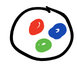
Journal club: Polyakov 1977

① The issue with quarks

→ Experiments of nucleon-nucleon scattering are well explained by "parton" models where  and quarks within a given nucleon are almost free (that's asymptotic freedom, cf GN).

→ However no free quarks are found alone → why?

Confinement: charges cannot be taken apart.

case 1: EM: $U(R) \sim \frac{1}{R} \Rightarrow \frac{U(R)}{R} \sim \frac{1}{R^2} \xrightarrow[R\to\infty]{} 0$ no confinement.

case 2: quarks?: $U(R) \sim R \longrightarrow \sigma := \frac{U(R)}{R} = \text{cst}$ confinement.

An aside: what is a charge, actually?

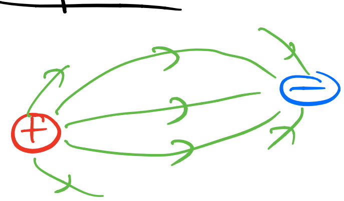
↳ one is charged under a certain gauge field.

ex: take $\mathcal{L} = \psi^\dagger (i\vec{D} - m)\psi$ invariant under $\psi \to e^{i\alpha}\psi$ global.

Make this local: $\psi \to e^{i\alpha(x)}\psi$ is still a symmetry, provided one replaces $\vec{D} \to \vec{D} - i\vec{A}$ and $\vec{A} \xrightarrow[g.t.]{} \vec{A} + \vec{D}\alpha$.

so: global ⇝ local is allowed provided that ψ is secretely charged under a gauge field $\vec{A}$

Interpretation:



linear growth $U(R) = \sigma R$ means that the gauge field $\vec{A}$ is massive → the issue is:

how does the g.f. get its mass?

② Why should we care?

→ cf lattice gauge theories (Kogut 1979).

Main idea: local constraints can be enforced by suitably defining a gauge theory on the lattice.

Indeed: $\psi(\vec{r}) \xrightarrow{g.t.} e^{i\alpha(\vec{r})}\psi(\vec{r})$

$\Phi(\vec{r}) \xrightarrow{g.t.} e^{i Q(\vec{r})\alpha(\vec{r})}\Phi(\vec{r})$

local charge of the field $\Phi(\vec{r})$

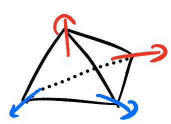
(ex: $\psi^{\dagger}(r)\psi^{\dagger}(r)$ has $Q(r)=-2$

$\psi^{\dagger}(r)\psi(r)$ has $Q(r)=0$).

basically $\int \mathcal{D}\alpha\, e^{i\alpha(\vec{r})Q(\vec{r})} = \prod_{\vec{r}} \delta(Q(\vec{r}))$.

And since only gauge-invariant quantities are meaningful (cf Elitzur's thm), $e^{iQ(r)}|\text{phys}\rangle = |\text{phys}\rangle$ → $\boxed{Q(r)=0.}$

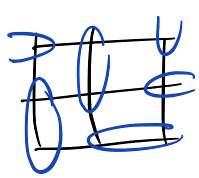
Ex 1: spin ice:



constraint: 2 in 2 out

$Q(r) = \text{out}(r) - \text{in}(r)$ works.

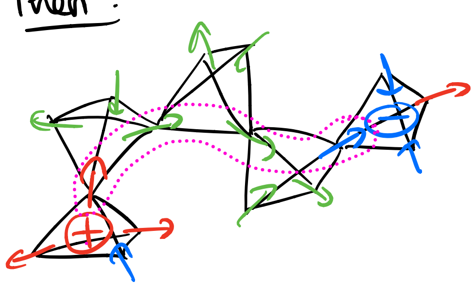
Ex 2: dimer models:



n_{ij} = # dimers on bond ij $= 0, 1$

$Q(r) = n_{r,r+\hat{x}} + n_{r,r+\hat{y}} + n_{r,r-\hat{x}} + n_{r,r-\hat{y}} - 1$.

Then:



Are these $Q(\vec{r})$ objects...

↙ confined? (i.e. boring)

↘ deconfined? (i.e. cool).

③ Phase diagram: confinement vs Coulomb

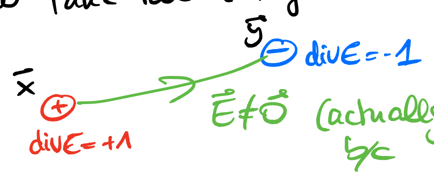
Take electromagnetism: $\left[H_{EM} = \frac{g}{2} \sum_{\vec r} \vec E^2_{(\vec r)} + \frac{1}{2g} \sum_{\vec r} \vec B^2_{(\vec r)} \right]$.

local g.t. defined by $e^{i\alpha(\vec r) Q(\vec r)}$ with $Q(\vec r) = \mathrm{div}\, \vec E(\vec r)$.

So-called physical states have $\vec\nabla \vec E = 0$ everywhere.

- Take $g \to \infty$. The GS has $\vec E(\vec r) = \vec 0 \ \forall \vec r$. (and is g-inv).

Now take two charges:



$\vec E \neq \vec 0$ (actually $E = \pm 1$ b/c compact EM)

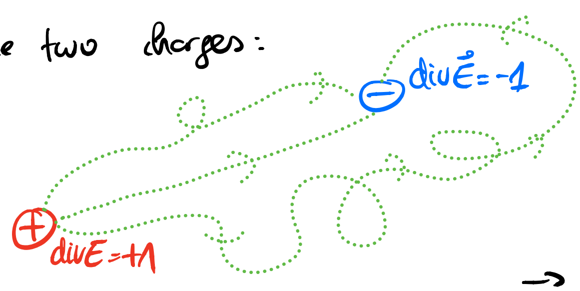
this state is gapped,

$\Delta(\vec x, \vec y) = \frac{g}{2} |\vec x - \vec y|$.

$\to$ confinement.

- Take $g \to 0$. The GS has $\vec B(\vec r) = \vec 0 \ \forall \vec r$ (and is g-inv)

Take two charges:

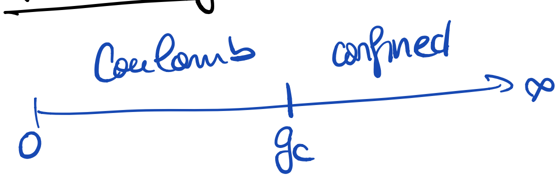


$\to$ degenerate w/ GS

$\to$ actually equal-weight superposition of all loop coverings, entanglement etc.

$\to$ deconfined (Coulomb) phase.

Phase diagram:



☹ "what's the value of g_c?"

☺ "how does g flow under RG?"

(Rq: $S = \int d^4x \left(-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + |D_\mu \phi|^2 \right)$

$[\phi] = 1 \quad [A^\mu] = 1 \quad [F^{\mu\nu}] = 2 \quad [e^2] = 0$

→ in $d = 3+1$, marginal → may depend on details.)

<u>Kadanoff '77</u>: in $d = 3+1$, $e^2 \nearrow \infty$ for nonabelian g.t.; $e^2 \searrow 0$ for abelian g.t.

⇝ strong coupling in lower dimensions

⇝ to understand quark confinement (and not simply "strong coupling") in $d = 3+1$, one idea (the one followed by Polyakov) is to consider EM in $d = 2+1$.

④ Analogies with superconductivity.

Now let's really try to answer: how does $\vec{A}$ become massive?

→ Anderson-Higgs mechanism: a charged field $\varphi(\vec{r})$ acquires an expectation value, $\varphi(\vec{r}) = \rho e^{i\theta(\vec{r})}$ ($\rho \neq 0$); but Elitzur forbids this, and as a consequence, the g.f. becomes massive.

(@youngsters: EM wave in a plasma → Klein-Gordon, $\omega^2 = k^2 + m^2$. Identify $\varphi(\vec{r}) = n_+(\vec{r})$, cation density).

→ Confinement of field lines ≅ the Meissner effect.

(in this case, ⊕ and ⊖ are rather magnetic monopoles).

So that's our mechanism.

Problem: there doesn't seem to be a condensate of $SU(3)$ charges filling the known universe.

→ How can one get a mass without a condensate?

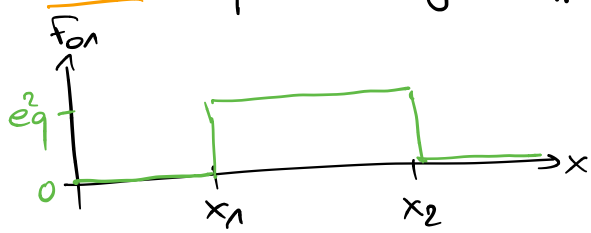
↳ Polyakov: consider pure $U(1)$ gauge theory.
i.e. without matter

⑤ Classical confinement

• It works in $d=1+1$

$$\left[S = \int d^2x \left(\frac{-1}{2e^2} F_{01} F^{01} + A_\mu j^\mu \right) \right]$$

e.o.m: $\partial_\mu F^{\mu\nu} = e^2 j^\nu$ applied to $\begin{cases} j^0 = q\,\delta(x-x_1) - q\,\delta(x-x_2). \\ \text{gauge } A_0 = 0. \end{cases}$



$$\left[H = \int dx \, \frac{F_{01}^2}{2e^2} = \frac{q^2 e^2}{2} |\vec{x}_1 - \vec{x}_2| \right]$$

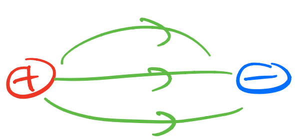
→ so in $d=1+1$, classical confinement is linear.

• It fails in $d=2+1$

$$\left[S = \int d^3x \left(\frac{-1}{4e^2} F_{\mu\nu} F^{\mu\nu} + A_\mu j^\mu \right) \right]$$

e.o.m $\partial_\mu F^{\mu\nu} = e^2 j^\nu \xrightarrow{j^0 = Q\,\delta^2(x)} \nabla^2 A_0 = Q\,\delta^2(x)$

$$A_0(r) = \frac{Q}{2\pi} \ln\left(\frac{r}{r_0}\right)$$



$V(r) \sim \ln(r)$

→ true confinement does not come from the classical solution of e.o.m.

Anyway in $d=3+1$ this can never be a mechanism for confinement so quarks being confined is a feature of strong coupling.

⑥ $d=2+1$: symmetries

$$S_{AH} = \int d^3x \left\{ \frac{-1}{4e^2} F_{\mu\nu}F^{\mu\nu} + |D_\mu \phi|^2 - m^2|\phi|^2 - \frac{\lambda}{2}|\phi|^4 \right\}.$$

- $U(1)$ gauge: $\phi \to e^{i\alpha(x)}\phi$, $A_\mu \to A_\mu + \partial_\mu \alpha(x)$.

  Conserved current $j_\mu^{gauge} = i(\partial_\mu \phi^\dagger)\phi - i\phi^\dagger \partial_\mu \phi$ as usual.

This is true of course in any dimension.

Now, in $d=2+1$, there is a very particular thing happening:

- Another conserved current: $j^\mu_{top} = \frac{1}{4\pi} \epsilon^{\mu\nu\rho} F_{\nu\rho}$ defined only in $d=2+1$!

which is always conserved (Bianchi).

$\rightarrow$ so there is another, global, symmetry $U(1)_{top}$.

$$Q_{top} = \int d^2x\, j^0_{top} = \frac{1}{2\pi}\int d^2x\, B(x)$$

: charges under $U(1)_{top}$ are magnetic monopoles.

Now what's an interpretation of this symmetry/current?

Idea: $Z = \int \mathcal{D}A_\mu\, e^{-\int d^3x \frac{-1}{4e^2} F_{\mu\nu}F^{\mu\nu}}$ and $A_\mu \equiv \{F_{\mu\nu}$; Bianchi$\}$

so rewrite $= \int \mathcal{D}F_{\mu\nu}\, \mathcal{D}\sigma\, e^{-\dots + \frac{i}{4\pi}\sigma \epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}}$ [Lagrange multiplier for Bianchi $\sigma \in [0, 2\pi]$ compact.

naively integrate out $F_{\mu\nu}$ $= \int \mathcal{D}\sigma\, e^{-\int d^3x \frac{e^2}{8\pi^2} \partial_\mu \sigma \partial^\mu \sigma}$

So: we've shown that in the Coulomb phase (where the ϕ field is massive), the EM field has a free boson σ.

Its e.o.m is $\partial_\mu \sigma = \frac{1}{\pi e^2} \epsilon_{\mu\nu\rho} F_{\nu\rho}$ so $J^\mu_{top} = \frac{e^2}{4\pi^2} \partial^\mu \sigma$

which shows that $U(1)_{top} : \sigma \to \sigma + \alpha$

and we thus learn that monopole operators are $e^{i\sigma(x)}$.

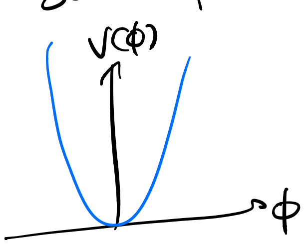
(or conversely: adding $e^{i\sigma(x)}$ to the action turns Bianchi into $\partial_\mu J^\mu_{top} = \delta(x - x_0)$ which tells us it's a monopole).

⑦ Phase diagram and duality

i.e. broken but no worries b/c it's global

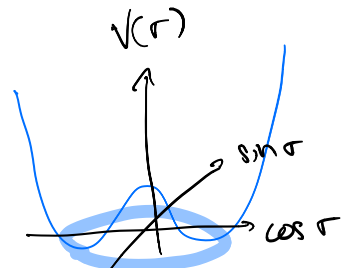
The GS has $\sigma = \text{cst}$; this is $U(1)_{top}$ degenerate (think $\sigma \to \sigma + \alpha$).

So our picture in the Coulomb phase is:



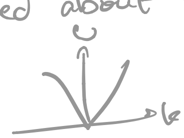
matter is massive

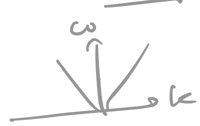
$$\xrightarrow[F_{\mu\nu} \text{ can be integrated out}]{U(1)_{gauge} \text{ unbroken}}$$



σ is a Goldstone mode

An aside: I've often wondered about the photon's dispersion relation.

Acoustic waves have dispersion  and are Goldstones of the broken transla° sym.

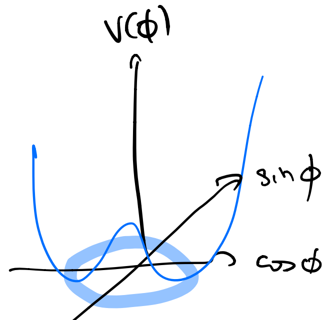
Spin waves have dispersion in an AFM  and are Goldstones of the broken $O(3)$ symmetry.

Photons have  : which broken sym has them as Goldstones?

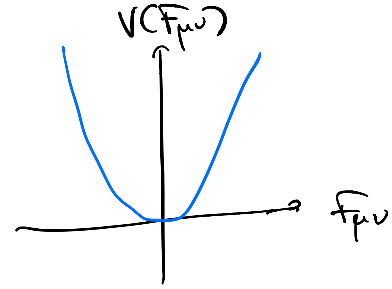
In $d = 2+1$* there is an answer: that's $U(1)_{top}$:)

* (and in the Coulomb phase)

Now, it may well be that $U(1)_{gauge}$ is broken (Higgs phase):
(well up to Elitzur but anyway)



Anderson-Higgs $\longrightarrow$ the photon becomes massive



matter is condensed

monopoles are massive so $U(1)_{top}$ unbroken.

Note that now Q_{top} is carried by matter vortices, such that

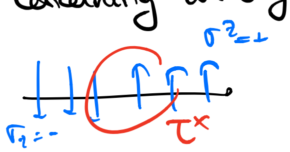
$$\oint dx^i \partial_i \phi = \frac{1}{2\pi}\int d^2x\, B = Q_{top} \qquad @ \text{ fixed } t.$$

that's because we are in the Higgs phase where basically $|D_\mu\phi|^2 \overset{!}{=} 0$.
of what happens in the London gauge with superconductivity.

So we see that there is a duality between Coulomb & Higgs.
This means that secretly they have the same physics.
(This implies that Anderson-Higgs is not responsible for confinement.)

NB: actually that's particle-vortex: vortex of $\phi \Leftrightarrow$ particle $e^{i\sigma}$

(Concerning duality: cf Kramers-Wannier duality: for $d=1+1$ Ising,



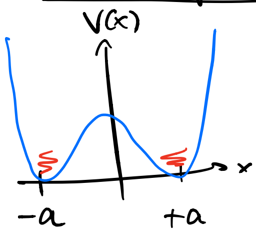
define "disorder operators" $\tau^{x,t}$ dual to the $\sigma^{z,x}$
Note this looks like instanton-particle duality.)

Morality: we can understand $d=2+1$ confinement just by looking at the Coulomb phase. No need for matter.
And anyway if matter is too simple (e.g. abelian Higgs) then the physics is not changed. (but there are examples where matter can help escape the Polyakov fate).

8) the main idea of Polyakov's paper

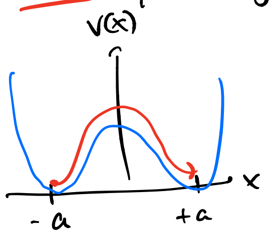
First sentence: he's referring to $U(1)_{top}$ which is broken in the Coulomb phase, yielding a massless photon.
The goal is to kill this Goldstone by restoring the symmetry.

The role of instantons: think of a symmetry-broken system.



Classically, the system stays in one of the wells. We can integrate out its fluctuations, and basically the physics is that of sym. breaking.

But quantumly there are instantons. (basically because in Euclidean time one takes $V(x) \to -V(x)$ so it becomes a perfectly legal "classical" trajectory).
in the sense of e.o.m.



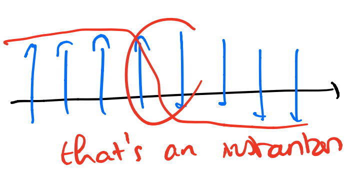
Now one has $\langle +a | e^{-H.T} | -a \rangle \propto T e^{-S_{instanton}}$

so $\langle +a | e^{-HT} | -a \rangle \propto \sinh(T e^{-S_{instanton}})$
$\langle -a | e^{-HT} | -a \rangle \propto \cosh(T e^{-S_{instanton}})$
{ (you cannot guess it, sca of a collective coordinate etc).

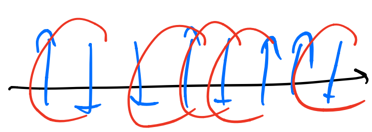
$\Rightarrow \lim_{T \to \infty} \langle x \rangle = 0$!

$\rightsquigarrow$ the symmetry has been restored by the instantons.

Analogy: think of Peierls' argument for no ph. tr. in d=1.



proliferation $\rightsquigarrow$ of instantons



magnons are condensed
instantons are gapped

instantons are condensed
magnons are gapped

$\longleftrightarrow$
cf spin duality
Kadanoff-Cevà

That's exactly the idea of Polyakov: make a certain kind of instantons proliferate, and restore the $U(1)_{top}$ symmetry.

↳ basically they're "diluted" monopoles $e^{i\sigma}$, with $\int B = \pm 1$ but they are delocalised all over the slice $\mathbb{R}^2$.

Now how does it work in practice?

⑨ Polyakov's treatment of d=2+1 pure EM

The idea is that, when computing Z, one cannot integrate out $F_{\mu\nu}$ as though it were sitting at the classical saddle point.

In sec. ⑥ I wrote this "naively". Now let's do better.

$$\left(S = S_{mp}\, \underset{\substack{\uparrow \\ \text{number of monopoles}}}{N} + \frac{1}{4\pi g^2} \sum_{i\neq j} \frac{\overset{\pm 1}{m_i m_j}}{|\vec{x}_i - \vec{x}_j|} . \right)$$

3d Coulomb gas

that's because monopoles interact like usual charges in d=3.

Sum over all possible N.

$$\Rightarrow Z = \sum_{N=0}^{\infty} \sum_{m_i = \pm 4\pi} \frac{1}{N!} \left(c\, e^{-S_{mp}} \right)^N e^{-\frac{1}{8\pi g^2} \sum_{i\neq j} \frac{m_i m_j}{|\vec{x}_i - \vec{x}_j|}}$$

(c captures interaction details of monopoles but we don't care)

Now big trick (actually would work $\forall d$): since the Coulomb potential is the Green's function for D^2, one can rewrite:

$$\int \mathcal{D}\sigma\, e^{-\int d^3x \left(\frac{g^2}{8\pi^2} (\partial_\mu \sigma)^2 + \frac{i}{2\pi} \sum_i m_i \sigma\, \delta^3(x - x_i) \right)} \overset{\text{Gauss}}{=} e^{-\frac{1}{8\pi g^2} \sum_{i\neq j} \frac{m_i m_j}{|\vec{x}_i - \vec{x}_j|}}$$

$$\Rightarrow \boxed{Z = \int \mathcal{D}\sigma\, \exp\left(- \int d^3x\, \frac{g^2}{8\pi^2} (\partial_\mu \sigma)^2 - c\, e^{-S_{mp}} \cos(2\sigma) \right)}.$$

→ the photon σ has now a mass $m_{ph}^2 = \frac{4\pi^2 c}{g^2} e^{-S_{mp}}$.

__Nb:__ nonperturbative wrt the small parameter $S_{mp}^{-1} \xrightarrow{\text{classical}} 0$.

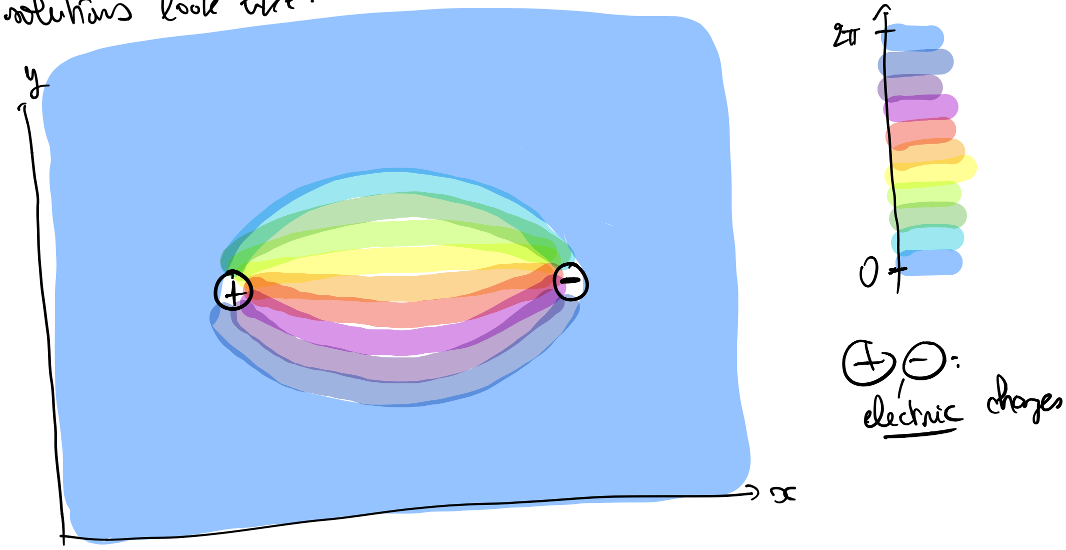
→ So we've found $S_{eff} = \int d^3x \left(\frac{g^2}{8\pi^2} (\partial_\mu \sigma)^2 + c\, e^{-S_{mp}} \cos(2\sigma) \right)$

which is Sine-Gordon.

Solutions to this equation are known to be "kinks", with energy per unit length $\frac{E}{R} = \frac{g}{\pi}\sqrt{2cg^2e^{-S_{mp}}}$ (obtained with the Bogomolnyi technique).

$\Rightarrow$ So confinement in $d=2+1$ EM is linear!

These solutions look like:



Conclusion

this corresponds to quarks but the mechanism is not known to study $\Rightarrow$ initial motivation

$d > 3+1$: everyone is deconfined.
$d = 3+1$: QED is deconfined, nonabelian g.th. are confined.
$d = 2+1$: QED is linearly confined, n.a.g.th. confined.
$d < 2+1$: everyone is confined

and that's very important
it means we must go to $d=3+1$
to find a pure deconfined QED.
(or we must add funny matter, cf work by Lee, Nagaosa, Wen etc)