CIS 301: Logical Foundations of Programming

Spring 2023

Exam 2 – 100 points

This test is closed-notes and closed-computers.

There are 8 questions worth 8-15 points each.

This exam was modified somewhat from the original version to match the new Logika format.

Name: ____

Score: ____________

1. (15 pts) Use natural deduction to prove the following sequent:

```
( q → p )  ⊢ ( ¬p → ¬q  )
Proof(
	//COMPLETE THE PROOF HERE
	1 (q → p) by Premise,
	2 SubProof(
		3 Assume(¬p),
		4 SubProof(
			5 Assume (q),
			6 (p) by ImplyE(1, 5),
			7 (F) by NegE(6, 3)
			//want: F
		),
		8 (¬q) by NegI(4)
		//want: !q
	),
	9 (¬p → ¬q) by ImplyI(2)
)
```

2. (15 pts) Consider the following questions about sets.

a) (3 pts) List the elements in the set:

$\{x : x = 6k, k \in \mathbb{Z}, -2 \le k \le 5\}$

{-12, -6, 0, 6, 12, 18, 24, 30}

b) (3 pts) List the elements in the set:

$\{y : y \in \mathbb{Z}, |4y| \le 12\}$

{-3, -2, -1, 0, 1, 2, 3}

c) (3 pts) Suppose $A = \{x : x = 3k+1, k \in \mathbb{Z}, 0 \le k \le 4\}$ and $B = \{y : y = 3k-2, k \in \mathbb{Z}, 0 \le k \le 4\}$. List (i.e., write them all out) the elements in $A \cap B$.

A = {1, 4, 7, 10, 13}

B = {-2, 1, 4, 7, 10}

$A \cap B$ = {1, 4, 7, 10}

d) (3 pts) Use set builder notation to describe a set with the elements below. Use only mathematical notation and not words to describe the set.

{1,4,9,16,25,36,49}

{x : x = k^2, k ∈ N, k <= 7 }

e) (3 pts) Use set builder notation to describe a set with the elements below. Use only mathematical notation and not words to describe the set.

{1, 3, 5, 7, 9, …}

{x : x = 2k - 1, k ∈ N }

3. (12 pts) Consider the following questions about set operations.

a) (6 pts) Suppose $A - B = \{4,5,7,11\}$, $B - A = \{1,12\}$, and $A \cap B = \{2,8,10\}$. Find the sets A and B.

$A = \{4,5,7,11,2, 8,10\}$

$B = \{1, 12,2,8,10\}$

b) (6 pts) Suppose A, B, and C are sets and that $A = B - C$. Must $B = A \cup C$? Either explain why this must be true or find a counterexample (with specific sets) where this property does not hold.

Not necessarily. Consider this counterexample:

$A = \{1\}$

$B = \{1,2,3\}$

$C = \{2,3,4\}$

Here, $A = B-C = \{1\}$ but B does not equal $A \cup C$. ($A \cup C = \{1,2,3,4\}$, while $B = \{1,2,3\}$.)

4. (8 pts) Consider the following statements over the domain of people, which use the predicates below:

B(x): person x plays baseball
L(x) : person x plays lacrosse
K(x): person x has kids
G(x): person x is in a gardening club
F(x): person x has flown on a plane
D(x): person x is a doctor

////////////////////////

∀ ∃ ∈ ∉ ∅ ∩ ∪ ⊂ ⊆ ∧ ∨ ¬ → ℕ ℤ ≠ ≤ ≥

////////////////////////

a) (2 pts) Translate to predicate logic: “There is a doctor who has kids and plays lacrosse.”

$\exists\, x\, (D(x) \wedge K(x) \wedge L(x))$

b) (2 pts) Assuming “Jose” is an individual in our domain, translate to predicate logic: “Jose plays lacrosse but not baseball.”

$L(Jose) \wedge \neg B(Jose)$

c) (2 pts) Translate to English: $\forall\, x\, (G(x) \rightarrow K(x))$

Everyone in gardening club has kids.

Don’t write like: For every person x if x is in gardening club then x has kids

d) (2 pts) Translate to English: $\exists\, x\, (D(x) \wedge \neg F(x))$

There is a person who is a doctor and hasn’t flown on a plane.

5. (10 pts) Consider the following questions about predicate logic and sets of numbers.

a) (5 pts) Is the statement $\forall x (x^2 > x)$ true or false in the domain of integers ($\mathbb{Z}$)? Explain.

Saying: for all integers, its square is greater than itself

False.

Counterexample: Consider 1. 1^2 = 1 which is not greater than 1.

b) (5 pts) Is the statement $\exists y \forall x (x + y = x - y)$ true or false in the domain of integers ($\mathbb{Z}$)? Explain.

True.

There is a number y where for each integer x, x + y = x-y.

Suppose y = 0. Then x + 0 = x – 0 for all integers x.

6. (10 pts) Negate each of the following predicate logic statements, **showing your work as you go**. Your final statement in each part should not have any negation applied to quantifiers or parenthetical expressions ( ). If your negated expression involves a $\forall$ quantifier and an $\vee$, rewrite that expression to use an $\rightarrow$ instead. **Put a box around your final statement**. (There are no predicates listed because you do not have to express these statements in words, but you are welcome to invent your own predicates if it helps you check your work.)

a) $\exists x\ (K(x) \land \neg L(x))$

$\neg\ (\ \exists x\ (K(x) \land \neg L(x))$

$\forall x\ \neg\ (\ K(x) \land \neg L(x))$

$\forall x\ (\neg K(x) \lor L(x))$

Answer: $\forall x\ (K(x) \rightarrow L(x))$

b) $\forall x\ \exists y\ (P(x) \rightarrow \neg B(x, y))$

$\neg(\forall x\ \exists y\ (P(x) \rightarrow \neg B(x, y)))$

$\exists x\ \forall y\ \neg(\ P(x) \rightarrow \neg B(x, y))$

$\exists x\ \forall y\ \neg(\ \neg P(x) \lor \neg B(x, y))$

Answer: $\exists x\ \forall y\ (P(x) \land B(x, y))$

7. (15 pts) Use natural deduction to prove the following sequent:

```
(
	∀ ((x: T) => (P(x) → K(x) ∧ Q(x))),
	∀ ((x: T) => (Q(x) → M(x))
)
⊢
(	∀ ((x: T) =>  (P(x) → M(x))	)
Proof(
	//COMPLETE THE PROOF HERE
	1 ( ∀ ((x: T) => (P(x) → K(x) ∧ Q(x))) ) by Premise,
	2 ( ∀ ((x: T) => (Q(x) → M(x)) ) by Premise,

	3 Let ((a: T) => SubProof(
		4 SubProof(
			5 Assume(P(a)),
			6 (P(a) → K(a) ∧ Q(a))) by AllE[T](1),
			7 (Q(a) → M(a)) by AllE[T](2)
			8 (K(a) ∧ Q(a)) by ImplyE(6, 5),
			9 (Q(a)) by AndE2(8),
			10 (M(a)) by ImplyE(7, 9)

			//want: M(a)
		),
		11 (P(a) → M(a)) by ImplyI(4)

		//want: P(a) → M(a)
	)),
	12 ( ∀ ((x: T) =>  (P(x) → M(x)) ) by AllI[T](3)
)
```

8. (15 pts) Use natural deduction to prove the following sequent (here, assume "indiv" is an individual of type T):

```
(
	∀ ((x: T) => (¬G(x) → H(x))),
	∀ ((x: T) => (K(x) → L(x) ∧ ¬G(x))),
	K(indiv)
)
⊢
(	∃ ((x: T) => H(x))		)
Proof(
	//COMPLETE THE PROOF HERE

	1 (∀ ((x: T) => (¬G(x) → H(x))))  by Premise,
	2 (∀ ((x: T) => (K(x) → L(x) ∧ ¬G(x)))) by Premise,
	3 ( K(indiv) ) by Premise,

	4 (¬G(indiv) → H(indiv)) by AllE[T](1),
	5 (K(indiv) → L(indiv) ∧ ¬G(indiv)) by AllE[T](2),

	6 (L(indiv) ∧ ¬G(indiv)) by ImplyE(5, 3),


	7 (¬G(indiv)) by AndE2(6),
	8 (H(indiv)) by ImplyE(4, 7),

	9 (∃ ((x: T) => H(x))) by ExistsI[T](8)
)
```