AND Cheat Sheet

1 Vocabulary

HYPERBOLIC EQ.POINTS $\mathcal{R}\{x\} \neq 0$
NONHYPERBOLIC PROBLEMS Equilibria with EV on the imaginary axis.

2 Basics on cont.-time dynamical systems

$$\dot{x} = f(t, x, p), \quad x(t_0) = x_0 \Leftrightarrow \dot{x} = f(x, p), \quad x(0) = x_0$$

2.1 Flow of a system

Solution is the flow $\phi_t(x_0)$

$$x(t; x_0, p) = \phi_t(x_0, p)$$

FLOW AXIOMS

$$\phi_0(x_0, p) = x_0$$
$$\phi_{t_1}[\phi_{t_2}(x_0, p), p] = \phi_{t_1+t_2}(x_0, p)$$

2.2 Existence and uniqueness

EXISTENCE AND UNIQUENESS OF LOCAL (IN TIME) SOLUTIONS
If $f(x)$ is smooth enough, then solutions exist and are unique. No guarantee that they exist forever - only guarantees to exist in a very short time interval around t_0.
If f is Lipschitz in x and piecewise continuous in t then the IVP has a unique solution $x(t) = \phi(t; t_0)x_0$ over a finite time interval $t \in [t_0 - \tau, t_0 + \tau]$.
Allows for finite-escape times/blow-up (reach infinity in finite time).

2.3 Stability

Def.: An equilibrium point is a state x^* s.t. $f(x^*) = 0$
Def.: An eq.point x^* is stable if for any $\epsilon > 0$ there exists a constant $\delta > 0$ such that

$$\forall x_0 : \|x_0 - x^*\| \leq \delta \Rightarrow \|x(t) - x^*\| \leq \epsilon \quad \forall t \geq 0$$

ATTRACTOR
x^* is attractive if $\lim_{t\to\infty} \|x(t) - x^*\| = 0 \quad \forall x_0 \in \mathcal{S}$ with $\mathcal{S}$ being the domain of attraction.
Eq points may be attractive without being stable (ex. Vinograd's system).

ASYMPTOTIC STABILITY
An eq.point x^* is asymptotically stable if it is stable and attractive.

EXPONENTIAL STABILITY
An eq.point x^* is exponentially stable in $\mathcal{S}$ if it is stable and there are constants $a, \lambda > 0$ such that
$\forall x_0 \in \mathcal{S} : \|x(t) - x^*\| \leq a\|x_0 - x^*\|e^{-\lambda t}$

3 Linear systems

3.1 LTI systems

3.2 LTV systems and Floquet theory

4 Nonlinear flows

4.1 Local Theory

HARTMAN-GROBMAN
The behavior of a nonlinear systems in a vicinity of hyperbolic eq. points is the same as in the linearized system around that point.
CENTER MANIFOLD THEOREM

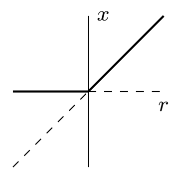
$$\frac{dh}{dx_z}[A_z x_z + f_z(x_z, h(x_z))] = A_s h(x_z) + f_s(h(x_z), x_z)$$
$$h(0) = 0, \quad \frac{dh(0)}{dx_z} = 0$$

4.2 Non-local phenomena

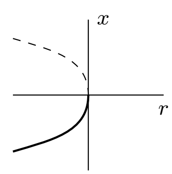
5 Bifurcations of vector fields

5.1 Bifurcations of SSs

TRANSCRITICAL BIFURCATION
Standard mechanism for change of stability for a fixed point which exists for all values of the parameter.
Ex.: $\dot{x} = rx - x^2$.
Fixed point at $x^* = 0 \quad \forall r$. For $r < 0$ unstable fixed point at $x^* = r$ and a stable at $x^* = 0$. For $r = 0^-$ the fixed points unify. For $r > 0$ the origin is unstable and $x^* = r$ becomes stable - **exchange of stabilities**

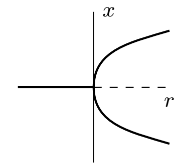


SADDLE NODE BIFURCATION
Fixed points are created and destroyed. As a parameter is varied, two fixed points move toward each other, collide and mutually annihilate.
Ex: $\dot{x} = r + x^2$.
$r < 0$: 2 FP (1 stable, 1 unstable),
$r = 0^-$: half-stable FP,
$r > 0$: no FP.
In this example the bifurcation occurred at $r = 0$.

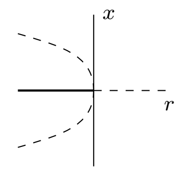


5.1.1 Pitchfork bifurcation

Common in physical systems that have (left/right) symmetry.
SUPERCRITICAL PITCHFORK BIFURCATION
Ex: $\dot{x} = rx - x^3$. If $r < 0$ the origin is the only fixed point and stable. $r = 0$ the origin is still stable, but weakly, since the linearization vanishes (solutions no longer decay exponentially fast - *critical slowing down*). For $r > 0$ the origin becomes unstable and two new stable fixed points appear at $x^* = \pm\sqrt{r}$.



SUBCRITICAL PITCHFORK BIFURCATION
Ex: $\dot{x} = rx + x^3$



5.2 Bifurcations of trajectories