There are (in general) two ways to calculate the spectra, here we take the tensor-TT spectrum as an example (the scalar-TT calculation involves a different function f, and BB calculation replaces the j_l hereafter by j_2). Its calculation goes like (ignoring lots of coefficients)

$$
\begin{aligned}
C_l^{TT} &\propto \int \mathcal{D}k\mathcal{D}k' \left\langle \left\{ \int_{\tau_{\text{rec}}}^{\tau_{\text{rei}}} d\tau\, h'_{ij}(k,\tau) \frac{j_l[(\tau_{\text{rei}}-\tau)\,k]}{(\tau_{\text{rei}}-\tau)^2\,k^2} \right\}^2 \right\rangle \\
&\propto \int \mathcal{D}k\mathcal{D}k' \int_{\tau_{\text{osc}}}^{\tau_{\text{rei}}} d\tau_1 d\tau_2 \langle h'_{ij}(k,\tau_1) h'_{ij}(k',\tau_2)\rangle J_l(k,\tau_1) J_l(k,\tau_2) \\
&= \int \mathcal{D}k \int_{\tau_{\text{osc}}}^{\tau_{\text{rei}}} d\tau_1 d\tau_2 \frac{1}{a(\tau_1)a(\tau_2)} \int_{\tau_{\text{osc}}}^{\tau_1} d\tau'_1 \int_{\tau_{\text{osc}}}^{\tau_2} d\tau'_2\, \mathcal{G}(k,\tau_1,\tau'_1)\mathcal{G}(k,\tau_2,\tau'_2)\Pi^2(k,\tau'_1,\tau'_2) J_l(k,\tau_1) J_l(k,\tau_2) \\
&= \int \mathcal{D}k \int_{\tau_{\text{osc}}}^{\tau_{\text{rei}}} d\tau_1 d\tau_2 \frac{1}{a(\tau_1)a(\tau_2)} \int_{\tau_{\text{osc}}}^{\tau_1} d\tau'_1 \int_{\tau_{\text{osc}}}^{\tau_2} d\tau'_2\, \mathcal{G}(k,\tau_1,\tau'_1)\mathcal{G}(k,\tau_2,\tau'_2) \\
&\quad \cdot 2 \int \mathcal{D}q\, \Theta_{++}(\mathbf{k}-\mathbf{q},\mathbf{k}) \mathcal{S}_{++}(\mathbf{q},\mathbf{k},\tau'_1) \mathcal{S}^*_{++}(\mathbf{q},\mathbf{k},\tau'_2) J_l(k,\tau_1) J_l(k,\tau_2)
\end{aligned} \tag{1}
$$

where $J_l(x) = j_l(x)/x^2$, $\mathcal{G}$ is the (modified-)Green's function, and $\mathcal{S}_{++}$ is defined as

$$
\mathcal{S}_{++}(\mathbf{q},\mathbf{k},\tau) = -\frac{1}{a^2(\tau)} \left[|\mathbf{q}||\mathbf{k}-\mathbf{q}| v_+(\mathbf{q},\tau) v_+(\mathbf{k}-\mathbf{q},\tau) + v'_+(\mathbf{q},\tau) v'_+(\mathbf{k}-\mathbf{q},\tau) \right] ,
$$

which, in spite of its arguments, is in fact a function of $\mathbf{q}$ and $\mathbf{p} = \mathbf{k} - \mathbf{q}$. In the third line the $\mathcal{D}k'$ is killed by the definition $\langle \Pi_{ij}(k,\tau'_1)\Pi_{ij}(k',\tau'_2)\rangle = (2\pi)^3\delta(k+k')\Pi^2(k,\tau'_1,\tau'_2)$.

Our current approach

Our current approach continues the calculation by switching the integration sequence of τ_i and τ'_i in Eq. (1) like

$$
\begin{aligned}
&\propto \int \mathcal{D}k\mathcal{D}q\, \Theta_{++}(\mathbf{k}-\mathbf{q},\mathbf{k}) \int_{\tau_{\text{osc}}}^{\tau_{\text{rei}}} d\tau'_1 d\tau'_2 \left\{ \int_{\tau'_1}^{\tau_{\text{rei}}} d\tau_1 \frac{J_l(k,\tau_1)}{a(\tau_1)} \mathcal{G}(k,\tau_1,\tau'_1) \right\} \left\{ \int_{\tau'_2}^{\tau_{\text{rei}}} d\tau_2 \frac{J_l(k,\tau_2)}{a(\tau_2)} \mathcal{G}(k,\tau_2,\tau'_2) \right\} \\
&\quad \cdot \mathcal{S}_{++}(\mathbf{q},\mathbf{k},\tau'_1) \mathcal{S}^*_{++}(\mathbf{q},\mathbf{k},\tau'_2) \\
&= \int \mathcal{D}k\mathcal{D}q\, \Theta_{++}(\mathbf{k}-\mathbf{q},\mathbf{k}) \int_{\tau_{\text{osc}}}^{\tau_{\text{rei}}} d\tau'_1 d\tau'_2 f_l(k,\tau'_1) f_l(k,\tau'_2) \mathcal{S}_{++}(\mathbf{q},\mathbf{k},\tau'_1) \mathcal{S}^*_{++}(\mathbf{q},\mathbf{k},\tau'_2) \\
&= \int \mathcal{D}k\mathcal{D}q\, \Theta_{++}(\mathbf{k}-\mathbf{q},\mathbf{k}) \int_{\tau_{\text{osc}}}^{\tau_{\text{rei}}} d\tau'_1 d\tau'_2 f_l(k,\tau'_1) f_l(k,\tau'_2) \\
&\quad \cdot \left(\mathrm{Re}\mathcal{S}_{++}(\mathbf{q},\mathbf{k},\tau'_1)\, \mathrm{Re}\mathcal{S}_{++}(\mathbf{q},\mathbf{k},\tau'_2) + \mathrm{Im}\mathcal{S}_{++}(\mathbf{q},\mathbf{k},\tau'_1) \mathrm{Im}\mathcal{S}_{++}(\mathbf{q},\mathbf{k},\tau'_2) \right) ,
\end{aligned}
$$

where in the last line the absence of the Re-Im cross terms can be easily seen by the symmetry between τ_1^1 and τ'_2 integration. I did the integration sequence switching for two reasons: it allows us to better organize the expressions in the scalar-TT case, and it saves storage space when dumping the intermediation step (*i.e.* the function f_l).

Keep up with the expressions above and move to the discretized form:

$$
\begin{aligned}
&= \int \mathcal{D}k\mathcal{D}q\, \Theta_{++}(\mathbf{k}-\mathbf{q},\mathbf{k}) \left\{ \left[\int_{\tau_{\rm osc}}^{\tau_{\rm rei}} d\tau'\, f_l(k,\tau') \mathrm{Re}\mathcal{S}_{++}(\mathbf{q},\mathbf{k},\tau') \right]^2 + \left[\int_{\tau_{\rm osc}}^{\tau_{\rm rei}} d\tau'\, f_l(k,\tau') \mathrm{Im}\mathcal{S}_{++}(\mathbf{q},\mathbf{k},\tau') \right]^2 \right\} \\
&\propto \int k^2 dk\, \frac{qdqpdp}{k} \Theta_{++}(k,p,q) \left\{ T^2_{l,r}(k,q,p,\tau') + T^2_{l,i}(k,q,p,\tau') \right\} \\
&\to \sum k\Delta k\, q\Delta q\, p\Delta p\, \Theta_{++}(k,p,q) \left(T^2_{l,r}(k,q,p,\tau') + T^2_{l,i}(k,q,p,\tau') \right) \\
&= \sum k\Delta k\, q\Delta q\, \left[p\Delta p\, \Theta_{++}(k,p,q) \left(T^2_{l,r}(k,q,p,\tau') + T^2_{l,i}(k,q,p,\tau') \right) \right] .
\end{aligned}
$$

In the second line we symmetrize the integration between p and q, and the last line shows how we organize our calculation on the cluster: *i.e.* we scan k and q over the momentum list, and within each combination of k and q (which is a job submitted to the cluster) we sum over all possible p in the range $|\mathbf{k}-\mathbf{q}| \leqslant p \leqslant |\mathbf{k}+\mathbf{q}|$ (as $\mathbf{p}$ is defined as $\mathbf{p} = \mathbf{k} - \mathbf{q}$).

straight-forward approach

If we don't switch the integration sequence of $\tau_{1(2)}$ and $\tau'_{1(2)}$, the calculation in Eq. (1) will continue as:

$$
\begin{aligned}
&\propto \int \mathcal{D}k\mathcal{D}q\, \Theta_{++}(\mathbf{k}-\mathbf{q},\mathbf{k}) \int_{\tau_{\rm osc}}^{\tau_{\rm rei}} d\tau_1 d\tau_2 \frac{J_l(k,\tau_1)J_l(k,\tau_2)}{a(\tau_1)a(\tau_2)} \\
&\quad \cdot \left\{ \int_{\tau_{\rm osc}}^{\tau_1} d\tau'_1\, \mathcal{G}(k,\tau_1,\tau'_1)\mathcal{S}_{++}(\mathbf{q},\mathbf{k},\tau'_1) \right\} \left\{ \int_{\tau_{\rm osc}}^{\tau_2} d\tau'_2\, \mathcal{G}(k,\tau_2,\tau'_2) \cdot \mathcal{S}^*_{++}(\mathbf{q},\mathbf{k},\tau'_2) \right\} \\
&= \int \mathcal{D}k\mathcal{D}q\, \Theta_{++}(\mathbf{k}-\mathbf{q},\mathbf{k}) \int_{\tau_{\rm osc}}^{\tau_{\rm rei}} d\tau_1 d\tau_2 \frac{J_l(k,\tau_1)J_l(k,\tau_2)}{a(\tau_1)a(\tau_2)} \cdot g(k,q,p,\tau_1) g^*(k,q,p,\tau_2) \\
&\to \sum k\Delta k\, q\Delta q\, p\Delta p\, \Theta_{++}(k,p,q) \int_{\tau_{\rm osc}}^{\tau_{\rm rei}} d\tau_1 d\tau_2 \frac{J_l(k,\tau_1)J_l(k,\tau_2)}{a(\tau_1)a(\tau_2)} \cdot g(k,q,p,\tau_1) g^*(k,q,p,\tau_2) \\
&= \sum k\Delta k\, q\Delta q\, p\Delta p\, \Theta_{++}(k,p,q) \int_{\tau_{\rm osc}}^{\tau_{\rm rei}} d\tau_1 d\tau_2 \frac{J_l(k,\tau_1)J_l(k,\tau_2)}{a(\tau_1)a(\tau_2)} \\
&\quad \cdot \left[\mathrm{Re}g(k,q,p,\tau_1)\mathrm{Re}g(k,q,p,\tau_2) + \mathrm{Im}g(k,q,p,\tau_1)\mathrm{Im}g(k,q,p,\tau_2) \right] \\
&= \sum k\Delta k\, q\Delta q\, p\Delta p\, \Theta_{++}(k,p,q) \left\{ \left[\int_{\tau_{\rm osc}}^{\tau_{\rm rei}} d\tau'\, \frac{J_l(k,\tau)}{a(\tau)} \mathrm{Re}g(k,q,p,\tau) \right]^2 + \left[\int_{\tau_{\rm osc}}^{\tau_{\rm rei}} d\tau'\, \frac{J_l(k,\tau)}{a(\tau)} \mathrm{Im}g(k,q,p,\tau) \right]^2 \right\} .
\end{aligned}
$$

Compared with the current approach, this straight-forward approach generates an auxiliary intermediate-step function $g(\tau)$ for each combination of k, q and p, while in the current approach we generates the intermediate-step function $f_l(\tau)$ for each k and l. Given that the list of angular size l is usually much shorter than that of momentum, our current approach seems to save some calculation.