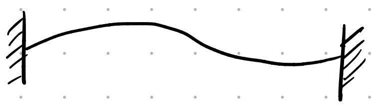
[A few surprising applications of number theory in condensed matter physics; at a very basic level because the speaker is very ignorant actually.]

I. The Casimir effect (that one is quite popular)

- Here let's just consider the $d=1$ version of the calculation.



$$\mathcal{L}_{\text{elasticity}} = \int_0^L dx \left[\frac{\mu}{2}(\partial_t y)^2 - \frac{K}{2}(\partial_x y)^2 \right]$$

$$\Rightarrow \quad \hat{H} = \sum_{n=1}^{\infty} \omega n \left(a_n^\dagger a_n + \frac{1}{2}\right). \qquad \text{where } \omega = \frac{\pi}{L}\sqrt{\frac{K}{\mu}}.$$

Ground state energy: $E_L = \frac{1}{2}\omega \sum_{n=1}^{\infty} n = \;?$ Problem.

- Mathematical solutions:

↳ Pedestrian one:

$$E_L = \frac{\omega}{2} \lim_{\alpha \to 0} \left(\sum_{n=1}^{\infty} n e^{-\alpha n} = -\frac{\partial}{\partial \alpha} \sum_{n=1}^{\infty} e^{-\alpha n} \right.$$
$$\left. = -\frac{\partial}{\partial \alpha} \frac{1}{1-e^{-\alpha}} = \frac{1}{\alpha^2} - \frac{1}{12} + O(\alpha) \right) = \;?$$

Then leap of faith: this $\frac{1}{\alpha^2}$ depends on the regularization α, therefore it's not physical. Let's keep this $-\frac{1}{12}$ which looks nice.

↳ "Official" one: zeta-regularization.

Def: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $\mathrm{Re}(s) > 1$.

Its analytic continuation to $\mathbb{C}$ is unique (I guess) and $\tilde{\zeta}(-1) = -\frac{1}{12}$.

Argument by Hawking (1977): « The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation. » Looks convincing but I don't understand.

↳ Physical (?) one: here we are summing the vacuum term $\frac{1}{2}$ for all <u>discrete</u> modes of the box, with $k_n = \frac{\pi n}{L}$ and $c = \sqrt{\frac{R}{\mu}}$.
Now, outside of the box, there are modes too, but they are continuous.

$$\Longrightarrow \quad \mathcal{E}_{\text{outside}} = \int_0^\infty d\nu \, \pi \nu c \frac{1}{2}.$$ which obviously is infinite.

However $\mathcal{E}_L - \mathcal{E}_{\text{outside}} = \sum_{n=0}^\infty \frac{1}{2}\pi c n - \int_0^\infty d\nu \frac{1}{2}\pi c \nu$ is finite.

Use the Euler-MacLaurin formula:

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + \underbrace{\sum_{p=1}^{\infty} \frac{B_p}{p!}\left(f^{(p)}(b) - f^{(p)}(a)\right)}.$$
$$= \frac{f(b)+f(a)}{2} + \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!}\left(f^{(2p-1)}(b) - f^{(2p-1)}(a)\right).$$

Here we have to assume an unspecified regularization $f(\infty) \stackrel{!}{=} 0$. (and all its derivatives)

$$\Rightarrow \Delta \mathcal{E} = -\frac{B_2}{2!}\frac{\pi c}{2} = -\frac{1}{12}\frac{\pi c}{2}.$$

- <u>Note</u>: $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}$, $B_6 = \frac{1}{42}$, etc, $B_{2p+1} \stackrel{p>0}{=} 0$.

are the Bernoulli numbers.

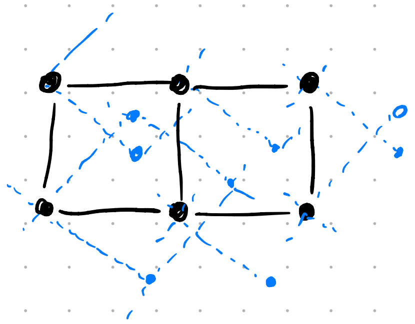
They verify $B_n = -n\zeta(1-n)$ and $B_{2n} = \frac{(-1)^{n+1} 2 (2n)!}{(2\pi)^{2n}} \zeta(2n)$.

That's nice but also quite cryptic.

II. <u>Quasicrystals</u> (there will be number theory at the end!).

(One possible) <u>def</u>: infinite set of points with dihedral D_n symmetry, which also has <u>translational symmetry</u> (in the sense that from every point it looks the same) so it has a (semi) group structure and can be called a <u>lattice</u>, but it's not a crystalline lattice because it does not have a finite unit cell that repeats itself.

Ex: take a square lattice, and at each point, paste another square lattice that is tilted at 45° (with origins that coincide).



This set is dense in the plane.

(Indeed a subgroup of $(\mathbb{R},+)$ is either dense or $\alpha\mathbb{Z}$, and $\mathbb{Z}+\frac{\mathbb{Z}}{\sqrt{2}}$ cannot be an $\alpha\mathbb{Z}$ because $\sqrt{2}$ is irrational.)

So there is no sense of a unit cell.

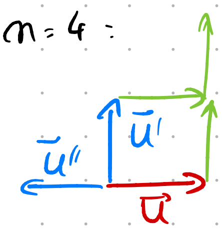
- Crystallographic restriction theorem: the only dihedral symmetries that a crystalline lattice can have are D_2, D_3, D_4 and D_6.

Proof: Since the lattice is periodic, with a finite unit cell, there is a lattice vector of minimal length, let's call it $\vec{u}$.

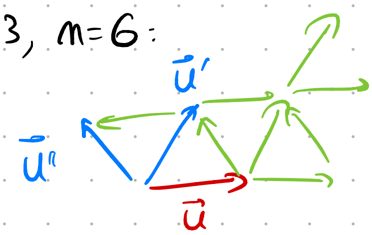
Then, all the $\vec{u}' = \hat{g}\vec{u}$, for $\hat{g} \in D_n$, are lattice vectors as well.

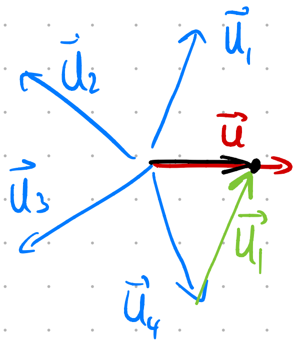
So let's see what this looks like:

$n=2$, $n=4$:



$n=3$, $n=6$:





These are the cases that work. And now e.g $n=5$:

The black vector belongs to the lattice, however it is shorter than $\vec{u}$. Contradiction.

(And for $n \geq 7$, it's even more obvious: $\|\vec{u}_1 - \vec{u}\| < \|\vec{u}\|$ clearly).

$\longrightarrow$ Structures that have D_N symmetry for $N \in \{5\} \cup [\![7, +\infty[\![$ are quasicrystals!

(In fact, an alternative definition requires the lattice to have a translation group instead of semigroup, therefore N is even).

[Now, question: for a given (even) N, how many different possibilities?]

Here use a more restrictive def of a quasicrystalline N-lattice:
We are looking for a structure that is stable by addition and N-fold rotation, and ⚠ such that there is a "generating vector" $\vec{w}$ such that any lattice vector $\vec{z}$ can be decomposed as
$\vec{z} = \sum_{p=1}^{N} c_p R_N^p \cdot \vec{w}$ with $c_p \in \mathbb{N}$ and R_N^p = rotation by $\frac{2\pi p}{N}$.

⚠ The case where all such combinations belong to the lattice, i.e. $\mathbb{Z}_N \stackrel{\text{def}}{=} \{ \sum_{p=1}^{N} c_p R_N^p \cdot \vec{w} \mid c_p \in \mathbb{N} \}$, is the "trivial" N-lattice.
The example I gave in the previous page is the $N=8$ trivial case.
→ Are there other solutions? (and how many?) → $S_N \subset \mathbb{Z}_N$?

Idea: describe vectors in the plane as complex numbers.
The rotation $e^{2i\pi/N} \equiv \xi$. In this language, $\mathbb{Z}_N = \underbrace{\mathbb{Z}[\xi]}$
integer-coeff' polynomials of ξ
We want $\mathbb{Z}_N \cdot S_N \stackrel{!}{=} S_N \Rightarrow S_N$ is an ideal of $\mathbb{Z}_N$.
Now, for any $z \in \mathbb{Z}_N$, the set $z\mathbb{Z}_N$ is an ideal; but it's just a deformation of $\mathbb{Z}_N$. This is called a principle ideal and we want to ignore them.

So the problem is: $\boxed{\text{how many non-principal ideals of } \mathbb{Z}_N \text{ are there?}}$

(⚠ Again, if there are $\alpha, \beta \in \mathbb{Z}_N$ such that $\alpha S_N^{(1)} = \beta S_N^{(2)}$ then $S_N^{(1)}$ and $S_N^{(2)}$ are equivalent ideals, we have to quotient.)

Thus *the number of distinct classes of reciprocal lattices with N-fold symmetry is the number of distinct classes of equivalent ideals in* $\mathbf{Z}_N$. *This number,* h_N, *is called the class number* of the cyclotomic field $\mathbf{Q}_N$, and has been and continues to be the object of much computational effort.

Beware of 46-Fold Symmetry: The Classification of Two-Dimensional Quasicrystallographic Lattices

N. David Mermin and Daniel S. Rokhsar
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853

and

David C. Wright
David Rittenhouse Laboratory, University of Pennsylvania, Philadelphia, Pennsylvania 19104
(Received 24 December 1986)

(3 pages)

A short, clear, funny paper with a very provocative style.

To answer to whether a general N-lattice is equivalent to $\mathbf{Z}_N$ for arbitrary N is this: *Overwhelmingly no; but for all practical purposes, yes.* There are only 29 even values of N for which there is a single class of N-lattices; but among these are all values of N up to including $N=44$.

The only cases in which there are just two classes of N-lattices are $N=56$ and $N=78$.[8] There are just three classes only for $N=46$, 52, and 72.[9] When there is not a unique N-lattice, things can be rather bizarre. The num-

So $N=46$ is the first nontrivial case. The two nonequivalent nonprincipal ideals of $\mathbb{Z}_{46}$ are

$$2\mathbb{Z}_{46}+\beta^{(*)}\mathbb{Z}_{46}$$

where $\beta=\frac{1}{2}(1+i\sqrt{23})$.

One has to check that $\beta\in\mathbb{Z}_{46}=\mathbb{Z}[\zeta]$ with $\zeta=e^{2i\pi/46}$.

It turns out that $-\beta=\zeta^{10}+\zeta^{14}+\zeta^{20}+\zeta^{22}+\zeta^{28}+\zeta^{30}+\zeta^{34}+\zeta^{38}+\zeta^{40}+\zeta^{42}+\zeta^{44}$,

which can easily be checked numerically, or given an elementary analytical proof[16] (which, however, this paper is too short to contain.)

[16]We are grateful to Keith Dennis for showing us how to do this.

The reader who cannot enjoy the proof can at least enjoy the joke ☺

In general the situation is quite horrendous. Although the number is finite for any N,[11] even for as "reasonable" a number as 128, there are 359057 distinct N-lattices. There are more than a hundred million distinct 158-lattices, more than ten billion distinct 178-lattices, and h_N grows astronomically as N gets still higher.

So «overwhelmingly» b/c:
- only a finite (small) set of integers N have $h_N=1$.
- other Ns have h_N very large.

[11]Ian Stewart and David Tall, *Algebraic Number Theory* (Chapman and Hall, London, 1979), Theorem 9.7, p. 165.

[12]See Harold M. Edwards, *Fermat's Last Theorem* (Springer-Verlag, New York, 1977), for a delightful historically oriented introduction to cyclotomic fields.

Also a few nice comments:

The tacit assumption that there is only one class of N-lattices for general N was the only fallacy in a sensational "proof" of Fermat's last theorem, announced by Lamé and avidly pursued by Cauchy in 1847. It was Kummer who discovered the multiplicity of 46-lattices, dashing cold water on these hopes.[12]

It is splendid and remarkable that this enormous but (for physicists) arcane branch of number theory, developed in an effort to prove Fermat's last theorem, should contain precisely the structures needed to formulate and answer one of the most fundamental crystallographic questions raised by the discovery of quasicrystals.

Also a cute footnote at the end of the preface of a book:

* For the second printing, which we prepared in May 1995, we have corrected a few misprints, and added some references to Chapters 3 and 12. The only significant modification of the text takes place in Chapter 3, page 234, and reports on the recent progress made by Wiles on the Taniyama-Weil conjecture, which provides a proof of Fermat's celebrated "Last Theorem".

III. Elliptic functions and bosonization

- Bosonization in $d=1$ (from S. Sachdev's books)

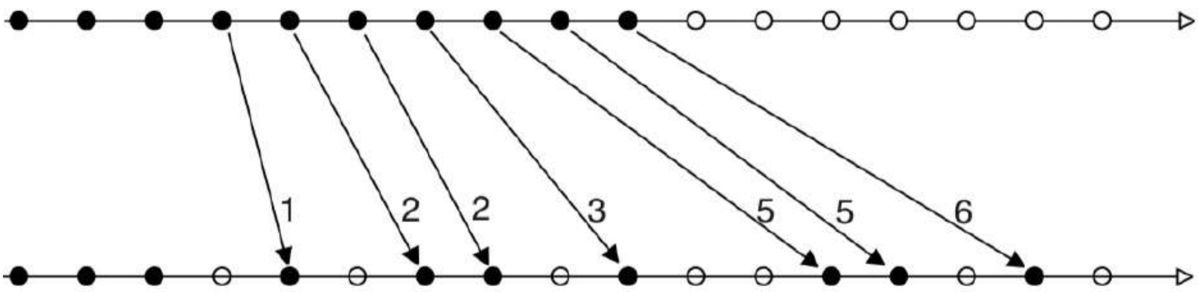
↳ Start from a theory for right-handed fermions: $\quad H_F = -i v_F \int dx\, \psi^\dagger(x)\partial_x \psi(x)$.

Expand in Fourier modes, $\quad \psi(x) = \frac{1}{\sqrt{L}} \sum_{m=-\infty}^{+\infty} \psi(x)\, e^{i(2m-1)\pi x/L}$ (choose antiperiodic boundary conditions)
with $\{\psi_m, \psi^\dagger_{m'}\} = \delta_{mm'}$.

$$\Rightarrow H_F = \frac{\pi v_F}{L} \sum_{m=-\infty}^{+\infty} (2m-1)\, \psi_m^\dagger \psi_m \quad + \text{cst}.$$

The partition function is $\quad Z_F = \prod_{m=1}^{\infty} (1+ q^{2m-1})^2 \quad , \quad q = e^{-\frac{\pi v_F}{LT}}$

↳ One can bosonize: (wrt the filled FS)
A state with charge Q has energy $\frac{\pi v_F}{L} \sum_{m=1}^{|Q|} (2m-1) = \frac{\pi v_F}{L} Q^2$.



Then generate all possible states with bosonic particle-hole excitations:

$$H_B = \frac{\pi v_F Q^2}{L} + \frac{2\pi v_F}{L} \sum_{m=1}^{\infty} m\, b_m^\dagger b_m \qquad \text{for a given } Q \text{ sector.}$$

Then the partition function is $\quad Z_B = \left(\prod_{m=1}^{\infty} \frac{1}{1-q^{2m}} \right) \sum_{Q=-\infty}^{+\infty} q^{(Q^2)}$.

↳ Puzzle: physically we want $\boxed{Z_B = Z_F}$. Is it true?

One can check the first terms (for an arbitrary $q \in \mathbb{C}$ in fact):

$$(1+q)^2(1-q^2)(1+q^3)^2(1-q^4)(1+q^5)^2(1-q^6)\ldots = 1 + 2q + 2q^4 + \ldots$$

Incroyable ! On dirait de la magie !

- **Solution**: this is a major result in analytic number theory. It is known as the Jacobi triple product; according to Wikipedia it's a bit more general: for two complex numbers x and y,

$$\boxed{\prod_{m=1}^{\infty}(1-x^{2m})(1+x^{2m-1}y^{2})(1+x^{2m-1}y^{-2}) = \sum_{n=-\infty}^{+\infty} x^{(n^2)}y^{2n}}$$

(and so for our problem we just need the case $x=q$, $y=1$).

A very nice elementary proof in a 2-page paper:

SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A SIMPLE PROOF OF JACOBI'S TRIPLE PRODUCT IDENTITY

GEORGE E. ANDREWS

Bellman remarks in [1, p. 42] that there are no simple proofs known of the complete triple product identity

However the two identities of Euler,

are rather easily established [1, p. 49]. It does not seem to have been noticed that Jacobi's triple product identity follows simply from Euler's identities.

1. R. Bellman, *A brief introduction to theta functions*, Holt, Rinehart and Winston, New York, 1961.

- **Jacobi θ function**: $\Theta(z;\tau) = \sum_{n=-\infty}^{+\infty} e^{i\pi n^2\tau + 2i\pi n z}$ $\quad z,\tau\in\mathbb{C}$.

It has all sorts of fascinating properties which I don't understand.

$$\left.\begin{aligned}\theta_{01}(z,\tau) &= \theta(z+\tfrac{1}{2},\tau)\\ \theta_{10}(z,\tau) &= e^{i\pi(z+\tau/4)}\theta(z+\tau/2,\tau)\end{aligned}\right\}\quad \theta_{01}(0,\tau)^4+\theta_{10}(0,\tau)^4 = \theta(0,\tau)^4.$$

This is the Jacobi identity, which defines the Fermat curve of degree 4, and more generally these auxiliary θ functions are useful to build modular forms.

I'm just repeating what Wikipedia told me, obviously.

Stay tuned! We're not done with those yet.

IV. The Poisson summation formula

- It's a formula of Fourier analysis (and distribution theory): for any function $g(t)$, whose FT is $\tilde{g}(\nu) = \int_{-\infty}^{+\infty} dt\, g(t)\, e^{-i2\pi\nu t}$, one has: $\left[\sum_{m=-\infty}^{+\infty} g(m) = \sum_{k=-\infty}^{+\infty} \tilde{g}(k) \right]$.

↳ In signal processing, it shows directly that the FT of the Dirac comb is the Dirac comb itself.

↳ Remember the thêta function? $\vartheta(\tau) \stackrel{\text{def}}{=} \theta(0,\tau) = \sum_{m=-\infty}^{+\infty} e^{i\pi\tau m^2}$.

Choose $g(x) = e^{-\pi x^2}$ whose FT is $\tilde{g}(y) = e^{-\pi y^2}$,

and the Poisson formula yields: $\left[\vartheta(-\tfrac{1}{\alpha}) = \sqrt{-i\alpha}\,\vartheta(\alpha) \right]$ $\alpha \in \mathbb{C}$.

which tells us that ϑ is a modular form or sthg like that.

More explicitly: $\sum_{m=-\infty}^{+\infty} \exp\left(-\frac{m^2}{2\beta} + im\varphi\right) = \sqrt{2\pi\beta} \sum_{p=-\infty}^{+\infty} \exp\left(-\frac{\beta}{2}(\varphi - 2\pi p)^2\right)$.

(also use $z \mapsto z+1$ periodicity)

- Application: classical XY chain a.k.a quantum rotor (from S. Sachdev).

↳ Starting point: $\hat{H} = -K \sum_{\ell=1}^{M} \underbrace{\hat{n}_\ell}_{\text{(rotor with angle } \varphi_\ell)} \hat{n}_{\ell+1} \xrightarrow{\text{continuum}} \hat{H}[\varphi] = \int_0^L d\tau\, \frac{\xi}{4}\left(\frac{d\varphi}{d\tau}\right)^2$. $(\xi = 2Ka)$

To compute $\int \mathcal{D}\varphi$, idea: sum over topological sectors: for every $\varphi(\tau)$ there is $p \in \mathbb{Z}$ such that $\varphi(\tau) = \frac{2\pi p \tau}{L} + \underbrace{\tilde{\varphi}(\tau)}_{\text{"spin wave"}}$ $\tilde{\varphi}(L) = \tilde{\varphi}(0)$.

$$\Rightarrow Z_{XY} = \sum_{p=-\infty}^{+\infty} \int_{\text{sector } p} \mathcal{D}\varphi\, e^{-H[\varphi]} = \underbrace{\sum_{p=-\infty}^{+\infty} e^{-\pi^2 p^2 \xi/L}}_{= \vartheta(\pi\xi/(Li))} \underbrace{\int \mathcal{D}\tilde{\varphi}\, \exp\left(-\frac{\xi}{4}\int_0^L d\tau \left(\frac{d\tilde{\varphi}}{d\tau}\right)^2\right)}_{\text{well-known path integral } = 2\pi\sqrt{\xi/4\pi L}}$$

↳ Now take a quantum rotor, $\hat{H} = -\Delta\, \partial^2/\partial\varphi^2$.

Solve $\hat{H}\Psi = E\Psi$: eigenstates $\Psi_m(\varphi) = e^{im\varphi}$ $m \in \mathbb{Z}$, energies $E_m = \Delta m^2$.

Thus $Z_{QR} = \sum_{m=-\infty}^{+\infty} e^{-\Delta m^2/T} = \vartheta(\Delta/\pi T i)$.

↳ Quantum-classical correspondence: identifying $\boxed{\frac{L}{\xi} = \frac{\Delta}{T}}$ we expect $Z_{XY} = Z_{QR}$.

This is true! Thanks to the "duality" formula $\vartheta(-\tfrac{1}{\alpha}) = \sqrt{-i\alpha}\,\vartheta(\alpha)$.

• <u>Similar but more involved:</u> for dimer gases (cf Fradkin's book) and more generally for lattice field theories.

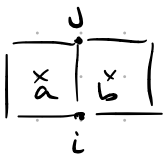
(<u>Refs</u>: from Chaikin & Lubensky and Itzykson & Drouffe – neither very clear)

Partition function of the 2d XY model: $\left[Z_{XY} = \int \prod_i \frac{d\varphi_i}{2\pi} \, e^{\beta \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j)} \right]$

Idea: aproximation $e^{\beta \cos(\varphi)} \simeq \sum_{m=-\infty}^{+\infty} \exp\left(-\beta \frac{1}{2} (\varphi - 2\pi m)^2\right)$.

and using the above explicit formula we obtain the Villain action:

$$Z_V = \int \prod_i \frac{d\varphi_i}{2\pi} \prod_{\langle ij \rangle} \sum_{m_{ij}=-\infty}^{+\infty} \exp\left(-\frac{m_{ij}^2}{2\beta} + i m_{ij} \varphi_{ij}\right) \stackrel{\int}{=} \sum_{\{m_{ij}\}, (\partial m)_i = 0} e^{-\frac{1}{2\beta} \sum_{\langle ij \rangle} m_{ij}^2}.$$

Finally  $m_{ij} \stackrel{\text{def}}{=} m_a - m_b = -m_{ji}$ (divergenceless ✓) yields $\sum_{\{m_a\}} \exp\left(-\frac{1}{2\beta} \sum_{\langle ab \rangle} (m_a - m_b)^2\right)$

$a \in$ dual <u>lattice</u>

→ The Poisson summation formula has been used to turn the XY model on a lattice into a gaussian model on the <u>dual</u> lattice.

Then, I'm not sure why we couldn't have done this all earlier, but people recommend to apply the Poisson formula a 2nd time.

One then gets: $Z = \sum_{\{m\}} \int \mathcal{D}\varphi \, \exp\left(-\frac{1}{2\beta} \sum_{\langle ab \rangle} (\varphi_a - \varphi_b)^2\right) \exp\left(2i\pi \sum_a m_a \varphi_a\right)$.

Then introduce the Green's function,

$G(r) = \int \frac{d^2q}{(2\pi)^2} \frac{e^{iqr}}{q^2}$, to get: $Z = \sum_{\{m\}} e^{-2\pi^2\beta \sum_{\langle ab \rangle} m_a G(r_{ab}) m_b}$

and $G(R) \xrightarrow[R\to\infty]{} 2\pi \ln(R/a)$ is the Coulomb potential in 2d.

To summarize:

$$\left[XY \xrightarrow[\text{low-T}]{\text{approx}} \text{Villain} \xrightarrow[\text{dual}]{\text{PSF}} \text{Gaussian} \xrightarrow[\text{Gauss} \int]{\text{PSF}} \text{Coulomb gas} \right]$$

One can understand the m variable as a vortex charge, I guess.

To be honest, we've made a few detours but perhaps it was useful because all these reasonings are pervasive in the literature about lattice gauge theories etc in the '80s and '90s.

<u>Morality</u>: the Poisson formula as a way to obtain dualities.