1-2. 1. How many ways you can order n objects? $N = n(n-1)\cdots(n-k+1)\cdots(3)(2)(1) = n!$ If not distinct $= \frac{n!}{n_1!n_2!n_r!}$ ways to arrange.

2. Select, order doesn't matter $= {}_nC_k = \binom{n}{k} = \frac{n!}{(n-k)!k!}$

3. $P[A\cup B] = P[A] + P[B] - P[A\cap B]$ $P(A\cup B) = P(A) + P(B)$ if $A\cap B = \emptyset$. $P[A] = P[A\cap \bar{B}] + P[A\cap B]$

4. P of exactly 5 spades (8 non spades) $= \binom{13}{5}\binom{39}{8}$ Hand exactly 5 hearts: $P(B) = \binom{13}{5}\binom{39}{8}/\binom{52}{13}$ Exactly 5 spades + 5 hearts $= \binom{13}{5}\binom{13}{5}\binom{26}{3}$

5. $P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$

3. Conditional probability:

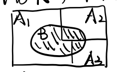
1. conditional probability of A given B is defined to be $P[A|B] = \frac{P[A\cap B]}{P[B]}$ if $P[B]\neq 0$

2. Independence: two events A and B are independent if $P[A\cap B] = P[A]P[B]$

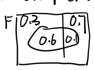
3. Mutually exclusive can not be independent, unless the special case that at least one of them $P=0$. $P(\bar{A}|B) = 1 - P(A|B)$

4. First is B, second is R, $P(B_1\cap R_2) = \frac{B}{B+R}\times\frac{r}{r+B-1}$, Red ball is selected on the second draw. $P(R_2) = P[B_1\cap R_2] + P[\bar{B}_1\cap R_2]$ $= P[B_1]P[R_2|B_1] + P[\bar{B}_1]P[R_2|\bar{B}_1]$ $= \frac{b}{r+b}\times\frac{r}{r+b-1} + \frac{r}{r+b}\times\frac{r-1}{r+b-1}$ $= \frac{r}{b+r}$

5. Total probability



$P(B) = P(A_1)P(B|A_1) + \dots P(A_k)P(B|A_k)$ Example:



$P(W) = P(F)\cdot P(W|F) + P(\bar{F})\cdot P(W|\bar{F})$

6. Bayes Rule $P[A|B] = \frac{P[A\cap B]}{P[B]} = \frac{P[A\cap B]}{P[A\cap B] + P[\bar{A}\cap B]}$ $P(F|W) = \frac{P(F\cap W)}{P(W)} = \frac{P(F)\cdot P(W|F)}{P(W)}$

$P[A|B] = \frac{P[A]P[B|A]}{P[A]P[B|A] + P[\bar{A}]P[B|\bar{A}]}$ $= P[A]P[B|A] \,/\, P[A]P[B|A] + P[\bar{A}]P[B|\bar{A}]$

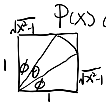
Example: 0.1 population has disease. If a person has a disease, test says he has disease 96% of time. If he doesn't have disease, the test he has the disease 2% of the time. Sensitivity $= P(\text{test}+|\text{disease}) = 0.96$ $P(\text{test}-|\text{no disease}) = 1 - 0.02 = 0.98$

A = the event that the person has the disease $= 0.001$ $\bar{A} = 0.999$ $\frac{P[A]P[B|A]}{}$ $= 0.0458$.

B = the event that test is positive $P[B|A] = 0.96$ $P[B|\bar{A}] = 0.02$ $P[A|B] = \frac{P[A]P[B|A]}{P[A]P[B|A] + P[\bar{A}]P[B|\bar{A}]}$

4. Random Variables: is a numerical quantity whose values is determined by a random experiment.

$P(x)$ of a RV $x = P[X=x] = P[\{X=x\}]$ Cumulative distribution function $F(x)$ of RV x $F(x) = P[X\le x] = P[\{X\le x\}]$ CDF



$\theta = \frac{\pi}{2} - 2\phi$ $\tan(\phi) = \sqrt{x^2-1}$ $\phi = \tan^{-1}(\sqrt{x^2-1})$

$A = 2\left(\frac{1\cdot\sqrt{x^2-1}}{2}\right) + \frac{\theta}{2\pi}(\pi x^2) = \sqrt{x^2-1} + \frac{\theta}{2}x^2 = \sqrt{x^2-1} + (\frac{\pi}{4} - \phi)x^2 = \sqrt{x^2-1} + (\frac{\pi}{4} - \tan^{-1}(\sqrt{x^2-1}))x^2$

Ex. $F_X(x) = \frac{1}{12}(x^2 + x^3)$, $x\in[0,2]$ $f(x) = ?$

Prob Density Function (PDF): 1. $f(x)\ge 0$ 2. $\int_{-\infty}^{\infty} f(x)dx = 1$ 3. $P[a\le x\le b] = \int_a^b f(x)dx$

$f(x) = F_X'(x) = \frac{1}{12}(x^2)' + \frac{1}{12}(x^3)' = \frac{1}{6}x + \frac{1}{4}x^2$

CDF: $F(x) = P[X\le x] = \int_{-\infty}^{x} f(t)dt$ Fundamental theorem of calculus: $F'(x) = \frac{dF(x)}{dx} = f(x)$

$[\tan^{-1}(\sqrt{x^2-1})]' =$

Ex. $[\sqrt{x^2-1} + (\frac{\pi}{4} - \tan^{-1}(\sqrt{x^2-1}))x^2]' = -\frac{1}{2}(\sqrt{x^2-1})^{\frac{3}{2}}(2x) + \frac{\pi x}{2} - 2x\tan^{-1}(\sqrt{x^2-1}) - x^2[\tan^{-1}(\sqrt{x^2-1})]' = \frac{\pi}{2}x - \frac{x}{(x^2-1)^{\frac{3}{2}}} - 2x\tan^{-1}(\sqrt{x^2-1}) - x^2[\tan^{-1}(\sqrt{x^2-1})]'$ $[\tan^{-1}(u)]' = \frac{1}{1+u^2}$ $\frac{1}{1+(x^2-1)}(-\frac{1}{2})(x^2-1)^{-\frac{3}{2}}(2x)$

Discrete Random Variables: 1. $0\le p(x)\le 1$ 2. $\sum p(x) = 1$ 3. $P(a\le x\le b) = \sum_a^b p(x) = \frac{\pi}{2}x - \frac{x}{(x^2-1)^{\frac{3}{2}}} - 2x\tan^{-1}(\sqrt{x^2-1}) + \frac{x}{(x^2-1)^{\frac{3}{2}}} = \frac{\pi}{2}x - 2x\tan^{-1}(\sqrt{x^2-1})$

Continuous Random Variables: 1. $f(x)\ge 0$ 2. $\int_{-\infty}^{\infty} f(x)dx = 1$ 3. $P[a\le X\le b] = \int_a^b f(x)dx$

Probability distribution ≈ distribution of mass   Discrete distribution ≈ a point distribution of mass

Continuous distribution ≈ continuous distribution of mass   Distribution function $F(x)$: $F(-\infty) = 0$ and $F(\infty) = 1$

$F(x)$ is non-decreasing, $x_1 < x_2 \Rightarrow F(x_1)\le F(x_2)$ $F(b) - F(a) = P[a < X\le b]$

5. Discrete Random Variables Ex. $p(x) = Cx^2$ $x = 1,2,3$ pmf Find C.

x	1	2	3
$p(x)$	c	$4c$	$9c$

$\Rightarrow c + 4c + 9c = 1$ $14c = 1$ $c = \frac{1}{14}$

1. The Bernoulli Distribution: 2 outcomes $P(2) = 1 - P(1)$ Ex. for $n=5$

x	0	1	2	3	4	5
$p(x) = P[X=x]$	q^5	$5pq^4$	…			p^5

for n, $P = p^kq^{n-k}$, $\binom{n}{k}$ sequences have k S's. $P(E) = P(X=k) = \binom{n}{k}p^kq^{n-k}$

2. Binomial Distribution: $(p+q)^n = \binom{n}{0}p^0q^n + \binom{n}{1}p^1q^{n-1} + \dots + \binom{n}{n}p^nq^0$ $P(x) = P[X=x] = \binom{n}{x}p^xq^{n-x}$ Ex1, Coin tossed $n=7$ times X = number of H then X has BD with $p = \frac{1}{2}$, $n = 7$

Ex2. $P(S) = 85\%$ $n = 20$ $P(x) = P[X=x] = \binom{n}{x}p^xq^{n-x} = \binom{20}{x}(0.85)^x(0.15)^{20-x}$ $x = 0,1,2,\dots 20$. So $p(x) = P[X=x] = \binom{n}{x}p^xq^{n-x} = \binom{7}{x}(\frac{1}{2})^x(\frac{1}{2})^{7-x} = \binom{7}{x}(\frac{1}{2})^7$ $x = 0,1,2,\dots 7$.

The probability that at least 16 success. $= P[X\ge 16] = p(16) + \dots + P(20) = 0.8298$

3. The Geometric Distribution: Bernoulli trial (S,F) until a success occurs $P[X=x] = p(x) = p(1-p)^{x-1} = pq^{x-1}$

A RV x has distribution is said to have GD. $a + ar + ar^2 + \dots = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ $\sum_{x=1}^{\infty} P(x) = p(1) + \dots = p + pq + pq^2 + \dots = \frac{p}{1-q} = 1$

The CDF for a geometric random variables $F_X(t) = P(X\le t) = 1 - (1-p)^{[t]}$ t means the greatest int t.

Ex. Die rolled until a 6 occurs. $P(S) = \frac{1}{6}$ $P(F) = \frac{5}{6}$ ① $P(\text{at most 5 rolls to get 6}) = P[X\le 5] = p(x) = pq^{x-1} = (\frac{1}{6})(\frac{5}{6})^{x-1}$ $x = 1,2,3\dots$

$a + ar + ar^2 + \dots ar^{n-1} = a\frac{1-r^n}{1-r}$ $a + ar + ar^2 + \dots = a\frac{1}{1-r}$ $= P(1) + \dots + P(5) = (\frac{1}{6})(\frac{5}{6})^0 + \dots + (\frac{1}{6})(\frac{5}{6})^4 = \frac{1}{6}[1 + (\frac{5}{6})^1 + \dots (\frac{5}{6})^4]$

② $P(\text{at least 10 rolls}) = P[X\ge 10]$ Or directly $F(t) = 1 - (1-\frac{1}{6})^{[5]} = 1 - \frac{5}{6}^5$ $= \frac{1}{6}[\frac{1-(\frac{5}{6})^5}{1-\frac{5}{6}}] = 1 - (\frac{5}{6})^5$

$= P(10) + P(11) + \dots = (\frac{1}{6})(\frac{5}{6})^9 + (\frac{1}{6})(\frac{5}{6})^{10} + \dots = (\frac{1}{6})(\frac{5}{6})^9[1 + (\frac{5}{6})^1 + (\frac{5}{6})^2 + \dots] = (\frac{1}{6})(\frac{5}{6})^9[\frac{1}{1-\frac{5}{6}}] = (\frac{5}{6})^9$

③ $P(X \text{ is divisible by } 2) = P(2) + P(4) + \dots = (\frac{1}{6})(\frac{5}{6})[1 + (\frac{5}{6})^2 + (\frac{5}{6})^4 + \dots] = (\frac{1}{6})(\frac{5}{6})[\frac{1}{1-(\frac{5}{6})^2}] = \frac{5}{36}[\frac{1}{1-\frac{25}{36}}] = \frac{5}{11}$

Poisson D Ex. 500 ppl At most one born on 9/1: $P(x\le 1) = \binom{n}{0}p^0q^n + \binom{n}{1}p^1q^{n-1}$ Approximation: $\lambda = np = \frac{500}{365} = 1.37$ $P = \frac{\lambda^0e^{-\lambda}}{0!} + \frac{\lambda^1e^{-\lambda}}{1!} = e^{-1.37}(1.37)^0 \dots$

④ $P(X \text{ is divisible by } 3 \mid X \text{ is divisible by } 2) = \frac{P[\{X/3\}\cap\{X/2\}]}{P[X/2]} = \frac{P[X/6]}{P[X/2]}$ First $P[X/6] = P(6) + P(12) + \dots = (\frac{1}{6})(\frac{5}{6})^5 + (\frac{1}{6})(\frac{5}{6})^{11} + \dots$

so $P(X/3 \mid X/2) = P[X/6]/P[X/2] = \dots$ $= (\frac{1}{6})(\frac{5}{6})^5[1 + (\frac{5}{6})^6 + \dots] = (\frac{1}{6})(\frac{5}{6})^5[\frac{1}{1-(\frac{5}{6})^6}] = \frac{5^5}{6^6 - 5^6}$

4. The Poisson Distribution: Events occur randomly and uniformly in time. Let X be # events occurs in fixed time, $p(x) = \frac{\lambda^x}{x!}e^{-\lambda}$ $x = 0,1,2,3\dots$

① $\sum_{x=0}^{\infty} P(x) = \sum_{x=0}^{\infty}\frac{\lambda^x}{x!}e^{-\lambda} = 1$ $= e^{-\lambda}(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots) = e^{-\lambda}(e^{\lambda}) = 1$ using $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$

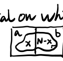
② PD approximation to Binomial: $P_{Bin}(x\mid p,n) = \binom{n}{x}p^x(1-p)^{n-x}$ $\lim_{n\to\infty, p\to 0} P_{Bin}(x\mid p,n) = P_{poisson}(x\mid\lambda) = \frac{\lambda^x}{x!}e^{-\lambda}$ Ex. X is PRV with λ

Ex. Hurricanes over period of a year, Poisson distribution $\lambda = 13.1$ ① prob function of X ② $P[X\le 8]$ ③ $P[X\ge 10]$ ④ $P[X\le 15 \mid X\ge 10]$ $P[X \text{ is even}] = \frac{1}{2}(1 + e^{-2\lambda})$ $\Rightarrow \sum_0^{\infty}\frac{\lambda^{2k}}{(2k)!}$

① $P(x) = \frac{\lambda^x}{x!}e^{-\lambda} = \frac{13.1^x}{x!}e^{-13.1}$ ② $P[X\le 8] = P(1) + \dots + P(8)$ ③ $P[X\ge 10] = 1 - P[X\le 9]$ ④ $P[X\le 15 \mid X\ge 10] = \frac{P[X\le 15\cap X\ge 10]}{P[X\ge 10]} = \frac{P[10\le X\le 15]}{P[X\ge 10]} = \frac{P(10) + P(11) + \dots P(15)}{0.74}$ Hint: $e^{\lambda} + e^{-\lambda} = \sum_0^{\infty}\frac{\lambda^k}{k!} + \sum_0^{\infty}\frac{(-\lambda)^k}{k!}$

5. Negative Binomial Distribution: X = the trial on which the k^{th} success occurs. $P(x) = P[X=x] = \binom{x-1}{k-1} p^k q^{x-k}$ Ex: $P(w) = 3\%$ I play until won $k=5$ times.

$p(x) = P[X=x] = \binom{x-1}{k-1} p^k q^{x-k} = \binom{x-1}{4} (0.003)^5 (0.97)^{x-5}$

6. Hypergeometric Distribution $p(x) = P[X=x] = \frac{\binom{a}{x}\binom{b}{n-x}}{\binom{N}{n}}$  $N = a+b$

Ex. $N=10$ $a=3$ bad $b=7$ good sample $n=4$ are selected : $P(x) = \frac{\binom{3}{x}\binom{7}{4-x}}{\binom{10}{4}}$, $x = 0,1,2,3 ..$

7. Bionomial Distn. : If sampling was done with replacement $p = \frac{a}{N}$ $q = 1-p = \frac{b}{N}$ $P_{Binom}(X) = P[X=x] = \binom{n}{x}\left(\frac{a}{N}\right)^x \left(\frac{b}{N}\right)^{n-x}$

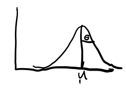
8. Hyper-geometric Distn: If sampling without replacement $P_{hyper}(X) = P[X=x] = \frac{\binom{a}{x}\binom{b}{n-x}}{\binom{N}{n}}$ $P_{hyper}(X) \approx P_{Binom}(X)$ for large value N, a, b.

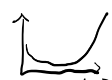
Geometric $\equiv$ Negative Binomial with $k=1$.

6. Continuous Distribution: (PDF): ① $f(x) \geq 0$ ② $\int_{-\infty}^{\infty} f(x)\,dx = 1$ ③ $P(a \leq x \leq b) = \int_a^b f(x)\,dx$ ④ $f(x) = F'(x)$, where $F(x)$ is CDF.

Ex. PDF $f(x) = cx^n$ where $0<x<1$ ① Find c : $\int_0^1 cx^n dx = 1 \Rightarrow c\frac{x^{n+1}}{n+1}\Big|_{x=0}^{1} = 1 \Rightarrow c\left(\frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1}\right) = 1 \Rightarrow c = n+1$

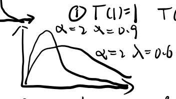
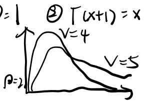
② $P[X>0.05] = \int_{0.05}^{1} (n+1)x^n dx = x^{n+1}\Big|_{0.05}^{1} = 1 - 0.05^{n+1}$

- Uniformed distribution $X \sim U(a,b)$. $f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$
- Cumulative DF. $\frac{x-a}{b-a}$.
- The Normal Distribution: mean μ, standard deviation σ. $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$  $f'(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $\left[-\frac{1}{\sigma^2}(x-\mu)\right] = 0$ if $x-\mu=0$ or $x=\mu$

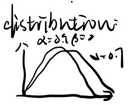
μ is an extremum point of $f(x)$ $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$ $X \sim N(\mu, \sigma)$ when $\mu=0$, $\sigma=1$ its standard normal $Z \sim N(0,1)$ 

CDF of $X \sim N(\mu,\sigma)$ is $F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$ CDF of $Z \sim N(0,1)$ is $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$

- Exponential Distribution : CDF $F(x) = 1 - e^{-\lambda x}$ To Find Density: $f(x) = F'(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x<0 \end{cases}$ $X \sim Exp(\lambda)$  ⑤ $\Gamma(n+\frac{1}{2}) = \frac{(2n)!}{n!\,2^{2n}}\sqrt{\pi}$
- Gamma Distribution: $\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du$  ① $\Gamma(1) = 1$ $\Gamma(1) = \int_0^{\infty} e^{-u} du = [-e^{-u}]_0^{\infty} = -e^{-\infty} - (-e^{0}) = 1$ ② $\Gamma(x+1) = x\Gamma(x)$ ③ $\Gamma(n) = (n-1)!$ ④ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$X \sim \Gamma(\alpha, \lambda)$ $f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x \geq 0 \\ 0 & x<0 \end{cases}$  The X^2 distribution: 

$\Gamma(1, \lambda) = Exp(\lambda)$ $\Gamma(\frac{v}{2}, \frac{1}{2}) = X_v^2$

- Weibull Distribution $F(x) = 1 - e^{-\frac{\alpha}{\beta}x^\beta}$ $f(x) = F'(x) = \alpha x^{\beta-1} e^{-\frac{\alpha}{\beta}x^\beta}$ $x \geq 0$ 

7. Expectation (Mean) of Random Variables : $E(x) = \sum_x x p(x) = \sum_i x_i p(x_i)$ $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

① The expected value of x, $E(x)$ is the center of gravity of the probability distribution of x. ② is the long-run average value

Ex. 7-game Series teams are evenly matched. ① Find distribution ② $E(x)$

possible value for x $\{4,5,6,7\}$, $P(\text{sequence of length } x) = (\frac{1}{2})^x$ $p(x) = P[X=x] = 2\binom{x-1}{x-4}(\frac{1}{2})^x$

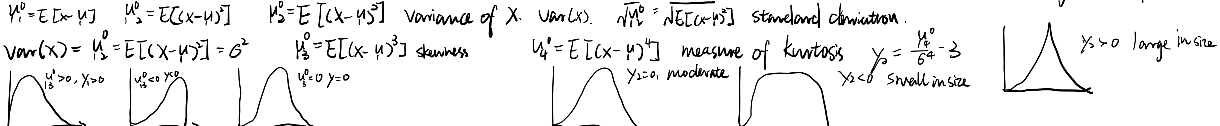
$E(x) = \sum_x x p(x) = \sum_i x_i p(x_i) = 4(\frac{1}{8}) + 5(\frac{1}{4}) + 6(\frac{5}{16}) + 7(\frac{5}{16}) = \frac{93}{16}$.

If x has an exponential distribution $E(x) = \frac{1}{\lambda}$.

Normal distribution $E(x) = \mu$

8. Moments, k^{th} moment of x $\mu_k = E(X^k) = \begin{cases} \sum_x x^k p(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x^k f(x)\,dx & \text{if } x \text{ is continuous.} \end{cases}$ Central Moment $\mu_k^0 = E[(x-\mu)^k] = \begin{cases} \sum_x (x-\mu)^k p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} (x-\mu)^k f(x)\,dx & \text{if Continuous.} \end{cases}$

$\mu_1^0 = E[x-\mu]$ $\mu_2^0 = E[(x-\mu)^2]$ $\mu_2^0 = E[(x-\mu)^2]$ Variance of X. $Var(x)$. $\sqrt{\mu_2^0} = \sqrt{E[(x-\mu)^2]}$ standard deviation.

$Var(x) = \mu_2^0 = E[(X-\mu)^2] = \sigma^2$ $\mu_3^0 = E[(X-\mu)^3]$ skewness $\mu_4^0 = E[(x-\mu)^4]$ measure of kurtosis $\gamma_2 = \frac{\mu_4^0}{\sigma^4} - 3$ 

Ex. X be a discrete R.V with $p(x)$

x	1	2	3
$p(x)$	$1/2$	$1/3$	$1/6$

① $Ex = 1 \cdot \frac{1}{2} + 2 * \frac{1}{3} + \cdots = 5/3 = \mu$ ② Standard deviation σ $Var(X) = \sum (x-\mu)^2 \cdot p(x) = (1-\frac{5}{3})^2(\frac{1}{2}) + (2-\frac{5}{3})^2(\frac{1}{3}) + \cdots = \frac{5}{9} \Rightarrow \sigma = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$

③ $\mu_3 = \sum x^3 p(x) = 1^3(\frac{1}{2}) + 2^3(\frac{1}{3}) + 3^3(\frac{1}{6}) = \frac{23}{3}$ ④ $\mu_3^0 = E[(x-\mu)^3] = (1-\frac{5}{3})^3 = \frac{7}{27}$

Moment generation function:

$m_x(t) = E[e^{tX}] = \begin{cases} \sum_x e^{tx} p(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if continuous} \end{cases}$ Ex. Bino D $X \sim Bin(n,p)$ $p(x) = \binom{n}{x} p^x (1-p)^{n-x}$ $= (a+b)^n = (e^t p + 1 - p)^n$

$m_x(t) = E[e^{tx}] = \sum_x e^{tx} p(x) = \sum_{x=0}^{n} e^{tx}\binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=0}^{n}\binom{n}{x} a^x b^{n-x}$

poisson $m_x(t) = E(e^{tx}) = \sum_x e^{tx} p(x) = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$

Uniform D: $m(t) = \int_0^1 e^{xt} \cdot 1 dx = \frac{1}{t}\int_0^1 e^{xt} d(xt) = \frac{1}{t} e^{xt}\Big|_{x=0}^{1} = \frac{e^t - 1}{t}$.

$X \sim Exp(\lambda)$ $m_x(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} f(x)\,dx = \begin{cases} \frac{\lambda}{\lambda - t} & t<\lambda \\ \text{undefined} & t \geq \lambda \end{cases}$

$X \sim N(0,1)$ $m_x(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2 - 2tx}{2}} dx$

$X \sim$ Gamma D $\Gamma(\alpha, \lambda)$ $m_x(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx$

① $m_x(0) = 1$ ② $m_x(t) = 1 + \mu_1 t + \frac{\mu_2}{2!}t^2 + \cdots$ ③ $\mu_x^{(k)}(0) = \frac{d^k}{dt^k} m_x(t)\Big|_{t=0} = \mu_k$

11. Joint PF