AND Cheat Sheet

1 Vocabulary

HYPERBOLIC EQ.POINTS $\Re\{\lambda\} \neq 0$
NONHYPERBOLIC PROBLEMS Equilibria with EV on the imaginary axis.

2 Basics on cont.-time dynamical systems

$$\dot{x} = f(t, x, p), \quad x(t_0) = x_0 \Leftrightarrow \dot{x} = f(x, p), \quad x(0) = x_0$$

2.1 Flow of a system

Solution is the flow $\phi_t(x_0)$

$$x(t; x_0, p) = \phi_t(x_0, p)$$

FLOW AXIOMS

$$\phi_0(x_0, p) = x_0$$
$$\phi_{t_1}[\phi_{t_2}(x_0, p), p] = \phi_{t_1+t_2}(x_0, p)$$

2.2 Existence and uniqueness

EXISTENCE AND UNIQUENESS OF LOCAL (IN TIME) SOLUTIONS
If $f(x)$ is smooth enough, then solutions exist and are unique. No guarantee that they exist forever - only guarantees to exist in a very short time interval around t_0.
If f is Lipschitz in x and piecewise continuous in t then the IVP has a unique solution $x(t) = \phi(t; t_0)x_0$ over a finite time interval $t \in [t_0 - \tau, t_0 + \tau]$.
Allows for finite-escape times/blow-up (reach infinity in finite time).

2.3 Stability

Def.: An equilibrium point is a state x^* s.t. $f(x^*) = 0$
Def.: An eq.point x^* is stable if for any $\epsilon > 0$ there exists a constant $\delta > 0$ such that

$$\forall x_0 : \|x_0 - x^*\| \leq \delta \Rightarrow \|x(t) - x^*\| \leq \epsilon \quad \forall t \geq 0$$

ATTRACTOR
x^* is attractive if $\lim_{t\to\infty} \|x(t) - x^*\| = 0 \quad \forall x_0 \in \mathcal{S}$ with $\mathcal{S}$ being the domain of attraction.
Eq points may be attractive without being stable (ex. Vinograd's system).

ASYMPTOTIC STABILITY
An eq.point x^* is asymptotically stable if it is stable and attractive.

EXPONENTIAL STABILITY
An eq.point x^* is exponentially stable in $\mathcal{S}$ if it is stable and there are constants $a, \lambda > 0$ such that
$\forall x_0 \in \mathcal{S} : \|x(t) - x^*\| \leq a\|x_0 - x^*\|e^{-\lambda t}$

3 Linear systems

3.1 LTI Systems

$$\dot{x} = f(x, u) \rightarrow \dot{x} = Ax$$

mit $A \in \mathbb{R}^{n\times n}$ linear map/transformation
Theorem: The origin of the linear system is stable if all EV λ_i have non positive real part. If all EV have negative Real part, the origin is exponentially stable.

$$\lambda^2 + tr(A)\lambda + det(A) = 0$$
$$tr(A) = a_{11} + a_{22}$$
$$det(A) = a_{11}a_{22} - a_{12}a_{21}$$

$$\lambda_{1,2} = \frac{1}{2}(tr(A) \pm \sqrt{tr(A)^2 - 4det(A)}$$

daraus folgt $tr(A) < 0 \rightarrow$ stable
$tr(A) > 0 \rightarrow$ unstable
$tr(A)^2 \geq 4det(A)$ node
$tr(A)^2 < 4det(A)$ spiral
Spezialfall: $tr(A)^2 = 4det(A) \rightarrow$ stable if $tr(A) < 0$ and if $det(A) < 0 \rightarrow$ saddle

3.2 LTV Systems and Floquet theory

3.2.1 Local Nonlinear Flows

HARTMAN-GROBMAN THEOREM
The local phase portrait of the linearization (stability) of the fixed point is captured by the linearization. (for hyperbolic fixed points) *Hyperbolic*: $EV of J(x^*, p) \neq 0$

3.2.2 Center Manifold theorem

$$\frac{dh}{dx_z}[A_z x_z + f_z(x_z, h(x_z))] = A_s h(x_z) + f_s(h(x_z), x_y)$$
$$h(0) = 0, \frac{dh(0)}{dx_z} = 0$$

3.2.3 Index theory

Fuer mehr globale information
stable, unstable node $I_C = -1$
s/u spiral $I_C = 1$
s/u limit cycle $I_C = 1 \rightarrow$ wegen EP im Kern

3.2.4 Limit Cycles

Isolated (no other closed trajectory, spiral towards or away from LC) closed trajectories
POINCARE BENDIXSON

3.3 Local Theory

HARTMAN-GROBMAN
The behavior of a nonlinear systems in a vicinity of hyperbolic eq. points is the same as in the linearized system around that point.
CENTER MANIFOLD THEOREM

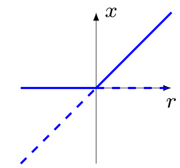
$$\frac{dh}{dx_z}[A_z x_z + f_z(x_z, h(x_z))] = A_s h(x_z) + f_s(h(x_z), x_z)$$
$$h(0) = 0, \quad \frac{dh(0)}{dx_z} = 0$$

3.4 Non-local phenomena

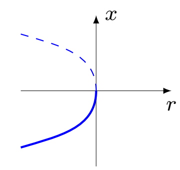
4 Bifurcations of vector fields

4.1 Bifurcations of SSs

TRANSCRITICAL BIFURCATION
Standard mechanism for change of stability for a fixed point which exists for all values of the parameter.
Ex.: $\dot{x} = rx - x^2$.
Fixed point at $x^* = 0 \quad \forall r$. For $r < 0$ unstable fixed point at $x^* = r$ and a stable at $x^* = 0$. For $r = 0^-$ the fixed points unify. For $r > 0$ the origin is unstable and $x^* = r$ becomes stable - **exchange of stabilities**

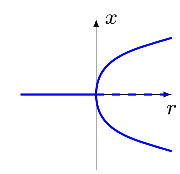


SADDLE NODE BIFURCATION
Fixed points are created and destroyed. As a parameter is varied, two fixed points move toward each other, collide and mutually annihilate.
Ex: $\dot{x} = r + x^2$.
$r < 0$: 2 FP (1 stable, 1 unstable),
$r = 0^-$: half-stable FP,
$r > 0$: no FP.
In this example the bifurcation occurred at $r = 0$.

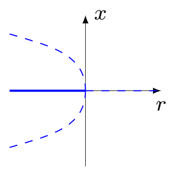


4.1.1 Pitchfork bifurcation

Common in physical systems that have (left/right) symmetry.
SUPERCRITICAL PITCHFORK BIFURCATION
Ex: $\dot{x} = rx - x^3$. If $r < 0$ the origin is the only fixed point and stable. $r = 0$ the origin is still stable, but weakly, since the linearization vanishes (solutions no longer decay exponentially fast - *critical slowing down*). For $r > 0$ the origin becomes unstable and two new stable fixed points appear at $x^* = \pm\sqrt{r}$.



Subcritical pitchfork bifurcation
Ex: $\dot{x} = rx + x^3$. Only for $r < 0$ two unstable fixed points $x^* = \pm\sqrt{-r}$ and stable origin. For $r > 0$ origin is unstable (finite escape)

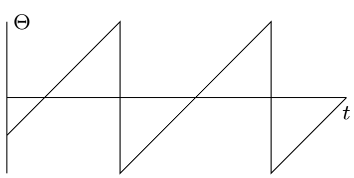


4.1.2 Bifurcations of SSs in dim $n > 1$

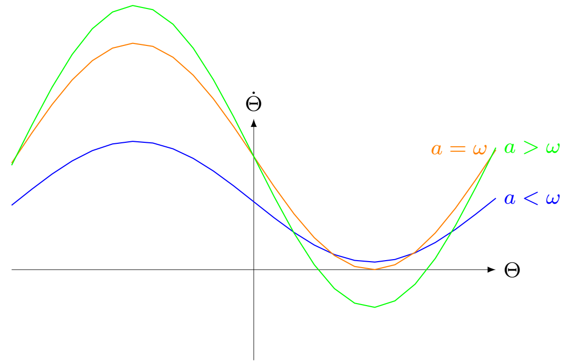
$$
\begin{aligned}
\text{transcritical bif: } \dot{x}_1 &= rx_1 - x_1^2 & \dot{x}_2 &= \pm x_2 \\
\text{saddle-node bif: } \dot{x}_1 &= r - x_1^2, & \dot{x}_2 &= \pm x_2 \\
\text{pitchfork bif: } \dot{x}_1 &= rx_1 - x_1^3, & \dot{x}_2 &= \pm x_2
\end{aligned}
$$

4.2 Bifurcations of trajectories

Flows on the circle
Vector field on the circle $\dot{\Theta} = f(\Theta)$, with Θ being a point on the circle. Particle can return to starting place - most basic model of oscillating systems.
Non-uniform oscillator
$\dot{\Theta} = \omega - a\sin\Theta$ with mean ω and amplitude a.
Uniform oscillator $p = 0$: $\dot{\Theta} = \omega$ and $\omega = \text{const.}$ with solution $\Theta(t) = \omega t + \Theta_0$. *Phase Θ changes uniformly.*



For $0 < p < \omega$ there is a non uniform oscillation, because the velocity depends on the angle. For $p = \omega$ a SS appears at $\Theta = \pi/2$ attracting the flow from one side and repelling the other side. For $p > \omega$ there are two SSs (1 repulsor, 1 attractor).
For $p \lesssim \omega$ there exists a saddle-note ghost. Time to pass through the bottleneck via local rescaling around the bottleneck: $\dot{x} = r + x^2$ where $0 < r = \frac{2(\omega - a)}{a} \ll 1$ is proportional to the distance of the bif. $T \approx \int_{-\infty}^{\infty} \frac{dx}{r + x^2} = \frac{\pi}{\sqrt{r}}$

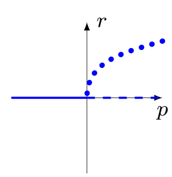


4.2.1 Andronov-Hopf bifurcation (2D)

No SSs appear/disappear. Rather the behaviour changes qualitatively for small variations, e.g. decaying oscillation vs. increasing oscillation with limiting amplitude after the bifurcation.
Can occur in phase spaces of any dimension $n \geq 2$.
Supercritical Hopf bifurcation
Decay rate dependent on control parameter p. Decay becomes slower and slower and finally changes to a growth at a critical p_c (equilibrium becomes unstable). In many cases there results a limit cycle about the former SS. **In phase plane:** Stable spiral becomes unstable spiral surrounded by a small nearly elliptical limit cycle. Example:

$$
\begin{aligned}
\dot{r} &= pr - r^3 \\
\dot{\Theta} &= \omega + br^2
\end{aligned}
$$

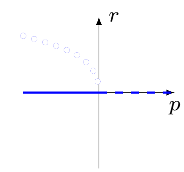
with control parameter p, ω frequency and b as dependence of frequency on amplitude. Stable spiral for $p < 0$ in origin $r = 0$ and sense of rotation depends on ω. For $p = 0$ the origin is very weakly stable. For $p > 0$ there is an unstable spiral at the origin and a stable limit cycle at $r = \sqrt{p}$.
EV analysis via cartesian coordinates $x_1 = r\cos\Theta$, $x_2 = r\sin\Theta$ yields $\dot{x}_1 = px_1 - \omega x_2, \quad \dot{x}_2 = \omega x_1 + px_2$ with the Jacobian $J = \begin{bmatrix} p & -\omega \\ \omega & p \end{bmatrix}$ and $\lambda_{1,2} = p \pm i\omega$. EVs cross the imaginary axis from left to right as p increases from - to +.
Rules of thumb: Size of limit cycle grows from 0 proportionally to $\sqrt{p - p_c}$. Frequency of limit cycle $\omega \approx \Im\{\lambda\}$ evaluated at $p = p_c$.



Subcritical Hopf bifurcation (More Dangerous)
Stable origin surrounded by unstable limit cycle which collapses at $p = 0$ leading to a repulsive spiral with no saving limit cycle around.
Example:

$$
\begin{aligned}
\dot{r} &= pr + r^3 \\
\dot{\Theta} &= \omega + br^2
\end{aligned}
$$

Cubic term is destabilizing yielding a stable origin and a unstable unstable limit cycle at $r = -\sqrt{-p}$ only existing for $p < 0$. Unstable for $p \geq 0$ (weakly at $p = 0$).



Degenerate Hopf bifurcation
Conj. complex EV pass through the imaginary axis, but do not isolated periodic orbits (limit cycle).
Linear system example:

$$
\begin{aligned}
\dot{x}_1 &= px_1 - x_2 \\
\dot{x}_2 &= x_1 + px_2
\end{aligned}
$$

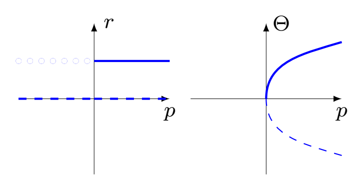
Nonlinear example, inverted pendulum with Coulomb friction $\ddot{\Theta} + \rho\dot{\Theta} + k\sin\Theta = 0$. For $p < 0$ stable spiral, a center (*no limit cycles*) for $p = 0$ (band of closed orbits) and unstable spiral for $p > 0$.

4.2.2 Bifurcation of limit cycles

Infinite period bifurcation
Example:

$$
\begin{aligned}
\dot{r} &= r - r^2 \\
\dot{\Theta} &= p - \Theta^2
\end{aligned}
$$

Two radial equilibrium solutions $r = 0$ and $r = 1$ (attractive). For $p < 0$ no angular eq.point exists so that the limit cycle $r = 1$ is a unique attractor, attracting trajectories from unstable origin. For $p = 0$ a SS at $(r, \Theta) = (1, 0)$ appears similar to a saddle point with radial attraction (x_1-direction) and angular saddle behaviour - the time through the bottleneck of $p < 0$ becomes infinite for $p = 0$. For $p > 0$ there are two SSs at $(1, \sqrt{p})$ (attractor) and $(1, -\sqrt{p})$ (repulsor) - looks like pacman.

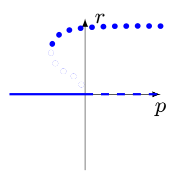


Saddle-node bifurcation of cycles

A bifurcation in which two limit cycles unite and annihilate.
Example

$$\dot{r} = pr + r^3 - r^5$$
$$\dot{\Theta} = \omega + br^2$$

Radial dynamics can be seen as a 1D system with saddle-node bifurcation of SS at $p_c = -1/4$. In the 2D system they correspond to limit cycles. At p_c a half-stable cycle is born. For $0 > p > p_c$ it splits into a pair of limit cycles, the inner unstable and the outer stable. The other way around two limit cycles collide and disappear as p decreases through p_c. The origin remains a stable spiral throughout.



Homoclinic bifurcation

Part of a LC moves closer to a saddle point. At the bif the LC touches the saddle and becomes a homoclinic orbit - another kind of infinite-period bif.
Example:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 + px_2 - x_1^2 + x_1 x_2$$

$p_c = -0.86$. For $p < -1$ there is a stable spiral and a saddle node. For $p = -1$ a Hopf bifurcation. For $-1 < p < p_c$ a stable LC passes close to a saddle at the origin. As $p = p_c$ the LC hits the saddle, creating a homoclinic orbit (connecting the saddle with itself). For $p > p_c$ the saddle connection breaks and the loop is destroyed. Branch of the manifold leaving the saddle either hit the origin after looping or spiral out to one side or the other.

5 Basics on discrete-time systems

5.1 Maps, orbits, cobwebs and fixed points

Maps: $x_{t_{n+1}} = f[x(t_n), p], \quad x(t_0) = x_0$, with state x and map f defining how the new state evolves from the current one. Sequence of states $x(t_0), x(t_1), ..., x(t_n)$ is called **orbit**. Orbit of a DTS is like the trajectory of a CTS only at discrete points. Visualization of the orbit via **Cobwebs**, where the subsequent state $x(t_{n+1})$ is plotted against the current $x(t_n)$.

Difference between CTS and DTS ist that DTS can have oscillations and very complex/chaotic behaviour already in 1D. Example $x(t_{n+1}) = -x(t_n), x(t_0) = x_0$ jumps between x_0 and $-x_0$ for all times with period $N = 2$. A **fixed point** of the system is $f(x_*) = x_* \Leftrightarrow x(t_{n+1}) = x(t_n) = x_*$. FP in DTSs are what EQPs or FPs are for CTSs. In DTSs a FP are period-1 solutions. There may be higher-period solutions - period-2 solutions are
$f^2 = f(f) = f \circ f, \quad x(t_{n+2}) = f^2(x(t_n))$.

5.2 Stability

Stability notions are the same as for CTSs. A set $\mathcal{M}$ is said to be attractive in $\mathcal{D}$ if $\forall x_0 \in \mathcal{D}: \quad x(t_n) \to \mathcal{M}$ for $n \to \infty$. For a 1D linear DTS: $x(t_{n+1}) = \lambda x(t_n), \quad x(t_0) = x_0$ the analytic solution is $x(t_n) = \lambda^n x_0$. For stability $|\lambda| \leq 1$ - for asymptotic stability $|\lambda| < 1$. This yields the characteristic step number $n_c = \left|\frac{1}{\ln(|\lambda|)}\right|$ and settling step number $n_s = 4n_c$.
A fixed point is exponentially stable if there exist $a > 0$ and $\lambda < 1$ so that $\|x(t_n) - x_*\| \leq a\lambda^n \|x_0 - x_*\|$.
A fixed point is practically stable
BLAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA.

5.3 Linear systems

$x(t_{n+1}) = Ax(t_n), \quad x(t_0) = x_0$ with solution $x(t_n) = A^n x_0$.
Convergence in finite time achieved by e.g. $A = 0$ - not possible in linear CTSs. Similar behaviour if A has zero EVs. Matrix-state product $x(t_n) = \sum_{i=1}^n \lambda_i^n \langle x_0, v_i \rangle v_i$. The origin $x = 0$ of the linear DTS is stable iff all EVs are contained in the unit circle, i.e. $|\lambda_i| \leq 1$. Asymptotically and exponentially iff $|\lambda_i| < 1$.

6 Nonlinear discrete-time systems

6.1 Hartman Grobman theorem for maps

6.2 Center manifold theorem for maps

7 Bifurcations in DTS

7.1 Transcritical bifurcation

$x(t_{n+1}) = rx(t_n) - x(t_n)^2, \quad x(t_0) = x_0$ with solutions $x_1 = 0, \quad x_2 = r - 1$. With Hartman Grobman for maps $x_1 = 0$ is locally asymptotically stable for $r < 1$ and unstable for $r > 1$. $x = r - 1$ is unstable for $r <$ and locally asymptotically stable for $r > 1$. Bifurcation at $r = 1$ where both FP coincide in a saddle and the slope of $f(x(t_n), r)$ is equal to one. FPs run along $x(t_{n+1}) = x(t_n)$ - **exchange of stabilites** at $r = 1$ (*parameter shift by 1 in comparison to CTSs*)

7.2 Saddle-node bifurcation

$x(t_{n+1}) = r - x^2(t_n), \quad x(t_0) = x_0$ with FP $x = \pm\sqrt{r-1}$. No FP for $r < 1$. Bifurcation at $r = 1$ with slope of $f(x, r)$ equal to one. For $r > 1$ 2 FPs, one stable, one unstable.

7.3 (Supercritical) Pitchfork bifurcation

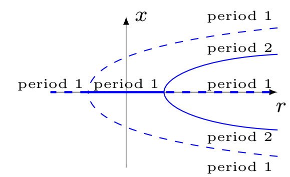
$x(t_{n+1}) = rx(t_n) - x^3(t_n), \quad x(t_0) = x_0$ with solution $x_1 = 0, \ x_{2,3} = \pm\sqrt{r-1}$. For $r < 1$ the origin is asymptotically stable and unstable for $r > 1$. For $r > 1$ two symmetric asymptotically stable solutions $x_{2,3}$ exist.

7.4 Period doubling bifurcation

If the slope of $f(x, r)$ is -1 the oscillation behaviour of the orbits changes.
$x(t_{n+1}) = -rx(t_n) + x^3(t_n), \quad x(t_0) = x_0$ with solutions $x_1 = 0, \ x_{2,3} = \pm\sqrt{r+1}$. $x_{2,3}$ only exist for $r > -1$. Pitchfork bif occurs at $r = -1$. For $r < -1$ and $r > 1$ unstable origin, for $-1 < r < 1$ asymptotically stable origin. For $r > -1$ both brances are unstable. So for $r > 1$ there are 3 unstable FPs $\Rightarrow$ orbits jump between FPs. No attractive period-1 solutions (or equivalently FPs) for $r > 1$ exist.
Period-2 solution using the map $f^2(x(t_n), r)$

$$x(t_{n+2}) = x(t_n)(r^2 - 1) - x^3(t_n)r(1 + r^2) + \mathcal{O}^4(x(t_n))$$

with solutions $x_{2,1} = 0, \ x_{2,(2,3)} = \pm\sqrt{\frac{r^2-1}{r(1+r^2)}}$ which only exist for $r > 1$. At $r = 1$ a pitchfork bif takes place making the origin unstable and creating 2 attractive period-2 branches.



7.5 (Supercritical) Neimark-Sacker bifurcation

A bifurcation when the FP changes stability via a pair of complex eigenvalues Discrete-time counterpart to the Hopf bifurcation.
$z(t_{n+1}) = (1 + \beta)e^{i\Theta(\beta)} z(t_n) + c(\beta) z(t_n)|z(t_n)|^2 + \mathcal{O}(|z(t_n)|^4)$.
For $\beta < 0$ the origin is a asymptotically stable FP (weakly stable at $\beta = 0$) and unstable for $\beta > 0$. Unique stable LC for $\beta >$ with radius $\sqrt{b}$.