Divide-and-Conquer Matrix Factorization

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Motivation: Low-rank Matrix Factorization

$$\begin{bmatrix} 1 & 3 & 3 & 1 & \dots & 3 \\ 5 & 4 & 4 & 5 & \dots & 4 \\ 3 & 5 & 5 & 3 & \dots & 5 \end{bmatrix}$$

Motivation: Low-rank Matrix Factorization

$$\begin{bmatrix} 1 & 3 & 3 & 1 & \dots & 3 \\ 5 & 4 & 4 & 5 & \dots & 4 \\ 3 & 5 & 5 & 3 & \dots & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 1 \end{bmatrix}$$

Motivation: Low-rank Matrix Factorization

$$\begin{bmatrix} 1 & 3 & 3 & 1 & \dots & 3 \\ 5 & 4 & 4 & 5 & \dots & 4 \\ 3 & 5 & 5 & 3 & \dots & 5 \end{bmatrix}$$

Motivation: Low-rank Matrix Factorization

$$\begin{bmatrix} 1 & ? & 3 & 1 & \dots & 3 \\ 5 & ? & 4 & ? & \dots & ? \\ ? & 5 & ? & 3 & \dots & 5 \end{bmatrix}$$

Motivation: Low-rank Matrix Factorization

$$\begin{bmatrix} \color{red}{5} & ? & 3 & 1 & \ldots & \color{red}{4} \\ 5 & ? & 4 & ? & \ldots & ? \\ ? & 5 & ? & 3 & \ldots & 5 \end{bmatrix}$$

Motivation: Low-rank Matrix Factorization

$$\begin{bmatrix} \color{red}{5} & ? & 3 & 1 & \ldots & \color{red}{4} \\ 5 & ? & 4 & ? & \ldots & ? \\ ? & 5 & ? & 3 & \ldots & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 & 1 & \ldots & 3 \\ 5 & 4 & 4 & 5 & \ldots & 4 \\ 3 & 5 & 5 & 3 & \ldots & 5 \end{bmatrix} ?$$

Motivation: Low-rank Matrix Factorization

Goal: Given a matrix $\mathbf{M} \in \mathbb{R}^{m \times n}$ formed by deleting and corrupting the entries of $\mathbf{L}_0$, recover the underlying low-rank matrix $\mathbf{L}_0$.

$$\mathbf{M} = \begin{bmatrix} 5 & ? & 3 & 1 & \ldots & 4 \\ 5 & ? & 4 & ? & \ldots & ? \\ ? & 5 & ? & 3 & \ldots & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 & 1 & \ldots & 3 \\ 5 & 4 & 4 & 5 & \ldots & 4 \\ 3 & 5 & 5 & 3 & \ldots & 5 \end{bmatrix} = \mathbf{L}_0$$

1. **Matrix completion (MC):** Small fraction of entries revealed
2. **Robust matrix factorization (RMF):** Fraction of entries grossly corrupted

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$$\mathbf{M} = \begin{bmatrix} 5 & ? & 3 & 1 & \ldots & 4 \\ 5 & ? & 4 & ? & \ldots & ? \\ ? & 5 & ? & 3 & \ldots & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 & 1 & \ldots & 3 \\ 5 & 4 & 4 & 5 & \ldots & 4 \\ 3 & 5 & 5 & 3 & \ldots & 5 \end{bmatrix} = \mathbf{L}_0$$

Examples: Matrix completion

- Collaborative filtering: How will user i rate movie j?
  - Netflix: 10 million users, 100K DVD titles
- Ranking on the web: Is URL j relevant to user i?
  - Google News: millions of articles, millions of users

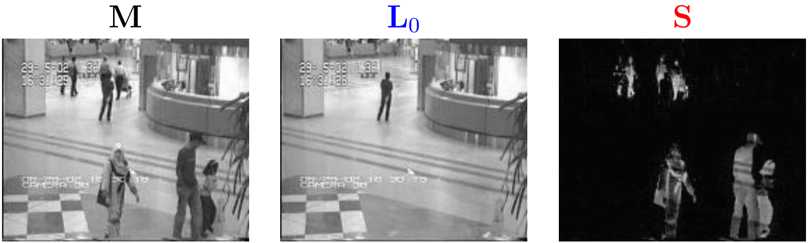
Motivation: Low-rank Matrix Factorization

Goal: Given a matrix $\mathbf{M} \in \mathbb{R}^{m \times n}$ formed by deleting and corrupting the entries of $\mathbf{L}_0$, recover the low-rank matrix $\mathbf{L}_0$.

$$\mathbf{M} = \begin{bmatrix} 5 & ? & 3 & 1 & \dots & 4 \\ 5 & ? & 4 & ? & \dots & ? \\ ? & 5 & ? & 3 & \dots & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 1 & \dots & 3 \\ 5 & 4 & 4 & 5 & \dots & 4 \\ 3 & 5 & 5 & 3 & \dots & 5 \end{bmatrix} = \mathbf{L}_0$$

Examples: Robust matrix factorization

- Background modeling/foreground activity detection



$\mathbf{M}$ $\mathbf{L}_0$ $\mathbf{S}$

(Candès, Li, Ma, and Wright, 2011)

Motivation: Low-rank Matrix Factorization

Goal: Given a matrix $\mathbf{M} \in \mathbb{R}^{m \times n}$ formed by deleting and corrupting the entries of $\mathbf{L}_0$, recover the low-rank matrix $\mathbf{L}_0$.

$$\mathbf{M} = \begin{bmatrix} 5 & ? & 3 & 1 & \dots & 4 \\ 5 & ? & 4 & ? & \dots & ? \\ ? & 5 & ? & 3 & \dots & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 3 & 1 & \dots & 3 \\ 5 & 4 & 4 & 5 & \dots & 4 \\ 3 & 5 & 5 & 3 & \dots & 5 \end{bmatrix} = \mathbf{L}_0$$

State of the art MF algorithms

- Strong recovery guarantees
- Plagued by expensive subroutines (e.g., truncated SVD)

This talk

- Present divide and conquer approaches for scaling up any MF algorithm while maintaining strong recovery guarantees

Roadmap

1. Introduction
2. Matrix Completion
    - Background
    - Divide-Factor-Combine
    - Simulations
    - Collaborative filtering
3. Robust Matrix Factorization
    - Background
    - Simulations
    - Video background modeling
4. Future Directions

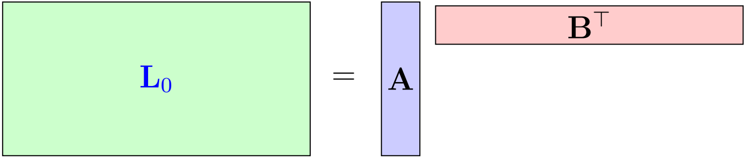
Exact Matrix Completion

Goal: Given entries from a matrix $\mathbf{L}_0 \in \mathbb{R}^{m\times n}$ with rank $r \ll m, n$, recover $\mathbf{L}_0$.

Noisy Matrix Completion

Goal: Given entries from a matrix $\mathbf{M} = \mathbf{L}_0 + \mathbf{Z} \in \mathbb{R}^{m \times n}$ where $\mathbf{L}_0$ has rank $r \ll m, n$ and $\mathbf{Z}$ is entrywise noise, recover $\mathbf{L}_0$.

- Good news: $\mathbf{L}_0$ has $\sim (m+n)r \ll mn$ degrees of freedom



  - Factored form: $\mathbf{A}\mathbf{B}^\top$ for $\mathbf{A} \in \mathbb{R}^{m \times r}$ and $\mathbf{B} \in \mathbb{R}^{n \times r}$

- Bad news: Not all low-rank matrices can be recovered

Question: What can go wrong?

What can go wrong?

Entire column missing

$$\begin{bmatrix} 1 & 2 & ? & 3 & \dots & 4 \\ 3 & 5 & ? & 4 & \dots & 1 \\ 2 & 5 & ? & 2 & \dots & 5 \end{bmatrix}$$

- No hope of recovery!

Solution: Uniform observation model

Assume that the set of s observed entries Ω is drawn uniformly at random:

$$\Omega \sim \text{Unif}(m, n, s)$$

What can go wrong?

Bad spread of information

$$\mathbf{L} = \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

- Can only recover $\mathbf{L}$ if $\mathbf{L}_{11}$ is observed

Solution: Incoherence with standard basis (Candès and Recht, 2009)

A matrix $\mathbf{L} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(\mathbf{L}) = r$ is (μ, r)-*coherent* if

Singular vectors are not too sparse: $\begin{cases}\max_i \|\mathbf{U}\mathbf{U}^\top \mathbf{e}_i\|^2 \le \mu r/m\\ \max_i \|\mathbf{V}\mathbf{V}^\top \mathbf{e}_i\|^2 \le \mu r/n\end{cases}$

and not too cross-correlated: $\|\mathbf{U}\mathbf{V}^\top\|_\infty \le \sqrt{\frac{\mu r}{mn}}$

How do we recover $\mathbf{L}_0$?

First attempt:

$$\begin{aligned} &\text{minimize}_{\mathbf{A}} && \operatorname{rank}(\mathbf{A}) \\ &\text{subject to} && \textstyle\sum_{(i,j)\in\Omega}(\mathbf{A}_{ij} - \mathbf{M}_{ij})^2 \le \Delta^2. \end{aligned}$$

Problem: Intractable to solve!

Solution: Solve **convex** relaxation (Fazel, Hindi, and Boyd, 2001; Candès and Plan, 2010)

$$\begin{aligned} &\text{minimize}_{\mathbf{A}} && \|\mathbf{A}\|_* \\ &\text{subject to} && \textstyle\sum_{(i,j)\in\Omega}(\mathbf{A}_{ij} - \mathbf{M}_{ij})^2 \le \Delta^2 \end{aligned}$$

where $\|\mathbf{A}\|_* = \sum_k \sigma_k(\mathbf{A})$ is the trace/nuclear norm of $\mathbf{A}$.

Questions:

- Will the nuclear norm heuristic successfully recover $\mathbf{L}_0$?
- Can nuclear norm minimization scale to large MC problems?

Noisy Nuclear Norm Heuristic: Does it work?

Yes, with high probability.

Typical Theorem

If $\mathbf{L}_0$ is (μ, r)-coherent, $s = O(\mu r n \log^2(n))$ entries of $\mathbf{M} \in \mathbb{R}^{m \times n}$ are observed uniformly at random, and $\hat{\mathbf{L}}$ solves the noisy nuclear norm heuristic, then

$$\|\hat{\mathbf{L}} - \mathbf{L}_0\|_F \leq f(m, n)\Delta$$

with high probability when $\|\mathbf{M} - \mathbf{L}_0\|_F \leq \Delta$.

- See Candès and Plan (2010); Mackey, Talwalkar, and Jordan (2011); Keshavan, Montanari, and Oh (2010); Negahban and Wainwright (2010)
- Implies **exact** recovery in the noiseless setting ($\Delta = 0$)

Noisy Nuclear Norm Heuristic: Does it scale?

Not quite...

- Standard interior point methods (Candès and Recht, 2009): O($|\Omega|(m+n)^3 + |\Omega|^2(m+n)^2 + |\Omega|^3$)
- More efficient, tailored algorithms:
  - Singular Value Thresholding (SVT) (Cai, Candès, and Shen, 2010)
  - Augmented Lagrange Multiplier (ALM) (Lin, Chen, Wu, and Ma, 2009a)
  - Accelerated Proximal Gradient (APG) (Toh and Yun, 2010)
  - All require rank-k truncated SVD on **every** iteration

Take away: Provably accurate MC algorithms are still too expensive for large-scale or real-time matrix completion

Question: How can we scale up a given matrix completion algorithm and still retain recovery guarantees?

Divide-Factor-Combine (DFC)

Idea: Divide and conquer

1. Divide $\mathbf{M}$ into submatrices.
2. Factor each submatrix **in parallel**.
3. Combine submatrix estimates to recover $\mathbf{L}_0$.

Advantages

- Factoring a submatrix is often much cheaper than factoring $\mathbf{M}$
- Multiple submatrix factorizations can be carried out in parallel
- DFC works with **any** base MC algorithm
- With the right choice of division and recombination, yields recovery guarantees comparable to those of the base algorithm

DFC-Nys: Generalized Nyström Decomposition

1. Choose a random column submatrix $\mathbf{C} \in \mathbb{R}^{m\times l}$ and a random row submatrix $\mathbf{R} \in \mathbb{R}^{d\times n}$ from $\mathbf{M}$. Call their intersection $\mathbf{W}$.

$$\mathbf{M} = \begin{bmatrix} \mathbf{W} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \mathbf{W} \\ \mathbf{M}_{21} \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \mathbf{W} & \mathbf{M}_{12} \end{bmatrix}$$

2. Recover the low rank components of $\mathbf{C}$ and $\mathbf{R}$ **in parallel** to obtain $\hat{\mathbf{C}}$ and $\hat{\mathbf{R}}$
   - Reduced cost: Expect $\min(n/l, m/d)$ speed-up per iteration
3. Recover $\mathbf{L}_0$ from $\hat{\mathbf{C}}$, $\hat{\mathbf{R}}$, and their intersection $\hat{\mathbf{W}}$

$$\hat{\mathbf{L}}^{nys} = \hat{\mathbf{C}}\hat{\mathbf{W}}^{+}\hat{\mathbf{R}}$$

   - Generalized Nyström method (Goreinov, Tyrtyshnikov, and Zamarashkin, 1997)
   - Minimal cost: O$(mk^2 + lk^2 + dk^2)$ where $k = \mathrm{rank}(\hat{\mathbf{L}}^{nys})$
4. **Ensemble:** Run p times in parallel and average estimates

DFC-Proj: Partition and Project

1. Randomly partition $\mathbf{M}$ into n/l column submatrices $\mathbf{M} = \begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_2 & \cdots & \mathbf{C}_{n/l}\end{bmatrix}$ where each $\mathbf{C}_i \in \mathbb{R}^{m \times l}$
2. Complete the submatrices **in parallel** to obtain
$$\begin{bmatrix}\hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2 & \cdots & \hat{\mathbf{C}}_{n/l}\end{bmatrix}$$
3. Recover a single factorization for $\mathbf{M}$ by projecting each submatrix onto the column space of $\hat{\mathbf{C}}_1$
$$\hat{\mathbf{L}}^{proj} = \hat{\mathbf{C}}_1\hat{\mathbf{C}}_1^+\begin{bmatrix}\hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2 & \cdots & \hat{\mathbf{C}}_{n/l}\end{bmatrix}$$
    - Minimal cost: $\mathrm{O}(mk^2 + lk^2)$ where $k = \operatorname{rank}(\hat{\mathbf{L}}^{proj})$
4. **Ensemble:** Project onto column space of each $\hat{\mathbf{C}}_j$ and average

Advantages over DFC-Nys

- Utilizes the entire matrix $\mathbf{M}$
- Column matrices are easier to factor when $n > m$

DFC: Does it work?

Yes, with high probability.

Theorem (Mackey, Talwalkar, and Jordan, 2011)

If $\mathbf{L}_0$ is (μ, r)-coherent and s entries of $\mathbf{M} \in \mathbb{R}^{m \times n}$ are observed uniformly at random, then

$$l = O\left(\frac{\mu^2 r^2 n^2 \log^2(n)}{s\epsilon^2}\right)$$

random columns suffice to have

$$\|\hat{\mathbf{L}}^{proj} - \mathbf{L}_0\|_F \leq (2+\epsilon) f(m,n)\Delta$$

with high probability when $\|\mathbf{M} - \mathbf{L}_0\|_F \leq \Delta$ and the noisy nuclear norm heuristic is used as a base algorithm.

- Can sample vanishingly small fraction of columns ($l/n \to 0$) whenever $s = \omega(n \log^2(n))$
- Implies exact recovery for noiseless ($\Delta = 0$) setting

DFC: Does it work?

Yes, with high probability.

Proof Ideas:

1. Uniform column/row sampling yields submatrices with low coherence (high spread of information) w.h.p.
2. Each submatrix has sufficiently many observed entries w.h.p.

$\Rightarrow$ Submatrix completion succeeds

3. Uniform sampling of columns/rows captures the full column/row space of $\mathbf{L}_0$ w.h.p.
   - Noisy analysis builds on randomized ℓ_2 regression work of Drineas, Mahoney, and Muthukrishnan (2008)

$\Rightarrow$ Generalized Nyström method and column projection succeed

DFC Noisy Recovery Error

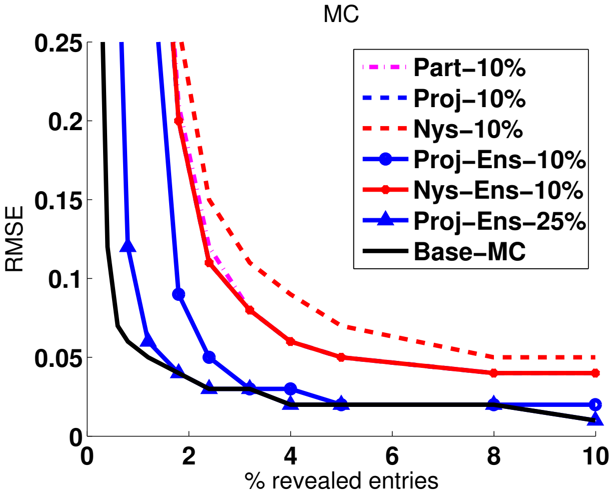


Figure: Recovery error of DFC relative to base algorithms with $(m = 10K, r = 10)$.

DFC Speed-up

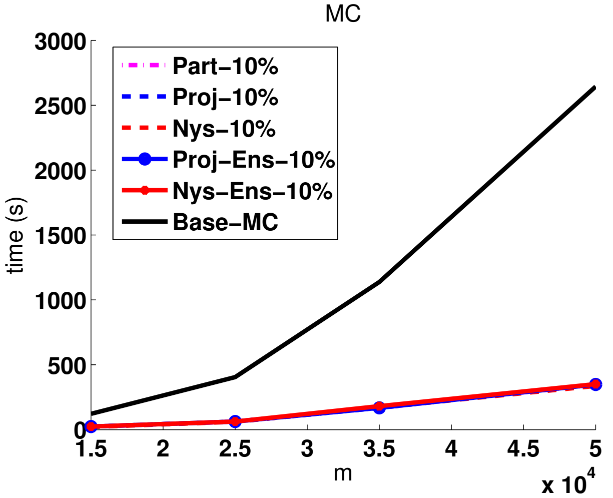


Figure: Speed-up over APG for random matrices with $r = 0.001m$ and 4% of entries revealed.

Application: Collaborative filtering

Task: Given a sparsely observed matrix of user-item ratings, predict the unobserved ratings

Issues

- Full-rank rating matrix
- Noisy, non-uniform observations

The Data

- **Netflix Prize Dataset**[1]
  - 100 million ratings in $\{1, \ldots, 5\}$
  - 17,770 movies, 480,189 users

[1] `http://www.netflixprize.com/`

Application: Collaborative filtering

Method	Netflix	
	RMSE	Time
APG	0.8433	2653.1s
DFC-Nys-25%	0.8832	890.9s
DFC-Nys-10%	0.9224	487.6s
DFC-Nys-Ens-25%	0.8486	964.3s
DFC-Nys-Ens-10%	0.8613	546.2s
DFC-Proj-25%	0.8436	689.5s
DFC-Proj-10%	0.8484	289.7s
DFC-Proj-Ens-25%	0.8411	689.5s
DFC-Proj-Ens-10%	0.8433	289.7s

Noisy Robust Matrix Factorization

Goal: Given a matrix $\mathbf{M} = \mathbf{L}_0 + \mathbf{S}_0 + \mathbf{Z}$ where $\mathbf{L}_0$ is low-rank, $\mathbf{S}_0$ is sparse, and $\mathbf{Z}$ is entrywise noise, recover $\mathbf{L}_0$

(Chandrasekaran, Sanghavi, Parrilo, and Willsky, 2009; Candès, Li, Ma, and Wright, 2011; Zhou, Li, Wright, Candès, and Ma, 2010)

- $\mathbf{S}_0$ can be viewed as an outlier/gross corruption matrix
  - Ordinary PCA breaks down in this setting
- **Harder than MC:** outlier locations are unknown
- **More expensive than MC:** dense, fully observed matrices

How do we recover L_0?

First attempt:

$$\begin{aligned} \text{minimize}_{\mathbf{L},\mathbf{S}} \quad & \operatorname{rank}(\mathbf{L}) + \lambda \operatorname{card}(\mathbf{S}) \\ \text{subject to} \quad & \|\mathbf{M} - \mathbf{L} - \mathbf{S}\|_F \leq \Delta. \end{aligned}$$

Problem: Intractable to solve!

Solution: Convex relaxation

$$\begin{aligned} \text{minimize}_{\mathbf{L},\mathbf{S}} \quad & \|\mathbf{L}\|_* + \lambda \|\mathbf{S}\|_1 \\ \text{subject to} \quad & \|\mathbf{M} - \mathbf{L} - \mathbf{S}\|_F \leq \Delta. \end{aligned}$$

where $\|\mathbf{S}\|_1 = \sum_{ij} \mathbf{S}_{ij}$ is the ℓ_1 entrywise norm of $\mathbf{S}$.

Question: Does it work? Yes!

Question: Does it scale? Not quite.

Idea: Divide and conquer

DFC Noisy Recovery Error

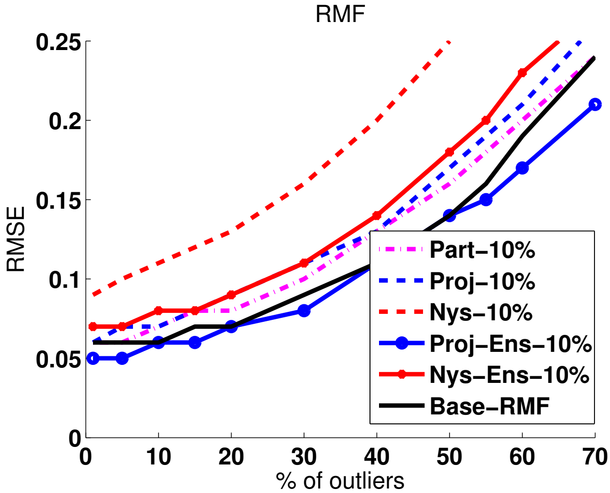


Figure: Recovery error of DFC relative to base algorithms with $(m = 1K, r = 10)$.

DFC Speed-up

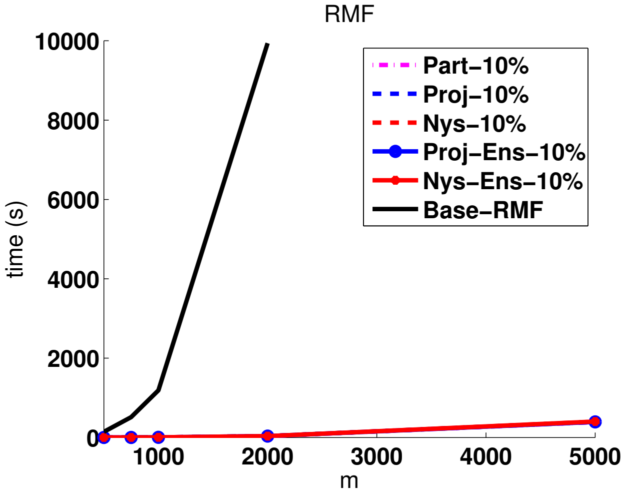


Figure: Speed-up over APG for random matrices with $r = 0.01m$ and 10% of entries corrupted.

Application: Video background modeling

Task

- Each video frame forms one column of matrix $\mathbf{M}$
- Decompose $\mathbf{M}$ into stationary background $\mathbf{L}_0$ and moving foreground objects $\mathbf{S}_0$

Issues

- Video is noisy
- Foreground corruption is often clustered, not uniform

Application: Video background modeling

Example: Changes in illumination

Specs

- 1.5 minutes of lobby surveillance (Li, Huang, Gu, and Tian, 2004)
- 1546 frames, 20480 pixels
- ALM running time: 5135.3s
- DFC-PROJ running time: 476.9s

Application: Video background modeling

Example: Significant foreground variation

Specs

- 1 minute of airport surveillance (Li, Huang, Gu, and Tian, 2004)
- 1000 frames, 25344 pixels
- ALM running time: 1871.0s
- DFC-PROJ running time: 392.8s

Future Directions

New Theory

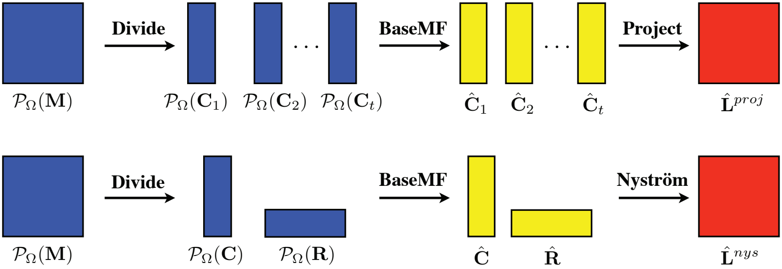
- Analyze statistical implications of divide and conquer algorithms
  - Trade-off between statistical and computational efficiency
  - Impact of ensembling
- Analysis of alternative optimization problems
  - Weighted trace norm for non-uniform sampling (Salakhutdinov and Srebro, 2010; Negahban and Wainwright, 2010)
  - Non-convex matrix completion with recovery guarantees (Keshavan, Montanari, and Oh, 2010)

New Applications

- Practical problems with large-scale or real-time MF requirements

The End

Thanks!


Divide
BaseMF
Project
$\mathcal{P}_\Omega(\mathbf{M})$
$\mathcal{P}_\Omega(\mathbf{C}_1)$
$\mathcal{P}_\Omega(\mathbf{C}_2)$
$\cdots$
$\mathcal{P}_\Omega(\mathbf{C}_t)$
$\hat{\mathbf{C}}_1$
$\hat{\mathbf{C}}_2$
$\cdots$
$\hat{\mathbf{C}}_t$
$\hat{\mathbf{L}}^{proj}$
Divide
BaseMF
Nyström
$\mathcal{P}_\Omega(\mathbf{M})$
$\mathcal{P}_\Omega(\mathbf{C})$
$\mathcal{P}_\Omega(\mathbf{R})$
$\hat{\mathbf{C}}$
$\hat{\mathbf{R}}$
$\hat{\mathbf{L}}^{nys}$

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Noisy Principal Component Pursuit: Does it work?

Yes, with high probability.

Theorem (Zhou, Li, Wright, Candès, and Ma, 2010)

If $\mathbf{L}_0$ is (μ, r)-coherent and $\mathbf{S}_0 \in \mathbb{R}^{m\times n}$ contains s non-zero entries with uniformly distributed locations, then if

$$r \leq \frac{\rho_r m}{\mu \log^2 n} \quad and \quad s \leq \rho_s mn$$

the minimizer to the problem

$$\begin{array}{ll} \text{minimize}_{\mathbf{L},\mathbf{S}} & \|\mathbf{L}\|_* + \lambda\|\mathbf{S}\|_1 \\ \text{subject to} & \|\mathbf{M}-\mathbf{L}-\mathbf{S}\|_F \leq \Delta. \end{array}$$

with $\lambda = 1/\sqrt{n}$ satisfies

$$\|\hat{\mathbf{L}} - \mathbf{L}_0\|_F \leq f(m,n)\Delta$$

with high probability when $\|\mathbf{M}-\mathbf{L}_0-\mathbf{S}_0\|_F \leq \Delta$.

Noisy Principal Component Pursuit: Does it scale?

Not quite...

- Standard interior point methods: O(n^6) (Chandrasekaran, Sanghavi, Parrilo, and Willsky, 2009)
- More efficient, tailored algorithms:
  - Accelerated Proximal Gradient (APG) (Lin, Ganesh, Wright, Wu, Chen, and Ma, 2009b)
  - Augmented Lagrange Multiplier (ALM) (Lin, Chen, Wu, and Ma, 2009a)
  - Require rank-k truncated SVD on **every** iteration
  - Best case SVD$(m, n, k) =$ O(mnk)

Take away: Provably accurate RMF algorithms are still too expensive for large-scale or real-time matrix factorization

Idea: Leverage the divide-and-conquer techniques developed for MC in the RMF setting

Divide-Factor-Combine (DFC)

Idea: Same divide-and-conquer strategies apply

1. Divide $\mathbf{M}$ into submatrices.
2. Factor each submatrix **in parallel**.
   - Using any base RMF algorithm
3. Combine submatrix estimates to recover $\mathbf{L}_0$.
   - Using DFC-NYS or DFC-PROJ

Question: Is it efficient?

- Yes, as in the MC case.

Question: Does it work?

DFC: Does it work?

Yes, with high probability.

Theorem (Mackey, Talwalkar, and Jordan, 2011)

If $\mathbf{L}_0$ is (μ, r)-coherent and $\mathbf{S}_0 \in \mathbb{R}^{m\times n}$ contains $s \leq \rho_s mn$ non-zero entries with uniformly distributed locations, then

$$l = O\left(\frac{\mu^2 r^2 \log^2(n)}{\epsilon^2}\right)$$

columns suffice to have

$$\|\hat{\mathbf{L}}^{proj} - \mathbf{L}_0\|_F \leq (2+\epsilon) f(m,n)\Delta$$

with high probability when $\|\mathbf{M} - \mathbf{L}_0 - \mathbf{S}_0\|_F \leq \Delta$ and noisy principal component pursuit is used as the base algorithm.

- Implies exact recovery for noiseless $(\Delta = 0)$ setting
- Similar result holds for DFC-Nys