbf{B\Sigma B}^T, \mathbf{B}\gamma)$.
3. $\mathbf{X}_2|\mathbf{X}_1 = \mathbf{x}_1 \sim \mathbf{MGH}_{d_2}(-(d_1 + 1)/2, \chi + (\mathbf{x}_1 - \mu_1)^T \mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1 - \mu_1), \psi + \gamma_1^T \mathbf{\Sigma}_{11}^{-1}\gamma_1, \mu_2 + \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1 - \mu_1)), \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}, \gamma_2 - \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\gamma_1)$.

In fact, the MNIG is a special case of MGH when $\lambda = -\frac{1}{2}$.

2.4. Multivariate hyperbolic distribution

When $\lambda = (d+1)/2$, we drop off "generalized" and obtain the so-called multivariate hyperbolic (MH) distribution. The density function of MH distribution can be written as

$$f(\mathbf{x}) = \frac{\sqrt{\psi}^{\frac{d+1}{2}} \exp(\gamma^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu) - \sqrt{(\chi + (\mathbf{x}-\mu)^T \mathbf{\Sigma}^{-1}(\mathbf{x}-\mu))(\psi + \gamma^T \mathbf{\Sigma}^{-1}\gamma)})}{(2\sqrt{\chi})^{\frac{d+1}{2}} \pi^{\frac{d-1}{2}} |\mathbf{\Sigma}|^{\frac{1}{2}} \sqrt{\psi + \gamma^T \mathbf{\Sigma}^{-1}\gamma} K_{\frac{d+1}{2}}(\sqrt{\chi\psi})}$$

and we use the notation $\mathbf{X} \sim \mathbf{MH}_d(\chi, \psi, \mu, \mathbf{\Sigma}, \gamma)$. Interestingly, the marginal distributions and linear combination of the MH are not hyperbolic distributed but an MGH distributed with $\lambda = (d+1)/2$and the MH distribution is close under conditioning; see Proposition 3.

Proposition 7. *Let $\mathbf{X} \sim \mathbf{MH}_d(\chi, \psi, \mu, \mathbf{\Sigma}, \gamma)$ and $(\mathbf{X}_1^T, \mathbf{X}_2^T)^T$ be a partitioning of $\mathbf{X}$ with dimensions d_1 and d_2, respectively. Correspondingly, let $(\mu_1^T, \mu_2^T)^T$ and $(\gamma_1^T, \gamma_2^T)^T$ be the partitions of μ and γ, and let*

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}$$

where $\mathbf{\Sigma}_{11}$ is $d_1 \times d_1$. Then the following statements hold:

1. $\mathbf{X}_1 \sim \mathbf{MGH}_{d_1}((d+1)/2, \chi, \psi, \mu_1, \mathbf{\Sigma}_{11}, \gamma_1)$.
2. $\mathbf{Y} = \mathbf{BX} + \mathbf{b}$ *is a regular affine transform of* $\mathbf{X}$. $\mathbf{Y} \sim \mathbf{MGH}_{d_1}((d+1)/2, \chi, \psi, \mathbf{B}\mu + \mathbf{b}, \mathbf{B\Sigma B}^T, \mathbf{B}\gamma))$.
3. $\mathbf{X}_2|\mathbf{X}_1 = \mathbf{x}_1 \sim \mathbf{MH}_{d_2}(\chi + (\mathbf{x}_1 - \mu_1)^T \mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1 - \mu_1), \psi + \gamma_1^T \mathbf{\Sigma}_{11}^{-1}\gamma_1, \mu_2 + \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1 - \mu_1)), \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}, \gamma_2 - \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\gamma_1)$.

2.5. Multivariate (generalized hyperbolic) skewed-t distribution

Another subclass that is often used to model financial assets is the so-called multivariate (generalized hyperbolic) skewed-*t* (MSt) distribution[3], in which the mixing random variable W is inverse gamma distributed. It is well known that the multivariate Student-*t* distribution with degree of freedom $\nu (\nu > 4)$[4] can be represented as a normal variance mixture, i.e., $\mathbf{X} \sim \mathbf{t}_d(\upsilon, \mu, \mathbf{\Sigma})$ if

$$\mathbf{X} = \mu + \sqrt{W}\mathbf{Z}, \mathbf{Z} \sim \mathbf{N}_d(\mathbf{0}, \mathbf{\Sigma})$$

and W follows the inverse gamma (IG) distribution, $W \sim IG(\upsilon/2, \upsilon/2)$. In fact, when $\lambda = -\nu/2, \chi = \nu, \psi = 0$, the GIG reduces to IG and thus according to (2), we obtain the so-called multivariate (generalized hyperbolic) skewed-*t* distribution as

$$\mathbf{X} = \mu + \gamma W + \sqrt{W}\mathbf{Z}$$

[3] To distinguish from another type of skewed-*t* distribution in Adcock (2010), we follow Aas and Haff (2006) and refer the skewed-*t* distribution in this paper as generalized hyperbolic skewed-*t* distribution.
[4] $\nu > 4$ is to ensure the existence of mean and variance.

Its density function can be written as

$$f(\mathbf{x}) = a_d \frac{K_{\frac{d+\nu}{2}}\left(\sqrt{(\nu + (\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu))(\gamma^T\mathbf{\Sigma}^{-1}\gamma)}\right)\exp(\gamma^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu))}{\left(\sqrt{(\nu + (\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu))(\gamma^T\mathbf{\Sigma}^{-1}\gamma)}\right)^{-\frac{\nu+d}{2}}\left(1+\frac{(\mathbf{x}-\mu)^T\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)}{\nu}\right)^{\frac{\nu+d}{2}}}$$

$$a_d = \frac{2^{1-\frac{\nu+d}{2}}}{\Gamma(\frac{\nu}{2})(\sqrt{\pi\nu})^d\,|\mathbf{\Sigma}|^{\frac{1}{2}}}$$

We denote it as $\mathbf{X} \sim \mathbf{MSt}_d(\nu, \mu, \mathbf{\Sigma}, \gamma)$. MSt distribution is close under margining and linear transform but its conditional distribution is MGH distributed.

Proposition 8. *Let* $\mathbf{X} \sim \mathbf{MSt}_d(\nu, \mu, \mathbf{\Sigma}, \gamma)$ *and* $(\mathbf{X}_1^T, \mathbf{X}_2^T)^T$ *be a partitioning of* $\mathbf{X}$ *with dimensions* d_1 *and* d_2*, respectively. Correspondingly, let* $(\mu_1^T, \mu_2^T)^T$ *and* $(\gamma_1^T, \gamma_2^T)^T$ *be the partitions of* μ *and* γ*, let*

$$\mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12} \\ \mathbf{\Sigma}_{21} & \mathbf{\Sigma}_{22} \end{pmatrix}$$

where $\mathbf{\Sigma}_{11}$ *is* $d_1 \times d_1$*. Then the following statements hold:*

1. $\mathbf{X}_1 \sim \mathbf{MSt}_d(\nu, \mu_1, \mathbf{\Sigma}_{11}, \gamma_1)$.
2. $\mathbf{X}_2|\mathbf{X}_1 = \mathbf{x}_1 \sim \mathbf{MGH}_{d_2}(-(\nu+d_1)/2, \nu + (\mathbf{x}_1-\mu_1)^T\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1-\mu_1), \gamma_1^T\mathbf{\Sigma}_{11}^{-1}\gamma_1, \mu_2 + \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}(\mathbf{x}_1-\mu_1), \mathbf{\Sigma}_{22}-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\mathbf{\Sigma}_{12}, \gamma_2-\mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}\gamma_1)$.

Note that for multivariate t-distribution ($\gamma = \mathbf{0}$), the conditional distribution is not necessarily a multivariate t -distributed; see Sec. 1.11 in Kotz and Nadarajah (2004).

2.6. Numerical examples

We further provide some interesting examples to these properties in the MGH subclasses.

- Let $(X_1, X_2)^T \sim \mathbf{MVG}_2(\lambda, \psi, (0,0)^T, \mathbf{I}_2, (0,0)^T)$, $\mathbf{I}_2$ is the 2×2 identity matrix, then

$$X_2|X_1 = x_1 \sim \mathbf{MGH}_1\left(\lambda - \frac{1}{2}, x_1^2, \psi, x_1, 1, 0\right)$$

- Let $(X_1, X_2)^T \sim \mathbf{MH}_2(\chi, \psi, (0,0)^T, \mathbf{I}_2, (0,0)^T)$, then

$$X_1 + X_2 \sim \mathbf{MGH}_1\left(\frac{3}{2}, \chi, \psi, 0, 2, 0\right)$$

- Let $(X_1, X_2)^T \sim \mathbf{MNIG}_2(\chi, \psi, (0,0)^T, A, (2,1)^T)$, where

$$A = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

then

$$X_2|X_1 = x_1 \sim \mathbf{MGH}_1\left(-1, \chi + x_1^2, \psi + 4, \frac{1}{2}x_1, \frac{3}{4}, 0\right) \tag{5}$$

We may observe that when MVG is symmetric, its conditional distribution is also symmetric. In fact, the symmetric MGH is a special case of the elliptical distribution family which is close for conditioning; see also Landsman, Vanduffel, and Yao (2015). Moreover, the MH distribution is generally not close under affine transform because the λ cannot vary accordingly when the dimension decreases. However, in the case that the affine transform $\mathbf{B}$ has rank d, MH is also close under affine transform as $\lambda = \frac{d+1}{2}$ is reserved. Example (5) is interesting. Although $(X_1, X_2)^T$ are asymmetric, the conditional distribution $X_2|X_1 = x_1$ is symmetric with respect to $\frac{1}{2}x_1$.

3. Stein-type inequality and a lower bound for Var[$h(X)$]

3.1. Stein-type inequality for MGH distribution

In this section, we extend the Stein-type inequality to MGH family and derive a lower bound for Var[$h(X)$]. For the simplicity and the readability of the formulae, we focus on univariate case in the sequel of the paper . Vanduffel and Yao (2017) extended Stein-type identity to MGH distribution.

Theorem 9 (Stein-type identity for MGH). *Let $X \sim MGH_1(\lambda, \chi, \psi, \mu, \sigma^2, \gamma)$ and let h be a function such that $E[|h'(X)|]$ exists. Then we have that*

$$Cov[X, h(X)] = \tau_1\sigma^2 E[h'(X^*)] + \tau_1(E[h(X^*)] - E[h(X)])\gamma \tag{6}$$

where $X^* \sim MGH_1(\lambda + 1, \chi, \psi, \mu, \sigma^2, \gamma)$.

The instrument to prove Theorem 9 is conditional expectation. A detailed proof can be found in Vanduffel and Yao (2017) . As the MGH distribution is both non normal and skewed, we can actually see that the Stein-type identity (6) involves both X^* (associated random variable) and an additional term (with γ). Based on (6), the Stein-type inequality is at hand. We show this in Theorem 10.

Theorem 10 (Stein-type inequality for MGH). *Let $X \sim MGH_1(\lambda, \chi, \psi, \mu, \sigma^2, \gamma)$, $\gamma \geq 0$, and a non negative function h be differentiable a.s. and non decreasing such that $E[|h'(X)|]$ and $E[|h(X)|]$ exist. Then we have that*

$$Cov[X, h(X)] \geq c_0\sigma^2 E[h'(X)] + \gamma(c_0 - \tau_1)E[h(X)]$$

where the constant $c_0 = K_{\lambda+\frac{1}{2}}(\sqrt{\chi(\psi + \gamma^2/\sigma^2)})/(\sqrt{(\psi + \gamma^2/\sigma^2)}K_{\lambda-\frac{1}{2}}(\sqrt{\chi(\psi + \gamma^2/\sigma^2)}))$.

Proof. First note that the Stein-type identity for X is

$$\mathrm{Cov}[X, h(X)] = \tau_1\sigma^2 \mathrm{E}[h'(X^*)] + \tau_1(\mathrm{E}[h(X^*)] - \mathrm{E}[h(X)])\gamma$$

Moreover,

$$\begin{aligned}
\mathrm{E}[h(X^*)] &= \mathrm{E}[h(X)C(X)], \text{ where} \\
C(x) &= f_{X^*}(x)/f_X(x) \\
&= \frac{\sqrt{\psi}K_\lambda\big(\sqrt{\chi\psi}\big)\sqrt{\big(\chi + \frac{(x-\mu)^2}{\sigma^2}\big)\big(\psi + \frac{\gamma^2}{\sigma^2}\big)}K_{\lambda+\frac{1}{2}}\left(\sqrt{(\chi + \frac{(x-\mu)^2}{\sigma^2})(\psi + \frac{\gamma^2}{\sigma^2})}\right)}{\sqrt{\chi}K_{\lambda+1}\big(\sqrt{\chi\psi}\big)\big(\psi + \frac{\gamma^2}{\sigma^2}\big)K_{\lambda-\frac{1}{2}}\left(\sqrt{\big(\chi + \frac{(x-\mu)^2}{\sigma^2}\big)(\psi + \frac{\gamma^2}{\sigma^2})}\right)}
\end{aligned}$$

Because $tK_{\nu+1}(t)/K_\nu(t)$ is non decreasing at $t > 0$ (Ismail 1977),

$$C(x) \geq \frac{\sqrt{\psi}K_\lambda\big(\sqrt{\chi\psi}\big)K_{\lambda+\frac{1}{2}}\big(\sqrt{\chi(\psi + \gamma^2/\sigma^2)}\big)}{\sqrt{\psi + \gamma^2/\sigma^2}K_{\lambda+1}\left(\sqrt{\chi\psi}\right)K_{\lambda-\frac{1}{2}}\left(\sqrt{\chi(\psi + \gamma^2/\sigma^2)}\right)}$$

If h is non negative and non decreasing,

$$\tau_1 \mathrm{E}[h'(X^*)] \geq c_0 \mathrm{E}[h'(X)] \text{ and } \tau_1 \mathrm{E}[h(X^*)] \geq c_0 \mathrm{E}[h(X)]$$

$$c_0 = \frac{\sqrt{\chi} K_{\lambda+\frac{1}{2}}\left(\sqrt{\chi(\psi+\gamma^2/\sigma^2)}\right)}{\sqrt{(\psi+\gamma^2/\sigma^2)} K_{\lambda-\frac{1}{2}}\left(\sqrt{\chi(\psi+\gamma^2/\sigma^2)}\right)}$$

Therefore,

$$\mathrm{Cov}[X, h(X)] \geq c_0 \sigma^2 \mathrm{E}[h'(X)] + \gamma(c_0 - \tau_1)\mathrm{E}[h(X)]$$

□

Remark 11. Stein-type inequality can be extended to its multivariate version straightforwardly by setting $h : \mathbb{R}^d \mapsto \mathbb{R}^+$ and h is non decreasing at each component. In case of symmetric MGH distribution, i.e., $\gamma = 0$, we may drop the condition that h is non negative; see Landsman, Vanduffel, and Yao (2015). Moreover, the inverse inequality holds if h is non increasing and non positive. In fact, as long as the sign of $\gamma E[h(X)]$ is the same as $E[h'(X)]$, we always have the inequality hold.

Stein-type inequality can be used to derive a lower bound for Var$[h(X)]$ straightforwardly, which is a problem of interest in the literature.

3.1.1. A lower bound for Var$[h(X)]$

Theorem 12. *Let $X \sim MGH_1(\lambda, \chi, \psi, \mu, \sigma^2, \gamma)$ and a non negative function h be differentiable a.s. and non decreasing. Further let us assume that $E[h'(X)]$ and $E[h(X)]$ exist. Then,*

$$Var[h(X)] \geq \frac{(c_0 \sigma^2 E[h'(X)] + \gamma(c_0 - \tau_1)E[h(X)])^2}{\tau_1 \sigma^2 + \gamma^2 \left(\tau_2 - \tau_1^2\right)} \tag{7}$$

where the constant $c_0 = K_{\lambda+\frac{1}{2}}(\sqrt{\chi(\psi+\gamma^2/\sigma^2)})/(\sqrt{(\psi+\gamma^2/\sigma^2)}K_{\lambda-\frac{1}{2}}(\sqrt{\chi(\psi+\gamma^2/\sigma^2)}))$.

Proof.

$$\begin{aligned}\mathrm{Var}[h(X)] &\geq \mathrm{Cov}^2[X, h(X)]/\mathrm{Var}[X] \\ &\geq \frac{(c_0 \sigma^2 \mathrm{E}[h'(X)] + \gamma(c_0 - \tau_1)\mathrm{E}[h(X)])^2}{\tau_1 \sigma^2 + \gamma^2 \left(\tau_2 - \tau_1^2\right)}\end{aligned}$$

□

Remark 13. Theorem 12 still holds if h is non increasing and non positive. When $\gamma = 0$, i.e., for symmetric MGH distribution, (7) reduces to the case in Landsman, Vanduffel, and Yao (2015), in which the monotonicity of h is enough to ensure the inequality holds.

We further list some special cases of this lower bound with respect to subclasses of MGH family.

- $X \sim MVG_1(\lambda, \psi, \mu, \sigma^2, \gamma)$

$$\mathrm{Var}[h(X)] \geq \frac{((1-2\lambda)\psi\sigma^4 \mathrm{E}[h'(X)] + (2\lambda\gamma^3 + \gamma\psi\sigma^2)\mathrm{E}[h(X)])^2}{2\lambda(\gamma^2+\psi\sigma^2)^2(2\gamma^2+\psi\sigma^2)} \tag{8}$$

- $X \sim MNIG_1(\chi, \psi, \mu, \sigma^2, \gamma)$

$$\mathrm{Var}[h(X)] \geq \frac{\psi\sqrt{\chi}\sigma^2(g_1 - g_2)^2 K_1^{-1}\left(\sqrt{\chi(\psi+\gamma^2/\sigma^2)}\right)}{\sqrt{\gamma^2+\psi\sigma^2}(\gamma^2\sqrt{\psi} + \sqrt{\psi^3}\sigma^2)} \tag{9}$$

$$g_1 = \sqrt{\psi}(\gamma\mathrm{E}[h(X)] + \sigma^2\mathrm{E}[h'(X)])K_0\left(\sqrt{\chi(\psi+\gamma^2/\sigma^2)}\right)$$

$$g_2 = \mathrm{E}[h(X)]\gamma\sqrt{\psi+\gamma^2/\sigma^2}K_1(\sqrt{\chi(\psi+\gamma^2/\sigma^2)})$$

- $X \sim MSt_1(\nu, \mu, \sigma^2, \gamma)$

$$\mathrm{Var}[h(X)] \geq \frac{\sigma^2(\nu-4)(g_1-g_2)^2 K^{-2}_{\frac{\nu+1}{2}}\left(\sqrt{\chi\gamma^2/\sigma^2}\right)}{\left(2\gamma^4\nu+\sigma^2\gamma^2(\nu-2)(\nu-4)\right)} \tag{10}$$

$$g_1 = \left(\gamma \mathrm{E}[h(X)] + \sigma^2 \mathrm{E}[h'(X)]\right)(\nu-2) K_{\frac{\nu-1}{2}}\left(\sqrt{\chi\gamma^2/\sigma^2}\right)$$

$$g_2 = \mathrm{E}[h(X)]\sqrt{\chi\gamma^4/\sigma^2} K_{\frac{\nu+1}{2}}\left(\sqrt{\chi\gamma^2/\sigma^2}\right)$$

- $X \sim MH_1(\chi, \psi, \mu, \sigma^2, 0)$

$$\mathrm{Var}[h(X)] \geq \frac{\left(\sqrt{\chi\psi}+1\right)^2 K_1\left(\sqrt{\chi\psi}\right)}{\sqrt{\chi\psi^3} K_2\left(\sqrt{\chi\psi}\right)} \sigma^2 \mathrm{E}^2[h'(X)] \tag{11}$$

Note that in the symmetric case, i.e., $\gamma = 0$, the above formulae can be simplified significantly; see Landsman, Vanduffel, and Yao (2015). For instance, (10) would simply be

$$\mathrm{Var}[h(X)] \geq \frac{\nu(\nu-2)}{(\nu-1)^2} \sigma^2 \mathrm{E}^2[h'(X)]$$

4. Conclusion

This note first provides a pedagogic review on the MGH distribution. We provide the proof for that this distribution probability family is close under margining, conditioning, and linear transforms. In doing so, we seem to be the first in literature to point out that such closing does *not* hold for its subclasses such as the MVG distribution, MNIG distribution, and MSt distribution. Considering the fact that these subclasses are frequently used in various fields, one needs to be more careful when applying these properties. Then we extend the Stein-type inequality to the MGH distribution. Moreover, we use the Stein-type inequality to prove a lower for $\mathrm{Var}[h(X)]$, where X follows the generalized hyperbolic distribution. As Landsman, Vanduffel, and Yao (2015) showed that, in the context of MGH distribution, some bounds for $\mathrm{Var}[h(X)]$ in the literature are not applicable, our work provides alternative results regarding this interesting question. In addition, we present some specific examples in the context of subclasses of MGH family.

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