AND Cheat Sheet

1 Vocabulary

Hyperbolic eq.points $\Re\{\lambda\} \neq 0$
Nonhyperbolic problems Equilibria with EV on the imaginary axis.

2 Basics on cont.-time dynamical systems

$$\dot{x} = f(t, x, p), \quad x(t_0) = x_0 \Leftrightarrow \dot{x} = f(x, p), \quad x(0) = x_0$$

2.1 Flow of a system

Solution is the flow $\phi_t(x_0)$

$$x(t; x_0, p) = \phi_t(x_0, p)$$

Flow axioms

$$\phi_0(x_0, p) = x_0$$
$$\phi_{t_1}[\phi_{t_2}(x_0, p), p] = \phi_{t_1+t_2}(x_0, p)$$

2.2 Existence and uniqueness

Existence and uniqueness of local (in time) solutions
If $f(x)$ is smooth enough, then solutions exist and are unique. No guarantee that they exist forever - only guarantees to exist in a very short time interval around t_0.
If f is Lipschitz in x and piecewise continuous in t then the IVP has a unique solution $x(t) = \phi(t; t_0)x_0$ over a finite time interval $t \in [t_0 - \tau, t_0 + \tau]$.
Allows for finite-escape times/blow-up (reach infinity in finite time).

2.3 Stability

Def.: An equilibrium point is a state x^* s.t. $f(x^*) = 0$
Def.: An eq.point x^* is stable if for any $\epsilon > 0$ there exists a constant $\delta > 0$ such that

$$\forall x_0 : \|x_0 - x^*\| \leq \delta \Rightarrow \|x(t) - x^*\| \leq \epsilon \quad \forall t \geq 0$$

Attractor
x^* is attractive if $\lim_{t\to\infty} \|x(t) - x^*\| = 0 \quad \forall x_0 \in \mathcal{S}$ with $\mathcal{S}$ being the domain of attraction.
Eq points may be attractive without being stable (ex. Vinograd's system).
Asymptotic stability
An eq.point x^* is asymptotically stable if it is stable and attractive.
Exponential stability
An eq.point x^* is exponentially stable in $\mathcal{S}$ if it is stable and there are constants $a, \lambda > 0$ such that
$\forall x_0 \in \mathcal{S} : \|x(t) - x^*\| \leq a\|x_0 - x^*\|e^{-\lambda t}$

3 Linear systems

3.1 LTI systems

3.2 LTV systems and Floquet theory

4 Nonlinear flows

4.1 Local Theory

Hartman-Grobman
The behavior of a nonlinear systems in a vicinity of hyperbolic eq. points is the same as in the linearized system around that point.
Center manifold theorem

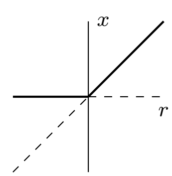
$$\frac{dh}{dx_z}[A_z x_z + f_z(x_z, h(x_z))] = A_s h(x_z) + f_s(h(x_z), x_z)$$
$$h(0) = 0, \quad \frac{dh(0)}{dx_z} = 0$$

4.2 Non-local phenomena

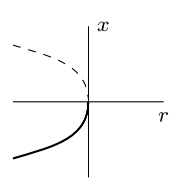
5 Bifurcations of vector fields

5.1 Bifurcations of SSs

Transcritical bifurcation
Standard mechanism for change of stability for a fixed point which exists for all values of the parameter.
Ex.: $\dot{x} = rx - x^2$.
Fixed point at $x^* = 0 \quad \forall r$. For $r < 0$ unstable fixed point at $x^* = r$ and a stable at $x^* = 0$. For $r = 0^-$ the fixed points unify. For $r > 0$ the origin is unstable and $x^* = r$ becomes stable - **exchange of stabilities**

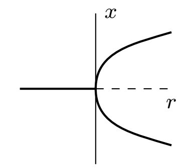


Saddle node bifurcation
Fixed points are created and destroyed. As a parameter is varied, two fixed points move toward each other, collide and mutually annihilate.
Ex: $\dot{x} = r + x^2$.
$r < 0$: 2 FP (1 stable, 1 unstable),
$r = 0^-$: half-stable FP,
$r > 0$: no FP.
In this example the bifurcation occurred at $r = 0$.

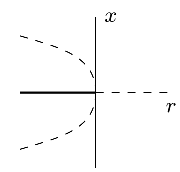


5.1.1 Pitchfork bifurcation

Common in physical systems that have (left/right) symmetry.
Supercritical pitchfork bifurcation
Ex: $\dot{x} = rx - x^3$. If $r < 0$ the origin is the only fixed point and stable. $r = 0$ the origin is still stable, but weakly, since the linearization vanishes (solutions no longer decay exponentially fast - *critical slowing down*). For $r > 0$ the origin becomes unstable and two new stable fixed points appear at $x^* = \pm\sqrt{r}$.



Subcritical pitchfork bifurcation
Ex: $\dot{x} = rx + x^3$. Only for $r < 0$ two unstable fixed points $x^* = \pm\sqrt{-r}$ and stable origin. For $r > 0$ origin is unstable (finite escape)

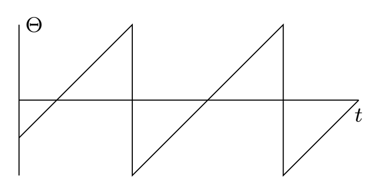


5.1.2 Bifurcations of SSs in dim $n > 1$

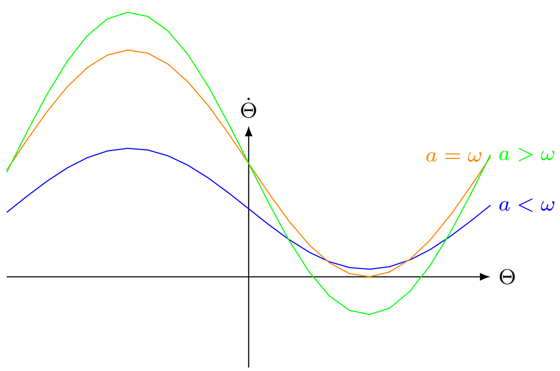
$$\begin{aligned} \text{transcritical bif: } & \dot{x}_1 = rx_1 - x_1^2 & \dot{x}_2 = \pm x_2 \\ \text{saddle-node bif: } & \dot{x}_1 = r - x_1^2, & \dot{x}_2 = \pm x_2 \\ \text{pitchfork bif: } & \dot{x}_1 = rx_1 - x_1^3, & \dot{x}_2 = \pm x_2 \end{aligned}$$

5.2 Bifurcations of trajectories

Flows on the circle
Vector field on the circle $\dot{\Theta} = f(\Theta)$, with Θ being a point on the circle. Particle can return to starting place - most basic model of oscillating systems.
Non-uniform oscillator
$\dot{\Theta} = \omega - a \sin \Theta$ with mean ω and amplitude a.
Uniform oscillator $p = 0$: $\dot{\Theta} = \omega$ and $\omega =$ const. with solution $\Theta(t) = \omega t + \Theta_0$. *Phase Θ changes uniformly.*



For $0 < p < \omega$ there is a non uniform oscillation, because the velocity depends on the angle. For $p = \omega$ a SS appears at $\Theta = \pi/2$ attracting the flow from one side and repelling the other side. For $p > \omega$ there are two SSs (1 repulsor, 1 attractor).
For $p \lesssim \omega$ there exists a saddle-note ghost. Time to pass through the bottleneck via local rescaling around the bottleneck: $\dot{x} = r + x^2$ where $0 < r = \frac{2(\omega - a)}{a} \ll 1$ is proportional to the distance of the bif. $T \approx \int_{-\infty}^{\infty} \frac{dx}{r+x^2} = \frac{\pi}{\sqrt{r}}$

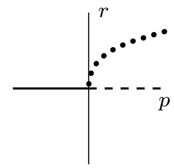


5.2.1 Andronov-Hopf bifurcation (2D)

No SSs appear/disappear. Rather the behaviour changes qualitatively for small variations, e.g. decaying oscillation vs. increasing oscillation with limiting amplitude after the bifurcation.
Can occur in phase spaces of any dimension $n \geq 2$.
Supercritical Hopf bifurcation
Decay rate dependent on control parameter p. Decay becomes slower and slower and finally changes to a growth at a critical p_c (equilibrium becomes unstable). In many cases there results a limit cycle about the former SS. **In phase plane:** Stable spiral becomes unstable spiral surrounded by a small nearly elliptical limit cycle. Example:

$$\dot{r} = pr - r^3$$
$$\dot{\Theta} = \omega + br^2$$

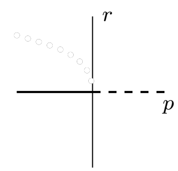
with control parameter p, ω frequency and b as dependence of frequency on amplitude. Stable spiral for $p < 0$ in origin $r = 0$ and sense of rotation depends on ω. For $p = 0$ the origin is very weakly stable. For $p > 0$ there is an unstable spiral at the origin and a stable limit cycle at $r = \sqrt{p}$.
EV analysis via cartesian coordinates $x_1 = r\cos\Theta$, $x_2 = r\sin\Theta$ yields $\dot{x}_1 = px_1 - \omega x_2$, $\quad \dot{x}_2 = \omega x_1 + px_2$ with the Jacobian $J = \begin{bmatrix} p & -\omega \\ \omega & p \end{bmatrix}$ and $\lambda_{1,2} = p \pm i\omega$. EVs cross the imaginary axis from left to right as p increases from - to +.
Rules of thumb: Size of limit cycle grows from 0 proportionally to $\sqrt{p - p_c}$. Frequency of limit cycle $\omega \approx \Im\{\lambda\}$ evaluated at $p = p_c$.



Subcritical Hopf bifurcation (More Dangerous)
Stable origin surrounded by unstable limit cycle which collapses at $p = 0$ leading to a repulsive spiral with no saving limit cycle around.
Example:

$$\dot{r} = pr + r^3$$
$$\dot{\Theta} = \omega + br^2$$

Cubic term is destabilizing yielding a stable origin and a unstable unstable limit cycle at $r = -\sqrt{-p}$ only existing for $p < 0$. Unstable for $p \geq 0$ (weakly at $p = 0$).



Degenerate Hopf bifurcation
Conj. complex EV pass through the imaginary axis, but do not isolated periodic orbits (limit cycle).
Linear system example:

$$\dot{x}_1 = px_1 - x_2$$
$$\dot{x}_2 = x_1 + px_2$$

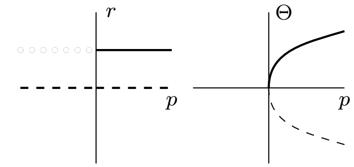
Nonlinear example, inverted pendulum with Coulomb friction $\ddot{\Theta} + \rho\dot{\Theta} + k\sin\Theta = 0$. For $p < 0$ stable spiral, a center (*no limit cycles*) for $p = 0$ (band of closed orbits) and unstable spiral for $p > 0$.

5.2.2 Bifurcation of limit cycles

Infinite period bifurcation
Example:

$$\dot{r} = r - r^2$$
$$\dot{\Theta} = p - \Theta^2$$

Two radial equilibrium solutions $r = 0$ and $r = 1$ (attractive). For $p < 0$ no angular eq.point exists so that the limit cycle $r = 1$ is a unique attractor, attracting trajectories from unstable origin. For $p = 0$ a SS at $(r, \Theta) = (1, 0)$ appears similar to a saddle point with radial attraction (x_1-direction) and angular saddle behaviour - the time through the bottleneck of $p < 0$ becomes infinite for $p = 0$. For $p > 0$ there are two SSs at $(1, \sqrt{p})$ (attractor) and $(1, -\sqrt{p})$ (repulsor) - looks like pacman.



Saddle-node bifurcation of cycles