APPH 4200
Physics of Fluids

Review (Ch. 3) & Fluid Equations of Motion (Ch. 4)
September 22, 2009

1. Review Chapter 3

2. Navier-Stokes Equation

Material or Convective Derivative

$$\frac{D}{Dt} \equiv \frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\bar{u} \cdot \bar{\nabla})$$

e.g. $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (\bar{u} \cdot \bar{\nabla}) f$

Velocity Gradient Tensor

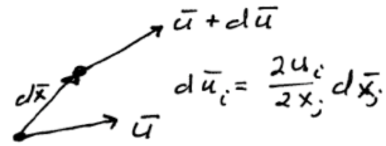
$$\frac{\partial u_i}{\partial x_j} = \epsilon_{ij} + \frac{1}{2} R_{ij}$$

DEFORMATION TENSOR (SYMMETRIC)

ROTATION TENSOR (ANTI-SYMMETRIC)

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$R_{ij} = \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$



$$R_{ij} = -\epsilon_{ijk}\Omega_k$$

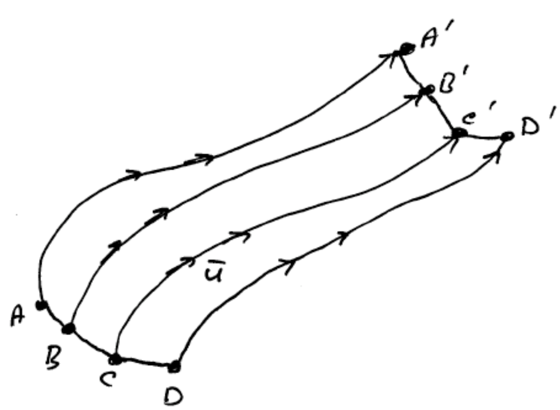
$$\Omega_k = -\frac{1}{2}\epsilon_{ijk} R_{ij}$$

= VORTICITY

Visualizing Flow

STREAMLINES: $\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z} = ds$

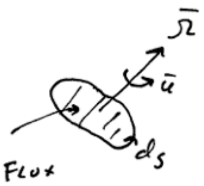
PATHLINES: $\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z} = dt$



Characterizing Flow

VORTICITY $\equiv$ $\bar{\Omega} = \bar{\omega} = \nabla \times \bar{u} = -\frac{1}{2}\epsilon_{ijk} R_{ij}$

CIRCULATION $=$
$$\Gamma \equiv \oint \bar{u} \cdot d\bar{s}$$
$$= \iint \nabla \times \bar{u} \cdot d\bar{A}$$
$$= \iint \bar{\Omega} \cdot d\bar{A}$$



Γ = FLUX OF VORTICITY THROUGH SURFACE.

Problem 3.1

1. A two-dimensional steady flow has velocity components

$$u = y \qquad v = x.$$

Show that the streamlines are rectangular hyperbolas

$$x^2 - y^2 = \text{const.}$$

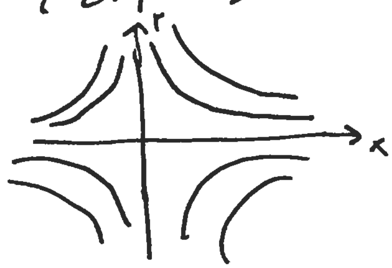
Sketch the flow pattern, and convince yourself that it represents an irrotational flow in a $90°$ corner.

Problem 3.1

FIND STREAMLINES WHEN $\overline{u} = (Y, X)$

$$\frac{dX}{Y} = \frac{dY}{X} = ds$$

$$X\,dX = Y\,dY \Rightarrow X^2 = Y^2 + c$$



Problem 3.2

2. Consider a steady axisymmetric flow of a compressible fluid. The equation of continuity in cylindrical coordinates (R, φ, x) is

$$\frac{\partial}{\partial R}(\rho R u_R) + \frac{\partial}{\partial x}(\rho R u_x) = 0.$$

Show how we can define a streamfunction so that the equation of continuity is satisfied automatically.

Problem 3.2

CONSIDER AXISYMMETRIC FLOW

CONSERVATION OF MASS IS

$$\frac{\partial}{\partial R}(\rho R u_R) + \frac{\partial}{\partial z}(\rho R u_z) = 0$$

$$\frac{\partial}{\partial \varphi} = 0$$

FIND A STREAM FUNCTION.

TRY $\rho \bar{u} = \nabla\varphi \times \nabla\psi(R,z)$ $\quad \nabla\varphi = \frac{\hat{\varphi}}{R}$

WHICH SATISFIES $\nabla\cdot(\rho\bar{u}) = 0$

THEN

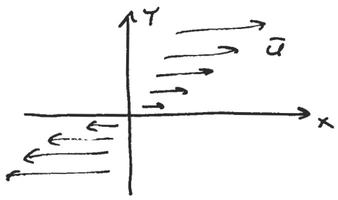
$$\frac{\partial\psi}{\partial z} = \rho R u_R \qquad \frac{\partial\psi}{\partial R} = -\rho R u_z$$

Problem 3.3

3. If a velocity field is given by $u = ay$, compute the circulation around a circle of radius $r = 1$ about the origin. Check the result by using Stokes' theorem.

Problem 3.3

LET $\bar{u} = (aY, 0, 0)$



$\Gamma = \oint \bar{u} \cdot d\bar{s}$

$= \iint \nabla \times \bar{u} \cdot d\bar{A}$

FIND CIRCULATION AROUND UNIT CIRCLE

$$\oint \bar{u} \cdot d\bar{s} = \int_0^{2\pi} d\theta \; \bar{u} \cdot \hat{t}$$

$$\hat{t} = \hat{y}\cos\theta - \hat{x}\sin\theta$$

$$\bar{u} \cdot \hat{t} = -aY\sin\theta = -a\sin^2\theta$$

SO $\oint \bar{u} \cdot d\bar{s} = -a \int_0^{2\pi} \sin^2\theta \, d\theta = -a\pi$

ALSO

$$\iint \nabla \times \bar{u} \cdot dA = -a \iint \hat{z} \cdot d\bar{A}$$

$$= -a\pi$$

SINCE $\nabla \times \bar{u} = -\hat{z}a$

Problem 3.4

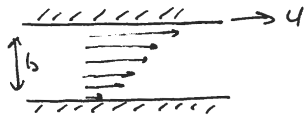
4. Consider a plane Couette flow of a viscous fluid confined between two flat plates at a distance b apart (see Figure 9.4c). At steady state the velocity distribution is

$$u = Uy/b \qquad v = w = 0,$$

where the upper plate at $y = b$ is moving parallel to itself at speed U, and the lower plate is held stationary. Find the rate of linear strain, the rate of shear strain, and vorticity. Show that the streamfunction is given by

$$\psi = \frac{Uy^2}{2b} + \text{const.}$$

Problem 3.4



$\bar{u} = (uy/b, 0, 0)$

LINEAR STRAIN RATE $= \frac{\partial u_i}{\partial x_i}$

$\epsilon_{xx} = \epsilon_{yy} = 0$

SHEAR STRAIN RATE $= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \epsilon_{ij}$

$\epsilon_{xy} = \epsilon_{yx} = \frac{1}{2}\, u/b$

VORTICITY $= \nabla \times \bar{u} = \hat{z}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)$

$\bar{\omega} = -\hat{z}\, u/b$

WHAT IS THE STREAM FUNCTION?

SINCE $\nabla \cdot \bar{u} = 0$, $\bar{u} = \hat{z} \times \nabla\psi = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right)$

$= \nabla z \times \nabla\psi$

$= \nabla \times (z \nabla\psi)$

SO

$\frac{uy}{b} = \frac{\partial\psi}{\partial y} \Rightarrow \psi(y) = \frac{uy^2}{2b} + \text{CONSTANT}$

Problem 3.5

5. Show that the vorticity for a plane flow on the xy-plane is given by

$$\omega_z = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right).$$

Using this expression, find the vorticity for the flow in Exercise 4.

Problem 3.5

FIND VORTICITY, $\bar{\omega}$, FOR FLOW ON X-Y PLANE

$\bar{\omega} = \nabla \times \bar{u}$ IF $\bar{u} = (u, v, 0)$ THEN

$\bar{u} = -\nabla z \times \nabla \psi = -\hat{z} \times \nabla \psi$

$\therefore \bar{\omega} = -\nabla \times (\hat{z} \times \nabla \psi) = -\hat{z}\, \nabla^2 \psi + \nabla \psi (\cancel{\nabla \cdot \hat{z}})^{0}$

FOR THE FLOW IN EXERCISE #4

$\omega_z = -u/b$

Problem 3.7

7. Determine an expression for ψ for a Rankine vortex (Figure 3.17b), assuming that $u_\theta = U$ at $r = R$.

Problem 3.7

FIND STREAMFUNCTION FOR A RANKINE VORTEX

$$u_\theta(r) = \begin{cases} u(r/R) & r < R \\ u(R/r) & r > R \end{cases}$$

$$\bar{u} = -\hat{z} \times \nabla\psi \Rightarrow \boxed{u_\theta = -\frac{\partial\psi}{\partial r}}$$

SO

$$\psi(r) = \begin{cases} -\frac{r^2}{2R}u & r \leq R \\ -uR\ln\left(\frac{r}{R}\right) - \frac{R\psi}{2} & r > R \end{cases}$$

NOTE:

$\nabla^2\psi \propto \omega_z$ FOR $r < R$ (ROTATIONAL)

$\nabla^2\psi = 0$ FOR $r > R$ (IRROTATIONAL)

Problem 3.8

8. Take a plane polar element of fluid of dimensions dr and $r\,d\theta$. Evaluate the right-hand side of Stokes' theorem

$$\int \boldsymbol{\omega} \cdot d\mathbf{A} = \int \mathbf{u} \cdot d\mathbf{s},$$

and thereby show that the expression for vorticity in polar coordinates is

$$\omega_z = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial \theta}\right].$$

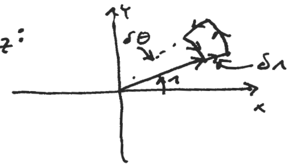
Also, find the expressions for ω_r and ω_θ in polar coordinates in a similar manner.

Problem 3.8

FIND EXPRESSIONS FOR VORTICITY IN CYLINDRICAL COORDINATES.

$\bar{\omega} = \nabla \times \bar{u}$ $\qquad \iint \bar{\omega} \cdot d\bar{A} = \oint \bar{u} \cdot d\bar{s}$

ω_z:

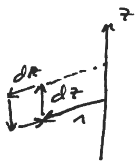


$$\oint \bar{u} \cdot d\bar{s} = u_r\,dr + (r+\delta r)\,d\theta\,u_\theta(r+\delta r) - u_r(\theta+d\theta)\,dr - r\,d\theta\,u_\theta$$

$$= -dr\,d\theta \frac{\partial u_r}{\partial \theta} + dr\,d\theta\left[u_\theta + r\frac{\partial u_\theta}{\partial r}\right]$$

$$= r\,dr\,d\theta\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]$$

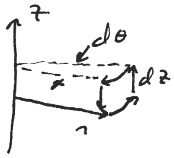
ω_θ:



$$\oint u \cdot ds = dz\,u_z + dr\,u_r(z+dz) - dz\,u_z(r+\delta r) - dr\,u_r$$

$$= -dr\,dz\frac{\partial u_z}{\partial r} + dr\,dz\frac{\partial u_r}{\partial z} = dr\,dz\left[\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right]$$

ω_r:



$$\oint u \cdot ds = r\,d\theta\,u_\theta + dz\,u_z(\theta+\delta\theta) - r\,d\theta\,u_\theta(z+\delta z) - dz\,u_z$$

$$= dz\,d\theta\frac{\partial u_z}{\partial \theta} - r\,d\theta\,dz\frac{\partial u_\theta}{\partial z}$$

$$= -r\,d\theta\,dz\left[\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right]$$

Q.E.D.

Problem 3.9

9. The velocity field of a certain flow is given by

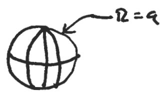
$$u = 2xy^2 + 2xz^2, \qquad v = x^2 y, \qquad w = x^2 z.$$

Consider the fluid region inside a spherical volume $x^2 + y^2 + z^2 = a^2$. Verify the validity of Gauss' theorem

$$\int \nabla \cdot \mathbf{u}\, dV = \int \mathbf{u} \cdot d\mathbf{A},$$

by integrating over the sphere.

VERIFY GAUSS' THEOREM FOR THE FLOW $u = (2xy^2 + 2xz^2,\ x^2y,\ x^2z)$



$$\iiint dV\, \nabla\cdot\bar{u} = \iint \bar{u}\cdot d\bar{A}$$

Problem 3.9

$$\nabla\cdot\bar{u} = 2y^2 + 2z^2 + x^2 + z^2 = 2r^2 - x^2 + z^2 = r^2\left(2 - \cos^2\varphi\sin^2\theta + \cos^2\theta\right)$$

$x = r\cos\varphi\sin\theta$
$z = r\cos\theta$) POLAR COORDINATES

$dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$ SO $\iiint dV\, \nabla\cdot\bar{u} = \frac{a^5}{5}\int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \sin\theta \times \left(2 - \cos^2\varphi\sin^2\theta + \cos^2\theta\right)$

$= \frac{8\pi}{5} a^5$

BUT AT THE $r = a$ SURFACE...

$d\bar{A} = \hat{n}\, a^2 \sin\theta\, d\theta\, d\varphi$

$\hat{n} = (x, y, z)/r$

$\hat{n}\cdot\bar{u} = (2x^2y^2 + 2x^2z^2 + x^2y^2 + x^2z^2)/r = \frac{x^2}{r}(3y^2 + 3z^2) = \frac{3x^2}{r}(r^2 - x^2)$

SO $\iint \bar{u}\cdot d\bar{A} = 3a^5 \iint d\theta\, d\varphi\, \cos^2\varphi \sin^3\theta\left(1 - \cos^2\varphi\sin^2\theta\right)$

$= \frac{8\pi}{5} a^5$ Q. E. D.

Problem 3.10

10. Show that the vorticity field for *any* flow satisfies

$$\nabla\cdot\omega = 0.$$

Problem 3.10

$\nabla \cdot \bar{\omega} = 0$ WHERE $\bar{\omega} = \nabla \times \bar{u}$

BUT $\nabla \cdot \nabla \times \bar{u} = 0$ FOR ALL $\bar{u}$

Problem 3.11

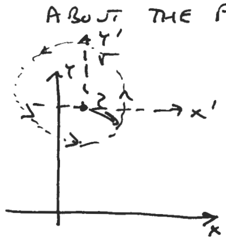
11. A flow field on the xy-plane has the velocity components

$$u = 3x + y \qquad v = 2x - 3y.$$

Show that the circulation around the circle $(x-1)^2 + (y-6)^2 = 4$ is 4π.

Problem 3.11

FIND THE CIRCULATION FOR THE FLOW $\bar{u} = (3x+y,\ 2x-3y)$
ABOUT THE POINT $(x,y)=(1,6)$ WITH RADIUS $=2$



LET'S TRANSLATE AXIS TO $(x,y)=(1,6)$
THEN

$$\oint d\bar{l}\cdot\bar{u} = \oint d\bar{l}'\cdot\bar{u}' = 2\int_0^{2\pi} d\theta\,(\bar{u}'\cdot\hat{t}')$$

$$\hat{t} = -\hat{x}\sin\theta' + \hat{y}\cos\theta'$$

$$\bar{u}' = \left(3(x'+1) + y'+6,\ 2(x'+1) - 3(y'+6)\right)$$

$$= (3x'+y'+9,\ 2x'-3y'-16)$$

$$x' = 2\cos\theta \qquad y' = 2\sin\theta$$

SO

$$\Gamma = 2\int_0^{2\pi} d\theta\left[-\sin\theta(3x'+y'+9) + \cos\theta(2x'-3y'-16)\right]$$

$$= 2\int_0^{2\pi} d\theta\left[12\cos\theta\sin\theta - 6\sin^2\theta + 4 - 9\sin\theta - 16\cos\theta\right]$$

$$= 4\pi$$

Problem 3.12

12. Consider the solid-body rotation

$$u_\theta = \omega_0 r \qquad u_r = 0.$$

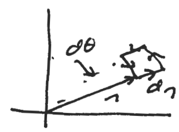
Take a polar element of dimension $r\,d\theta$ and dr, and verify that the circulation is vorticity times area. (In Section 11 we performed such a verification for a circular element surrounding the *origin*.)

Problem 3.12

VERIFY THAT CIRCULATION IS VORTICITY TIMES AREA FOR THE FLOW $U_\theta = \omega_0 r$ (SOLID BODY ROTATION)

NOTE: VORTICITY $= \nabla \times u = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \omega_0\right) = 2\omega_0$

$$\Gamma = \oint \bar{u} \cdot d\bar{l}$$



$$\Gamma = \omega_0 \underbrace{(r+\delta r)^2}_{r^2 + 2\partial r} d\theta - \omega_0 r^2 d\theta$$

$$= 2\omega_0 r\, dr\, d\theta = \underbrace{2\omega_0}_{\text{VORTICITY}} \times \underbrace{r\, dr\, d\theta}_{\text{AREA}}$$

Problem 3.13

13. Using the indicial notation (and without using any vector identity) show that the acceleration of a fluid particle is given by

$$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) + \boldsymbol{\omega} \times \mathbf{u},$$

where q is the magnitude of velocity $\mathbf{u}$ and $\boldsymbol{\omega}$ is the vorticity.

Problem 3.13

USING INDICIES, SHOW $\frac{d\bar{u}}{dt} = \frac{\partial \bar{u}}{\partial t} + \nabla\left(\frac{1}{2}\bar{u}\cdot\bar{u}\right) + \bar{\omega}\times\bar{u}$

BUT $\frac{d\bar{u}}{dt} = \frac{\partial \bar{u}}{\partial t} + \bar{u}\cdot\nabla\bar{u} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$

BUT $u_j \frac{\partial u_i}{\partial x_j} = \underbrace{u_j\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\bar{\omega}\times\bar{u}} + \underbrace{u_j \frac{\partial u_j}{\partial x_i}}_{\frac{\partial}{\partial x_i}\left(\frac{1}{2}u_j^2\right)}$

TO SHOW THIS...

$$\left.\begin{aligned} \bar{\omega} &= \nabla\times u \\ &= \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} \\ (\bar{\omega}\times\bar{u})_i &= \epsilon_{ijk}\,\omega_j u_k \end{aligned}\right\} = \epsilon_{ijk}\,\epsilon_{jlm}\,u_k\frac{\partial u_m}{\partial x_l} = -\epsilon_{ikj}\,\epsilon_{jlm}\,u_k\frac{\partial u_m}{\partial x_l}$$

$$= -\left(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}\right)u_k\frac{\partial u_m}{\partial x_l}$$

$$= -u_m\frac{\partial u_m}{\partial x_i} + u_l\frac{\partial u_i}{\partial x_l}$$

WHICH CAN BE RE-WRITTEN AS ABOVE. (QED)

Problem 3.14

14. The definition of the streamfunction in vector notation is

$$\mathbf{u} = -\mathbf{k}\times\nabla\psi,$$

where $\mathbf{k}$ is a unit vector perpendicular to the plane of flow. Verify that the vector definition is equivalent to equations (3.35).

Problem 3.14

VERIFY THAT $\bar{u} = -\bar{k} \times \nabla \psi$ (WITH $\bar{k} = \hat{z}$) IS EQUIVALENT TO THE CONDITION $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ WITH $\bar{u} = (u, v)$ ....

BUT $(\bar{u})_i = (-\hat{z} \times \nabla \psi)_i = -\epsilon_{i3k} \frac{\partial \psi}{\partial k}$ OR $\bar{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)$

THUS $\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0$ IS SATISFIED.

Equations of Fluid Dynamics

(Conservation Laws)

- Continuity (Mass)
- Navier-Stokes (Force, Momentum)
- Energy

Continuity

CONSERVATION OF Mass

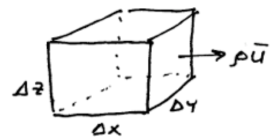
$$\frac{\partial \rho}{\partial t} + \bar{\nabla}\cdot(\rho\bar{u}) = 0$$

$$\underbrace{\frac{\partial \rho}{\partial t} + (\bar{u}\cdot\bar{\nabla})\rho}_{} = -\rho\nabla\cdot\bar{u}$$

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\bar{u}$$

$$\frac{D\rho}{\rho} = -Dt(\nabla\cdot\bar{u})$$

e.g. $\rho \sim e^{-\int dt'\nabla\cdot\bar{u}}$



$$\frac{\partial}{\partial t}(\Delta V\rho) = -\sum_{\substack{\text{ALL}\\\text{SURFACES}}}\rho\bar{u}\cdot\Delta\bar{A}$$

$$= -\frac{\partial}{\partial x_i}(\rho u_i)\Delta x_i\Delta$$

$$= -\Delta V\,\nabla\cdot(\rho\bar{u})$$

Newton's Law

NEWTON'S LAW FOR A PARTICLE

$$\bar{F} = m\bar{a}$$

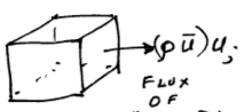
$$\bar{F} = \frac{d}{dt}(m\bar{v})$$

NEWTON'S LAW FOR A FLUID:

$$\bar{F} = \frac{D}{Dt}(\rho\bar{u})$$

$$= \frac{\partial}{\partial t}(\rho\bar{u}) + \bar{\nabla}\cdot(\rho\bar{u}\,\bar{u})$$

$$= \frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j)$$

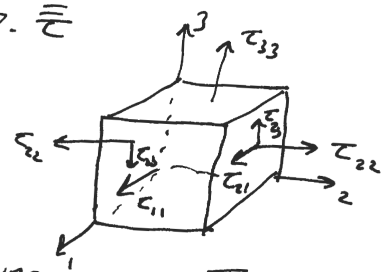


$$= \rho\left[\frac{\partial\bar{u}}{\partial t} + (\bar{u}\cdot\bar{\nabla})\bar{u}\right] + \bar{u}\underbrace{\left[\frac{\partial\rho}{\partial t} + \bar{\nabla}\cdot(\rho\bar{u})\right]}_{=0 \text{ CONSERVATION OF MASS}}$$

$$= \rho\frac{D\bar{u}}{Dt}$$

Momentum

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}\right) = \rho\bar{g} + \nabla\cdot\bar{\bar{\tau}}$$



$\bar{\bar{\tau}}$ = STRESS TENSOR

= USUALLY SYMMETRIC

- = HAS NORMAL STRESS ~ PRESSURE
- = HAS SHEAR STRESS ~ (OFF DIAGONAL)

GRADIENTS OF STRESS PRODUCE FORCE

τ_{ij} — i WHICH FACE, j DIRECTION

τ_{ij} IS POSITIVE WHEN DIRECTED IN DIRECTION OF AXIS

$\tau_{ii} > 0$ IMPLIES TENSILE STRESS

$\tau_{ii} < 0$ IMPLIES COMPRESSIVE STRESS

τ_{ij} $(i \neq j)$ ARE SHEAR STRESSES

Models for Stress

- ISOTROPIC PRESSURE

$$\bar{\bar{\tau}} = -p\bar{\bar{\delta}} \qquad \nabla\cdot\bar{\bar{\tau}} = -\nabla p$$

- MOVING FLUID WITH VISCOSITY

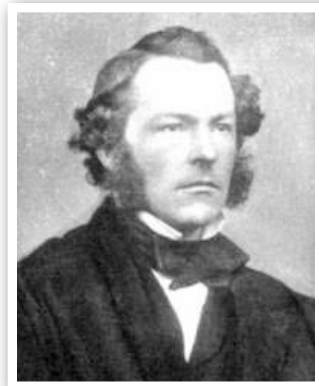
$$\bar{\bar{\tau}} = -p\bar{\bar{\delta}} + \bar{\bar{\sigma}}$$

$\bar{\bar{\sigma}}$ — VISCOUS STRESS

WHAT IS $\bar{\bar{\sigma}}$?

Navier & Stokes

Claude-Lewis Henri Navier
(1785-1836)



George Stokes
(1819-1903)

Stokesian Fluid

MATERIAL ISOTROPY AND STRESS SYMMETRY

(e.g. AIR, WATER BUT <u>NOT</u> MAGNETIZED PLASMA)

$$\bar{\bar{\sigma}} = 2\mu\bar{\bar{\epsilon}} + \lambda(\nabla\cdot\bar{u})\bar{\bar{\delta}}$$

VISCOSITY

~ BULK VISCOSITY

STOKES MODELED VISCOSITY VIA KINETIC THEORY OF MONATOMIC ATOMS AND SHOWED $\lambda = -\frac{2}{3}\mu$.

THEN, STRESS TENSOR

$$\bar{\bar{\tau}} = -p\bar{\bar{\delta}} + 2\mu\bar{\bar{\epsilon}} - \frac{2}{3}\mu(\nabla\cdot\bar{u})\bar{\bar{\delta}}$$

Navier-Stokes Equation

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}\right) = -\nabla p + \rho\bar{g} + \nabla\cdot\left[2\mu\bar{\bar{\epsilon}} - \frac{2}{3}\mu(\nabla\cdot\bar{u})\bar{\bar{\delta}}\right]$$

ASSUME $\mu \sim$ INDEPENDENT OF $\bar{x}$. THEN,

$$\nabla\cdot 2\mu\bar{\bar{\epsilon}} = 2\mu(\nabla\cdot\bar{\bar{\epsilon}}) \qquad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$(\nabla\cdot\bar{\bar{\epsilon}})_i = \frac{1}{2}\left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j}\right) = \frac{1}{2}\nabla^2\bar{u} + \frac{1}{2}\bar{\nabla}(\nabla\cdot\bar{u})$$

Navier-Stokes & Euler

$$\rho\frac{D\bar{u}}{Dt} = -\nabla p + \rho\bar{g} + \begin{cases} \mu\left[\nabla^2\bar{u} + \frac{1}{3}\bar{\nabla}(\nabla\cdot\bar{u})\right] & \text{NAVIER STOKES EQUATION} \\ \mu\nabla^2\bar{u} & \text{INCOMPRESSIBLE N.S.} \\ 0 & \text{EULER EQUATION} \end{cases}$$

Energy

$$\rho \frac{D\bar{u}}{Dt} = \rho \bar{g} + \bar{\nabla} \cdot \bar{\bar{\tau}}$$

$$\rho \frac{D}{Dt}\left(\tfrac{1}{2} u^2\right) = \underbrace{\rho\, \bar{u} \cdot \bar{g}}_{\substack{\text{WORK DONE}\\ \text{BY}\\ \text{BODY FORCES}}} + \underbrace{\bar{u} \cdot (\nabla \cdot \bar{\bar{\tau}})}_{\substack{\text{WORK DONE}\\ \text{BY}\\ \text{SURFACE FORCES}}}$$

The Importance of Viscosity

INCOMPRESSABLE EULER EQUATION

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u} \cdot \nabla)\bar{u} = -\nabla p/\rho + \bar{g}$$

$$\nabla \cdot \bar{u} = 0$$

LET $\bar{\Omega} = \nabla \times \bar{u}$. THEN

$$(\bar{u} \cdot \bar{\nabla})\bar{u} = \bar{\Omega} \times \bar{u} + \tfrac{1}{2}\bar{\nabla} u^2$$

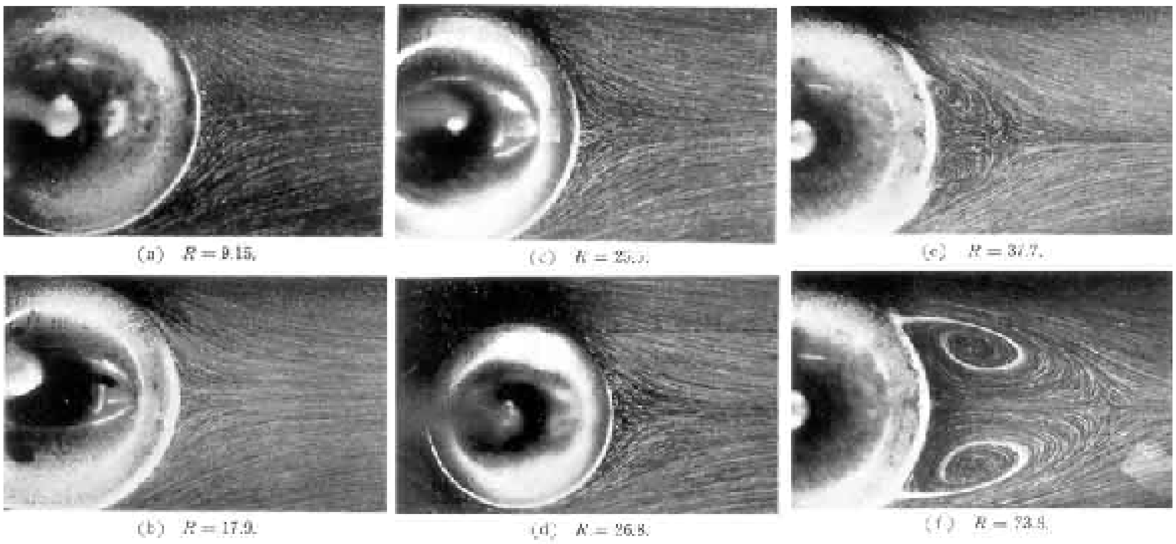
$$\frac{\partial \bar{u}}{\partial t} + \bar{\Omega} \times \bar{u} = -\frac{\nabla p}{\rho} + \bar{g} - \tfrac{1}{2}\bar{\nabla} u^2$$

TAKE CURL OF THIS EQUATION

$$\frac{\partial}{\partial t}\bar{\Omega} + \nabla \times (\bar{\Omega} \times \bar{u}) = 0 \qquad (\text{IF } \bar{g} = -\nabla\varphi)$$

IF $\bar{\Omega} = 0$ at $t = 0$, THEN $\bar{\Omega} = 0$ FOREVER!

Creation of Vorticity



(a) $R = 9.15$, (c) $R = 25.5$, (e) $R = 31.7$, (b) $R = 17.9$, (d) $R = 26.8$, (f) $R = 73.6$

(Note: Flow at thin layer at surface of cylinder vanishes.)

Summary

- The equations of fluid dynamics are dynamical conservation equations:
- Mass conservation
- Momentum changes via total forces (body and surface forces)
- Energy conservation