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1. a) $L(\theta) = \prod_{i=1}^{n} \left(\frac{1}{1-\theta}\right) \cdot I(\theta \le x_i \le 1)$

$$= \frac{1}{(1-\theta)^n} I[\theta \le \min\{x_1, \cdots x_n\} \le 1]$$

θ will increase in $(0, \min\{x_1, \dots, x_n\}$

and $\theta = 0$ on $(\min\{x_1, \dots, x_n\}$

So MLE of θ is $\hat{\theta} = \min\{x_1, \dots, x_n\}$.

b) $E(\hat{\theta}) = \int_{\theta}^{1} \frac{t(1-t)^2}{(1-\theta)^3} dt = \frac{1}{(1-\theta)^3} \int_{\theta}^{1} t(1-t)^2 dt$

$$= \frac{1}{(1-\theta)^3} \left(\frac{t^4}{3} - \frac{2t^3}{3} + \frac{t^2}{2}\right) \Big]_{\theta}^{1}$$

$$= \frac{1}{(1-\theta)^3} \left(\frac{1}{12} - \frac{\theta^4}{4} + \frac{2\theta^3}{3} - \frac{\theta^2}{2}\right) = \frac{3\theta+1}{12}$$

So $Bias(\hat{\theta}) = E(\hat{\theta}) - \theta = \frac{1}{12}(3\theta+1) - \theta = \frac{1}{12} - \frac{3}{4}\theta$

c)

2. a) $Z_{0.035} = 1.81$

$58842 \pm \frac{1577}{\sqrt{30}} = (58320.866, 59363.13)$

b) $H_0: \mu = 60000$ vs. $H_a: \mu < 60000$

$$Z = \frac{58842 - 60,000}{\frac{1577}{\sqrt{30}}} = \frac{-1158}{287.9195} = -4.02$$

$Z_{0.07} = 1.47$

$\because -4.02 < 1.47$

So reject H_0

Salary is less than 60,000 on average.

c) p-value $= \Phi(-4.02) < 0.0001$

d) Power $= p(Z$

3. a) $E(x) = \sum_{x=1}^{\infty} x \cdot p(1-p)^{x-1}$

$\Rightarrow (1-p)E(x) = \sum_{x=1}^{\infty} x \cdot p(1-p)^{x} = \sum_{x=2}^{\infty} (x-1)\, p(1-p)^{x-1}$

$\Rightarrow p \cdot E(x) = p + \frac{p(1-p)}{1-(1-p)}$

$\Rightarrow E(x) = \frac{1}{p} = \bar{x} \quad \hat{p}_{mm} = \frac{1}{\bar{x}}$

$\therefore \hat{p} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$

b) $L(p;x_1 \ldots x_n) = \prod_{i=1}^{n} p(1-p)^{x_i - 1} = p^n (1-p)^{\sum_{i=1}^{n} x_i - n}$

$L(p) = \log l(p) = n\log(p) + \left(\sum_{i=1}^{n} x_i - n\right)\log(1-p)$

$\frac{\partial L(p)}{l(p)} = \frac{n}{p} - \frac{\sum_{i=1}^{n} x_i - n}{1-p} = 0$

So $p = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$

c) $H_0 = p = \frac{1}{2}$ $\quad H_a: p = \frac{1}{4}$

$\frac{L(\theta_a)}{L(\theta_0)} = \frac{(\frac{1}{4})^n (\frac{3}{4})^{\Sigma(x_j - 1)}}{(\frac{1}{2})^n (\frac{1}{2})^{\Sigma(x_j - 1)}} \geq K$

$= (\frac{1}{2})^n (\frac{3}{2})^{\Sigma(x_j - 1)} \geq k$

$\Rightarrow (\frac{3}{2})^{\Sigma(x_j - 1)} \geq k_1$

$\Rightarrow \Sigma(x_j - 1) \geq k_2$

$\therefore$ RR: $\{\Sigma(x_j - 1) \geq k_2\}$

d) $\lambda(x) = \dfrac{L(p_0)}{L(\hat{p})}$

$$= \frac{p_0^{\,n} \cdot (1-p_0)^{\sum(x_i - 1)}}{(\frac{1}{\bar{x}}) \cdot (1-\frac{1}{\bar{x}})^{\sum x_i - 1}} \le k .$$

4. a) $Var(Y) = 2k$ if $Y \sim \chi^2_k$

$\frac{(n-1)S^2}{\sigma^2} = \chi^2_{26}$ $\quad \chi^2_n \sim N(n, \sqrt{2n})$

$SE(S^2) = \sqrt{Var(S^2)} = \sqrt{26 \times 2} = \sqrt{52}$

b) 90% CI $\quad n = 27-1$

$\chi^2_{27-1} < 35.563 = 0.9$

$CI = \bar{x} \pm 35.56 \cdot 3.6/\sqrt{27} = \bar{x} \pm 24.639$

c)

$P(\chi^2_{27} \le 12) = P\left(Z < \dfrac{\frac{12 - 3.6^2}{3.6}}{\sqrt{27}}\right) = P(Z < -1.38)$

$= 0.0838$

5.

b) $L(\theta) = \prod_{i=1}^{n} f(x_i \mid \theta) = \prod_{i=1}^{n} \frac{1}{\theta} x_i^{(1-\theta)/\theta} = \theta^{-n} \left(\prod_{i=1}^{n} x_i \right)^{\frac{1-\theta}{\theta}}$

$$\log L(\theta) = -n\log\theta + \frac{1-\theta}{\theta} \sum_{i=1}^{n} \log x_i = -n\log\theta + \frac{1}{\theta} \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \log x_i$$

$$\frac{d}{d\theta} \log L(\theta) = -\frac{n}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \log x_i$$

$$= -\frac{n}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^{n} \log x_i = 0$$

$$\hat{\theta} = -\frac{1}{n} \sum_{i=1}^{n} \log x_i$$

c) $\hat{\theta} > 0$ since $\log x_i < 0$ since $0 < x_i < 1$

$$\frac{d^2}{d\theta^2} \log L(\theta) = \frac{n}{\theta^2} + \frac{2}{\theta^3} \sum_{i=1}^{n} \log x_i$$

$$\frac{d^2}{d\theta^2} \log L(\hat{\theta}) = \frac{n}{\hat{\theta}^2} + \frac{2}{\hat{\theta}^3}(-n\hat{\theta}) = \frac{n}{\hat{\theta}^2} - \frac{2n}{\hat{\theta}^2} = -\frac{n}{\hat{\theta}^2} < 0$$

$$\hat{\theta} = -\frac{1}{n} \sum_{i=1}^{n} \log x_i = -\frac{1}{4} \log (0.41 \times 0.22 \times 0.54 \times 0.36)$$

$$= 0.4390$$