About the SYK model

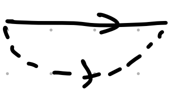
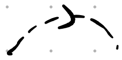
I. Motivation: non-Fermi liquids

1) Some experimental signatures of a Fermi liquid:

- Specific heat: $C_V = \frac{\pi^2}{3} k_B^2 T\, g(0)$ (equal to that of the free electron gas)
  (at least to the order T^1).
- Conductivity:
  - Wiedemann-Franz law: $\frac{K}{T\sigma} = \mathcal{L} = \frac{\pi^2}{3}\frac{k_B^2}{e^2}$: charge and energy carriers are the same.
  - Bloch-Grüneisen formula:

$$\rho(T) = \rho(0) + A\left(\frac{T}{\Theta_R}\right)^m \int_0^{\Theta_R/T} \frac{x^m}{(e^x-1)(1-e^{-x})}\,dx$$

where $m = 2, 3, 5$ for FL interactions, s-d scattering and $\bar{e}$-ph scattering, respectively.

Rk: that's "just" Drude, $\rho = \frac{m}{ne^2\tau}$, with τ^{-1} given by the RPA approx $\tau^{-1} = \mathrm{Im}$  where  can be another  $(m=5)$ or a d-electron $(m=3)$ or a phonon $(m=2)$.

→ These are predictions which can be tested.
In some experiments, deviations from these behaviors.
↳ theory of non-Fermi liquids.

2) Some theoretical ideas

- In a Fermi liquid, the propagator becomes

$$G(p, i\omega) = \frac{1}{i\omega - \epsilon_p - \Sigma(p, i\omega)} = \frac{Z_p}{i\omega - \tilde{\epsilon}_p} + G_{inch.}(p, i\omega)$$

qp weight

As long as $Z_p \neq 0$, there are quasiparticles hence a FL.
(NB even when $Z_p = 0$, it only means that electrons are not good qps, but there can still be other qps - cf the Luttinger liquid).

- Quasiparticle lifetime in a FL: consider a FL with qp repulsion U; then estimate the qp lifetime γ_ε at an energy ε close to the FS using Fermi's golden rule. It reads:

$$\gamma_\varepsilon = \frac{U^2 \rho(0)^3}{k_F^d} \int_{-\infty}^{+\infty} d\varepsilon_1 \, d\varepsilon_2 \, d\varepsilon_3 \, f(\varepsilon_1)(1-f(\varepsilon_2))(1-f(\varepsilon_3)) = \frac{U^2 \rho(0)^3}{k_F^d} \times \begin{cases} \varepsilon^2/2 & \text{si } T=0 \\ \pi^2 T^2/4 & \text{si } \varepsilon = 0 \end{cases}$$

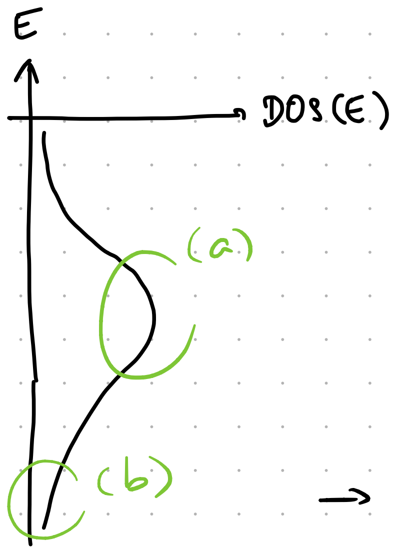
(x geometric factors)

"In a FL, equilibration takes a long time."

↳ This is the usual argument for the stability of the FL.
⚠ However there is a loophole: it does not depend on dimensionality d.
But we know that in $d=1$ the FL becomes unstable → look @ ReΣ too!

- Density of manybody states: take a FL with single-particle energies $\{\varepsilon_i\}$.

Then the manybody energy is $E = \sum_{i=1}^{N} n_i \varepsilon_i$ with $n_i = 0, 1$ (fermions).



(a) The energies vary over $\sim E$, and there are roughly 2^N states in this region, so level spacing $\sim 2^{-N}$: high-T region.

(b) Very few states (typically $O(1)$) populated, hence level spacing $\sim 1/N$: the quasiparticle limit.

→ A Fermi liquid has a manybody level spacing $\sim 1/N$.

The SYK model provides an exactly solvable model where there are no qps whatsoever (so it is "more NFL than the TLL").

II Technical background: disordered metal

→ Use random matrices to model mesoscopic transport.

Starting point: $\left[H = \frac{1}{\sqrt{N}} \sum_{ij}^{N} t_{ji} c_i^\dagger c_j - \mu \sum_i c_i^\dagger c_i \right]$

t_{ji}: random matrix (gaussian):

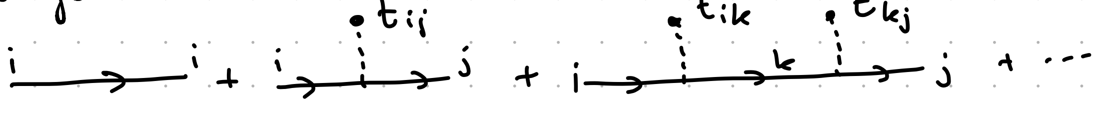
$\overline{}$ means: disorder average → $\overline{t_{ij}} = 0$, $\overline{t_{ij} t_{kl}^*} = t^2 (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.

(Standard methods from the study of spin glasses etc).

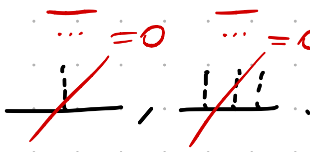
Diagrammatic representation: $t_{ij} c_j^\dagger c_i \equiv$

Green's function:

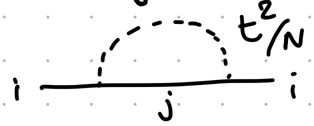
$G_{ij} =$



Great simplification comes from disorder averaging:

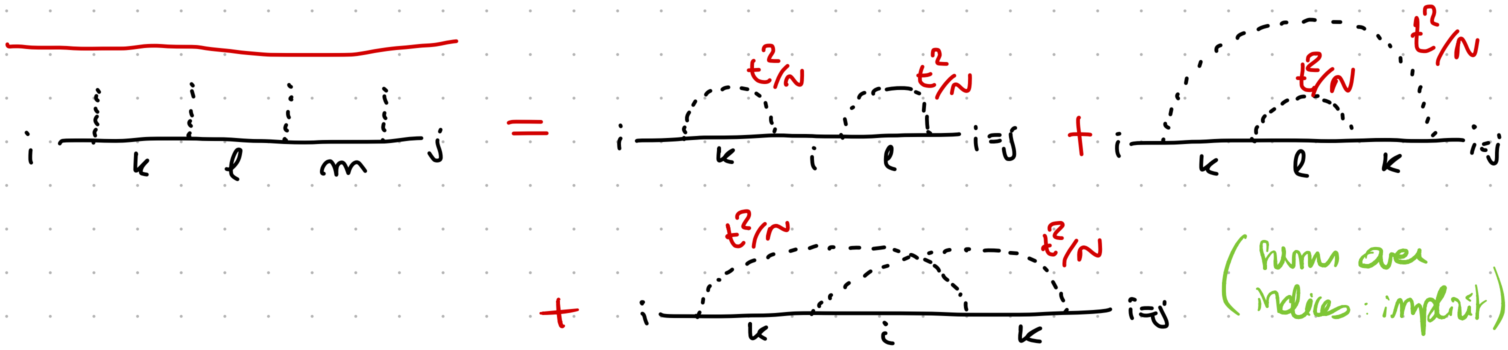


and no "loose ends":



etc.

Even greater simplification comes from the large-N expansion. (number of quantum dots)
Let's look at the 4-th order term:



(sums over indices implicit)

The last diagram is $\sum_k^N \mathcal{O}(1/N^2) \xrightarrow[N\to\infty]{} 0$ negligible wrt the others.

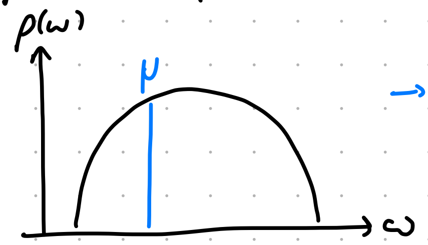
So: self energy: $\Sigma_{ii} = \sum_j \frac{t^2}{N} G_{jj} = t^2 G_{ii}$. $\Sigma =$

$$\begin{cases} G(i\omega) = \dfrac{1}{i\omega + \mu - \Sigma(i\omega)} \\ \Sigma(\tau) = t^2 G(\tau) \end{cases}$$

and this is easily solved.

$\Rightarrow \rho(\omega) = -\frac{1}{\pi} \operatorname{Im} G(\omega) = \frac{1}{2\pi t^2}\sqrt{4t^2 - \omega^2}$.

(μ absorbed)



→ a band of fermions

→ metal

NB: an important feature is the time-dependence of $G(\tau)$ at long τ.

Fastest way: assume $G(\tau) \sim 1/\tau^{\alpha} \Rightarrow \Sigma(\tau) \sim 1/\tau^{\alpha}$

$$\Rightarrow \Sigma(i\omega) = \int d\tau\, e^{-i\omega\tau}\, \Sigma(\tau) \sim \omega^{\alpha-1} \Rightarrow G(i\omega) \sim \omega^{1-\alpha} \Rightarrow G(\tau) \sim \tau^{\alpha-2}$$

ssi $\alpha - 1 < 1$ et $\Sigma(0) = \mu$

hence: $\boxed{G(\tau) \sim 1/\tau}$ → a signature of a FL from a random matrix model

III The SYK model

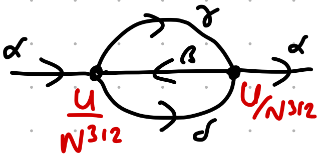
$\overline{U} = 0$, $\overline{|U_{\alpha\beta\gamma\delta}|^2} = U^2$

It's a model of pure interaction:
$$\boxed{H = \frac{1}{N^{3/2}} \sum_{\alpha\beta\gamma\delta} U_{\alpha\beta\gamma\delta}\, c^\dagger_\alpha c^\dagger_\beta c_\gamma c_\delta - \mu \sum_\alpha c^\dagger_\alpha c_\alpha}$$

1) Diagrammatic solution

Same reasoning as previously:

$$\overline{G}_{\alpha\beta} = \xrightarrow{\alpha\qquad\alpha} + \;0\; + \;\cdots\; + 0 + \cdots$$



Similarly, all diagrams with "line crossings" vanish in the $N \to \infty$ limit.

We have: $\Sigma(\tau) = -U^2 G^2(\tau) G(-\tau)$.

2) Some qualitative features of the solution

- It describes a gapless phase: indeed, assume $\mathrm{Im}\, G(\omega) = 0 \;\forall \omega \in [-\Delta, \Delta]$. ← $\rho(\omega)$: DOS

Since $G(\omega) = \dfrac{1}{\omega - \mu - \Sigma(\omega)}$, this is equivalent to $\mathrm{Im}\, \Sigma(\omega) = 0$.

But because of energy conservation at a vertex, this implies $\mathrm{Im}\, G(\omega) = 0 \;\forall \omega \in [-3\Delta, 3\Delta]$.

... etc: one can have $G(\omega) \neq 0$ only if $\boxed{\Delta = 0}$: gapless phase.

- Now to check it is not just a metal, find the long-τ decay of $G(\tau)$.

Assume $G(\tau) \sim 1/\tau^{\alpha} \Rightarrow \Sigma(\tau) \sim 1/\tau^{3\alpha} \Rightarrow \Sigma(i\omega) \sim \omega^{3\alpha - 1}$

$\Rightarrow G(i\omega) \sim \omega^{1-3\alpha} \Rightarrow G(\tau) \sim \tau^{3\alpha-1}$ hence: $\boxed{G(\tau) \sim 1/\sqrt{\tau}}$

ssi $3\alpha - 1 < 1$ et $\Sigma(0) = \mu$

→ slower decay of correlations @ long time than in a metal
→ NFL.

3) Exact solution of the problem

- Recall $G(i\omega) = \frac{1}{i\omega + \mu - \Sigma(i\omega)}$ and we consider only the singular contribution where Σ dominates in the denominator.

$G(i\omega)\Sigma_{sing}(i\omega) = -1$.

Rewrite the "new" problem we have to solve:

$$(*)\begin{cases} \int_0^\beta d\tau_2\, \Sigma_{sing}(\tau_1,\tau_2)\, G(\tau_2,\tau_3) = -\delta(\tau_1-\tau_3) \\ \Sigma_{sing}(\tau_1,\tau_2) = -U^2 G^2(\tau_1,\tau_2)\, G(\tau_2,\tau_1) \end{cases}$$

- It turns out that this problem has an exact solution:

$$\left[G(z) = \frac{A}{\sqrt{z}} \quad , \quad A = e^{-i\pi/4}\left(\pi/U^2 \cos(2\theta)\right)^{1/4} e^{-i\theta} \right]$$

where θ is a parameter fixed by the filling. (that's @ $T=0$).

(NB this is an approximate solution of the "true" problem for $|z| \ll U$).

- We want to know $Q = \frac{1}{N}\sum_\alpha c_\alpha^\dagger c_\alpha = G(\tau = 0^-)$ → how does one do that?

Method: rewrite it as in the proof of Luttinger's thm, then a series of clever tricks, and the result: $Q = \frac{1}{2} - \frac{\theta}{\pi} - \frac{\sin(2\theta)}{4}$.

Also denote $e^{2\pi\mathcal{E}} = \frac{\sin(\pi/4 + \theta)}{\sin(\pi/4 - \theta)}$. ($\theta$ or $\mathcal{E}$ parameterize particle-hole discrepancy)

- Important feature: $G(\tau) \sim \begin{cases} -e^{\pi\mathcal{E}}/|\tau|^{1/2} & \tau \gg 1/U \\ +e^{-\pi\mathcal{E}}/|\tau|^{1/2} & \tau \ll -1/U \end{cases}$

does not depend on U @ long times.

Thus the typical dissipation time can only be $\tau_p = \hbar/k_B T$

↳ "Planckian metal": information propagates at maximal speed.

↳ "No quasiparticles whatsoever": now proved! (can be made rigorous, Lieb-Robinson etc).

- The solution at finite T also exists!

A.G. & O.P. found it by recycling results from multichannel Kondo etc, and their ansatz appeared to yield a correct solution! ◡

4) Reparameterization (& gauge) symmetries

- The low-energy problem (*) has a full Diff(ℝ) sym. group:

$$(\#)\begin{cases} \tau = f(\sigma) \\ G(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-1/4}\,\tilde{G}(\sigma_1,\sigma_2) \\ \Sigma(\tau_1,\tau_2) = [f'(\sigma_1)f'(\sigma_2)]^{-3/4}\,\tilde{\Sigma}(\sigma_1,\sigma_2) \end{cases}$$

leaves the problem invariant.

(There is also a $U(1)$ gauge invariance $\begin{matrix} G(\tau_1,\tau_2) \\ \Sigma \end{matrix} \to \frac{g(\tau_1)}{g(\tau_2)} \begin{matrix} G(\tau_1,\tau_2) \\ \Sigma \end{matrix}$
which had to be there / cf Luttinger's thm.)

- In particular, the conformal map $\underset{\in\mathbb{R}}{\tau} = \frac{1}{\pi T}\tan(\pi T \underset{\in S^1}{\sigma})$

gives the finite-T solution from the T=0 one!

Thus, the "miracle" of A.G. & O.P. comes from this Diff(ℝ) hidden symmetry.

- Now, recall the solution we found: (here, $\theta = 0 = \mathcal{E}$ and $T=0$ for simplicity)

$G_{sol}(\tau_1,\tau_2) \propto |\tau_1-\tau_2|^{-1/2}$ and $\Sigma_{sol}(\tau_1,\tau_2) \propto |\tau_1-\tau_2|^{-3/2}$.

This has a much smaller sym. grp. than (#)!

Actually it is only $SL(2,\mathbb{R})$, i.e. $f(\sigma) = \frac{a\sigma+b}{c\sigma+d}$ $ad-bc=1$.

Hence the gaplessness ("reparameterization soft modes").

(Of course not goldstones: fermionic, and not quasiparticles)

5) T=0 entropy

Idea: take free fermions with $\left\{\begin{matrix}\text{energy } 0 \\ \text{chem. pot. } \delta\mu\end{matrix}\right.$; they have $\frac{G_0(\tau)}{G_0(\beta-\tau)} = e^{-\frac{\delta\mu}{T}}$.

And here, we have (cf prev. page) $\frac{G(\tau)}{G(\beta-\tau)} = e^{2\pi\mathcal{E}}$

so: $\boxed{\mu-\mu_0 = -2\pi\mathcal{E}T}$ + corrections.

Maxwell: $\frac{\partial\mu}{\partial T} = -\frac{1}{N}\frac{\partial S}{\partial Q} \overset{so}{=} 2\pi\mathcal{E}$.

↳ $\neq 0$: breaks the 3rd law of thermodynamics.

Even stronger: $S_0 = \lim_{T\to 0} S = 2\pi\int_0^Q dq\,\mathcal{E}(q) \neq 0$ (and actually there is a formula)

↳ low-energy level spacing $\sim e^{-N \underset{\neq 1/2 \text{ but still...}}{S_0}}$: that's $\neq$ quasiparticle physics ☺