Notes about the Anderson-Higgs mechanism

I. Electromagnetic wave in a plasma

$$m\frac{d\vec{v}}{dt} = q\vec{E} \implies \frac{d\vec{j}}{dt} = \frac{q^2}{m}\vec{E}$$

$\equiv \omega_p^2$

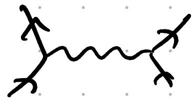
$$\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0}) - \vec{\Delta}\vec{E} = \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{\partial}{\partial t}\left(\vec{j} + \frac{\partial\vec{E}}{\partial t}\right) = -\left(\frac{q^2}{m} + \frac{\partial^2}{\partial t^2}\right)\vec{E}$$

→ KG equation, dispersion $\underline{\omega^2 = k^2 + \omega_p^2}$.

→ The photon in a plasma is massive.

II. Equivalence mass ↔ lengthscale

Understand the photon mass as a typical lengthscale (beyond just dim. analysis).

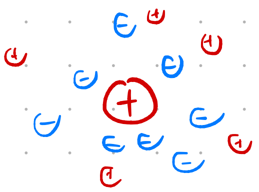
Coulomb interaction between charged particles: 

The Coulomb force as the susceptibility:

$$\chi(\omega=0, \vec{q}) =$$  $$= \frac{1}{q^2 + \omega_p^2} \xrightarrow[d=3]{FT} \chi(r) = \frac{e^{-\omega_p r}}{r}.$$

Massless case ($\omega_p = 0$) recovers Coulomb in the vacuum, while the massive case ($\omega_p > 0$) describes screened interaction.

III. Link to Debye (or Thomas-Fermi) screening.



$$\begin{cases} \rho(r) = -\nabla^2 V(r) \\ \rho(r) = +Q\,\delta(r) + \sum_i \rho_0^i \exp\left(-\frac{q_i V(r)}{k_B T}\right) \end{cases}$$

$= \sum_i \rho_0^i q_i / k_B T$

$$\Rightarrow \nabla^2 V(r) \simeq -Q\,\delta(r) - \sum_i \rho_0^i \left(1 - \frac{q_i V(r)}{k_B T}\right) \underset{\uparrow}{=} -Q\,\delta(r) + \lambda_D^{-2} V(r)$$

neutral electrolyte

Solution: $V(r) = \frac{Q}{4\pi r} e^{-r/\lambda_D}$ → identify $\omega_p \equiv \lambda_D^{-1}$.

⇒ EM in a charged medium is screened i.e. massive.

IV. This is the physics of superconductors

1) Justification:

in other words $\vec{J} \sim \vec{A}$.

The KG equation appears because $\frac{d\vec{J}}{dt} \propto \vec{E}$, i.e. $\sigma(\omega) \sim \frac{1}{j\omega}$.

This is the SC limit of the Drude conductivity $\sigma(\omega) = \frac{\sigma_0}{j\omega + 1/\tau}$ $(\tau \to \infty)$.

(cf note about transport for more details).

2) Ginzburg-Landau for a charged field

- Neutral GL free energy: $f_{GL} = \frac{1}{2m}|\nabla\psi|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4$

Rewrite $\psi = |\psi| e^{i\phi}$

$$\Rightarrow \quad f_{GL} = \underbrace{\frac{|\psi|^2}{2m}(\nabla\phi)^2}_{\text{phase stiffness in the ordered phase}} + \left[\frac{1}{2m}(\nabla|\psi|)^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4\right]$$

- Charged case: now gauge invariance $\psi \to e^{i\alpha}\psi$, $\vec{A} \to \vec{A} + \frac{1}{q}\vec{\nabla}\alpha$.

$$f[\psi, \vec{A}] = \frac{1}{2M}\left|(\vec{\nabla} - iq\vec{A})\psi\right|^2 + r|\psi|^2 + \frac{u}{2}|\psi|^4 + \frac{1}{2\mu_0}|\vec{\nabla}\times\vec{A}|^2 .$$

Equations of motion: $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$ and non-linear Schrödinger.

$\mu_0|\psi|^2 q^2/M$

$$\hookrightarrow = \frac{-iq}{2M}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) - \underbrace{\frac{q^2}{M}|\psi|^2\vec{A}}_{\text{this term appears in the ordered phase}}$$

!

and ultimately $\nabla^2\vec{B} = \lambda_L^{-2}\vec{B}$

↳ the Meissner effect ~ Debye screening ☺

3) Slightly more formally

Action of a charged superfluid in the ordered phase $\psi(\vec{r},t) = n_s e^{i\phi(\vec{r},t)}$:

require gauge invariance $\phi \to \phi + \alpha(\vec{r},t)$; $\vec{A} \to \vec{A} + \frac{1}{q}\vec{\nabla}\alpha$; $\varphi \to \varphi - \frac{1}{q}\partial_t\alpha$.

then $S = \int dt d^3r \, \frac{n_s}{2M}\left[(\dot{\phi} + q\varphi)^2 - (\vec{\nabla}\phi - q\vec{A})^2\right] + \frac{1}{2\mu_0}\left[\vec{E}^2 - \vec{B}^2\right]$

Now idea: absorb all fluctuations $\dot{\phi}, \vec{\nabla}\phi$ into $(\varphi, \vec{A})$. (does not change $\vec{E}, \vec{B}$)

$$\Rightarrow \quad S = \int dt d^3r \left\{ \underline{\frac{n_s q^2}{2M}(\varphi^2 - \vec{A}^2)} + \frac{1}{2\mu_0}\left[E^2 - B^2\right]\right\}$$

↳ now mass terms for φ and $\vec{A}$! this is Anderson-Higgs.

V. Really more formally ~ (but nothing really new)

Here I want to give the "full" picture, including condensation.

$$\mathcal{L} = (\partial^\mu \varphi^\dagger - iqA^\mu \varphi^\dagger)(\partial_\mu \varphi + iqA_\mu \varphi) + \mu^2 \varphi^\dagger \varphi - \lambda (\varphi^\dagger \varphi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

Gauge symmetry $\begin{cases} \varphi \to e^{i\alpha(x)} \varphi \\ A_\mu \to A_\mu - \frac{1}{q} \partial_\mu \alpha(x) \end{cases}$ (NB: can be generalized to nonabelian)

Write $\varphi(x) = \rho(x)\, e^{i\theta(x)}$ and replace $A_\mu + \frac{1}{q} \partial_\mu \theta =: C_\mu$

$$\Rightarrow \mathcal{L} = (\partial_\mu \rho)^2 + \rho^2 q^2 C_\mu C^\mu + \mu^2 \rho^2 - \lambda \rho^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

In the SB manifold $\rho_0 = \sqrt{\mu^2/2\lambda}$, and spontaneously chosen $\theta = 0$:
$(\rho = \rho_0 + \chi)$

$$\mathcal{L}_{SSB} = \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2 \chi^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \underbrace{\frac{q^2 \mu^2}{2\lambda} C_\mu C^\mu}_{} + \dots$$

→ mass term for the gauge field.

VI. Remark: counting degrees of freedom

1) The photon polarizations

The photon has spin $S = 1$ so in principle $S^z = +1, 0, -1$. "forbidden"! why?

Answer: first consider massive EM, $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu$.

Then there is 1 constraint, $\partial_\mu A^\mu = 0$, whence 3 copies of the KG equation.

Now massless: since $\cancel{m^2 A_\mu A^\mu}$ is no longer there to break gauge invariance, one can fix the gauge to remove one additional d.o.f, whence, 2 polarizations.

2) Before/after Anderson-Higgs

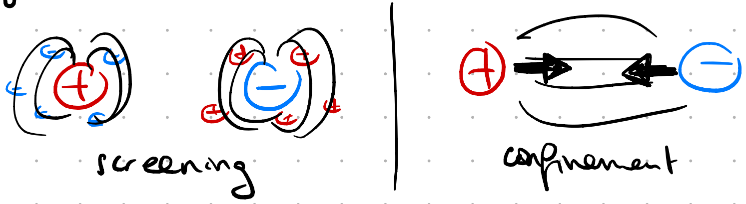
$\begin{cases} 2 \text{ massive scalars } \varphi^\dagger, \varphi \\ \text{massless photon: 2 polarizations} \end{cases} \xrightarrow{AH} \begin{cases} 1 \text{ massive scalar } \chi \\ \text{massive photon: 3 polarizations.} \end{cases}$

→ the composite object $C_\mu = A_\mu + \frac{1}{q} \partial_\mu \theta$ as the new massive photon.

VII. This is not the same as confinement!

True: in both cases, field lines become costly and there is a mass generated for the gauge field. But:

↳ Consequences for two test charges and the field lines:



↳ Confinement occurs for pure gauge theories (cf Polyakov '77 for $U(1)$) e.g. from the proliferation of instantons (which are pure g.t.).

$\neq$ Screening from Anderson-Higgs arises from interaction w/ matter.

(well, at least in the abelian case, see later).

VIII. Reminder about gauge theories.

- Assume we have a theory of fermions $f_\uparrow, f_\downarrow$ with Lagrangean $\mathcal{L}_f$, and we want to enforce the constraint $\sum_{\alpha=\uparrow\downarrow} f^\dagger_{r\alpha} f_{r\alpha} = 1 \quad \forall r, \forall t$.

$\Rightarrow$ Lagrange multiplier: $\mathcal{L}_{tot} = \mathcal{L}_f - a_0(r,t)\left[\sum_\alpha f^\dagger_{r\alpha} f_{r\alpha} - 1\right]$.

Indeed $\int [\mathcal{D}a_0(r,t)]\, e^{-\int \mathcal{L}_{tot}} \propto \prod_{r,t} \delta\left(\sum_\alpha f^\dagger_{r\alpha} f_{r\alpha} - 1\right)$.

→ The "dynamical potential" $a_0(r,t)$ enforces the constraint $\forall r,t$!

Now what does this have to do with a gauge theory?

- In $\mathcal{L}_f$, the term $J_{ij} f^\dagger_{i\alpha} f_{j\alpha} e^{-i a_{ij}}$ (i,j = positions, α = "spin")

is invariant under $f^\dagger_{i\alpha} \to f^\dagger_{i\alpha} e^{i\theta_i}$ provided that $a_{ij} \to a_{ij} + \theta_i - \theta_j$.

This is a $U(1)$ gauge theory all right.

Now look at a given state $f^\dagger_{1,\uparrow} f^\dagger_{2,\uparrow} f^\dagger_{2,\downarrow} f^\dagger_{3,\downarrow} \cdots f^\dagger_{N-1,\uparrow} f^\dagger_{N-1,\downarrow} f^\dagger_{N,\uparrow} |0\rangle$.

Which phase does it pick up under such a gauge transformation?

Answer: $\prod_i \exp\left(i\theta_i [Q_i + 1]\right)$ where $Q_i = \sum_\alpha f^\dagger_{\alpha i} f_{\alpha i} - 1$.

in other words: the local charge generates gauge transformations.

IX – Parton constructions, from $SU(2)$ to $U(1)$

- Representation of spins using Abrikosov fermions:

$$\underbrace{\vec{S}_r = \frac{1}{2} f^\dagger_{r\alpha} \vec{\tau}_{\alpha\beta} f_{r\beta}}_{\text{recovers the spin algebra}} \quad \text{and constraint} \quad \underbrace{\sum_{\alpha=\uparrow,\downarrow} f^\dagger_{r\alpha} f_{r\alpha} = 1 \quad \forall r.}_{\text{recovers the } S=\frac{1}{2} \text{ Hilbert space.}}$$

Redundancy in this description: $f^\dagger_\uparrow$ and $f_\downarrow$ mean the same thing.

$\to$ Doublet $\psi = \begin{pmatrix} f_\uparrow \\ f^\dagger_\downarrow \end{pmatrix}$. Idea: $\alpha f_\uparrow + \beta f^\dagger_\downarrow$ works as well.

i.e.: acting on ψ with an $SU(2)$ matrix preserves $\vec{S}_r$.

$\longrightarrow$ Gauge symmetry of the parton construction.

- Usually perform some sort of mean-field, with param $\underbrace{U_{ij}}_{}$ (2x2),

and $H_{MF} = \sum_{\langle ij \rangle} J_{ij} \left(\psi_i^\dagger \underbrace{\bar{U}_{ij}\, e^{i a^\ell_{ij} \tau^\ell}}_{\in SU(2)} \psi_j + h.c. \right) + a_0^\ell \psi_i^\dagger \tau^\ell \psi_i$.

This is a theory of fermions coupled to an $SU(2)$ gauge field.

Now, usually we don't have this but just $U(1)$. How come?

- Possible answer to the above: the $SU(2)$ g.f has become massive except for one $U(1)$ subgroup of it.

But is it possible to do Anderson-Higgs without a condensate?

- Idea: recall gauge charges are generators of g.t.

Indeed they do not commute trivially with the gauge field e^{iA}, and in the abelian case they alone have this property.

But in the nonabelian case, "e^{iA}" itself has nontrivial commutation rules with itself!

$\longrightarrow$ In the nonabelian case, there can be Anderson-Higgs without a condensate since different components of the gauge group are charged objects for each other!

X. Anderson-Higgs of pure $SU(2)$

- By definition, energy is gauge-invariant: $E[\{U_{ij}\}] = E[\{W_i U_{ij} W_j^+\}]$; $U_{ij} = \bar{U}_{ij} e^{i a_{ij}^\ell \tau^\ell}$; $W \in SU(2)$

Question: are mass terms $\propto (a_{ij}^\ell)^2$ allowed by gauge invariance?

To answer it, consider $P_{C_i} = \bar{U}_{ij}\bar{U}_{jk}\dots\bar{U}_{\ell i}$; i, C_i

And $P_{C_i} = \chi_{C_i}^0 \tau^0 + i\chi_{C_i}^\ell \tau^\ell$ is charged: $P_{C_i} \to W_i P_{C_i} W_i^+$.

- Now consider the effect of g.t. $e^{i a_{ij}^\ell \tau^\ell} \to e^{i\theta_i^\ell \tau^\ell} e^{i a_{ij}^\ell \tau^\ell} e^{-i\theta_j^\ell \tau^\ell}$.

$\Rightarrow$ Several phases depending on the flux of our MF ansatz $\bar{U}_{ij}$:

↳ Trivial flux $P_{C_i} \propto \tau^0$. (and $\bar{U}_{k\ell} \propto \tau^0$ up to free g.t.)

Then $SU(2)$ is preserved, and $a_{ij}^\ell \to a_{ij}^\ell + \theta_i^\ell - \theta_j^\ell$ (at least) $\forall \ell = 1,2,3$

so the mass term $(a_{ij}^\ell)^2$ is not allowed. $\forall \ell = 1,2,3$. ↳ we could mix them

↳ Collinear flux: say $\bar{U}_{k\ell} = i e^{i\varphi_{k\ell}\tau^3}$ and $P_{C_i} = \chi_{C_i}^0 \tau^0 + \chi_{C_i}^3 \tau^3$.

Then $SU(2)$ is broken down to $U(1)$ and $a_{ij}^3 \to a_{ij}^3 + \theta_i^3 - \theta_j^3$

is the only remaining gauge invariance. So $(a_{ij}^3)^2$ is not allowed,

but the two other degrees of freedom $a_{ij}^{1,2}$ are now gapped!

↳ Noncollinear: then all bosons are gapped $\to$ discrete g.t. (?)

- NB: above we have just said that the $(a_{ij}^3)^2$ term is not allowed, while no such statement could be made about $(a_{ij}^{1,2})^2$.

Can we be more explicit and show a mass term for $a_{ij}^{1,2}$?

$\Leftrightarrow$ a term invariant under $SU(2)$ whose quadratic order is $\sim (a_{ij}^{1,2})^2$.

↳ It turns out that $E = \kappa \operatorname{Tr}[P_{C_i} U_{i,i+x} P_{C_{i+x}} U_{i+x,i}]$ works.

$U_{i,i+x} \to \underbrace{\chi e^{i\phi_x \tau^3}}_{=\bar{U}_{ij}} e^{i a_x^\ell \tau^\ell}$

At quadratic order, $E = \frac{\kappa}{2}(-\chi^2)\operatorname{Tr}\left([P_{C_i}, a_x^\ell \tau^\ell]^2\right) + \dots$

↳ if $\propto (\tau^0, \tau^3)$ then $E \sim (a_x^{1,2})^2$.

XI – An "application" (→ disputable!): the Peierls mechanism → polyacetylene.

Basically a 1d molecule with right-movers, left-movers, and the hopping amplitude depends on the distance. elastic energy $\propto \Delta(x)^2$.

→ Role of staggered displacement $\Delta_n = (-1)^n u_n \to \widehat{\Delta(x)}$.

$$\left[\mathcal{L} = i\psi^\dagger(\partial_t + v_F \partial_x \hat{\sigma}^z)\psi - \frac{1}{2}\Delta^2 - \alpha\, \Delta(x)\, \psi^\dagger \hat{\sigma}^x \psi \right] \quad \psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

Now if we give a $U(1)$ charge $+1$ to R and -1 to L, we see that $\Delta(x)$ is a charge-2 field. (NB this is not the usual way to see it).

And it turns out that $\Delta(x)$ condenses $\to \bar{\Delta}$. (integrate out the ψ's etc).

Consequence of "Anderson-Higgs": the "gauge field" (here, the fermions) acquires a mass → Here it means $H = i v_F \psi^\dagger \partial_x \hat{\sigma}^z \psi + \underbrace{\alpha \bar{\Delta}\, \psi^\dagger \hat{\sigma}^x \psi}_{\text{mass term}}$.

XII – Another application: heavy fermions

Basic idea: local moments $\Uparrow \Downarrow$ interact with itinerant $\bar{e}$s $\uparrow \downarrow \uparrow$ to form mobile (but heavy) fermions $\Uparrow\downarrow$.
↳ enlarged FS (cf talk about Luttinger's theorem).

Declare $\Uparrow = f^\dagger \vec{\sigma} f$ and $\uparrow = c^\dagger \vec{\sigma} c$. There are 2 fermion species.

The $c^\dagger$ fermions are charged under usual EM: $c^\dagger \to c^\dagger e^{-i\alpha}$, $A_\mu \to A_\mu + (\vec{\nabla}\alpha)\frac{1}{e}$.

The $f^\dagger$ fermions, under an emergent EM: $f^\dagger \to f^\dagger e^{-i\chi}$, $\mathcal{A}_\mu \to \mathcal{A}_\mu + \vec{\nabla}\chi$.
(because their number is fixed)

Condensation of the hybridization $V = \langle f^\dagger c \rangle$ charged under $\begin{cases} \vec{\mathcal{A}} - e\vec{A} \\ \lambda - e\Phi \end{cases}$ i.e. $\mathcal{A}_\mu - eA_\mu$.
→ the latter becomes massive!

The action for ϕ, the phase of the hybridization V, is now:

$$S_{eff} = \int d^3x\, d\tau \left\{ \frac{\rho_s}{2}(\vec{\nabla}\phi - \vec{\mathcal{A}} + e\vec{A})^2 - \frac{\chi_c}{2}(i\partial_\tau \phi + \lambda - e\Phi)^2 \right\}.$$

The "Meissner effect" here is: $\begin{cases} \vec{\mathcal{A}} = e\vec{A} \\ \lambda = e\Phi \end{cases}$ ⇝ the 2 g.f's are locked together.

⇒ The f fermions have become charged under usual EM.
Besides, there is now only 1 surviving $U(1)$ gauge theory.
And all the fermions are charged under it → enlarged FS.