Introduction to Fixed Income Valuation

Abstract

12a	Calculate and interpret price, income and cross-price elasticities of demand and describe factors that affect each measure
12b	Compare substitution and income effects
12c	Distinguish between normal goods and inferior goods
12d	Describe the phenomenon of diminishing marginal returns
12e	Determine and interpret breakeven and shutdown points of production
12f	Describe how economies of scale and diseconomies of scale affect costs

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1. Bonds Price and the Time Value of Money

1.1 Bond Pricing with a Market Discount Rate

The price of a bond is the present value of the promised cash flows. The **market discount rate** is the rate corresponding to the internal rate of return used in order to obtain that present value. The market discount rate is considered the **required yield** or **required rate of return** by the investors. A bond with time-to-maturity is five years and the market discount rate is 6% the price of the bond is 91.575 per 100 of par value. A bond is trading at **discount** if its market price is below par value and is said to be trading at **premium** if its price is above par value:

- When the coupon-rate is less than the market discount rate, the bond is *priced at discount below par value*
- When the coupon-rate is more than the market discount rate the bond is *trading at premium above par value*
- When the coupon-rate is equal to the market discount rate, the bond is *priced at par value*

The general formula for calculating a bond price given the market discount rate is:

$$PV = \frac{PMT}{(1+r)^1} + \frac{PMT}{(1+r)^2} + ... + \frac{PMT+FV}{(1+r)^n} \tag{1}$$

where,
PV = present value, or the market value of the bond (bond's price)
PMT = coupon payment per period (payment per period)
FV = par value of the bond (face value)
n = number of periods to maturity

1.2 Yield-to-Maturity

The **yield-to-maturity** is the internal rate of return of the cash flows or the implied market discount rate. It has three fundamental assumptions:

- The investor holds the bond to maturity

- The issuer makes all coupon and principal payments on time
- The investor is able to reinvest coupon payments at the same yield (this is a characteristeric of an internal rate of return)

For example, suppose that a four-year, 5% annual coupon payment bond is priced at 105 per 100 of par value, the yield-to-maturity is the solution for the rate r:

$$105 = \frac{5}{(1+r)^1} + \frac{5}{(1+r)^2} + \frac{5}{(1+r)^3} + \frac{105}{(1+r)^4} \quad (2)$$

The yield-to-maturity would be r=0.03634 (3.364%).

1.3 Bond Characteristics

- The bond price is inversely related to market discount rate. The more market discount rate increases, the more bond price decreases
- For the same time-to-maturity, a lower coupon bond has greater percentage price change than a higher coupon bond when their market discount rates change by the same amount (coupon effect)
- For the same coupon rate, a longer-term bond has greater percentage price change than a shorter-term bond when their discount rates change for the same amount. (maturity effect)

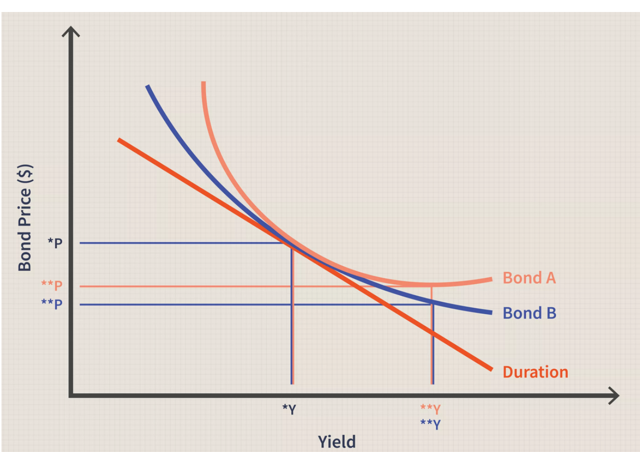
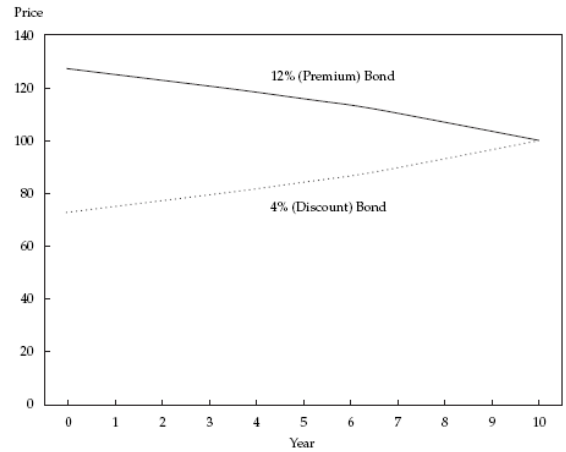


Figure 1. Bond convexity (convexity effect)

Constant-yield price trajectory This illustrates the change in the price of fixed-income bonds over time. The trajectory of the price approaches towards the par value as it's tenor approaches to zero.



Discount Bond	73.160	75.012	77.013	79.175	81.508	84.029	86.751	89.692	92.867	96.296	100.00
Premium Bond	126.84	124.98	122.98	120.82	118.49	115.97	113.24	110.30	107.13	103.70	100.00

Figure 2. Constant-Yield Price Trajectories

1.4 Spot rates

Spot rates are yield-to-maturity on zero-coupon bonds maturing at the date of each cash flow. That is, the yield-to-maturity at each date of coupon payment. The spot rate of each period should comprehensibly be different given the change in conditions affecting the bond pricing such as interest rates or inflation expectations.

$$PV = \frac{PMT}{(1+Z_1)^1} + \frac{PMT}{(1+Z_2)^2} + \ldots + \frac{PMT+FV}{(1+Z_N)^n} \quad (3)$$

where,
Z_1 = spot rate for period 1
Z_2 = spot rate for period 2
Z_N = spot rate for period N

2. Conventions for Quotes and Calculations

2.1 Flat Price, Accrued Interest and the Full Price

The price of bond between coupon payment dates has two parts: the **flat price** (PV^{flat}) and the **accrued interest** (AI). The sum of those parts is the **full price** (PV^{full}).

$$PV^{full} = PV^{flat} + AI \tag{4}$$

The price quoted is the flat price because if the full price were quoted, this could be misleading to investors as the price would increase day after day even if the yield-to-maturity didn't change - this would be due to the accumulation of accrued interest.
The accrued interest is the proportional share of the next coupon. Accrued interest is not affected by the market discount rate, but flat price is.

$$AI = \frac{t}{T} PMT \tag{5}$$

where,
T = days between payment dates (coupon period)
t = days gone by since the last coupon payment

The day to day conventions vary from market to market. The full price is given by:

$$PV^{full} = PV(1+r)^{\frac{t}{T}} \tag{6}$$

2.2 Matrix Pricing

Matrix pricing is useful to price bonds that are not actively traded. Because there is no market price available to calculate the rate of return required by investors, investors estimate the market discount rate and price based on the quoted prices of traded comparable bonds. These bonds have similar tenors (time to maturity), coupon rates and credit quality.

	2% Coupon	3% Coupon	4% Coupon	5% Coupon
Two Years		98.500		102.250
		3.786%		3.821%
Three Years			Bond X	
Four Years				
Five Years	90.250		99.125	
	4.181%		4.196%	

Figure 3. Matrix Pricing Example

The estimated bond prices can be obtain through a linear interpolation from the reference values.

Underwriting Matrix pricing can also be used in underwriting new bonds to get an estimate of the *required yield spread* over the *benchmark rate*. The benchmark rate is usually the YTM on a government bond with similar time-to-maturity. This spread can also be called **spread over the benchmark**.

2.3 Annual Yields for Varying Compounding Periods

There are different ways to measure the yield. However, investors want a yield measure that is standardized to allow for comparison between bonds of different tenors.
The annualized and compounded yield on a fixed-rate bond depends on the assumed number of periods of capitalization in the year, which is called the **periodicity** of the annual rate. Hence, a bond that pays on a semiannual basis has a periodicity of two; a bond that pays quarterly has a periodicity of four, etc. The **effective annual rate** has the periodicity of one because it just compounds once per year.
The most common periodicity for USD-denominated bond yields is two because the bonds in US markets usually make semiannual coupon payments. Therefore, it is important to be able to convert an annual yield from one periodicity to another.
The general formula to convert annual percentage rate for m periods in a year (APR_m) to an annual percentage for n periods per year APR_n is:

$$(1+\frac{APR_m{}^{m}}{m}) = (1+\frac{APR_n{}^{n}}{n}) \tag{7}$$

2.4 Yield Measures for Fixed-Rate Bonds

Yield measures that neglect weekends are called **street convention**. The true yield is the YTM on the cash flow that considers both weekends and bank holidays. Because weekends and holidays delay the time to payment, the true yield can never be higher than the street convention yield.
A **government equivalent yield** is a quote for corporate bonds that restates YTM based on a 30/360 day-count (30 days per month and 360 days a year). The **current yield** or running yield is the sum of the payments received within a year divided by the flat price.

2.4.1 Bonds with embedded options

If a bond contains an embedded option, other yield measures are required. For example, the *callable bond* contains an embedded call option that gives the issuer the right to buy back the bond from the investor at a specified price after the period of *call protection*.
Therefore, the outcome of the bond is not fixed anymore as it opens new possibilities, like the recalling of the bonds prematurely. In this scenario, the investor should calculate the yields for every periods after the call protection period. That is, *the yield to first call*, *the yield to second call*, ... until the *yield to maturity*. The lowest of the sequence is called **yield-to-worst** and is the most conservative assumption for the rate of return.

$$105 = \frac{8}{(1+r)^1} + \frac{8}{(1+r)^2} + \frac{8}{(1+r)^3} + \frac{8+102}{(1+r)^4}, \quad r = 0.06975$$

The yield-to-second-call in five years is 6.956%.

$$105 = \frac{8}{(1+r)^1} + \frac{8}{(1+r)^2} + \frac{8}{(1+r)^3} + \frac{8}{(1+r)^4} + \frac{8+101}{(1+r)^5}, \quad r = 0.06956$$

The yield-to-third-call is 6.953%.

$$105 = \frac{8}{(1+r)^1} + \frac{8}{(1+r)^2} + \frac{8}{(1+r)^3} + \frac{8}{(1+r)^4} + \frac{8}{(1+r)^5} + \frac{8+100}{(1+r)^6}, \quad r = 0.06953$$

Finally, the yield-to-maturity is 7.070%.

$$105 = \frac{8}{(1+r)^1} + \frac{8}{(1+r)^2} + \frac{8}{(1+r)^3} + \frac{8}{(1+r)^4} + \frac{8}{(1+r)^5} + \frac{8}{(1+r)^6} +$$

Figure 4. Yield measures on a callable bond

Other alternatives include the valuation of the embedded call option separately and adding it to the flat price of the bond to get the **option-adjusted price** and **option-adjusted yield**.

2.4.2 Bonds with Floating Rates

The structure of floaters interest rates is the specified yield spread over (or subtracted from) the reference rate. For example, a floater interest rate might reset every 3 months at three month Libor plus a 0.50% **quoted margin** (yield over the reference rate).
The FRN pricing model is as follows.

$$PV = \frac{\frac{(RR+QM)\cdot FV}{m}}{(1+\frac{RR+DM}{m})^1} + ... + \frac{\frac{(RR+QM)\cdot FV}{m} + FV}{(1+\frac{RR+DM}{m})^N} \tag{8}$$

where,
PV = present value of the FRN
RR = reference rate, stated as annual percentage rate
FV = face value paid at maturity
m = periodicity (number of payment periods a year)
DM = discount margin, stated as annual percentage rate
N = number of evenly spaced periods to maturity

2.5 Yield Measures for Money Market Instruments

There are several important differences between the yield measures of money market instruments and the bond market:

- Bond yields-to-maturity are annualized and compounded. Yield measures in the money market are annualized but no compounded
- Bond YTM can be calculated using standard time-value-of-money equations. Money market instruments often are quoted using nonstandard interest rates and require different pricing equations
- Bond YTM are usually stated for a common periodicity for all times-to-maturity; money market instruments have different times-to-maturity and different periodicities for the annual rate.

In general, quoted money market rates are either **discount rates** or **add-on rates**.

Discount rate basis

$$PV = FV(1 - \frac{Days}{Year} \cdot DR) \tag{9}$$

where,
PV = present value of the money market instrument
FV = face value
Days = number of days between settlement and maturity
Year = number of days in the year
DR = discount rate, stated as annual percentage rate

Add-on rate basis

$$PV = \frac{FV}{(1+\frac{Days}{Year} \cdot AOR)} \tag{10}$$

where, AOR = add-on rate, stated as annual percentage rate

Investment analysis is made difficult for money market securities because:

- Some instruments are quoted on a discount rate basis and other on a add-on rate basis
- Some are quoted for 360-day year, others for 365-day year
-

The **bond equivalent yield** or investment yield is the money market rate stated on 365-day add-on basis.

3. Maturity Structure of Interest Rates

Some of the possible reasons for the difference between yields are:

- *Currency*: different currency denominated bonds bear different expected inflation rates
- *Credit risk*: different bonds have different credit risks and credit ratings
- *Liquidity*: different bonds have different liquidities (some bonds can even be iliquid)
- *Tax Status*
- *Periodicity*
- *Maturity*: different bonds have different times-to-maturity

This factor - yield-to-maturity - explaining the differences in yields is called **maturity structure** or **term structure** of interest rates. This involves analyzing the *yield curves* which are the relationships between yields-to-maturity and times-to-maturity.
The maturity structure should be analyzed for bonds that have the same properties other than TTM: they should be denominated in the same currency, have the same credit risk, liquidity and tax status, etc.
The dataset used for this analysis is the *government bond spot curve*, also called as *zero curve* or *strip curve*. This curve is a sequence of yields-to-maturity on zero-coupon bonds. These bonds are interpreted as *risk free* as to default risk free, but they aren't free of risk from liquidity risks or inflation risks. Some of this data is interpolated as there is a limited range of maturities.

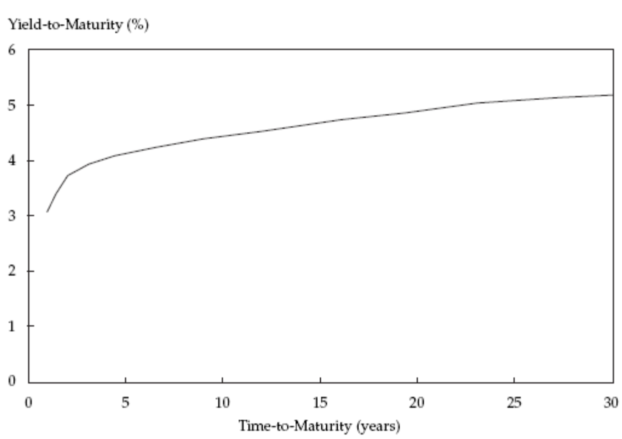


Figure 5. Government bond spot curve (interpolated)

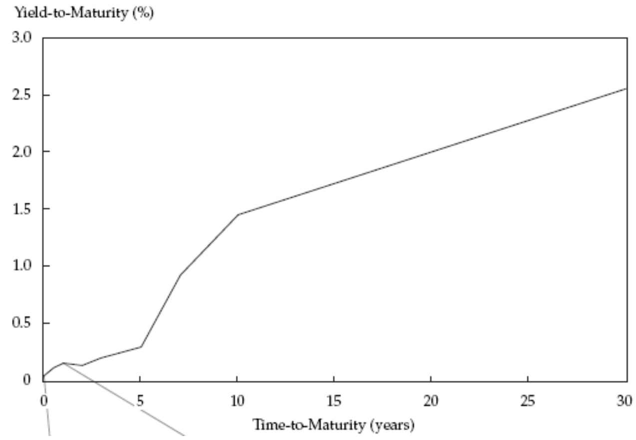


Figure 6. Government bond spot curve (real)

Par curve The maturity structure can be assess using a **par curve**. A par curve is a sequence of YTM such that each bond is priced at par value. The par curve is obtained from the spot curve.

Forward market A forward rate is for future delivery, beyond the usual settlement time of spot markets. The **forward rate** is the interest rate that the fixed income instrument is traded in a forward market.
The common market practice is to name *2y5y*. This is pronounced "the two-year into five-year rate". The first number (two years) refers to the length of the forward period in years and the other (five years) refers to the tenor of the underlying bond. This means the dealer is selling a 5 year bond two years into the future.
An **implied forward rate** is calculated from spot rates and it is the rate of break-even reinvestment rate. It links the return on an investment in a shorter-term zero-coupon bond to the return on an investment in a loger-term coupon bond.
Suppose that the shorter term bonds matures in A periods and longer term in B periods. The YTM of the bonds are z_A and z_B. The implied forward rate between period A and B is denoted as $IFR_{A,B-A}$.
The relationship between the two spot rates is given as

$$(1+z_A)^A \cdot (1+IRF^B_{A,B-A}-A) = (1+z_B)^B \tag{11}$$

A **forward curve** can be constructed from forward rates, each having the same time frame. A forward rate can be interpreted as an incremental or marginal return for extending the time-to-maturity for an additional time period.

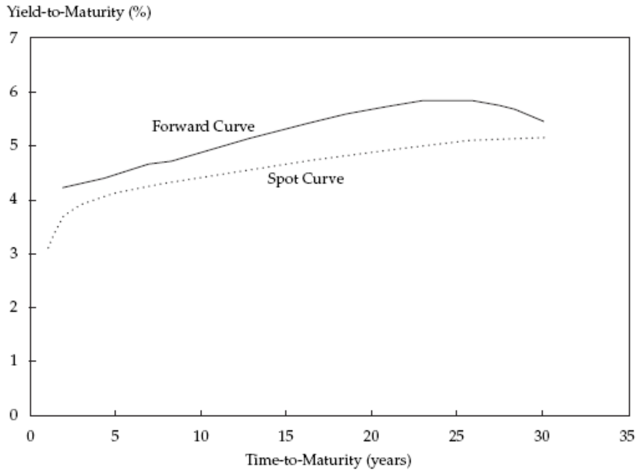


Figure 7. Utility function

4. Yield Spreads

A yield spread is the difference in yield between different fixed income securities.

4.1 Yield Spreads over Benchmark Rates

In order to analyze bond prices and YTM, it is useful to separate YTM into two components: the **benchmark** and the **spread**. The spread is the difference between the YTM and the benchmark. while the benchmark is the yield for a fixed-income security with similar time-to-maturity (the *base rate*). The reason why we separate between the spread and benchmark is to distinguish macroeconomic and microeconomic factors that affect the bond price, and therefore, its yield-to-maturity. That being said, the benchmark captures the macroeconomic factors: expected rate of inflation, economic growth, business cycle, the impact of monetary and fiscal policies. On the other hand, the spread captures microeconomic factors specific to the bond issuers such as the credit risk, liquidity, tax status, prospects for the future performance and changes in rating.

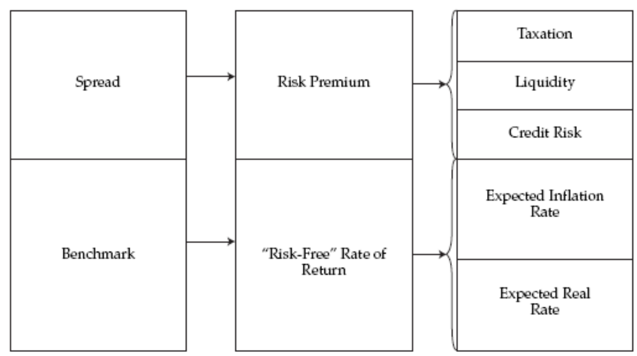


Figure 8. The building blocks of Yield-to-Maturity

The most common benchmark rate for fixed-rate bonds is a government yield (for domestic non-euro bonds), although another frequently used benchmark is the Libor composite interbank rate. When using the government basis, the yield spread in basis points over an actual or interpolated government bond is known as the **G-spread**. Euro-denominated corporate bonds are priced over a Euro interest rate swap benchmark (Euribor). The yield spread over a standard swap rate of that currency of the same tenor (over interest rate swap benchmark) is called the **I-spread** or **interpolated spread**.

4.2 Yield Spreads over the Benchmark Yield Curve

There are essentially two cases of yield curves: the benchmark yield curves for government bonds which is the relationship between the yields of *on-the-run* government bonds and its times-to-maturity and the **swap yield curve** which is the relationship between fixed Libor swap rates and their times-to-maturity.

The benchmark yield curve tens to be upward sloping because investors typically demand a premium for holding longer term

securities.
Another alternative is to used the **zero volatility spread** (**Z-spread**) of a bond over the benchmark rate. The Z-spread is calculated from the spot curve.

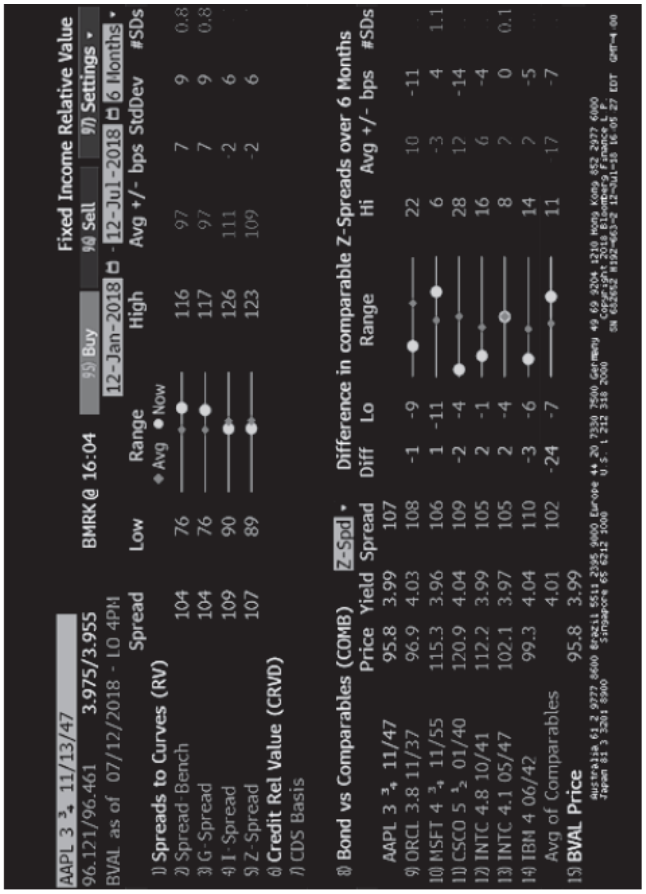


Figure 9. Bloomberg Fixed Income Relative Value (FIRV) page for Apple's bond

The Z-spread is also used to calculate the option-adjusted spread (OAS) on a callable bond.

$$OAS = Z - spread - OptionValue \tag{12}$$

References

[cfa, 2019] 2019. *CFA program curriculum*. CFA Institute.