$$\operatorname*{Max}_{\mathbf{x}} S(\mathbf{x}) = \sum_{i=1}^{I}\sum_{j=1}^{J(t)} s_{ij}(\mathbf{x})x_{ij}, \tag{1}$$

$$\text{s.t.} \sum_{j=1}^{J(t)} \rho_{ij}x_{ij} \le K_i, \forall i \in \mathcal{I}, \tag{2}$$

$$\sum_{i=1}^{I} x_{ij} \le 1, \forall j \in \mathcal{J}(t), \tag{3}$$

$$x_{ij} \in \{0,1\}, s_{ij}(\mathbf{x}) \in [0,1], \forall i \in \mathcal{I}, \forall j \in \mathcal{J}(t). \tag{4}$$

$$\operatorname*{Max}_{\mathcal{R}\subseteq\mathcal{G}} S(\mathcal{R}) = \sum_{(i,j)\in\mathcal{R}}\sum_{n=0}^{N_{worst}} P(\mathcal{N}_i^j(\mathcal{R}), n)\rho_{ij} f(c_i^{n+1}), \tag{5}$$

$$\text{s.t.} \sum_{j:(i,j)\in\mathcal{R}} \rho_{ij} \le K_i, \forall i \in \mathcal{I}, \tag{6}$$

$$\sum_{i:(i,j)\in\mathcal{R}} \mathbb{1}_{(i,j)\in\mathcal{R}} \le 1, \forall j \in \mathcal{J}(t), \tag{7}$$

Definition 1. (**Non-negativity**, **Monotonicity**, **Submodularity** [1]): A set function $S(\cdot) : 2^{\mathcal{G}} \to \mathbb{R}$ is i) *non-negative*, if $S(\emptyset) = 0$ and $S(\mathcal{R}) \ge 0$ ($\forall \mathcal{R} \subseteq \mathcal{G}$); ii) *monotone*, if for $\forall \mathcal{R} \subseteq \mathcal{G}$ and $\forall g_1 \in \mathcal{G} \setminus \mathcal{R}$, $S(\mathcal{R} \cup \{g_1\}) \ge S(\mathcal{R})$; iii) *submodular*, if and only if $\forall \mathcal{R}_1 \subseteq \forall \mathcal{R}_2 \subseteq \mathcal{G}$ and $\forall g_1 \in \mathcal{G}\backslash\mathcal{R}_2$, $S(\mathcal{R}_1 \cup \{g_1\}) - S(\mathcal{R}_1) \ge S(\mathcal{R}_2 \cup \{g_1\}) - S(\mathcal{R}_2)$.

Definition 2. (**m-Knapsack Constraint** [2]): Given m cost functions $C_1, C_2, \ldots, C_m$, the associated constraint is called *m-multiple knapsack constraint* if the solution $\mathcal{R} \subseteq \mathcal{G}$ satisfies $\forall i \in [m], C_i(\mathcal{R}) \le 1$.

Definition 3. (**Matroid** [3]): Consider a finite ground set $\mathcal{G}$ and a non-empty collection of subsets of $\mathcal{G}$ which is represented as $\mathcal{V}$. The pair $(\mathcal{G}, \mathcal{V})$ is a *matroid*, if and only if three conditions hold: i) $\emptyset \in \mathcal{V}$; ii) If $\forall \mathcal{R}_1 \subseteq \forall \mathcal{R}_2 \in \mathcal{V}$, $\mathcal{R}_1 \in \mathcal{V}$; iii) If $\forall \mathcal{R}_1 \in \mathcal{V}, \forall \mathcal{R}_2 \in \mathcal{V}$ and $|\mathcal{R}_1| < |\mathcal{R}_2|$, $\exists g_1 \in \mathcal{R}_2$ satisfies $\mathcal{R}_1 \cup \{g_1\} \in \mathcal{V}$.

Theorem 1. *Given the acceptance probability, the SSC problem is NP-hard.*

Proof. Consider an instance of this problem, where the total number of batteries in the BSS network (*i.e.,* $\sum_{i\in\mathcal{I}} K_i$) is larger than the number of recommended e-scooter drivers (*i.e.,* $J(t)$). Thus, constraint (3) can be relaxed as $\forall j \in \mathcal{J}(t), \sum_{i=1}^{I} x_{ij} = 1$. In this case, given the acceptance probability $\rho = [\rho_{ij}]_{I\times J_{(t)}}$, the SSC problem can be reduced into the classical NP-hard *Generalized assignment problem* [4] as follows: given I knapsacks (*i.e.,* BSSs), each with capacity K_i ($\forall i \in \mathcal{I}$), assign each item j (*i.e.,* driver $j \in \mathcal{J}(t)$) to exactly one knapsack i ($\forall i \in \mathcal{I}$) with a profit s_{ij} and a weight ρ_{ij}, so as to maximize the total profit under the limited capacity of each knapsack. As a result, the SSC problem is NP-hard, and Theorem 1 is proved. □

Lemma 1. The objective function $S(\mathcal{R})$ ($\forall \mathcal{R} \subseteq \mathcal{G}$) is *non-negative*, *monotone*, and *submodular*.

Proof. i) *Non-negativity*. If no station-driver pair is recommended for satisfying drivers' demand, $\mathcal{R} = \emptyset$ and $\mathbf{x} = 0$. Hence, according to Eq. (1), $S(\emptyset) = 0$. Moreover, since $S(\mathcal{R})$ represents the DSAT score of drivers which is non-negative, $\forall \mathcal{R} \subseteq \mathcal{G}, S(\mathcal{R}) \ge 0$. Thus, according to Def. 1, $S(\mathcal{R})$ is a non-negative function.

ii) *Monotonicity*. When $S(\mathcal{R})$ is calculated in terms of each station, Eq. (5) can be equivalently transformed as

$$\operatorname*{Max}_{\mathcal{R}\subseteq\mathcal{G}} S(\mathcal{R}) = \sum_{i:(i,j)\in\mathcal{R}}\sum_{n=1}^{N_{max}} P(\mathcal{N}_i(\mathcal{R}), n)\sum_{k=1}^{n} f(c_i^k), \tag{8}$$

where $\mathcal{N}_i(\mathcal{R})$ denote the set of drivers who are recommended to station i under recommendation set $\mathcal{R}$, *i.e.,* $\mathcal{N}_i(\mathcal{R}) = \{j|x_{ij} = 1, \forall j \in \mathcal{J}\}$, and $N_{max} = min(|\mathcal{N}_i(\mathcal{R})|, K_i)$. Note that, given recommendation set $\mathcal{R}$, the arrival sequences of drivers do not affect the total DSAT score $S(\mathcal{R})$. Hence, $\forall \mathcal{R} \subseteq \mathcal{G}$, $\forall g_1 = (i_1, j_1) \in \mathcal{G}\backslash\mathcal{R}$, we assume that driver j_1 is the last to arrive at station i_1 compared to the drivers in $\mathcal{R}$. Then according to Eq. (5), we have $S(\mathcal{R} \cup \{g_1\}) - S(\mathcal{R}) = \sum_{n=0}^{N_{worst}} P(\mathcal{N}_{i_1}(\mathcal{R}), n) f(c_{i_1}^{n+1})\rho_{i_1 j_1} \ge 0$, where $N_{worst} = min(|\mathcal{N}_{i_1}(\mathcal{R})|, K_{i_1} - 1)$. As a result, $S(\mathcal{R})$ is a monotone function.

iii) *Submodularity*. $\forall \mathcal{R}_1 \subseteq \forall \mathcal{R}_2 \subseteq \mathcal{G}$, $\forall g_1 = (i_1, j_1) \in \mathcal{G}\backslash\mathcal{R}_2$, according to Eq. (5), we can obtain

$$S(\mathcal{R}_1 \cup \{g_1\}) - S(\mathcal{R}_1) = \sum_{n=0}^{N_{worst}^1} P(\mathcal{N}_{i_1}(\mathcal{R}_1), n) f(c_{i_1}^{n+1})\rho_{i_1 j_1}, \tag{9}$$

$$S(\mathcal{R}_2 \cup \{g_1\}) - S(\mathcal{R}_2) = \sum_{n=0}^{N_{worst}^2} P(\mathcal{N}_{i_1}(\mathcal{R}_2), n) f(c_{i_1}^{n+1})\rho_{i_1 j_1}, \tag{10}$$

where $N_{worst}^1 = min(|\mathcal{N}_{i_1}(\mathcal{R}_1)|, K_{i_1} - 1)$ and $N_{worst}^2 = min(|\mathcal{N}_{i_1}(\mathcal{R}_2)|, K_{i_1} - 1)$.

Firstly, $\forall i \in \mathcal{I}$, $\forall \mathcal{R} \subseteq \mathcal{G}$, let $\mathcal{N}_i(\mathcal{R}) = \{1, 2, \ldots, N\}$ ($1 \le N \le J$), and we define a function

$$H(\mathcal{N}_i(\mathcal{R})) = \sum_{n=0}^{N} P(\mathcal{N}_i(\mathcal{R}), n) s_{n+1}, \tag{11}$$

where $0 \le s_{n+1} \le 1$ and $s_{n+1} \ge s_{n+2}$. Specifically, we define $H(\mathcal{N}_i(\mathcal{R}))_m$ as the sum of the m ($0 \le m \le N$) items, *i.e.,* $H(\mathcal{N}_i(\mathcal{R}))_m = \sum_{n=0}^{m} P(\mathcal{N}_i(\mathcal{R}), n) s_{n+1}$. Moreover, for $\forall j_+ \in \mathcal{J}(t) \setminus \mathcal{N}_i(\mathcal{R})$, let $\mathcal{N}_i(\mathcal{R})^+ = \mathcal{N}_i(\mathcal{R}) \cup \{j_+\}$, we have

$$H(\mathcal{N}_i(\mathcal{R})^+) = \sum_{n=0}^{N+1} P(\mathcal{N}_i(\mathcal{R})^+, n) s_{n+1}. \tag{12}$$

In the following, our goal is to compare $H(\mathcal{N}_i(\mathcal{R}))$ and $H(\mathcal{N}_i(\mathcal{R})^+)$. Initially, according to Eqs. (11)(12), when $m = 0$, we have

$$H(\mathcal{N}_i(\mathcal{R}))_0 = s_1 \prod_{j_2\in\mathcal{N}_i(\mathcal{R})} (1 - \rho_{ij_2}), \tag{13}$$

$$H(\mathcal{N}_i(\mathcal{R})^+)_0 = s_1(1 - \rho_{ij_+}) \prod_{j_2\in\mathcal{N}_i(\mathcal{R})} (1 - \rho_{ij_2}), \tag{14}$$

$$H(\mathcal{N}_i(\mathcal{R}))_0 - H(\mathcal{N}_i(\mathcal{R})^+)_0 = s_1\rho_{ij_+} \prod_{j_2\in\mathcal{N}_i(\mathcal{R})} (1 - \rho_{ij_2}). \tag{15}$$

Furthermore, for $1 \le m \le N$, suppose
$H(\mathcal{N}_i(\mathcal{R}))_m - H(\mathcal{N}_i(\mathcal{R})^+)_m =$

$$s_{m+1} \sum_{\mathcal{A}\in\mathbb{A}(m)} \rho_{ij_+} \prod_{j_1\in\mathcal{A}} \rho_{ij_1} \prod_{j_2\in\mathcal{N}_i(\mathcal{R})\backslash\mathcal{A}} (1 - \rho_{ij_2})$$

$$+\sum_{\xi=1}^{m}(s_\xi - s_{\xi+1})\sum_{\mathcal{A}\in\mathbb{A}(\xi-1)}\rho_{ij_+}\prod_{j_1\in\mathcal{A}}\rho_{ij_1}\prod_{j_2\in\mathcal{N}_i(\mathcal{R})\backslash\mathcal{A}}(1-\rho_{ij_2}), \tag{16}$$

where $\mathbb{A}(\cdot)$ denotes $\mathcal{A}(\mathcal{N}_i(\mathcal{R}),\cdot)$. Based on mathematical induction, we prove Eq. (16) by induction on m.

For $m=1$, based on Eq. (15), we have

$$\begin{aligned}&H(\mathcal{N}_i(\mathcal{R}))_1 - H(\mathcal{N}_i(\mathcal{R})^+)_1 = \\ &s_2\sum_{\mathcal{A}\in\mathbb{A}(1)}\rho_{ij_+}\prod_{j_1\in\mathcal{A}}\rho_{ij_1}\prod_{j_2\in\mathcal{N}_i(\mathcal{R})\backslash\mathcal{A}}(1-\rho_{ij_2}) \\ &\quad+(s_1-s_2)\rho_{ij_+}\prod_{j_2\in\mathcal{N}_i(\mathcal{R})}(1-\rho_{ij_2}).\end{aligned} \tag{17}$$

Thus, Eq. (16) holds for $m=1$. Assume that the conclusion holds for $m=u, u=1,2,\ldots,N$. Then, for $m=u+1$, we can obtain

$$\begin{aligned}&H(\mathcal{N}_i(\mathcal{R}))_{u+1} - H(\mathcal{N}_i(\mathcal{R})^+)_{u+1} = \\ &s_{u+2}\sum_{\mathcal{A}\in\mathbb{A}(u+1)}\prod_{j_1\in\mathcal{A}}\rho_{ij_1}\prod_{j_2\in\mathcal{N}_i(\mathcal{R})\backslash\mathcal{A}}(1-\rho_{ij_2}) \\ &- s_{u+2}\Bigg(\sum_{\mathcal{A}\in\mathbb{A}(u+1)}(1-\rho_{ij_+})\prod_{j_1\in\mathcal{A}}\rho_{ij_1}\prod_{j_2\in\mathcal{N}_i(\mathcal{R})\backslash\mathcal{A}}(1-\rho_{ij_2}) \\ &+\sum_{\mathcal{A}\in\mathbb{A}(u)}\rho_{ij_+}\prod_{j_1\in\mathcal{A}}\rho_{ij_1}\prod_{j_2\in\mathcal{N}_i(\mathcal{R})\backslash\mathcal{A}}(1-\rho_{ij_2})\Bigg) \\ &+\big(H(\mathcal{N}_i(\mathcal{R}))_m - H(\mathcal{N}_i(\mathcal{R})^+)_m\big) \\ &= s_{u+2}\sum_{\mathcal{A}\in\mathbb{A}(u+1)}\rho_{ij_+}\prod_{j_1\in\mathcal{A}}\rho_{ij_1}\prod_{j_2\in\mathcal{N}_i(\mathcal{R})\backslash\mathcal{A}}(1-\rho_{ij_2}) \\ &+\sum_{\xi=1}^{u+1}(s_\xi - s_{\xi+1})\sum_{\mathcal{A}\in\mathbb{A}(\xi-1)}\rho_{ij_+}\prod_{j_1\in\mathcal{A}}\rho_{ij_1}\prod_{j_2\in\mathcal{N}_i(\mathcal{R})\backslash\mathcal{A}}(1-\rho_{ij_2}).\end{aligned} \tag{18}$$

As a result, Eq. (16) holds for $m=u+1$. The mathematical induction is completed.

Hence, according to Eqs. (15)(16), we have

$$\begin{aligned}&H(\mathcal{N}_i(\mathcal{R})) - H(\mathcal{N}_i(\mathcal{R})^+) = \\ &\sum_{\xi=1}^{N+1}(s_\xi - s_{\xi+1})\sum_{\mathcal{A}\in\mathbb{A}(\xi-1)}\rho_{ij_+}\prod_{j_1\in\mathcal{A}}\rho_{ij_1}\prod_{j_2\in\mathcal{N}_i(\mathcal{R})\backslash\mathcal{A}}(1-\rho_{ij_2}).\end{aligned} \tag{19}$$

Moreover, since $s_\xi \geq s_{\xi+1}$ $(1\leq\xi\leq N+1)$, $\forall i\in\mathcal{I}, \forall j\in\mathcal{J}(t)$, $0\leq\rho_{ij}\leq 1$, based on Eq. (19), we have

$$H(\mathcal{N}_i(\mathcal{R})) \geq H(\mathcal{N}_i(\mathcal{R})^+). \tag{20}$$

Similarly, according to Eqs. (15)(16), we have

$$H(\mathcal{N}_i(\mathcal{R}))_m \geq H(\mathcal{N}_i(\mathcal{R})^+)_m, \tag{21}$$

where $0\leq m\leq N$.

Let $s_{n+1}=\rho_{i_1j_1}f(c_{i_1}^{n+1})$. According to Eqs. (9)(10), since $\mathcal{N}^1_{worst}$ and $\mathcal{N}^2_{worst}$ is dependent on the number of recommended drivers and the station capacity, we consider the following three cases:

Case 1: $|\mathcal{N}_{i_1}(\mathcal{R}_2)|\leq K_{i_1}-1$. Thus, $\mathcal{N}^1_{worst}=|\mathcal{N}_{i_1}(\mathcal{R}_1)|$ and $\mathcal{N}^2_{worst}=|\mathcal{N}_{i_1}(\mathcal{R}_2)|$. Based on Eqs. (9)(10)(11), we have

$$\text{LHS of } (9) = H(\mathcal{N}_{i_1}(\mathcal{R}_1)), \tag{22}$$

$$\text{LHS of } (10) = H(\mathcal{N}_{i_1}(\mathcal{R}_2)). \tag{23}$$

Case 2: $|\mathcal{N}_{i_1}(\mathcal{R}_1)|\leq K_{i_1}-1<|\mathcal{N}_{i_1}(\mathcal{R}_2)|$. Hence, $\mathcal{N}^1_{worst}=|\mathcal{N}_{i_1}(\mathcal{R}_1)|$ and $\mathcal{N}^2_{worst}=K_{i_1}-1$, we have

$$\text{LHS of } (9) = H(\mathcal{N}_{i_1}(\mathcal{R}_1)), \tag{24}$$

$$\text{LHS of } (10) = H(\mathcal{N}_{i_1}(\mathcal{R}_2))_{K_{i_1}-1}. \tag{25}$$

Case 3: $K_{i_1}-1<|\mathcal{N}_{i_1}(\mathcal{R}_1)|$. Hence, $\mathcal{N}^1_{worst}=\mathcal{N}^2_{worst}=K_{i_1}-1$, we have

$$\text{LHS of } (9) = H(\mathcal{N}_{i_1}(\mathcal{R}_1))_{K_{i_1}-1}, \tag{26}$$

$$\text{LHS of } (10) = H(\mathcal{N}_{i_1}(\mathcal{R}_2))_{K_{i_1}-1}. \tag{27}$$

Since $\mathcal{R}_1\subseteq\mathcal{R}_2$, $\mathcal{N}_{i_1}(\mathcal{R}_1)\subseteq\mathcal{N}_{i_2}(\mathcal{R}_2)$. When $\mathcal{N}_{i_1}(\mathcal{R}_1)\subset\mathcal{N}_{i_2}(\mathcal{R}_2)$, according to Eqs. (20)-(27), we have

$$S(\mathcal{R}_1\cup\{g_1\})-S(\mathcal{R}_1)\geq S(\mathcal{R}_2\cup\{g_1\})-S(\mathcal{R}_2). \tag{28}$$

Moreover, when $\mathcal{N}_{i_1}(\mathcal{R}_1)=\mathcal{N}_{i_2}(\mathcal{R}_2)$, it is clear that the above relationship also holds true. As a result, according to Def. 1, $S(\mathcal{R})$ is a submodular function. Thus, based on the above proofs, Lemma 1 is proved.

□

Lemma 2. Constraints (6) and (7) are an *I-knapsack* constraint and a *matroid* constraint, respectively.

Proof. According to Def. 2, we first prove that constraint (6) is an *I-knapsack* constraint. Specifically, $\forall\mathcal{R}\subseteq\mathcal{G}$, consider that each BSS $i\in\mathcal{I}$ is a knapsack with the cost function $C_i(\mathcal{R})=\frac{\sum_{j:(i,j)\in\mathcal{R}}\rho_{ij}}{K_i}$. Hence, based on Eqs. (6), we have $\forall i\in[I], C_i(\mathcal{R})\leq 1$.

Moreover,we prove that constraint (7) is a matroid constraint. Specifically, we prove that the pair $(\mathcal{G},\mathcal{V})$ constructed by constraint (7) satisfies the three conditions of matroid.

i) $\emptyset\in\mathcal{G}$, and $\emptyset$ satisfies constraint (7). Hence, $\emptyset\in\mathcal{V}$, and the first condition is true.

ii) Since $\mathcal{R}_2\in\mathcal{V}$, according to Eq. (7), for $\forall j_1:(i,j_1)\in\mathcal{R}_2$, we can obtain

$$\sum_{i:(i,j_1)\in\mathcal{R}_2}\mathbb{1}_{(i,j_1)\in\mathcal{R}_2}\leq 1. \tag{29}$$

Furthermore, since $\mathcal{R}_1\subseteq\mathcal{R}_2$, for $\forall j_2:(i,j_2)\in\mathcal{R}_1$, we have

$$\sum_{i:(i,j_2)\in\mathcal{R}_1}\mathbb{1}_{(i,j_2)\in\mathcal{R}_1}\leq 1. \tag{30}$$

Thus, we have $\mathcal{R}_1\in\mathcal{V}$, and the second condition is true.

iii) Owing to $\mathcal{R}_1\in\mathcal{V}$, $\mathcal{R}_2\in\mathcal{V}$, and $|\mathcal{R}_1|<|\mathcal{R}_2|$, $\exists j_1:(i_1,j_1)\in\mathcal{R}_2\backslash\mathcal{R}_1$ which satisfies the following inequation:

$$\sum_{i:(i,j_1)\in\mathcal{R}_1}\mathbb{1}_{(i,j_1)\in\mathcal{R}_1}<\sum_{i:(i,j_1)\in\mathcal{R}_2}\mathbb{1}_{(i,j_1)\in\mathcal{R}_2}=1. \tag{31}$$

Moreover, since $\mathcal{R}_1\in\mathcal{V}$, according to Eqs. (31), for $\forall j_2:(i,j_2)\in\mathcal{R}_1\cup\{(i_1,j_1)\}$, the following inequation holds:

$$\sum_{i:(i,j_2)\in\mathcal{R}_1\cup\{(i_1,j_1)\}}\mathbb{1}_{(i,j_2)\in\mathcal{R}_1\cup\{(i_1,j_1)\}}\leq 1. \tag{32}$$

As a result, $\mathcal{R}_1\cup\{(i_1,j_1)\}\in\mathcal{V}$, and the third condition is true. As constraint (7) satisfies all the three conditions, it is a matroid constraint. Thus, Lemma 2 is proved. □

Theorem 2. Alg. 1 achieves a near-optimal solution with a $1/[(1+\epsilon)(I+2)]$-approximation ratio ($\epsilon>0$) in polynomial time $O(I^2J^2)$, where I and J denote the number of BSSs and drivers, respectively.

Proof. According to Lemmas 1 and 2, the optimal BSS recommendation problem is maximizing a monotone and submodular objective function with an *I-knapsack* constraint and a *matroid* constraint. As a result, referring to [5], Alg. 2 which uses a partial enumeration technique and the simultaneous greedy framework can achieve a $1/[(1+\epsilon)(I+2)]$-approximation ratio.

Alg. 2 has at most IJ iterations in the outer loop (*i.e.,* lines 2-22). The time complexity of the binary search in each iteration costs $O(log(d_0/d_1)) = O(log(logIJ/\delta)) = O(1)$. The process that consumes the most time is the simultaneous greedy search (*i.e.,* lines 11-17), and the complexity of each search is $O(\ell IJ/\epsilon)$ according to [6]. Thus, the complexity of Alg. 2 is $O(I^2J^2)$. To sum up, Theorem 2 is proved. □

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