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HUDM 4125 Due December 2, 2021

HW 8

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1. Let $X_1, \ldots, X_n$ represent a random sample from a distribution with pdf

$$f(x;\theta) = \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}, \qquad x > 0$$

a) Show that $E(X^2) = 2\theta$.

Hint: Do a change of variable $y = x^2$ in the integral, and then use the pdf of the Gamma distribution.

b) Show that $\hat{\theta} = \frac{\sum_{i=1}^{n} X_i^2}{2n}$ is an unbiased estimator of θ.

2. From Devore's textbook:

p. 348: Q10

p. 350: Q20

1. a) $E[x^2] = \int_0^{\infty} x^2 \frac{x}{\theta} \cdot e^{-\frac{x^2}{2\theta}} \, dx$

Let $\frac{-x^2}{2\theta} = t \Rightarrow \frac{-2x}{2\theta} dx = dt$

$x\,dx = -\theta\,dt$

$E[x^2] = \int_0^{-\infty} -2\theta t e^t (-dt)$

$= 2\theta \int_0^{-\infty} te^t \, dt = 2\theta \left(te^t - e^t\right)\Big|_0^{-\infty}$

$E[x^2] = 2\theta(0-(-1)) = 2\theta.$

b) since $E[x^2] = 2\theta$.

$V(x) = E[x^2] - (E[x])^2 \Rightarrow E(x^2) = V(x) + [E(x)]^2$

$\Rightarrow 2\theta = \frac{\Sigma x_i^2}{n} - \left[\frac{\Sigma x_i}{n}\right]^2 + \left[\frac{\Sigma x_i}{n}\right]^2$

$\Rightarrow 2\theta = \frac{\Sigma x_i^2}{n} \Rightarrow \theta = \frac{\Sigma x_i^2}{2n}$

2. a) $E(\bar{x}) = \mu \quad V(\bar{x}) = \frac{\sigma^2}{n} \quad V(x) = E(x^2) - [E(x)]^2$

$\Rightarrow E(x^2) = Vx + [E(x)]^2$

$\therefore E(\bar{x}^2) = V(\bar{x}) + [E(\bar{x})]^2$

$= \frac{\sigma^2}{n} + \mu^2 \neq \mu$

so $\bar{x}^2$ is not an unbiased estimator for μ^2.

b) Now $E[\bar{x}^2 - kS^2] = E(\bar{x}^2) - kS(S^2) = \frac{\sigma^2}{n} + \mu^2 - k\sigma^2$

$E[\bar{x}^2 - kS^2] = \mu^2$ So $\frac{\sigma^2}{n} + \mu^2 - k\sigma^2 = \mu^2$

$\Rightarrow \frac{\sigma^2}{n} = k\sigma^2 \Rightarrow k = \frac{1}{n}$

3. a) $\hat{p} = \frac{x + \sqrt{\frac{n}{4}}}{n + \sqrt{n}} \quad MSE = E((\hat{p}-p)^2) = var(\hat{p}) + (E(\hat{p}) - p)^2$

$E(\hat{p}) - p = E\left(\frac{x + \sqrt{n/4}}{n + \sqrt{n}}\right) - p = \frac{np + \sqrt{n/4}}{n + \sqrt{n}} - p$

$= \frac{\sqrt{n/4} - p\sqrt{n}}{n + \sqrt{n}} = \frac{\sqrt{n-4} - p\sqrt{n}}{n + \sqrt{n}} \left(\frac{n - \sqrt{n}}{n - \sqrt{n}}\right)$

$= \frac{n\sqrt{n}(\frac{1}{2} - p) + n(p - \frac{1}{2})}{n^2 - n}$

$$= \frac{\sqrt{n}\left(\frac{1}{2}-p\right)+\left(p-\frac{1}{2}\right)}{n-1} = \frac{(1-\sqrt{n})\left(p-\frac{1}{2}\right)}{n-1}$$

$$= \frac{(\sqrt{n}-1)\left(\frac{1}{2}-p\right)}{(\sqrt{n}-1)(\sqrt{n}+1)} = \frac{\frac{1}{2}-p}{\sqrt{n}+1}$$

$$\mathrm{Var}(\hat{p}) = \mathrm{Var}\left(\frac{X+\sqrt{n/4}}{n+\sqrt{n}}\right) = \frac{1}{(n+\sqrt{n})^2}\left(np(1-p)\right)$$

$$MSE = \mathrm{Var}(\hat{p}) + (E(\hat{p})-p)^2$$

$$= \frac{1}{(n+\sqrt{n})^2}\left(np(1-p)\right) + \left(\frac{\frac{1}{2}-p}{\sqrt{n}+1}\right)^2$$

$$= \frac{np(1-p)}{n^2+2n\sqrt{n}+n} + \frac{\frac{1}{4}-p+p^2}{n+2\sqrt{n}+1} = \frac{p(1-p)}{n+2\sqrt{n}+1} + \frac{\frac{1}{4}-p+p^2}{n+2\sqrt{n}+1}$$

$$= \frac{p-p^2+\frac{1}{4}-p+p^2}{n+2\sqrt{n}+1} = \frac{1}{4(n+\sqrt{n})^2}$$

b) $MSE = \mathrm{Var}(\hat{p}) + (E(\hat{p})-p)^2$

$$= \mathrm{Var}\left(\frac{X}{n}\right) + \left(E\left(\frac{X}{n}\right)-p\right)^2$$

$$= \frac{1}{n^2}np(1-p) + \left(\frac{1}{n}np-p\right)^2$$

$$= \frac{1}{n}p(1-p) + (p-p)^2$$

$$= \frac{1}{n}p(1-p)$$