Numerical Methods in ESP

Numerical Methods II

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/afs/ictp.it/public/g/ggiulian/WORLD/num2_lesson7.pdf

The linear advection equation

Linear advection problem:

$$\begin{aligned} \frac{\partial \phi}{\partial t} &+ u\frac{\partial \phi}{\partial x} = 0 \\ \phi(x,t) &= F(x-ut) \end{aligned} \qquad (1)$$

Numerical solutions so far:

$$\begin{aligned} \phi_j^{n+1} &= (1+c)\phi_j^n - c\phi_{j+1}^n \\ \phi_j^{n+1} &= (1-c)\phi_j^n + c\phi_{j-1}^n \\ \phi_j^{n+1} &= \phi_j^{n-1} - c(\phi_{j+1}^n - \phi_{j-1}^n) \end{aligned}$$

where:

$$c = u\frac{\Delta t}{\Delta x}$$

Diffusion equation

- Consider the linear diffusion equation:

$$\frac{\partial \phi}{\partial t} = K \frac{\partial^2 \phi}{\partial x^2} \tag{2}$$

  where K is a constant.

- Using a second-order accurate centered difference scheme, with $j \in [0, N]$: we have discretized [2] in the last lesson as:

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = K \left(\frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta x^2} \right) \tag{3}$$

  or fully implicit:

$$-\alpha\phi_{j+1}^{n+1} + (1 + 2\alpha)\phi_j^{n+1} - \alpha\phi_{j-1}^{n+1} = \phi_j^n \tag{4}$$

  with Dirichlet boundary condition:

$$\begin{aligned} x_N &= x_0 + N\Delta x \\ \phi(x_0) &= \phi_0 \qquad \forall t \\ \phi(x_N) &= \phi_N \qquad \forall t \end{aligned} \tag{5}$$

An hindsight 1

Let us now make a small diversion. Let us start with the forward first order accurate formulation for the first derivative:

$$\frac{\partial \phi}{\partial x} = \frac{\phi_{j+1} - \phi_j}{\Delta x} \tag{6}$$

If we add the null term:

$$\frac{2}{2}\left(\phi_{j-1} - \phi_{j-1}\right) = 0 \tag{7}$$

to [6] the equation does not change.

$$\frac{\phi_{j+1} - \phi_j}{\Delta x} = 2\left(\frac{\phi_{j+1} - \phi_j + \phi_{j-1} - \phi_{j-1}}{2\Delta x}\right) \tag{8}$$

An hindsight 2

If we now split in two terms:

$$\frac{\partial \phi}{\partial x} = \frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x} + \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{2\Delta x} \tag{9}$$

and we multiply the second term by $\Delta x / \Delta x$, we finally get:

$$\frac{\partial \phi}{\partial x} = \frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x} + \left(\frac{\Delta x}{2}\right) \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta x^2} \tag{10}$$

We have here that the forward scheme is a centered scheme with diffusion added (!). That is why one of the major error in upstream was its diffusivity.

Advection and diffusion problem

- Consider now a problem with BOTH advection and diffusion:

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = K\frac{\partial^2 \phi}{\partial x} \tag{11}$$

where K is a constant.

- we have seen that leapfrog is conditionally stable for advection and forward approximation is conditionally stable for the diffusion scheme
- we can use the combination of second order accurate approximation to the second derivative for diffusion and leapfrog for the advection
- Using a $2\Delta t$ time step for the diffusion, we can discretize [11] as:

$$\frac{\phi_j^{n+1} - \phi_j^{n-1}}{2\Delta t} + u\frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x} = K\left(\frac{\phi_{j+1}^{n-1} - 2\phi_j^{n-1} + \phi_{j-1}^{n-1}}{\Delta x^2}\right) \tag{12}$$

where the diffusion term is calculated at a previous time step.

Non Linear Advection

- What if the wind is not constant, and the physical quantity to advect is the wind intensity itself?

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{13}$$

  In the equation above we have a non-linear term.

- The non linearity allows more interesting dynamics but can cause numerical problems both through truncation and stability.
- It generates large gradients.

The simplest solution to use a linear formulation to solve a nonlinear problem is to include a diffusion process to prevent the formations of sharp gradients.

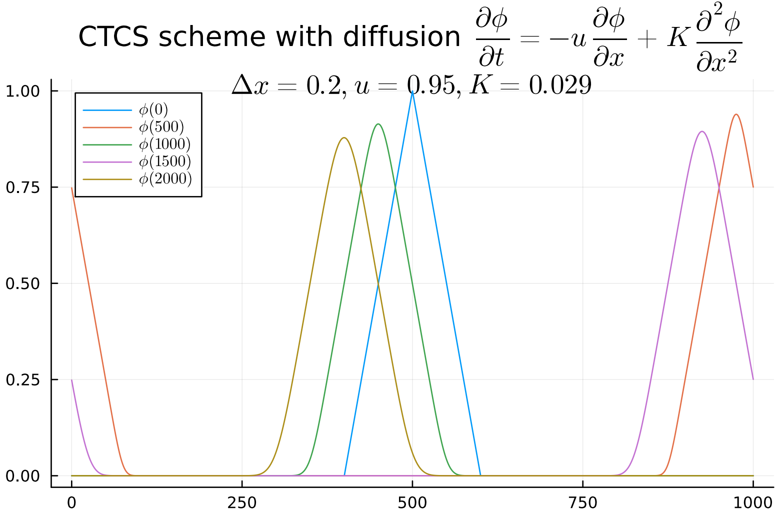
Exercise on advection-diffusion

- Write a program to integrate the advection-diffusion equation in [11] using the scheme in [12] in the domain $0 \leq x \leq 1000m$ with advection velocity $u = 0.95m/s$ and diffusion coefficient $K = 0.029$ Let $\Delta x = 0.2m$ and assume periodic boundary conditions. Assume the initial shape to be:

$$\phi(x,0) = \begin{cases} 0.0 & for \quad x < 400 \\ 0.01(x-400.0) & for \quad 400 \leq x < 500 \\ 2.0 - 0.01(x-400.0) & for \quad 500 \leq x \leq 600 \\ 0.0 & for \quad x > 600 \end{cases} \tag{14}$$

- Integrate forward and show solutions from $t = 0s$ up to $t = 2000s$ every $500s$. What happens if you increase the spatial Δx resolution? Set $\Delta x = 0.05m$.
- Apply a RAW filter with $\alpha = 0.1$ and $\beta = 0.53$.

Expected result



Julia Code

```
c = u * dt/dx;
dc = K * dt/(dx^2);

function ftcs_diff(p_now)
  p_now - 0.5 * c * (circshift(p_now,-1) - circshift(p_now,1)) +
     dc * (circshift(p_now,-1)-2.0*p_now+circshift(p_now,1))
end;
function ctcs_diff(p_old,p_now)
  p_old - c * (circshift(p_now,-1) - circshift(p_now,1)) +
     2.0 * dc * (circshift(p_old,-1)-2.0*p_old+circshift(p_old,1))
end;

p_now = ftcs_diff(p_old);
t = t0+dt;
while (t < t1)
  p_new = ctcs_diff(p_old,p_now);
  d = alpha * (p_old+p_new-2.0*p_now);
  global p_old = p_now + beta*d;
  global p_now = p_new + (1-beta)*d;
  global t = t + dt;
  if mod(t,tp) < dt
    plot!(...)
  end;
end;
```