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HUDM 4125 Due October 14, 2021

HW 5

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1. Consider the location-scale Cauchy distribution with pdf

$$f(x) = \frac{1}{\sigma\pi\left(1 + \left(\frac{x-\mu}{\sigma}\right)^2\right)}, -\infty < x < \infty$$

   Show that μ is the median of the distribution. That is, $P(X \le \mu) = P(X \ge \mu) = \frac{1}{2}$.

2. An urn contains four chips numbered 1 through 4. Two are drawn without replacement. Let the random variable X denote the larger of the two numbers on the chips. Find $E(X)$.

3. The hypotenuse of an isosceles right triangle is a random variable having U(6, 10) distribution. Calculate the expected value of the triangle's area.

4. The random variable X has probability function

$$P(X = k) = \frac{1}{15}(6 - k), k = 1, 2, 3, 4, 5.$$

   Find:

   a) $E(X)$

   b) $Var(X)$

   c) Skewness coefficient of X

1. $f(x) = \dfrac{1}{\sigma\pi\left(1+\left(\frac{x-\mu}{\sigma}\right)^2\right)}, \quad -\infty < x < \infty$

$$= \frac{1}{\sigma\pi + \frac{\pi(x-\mu)^2}{\sigma}} = \frac{\sigma}{\sigma^2\pi + \pi(x-\mu)^2} = \frac{\sigma}{\pi(\sigma^2+(x-\mu)^2)}$$

$$F(x) = \int_{-\infty}^{x} \frac{\sigma}{\pi(\sigma^2+(x-\mu)^2)}\,dx = \frac{1}{\pi}\int_{-\infty}^{x} \frac{\sigma}{\sigma^2+(x-\mu)^2}\,dx$$

$$= \frac{1}{\pi}\tan^{-1}\frac{x-\mu}{\sigma}\Big]_{-\infty}^{x} = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-\mu}{\sigma}\right)$$

$$\therefore P(x \le \mu) = P(x \ge \mu) = F(\mu) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{\mu-\mu}{\sigma}\right) = \frac{1}{2}$$

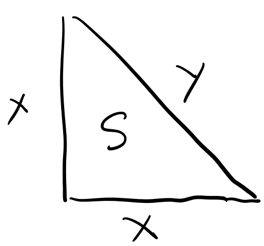
2.

x	1	2	3	4
$p(x)$	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

$\Rightarrow$ (1,2) (1,3) (1,4)
(2,3) (2,4) (3,4)

$$E(x) = 1\times 0 + 2\times\frac{1}{6} + 3\times\frac{2}{6} + 4\times\frac{3}{6} = 3.333$$

3.



$y^2 = 2x^2 \Rightarrow y = \sqrt{2}x, \quad y \in [6, 10]$

$x = \frac{\sqrt{2}}{2}y$

$S = \frac{1}{2}x^2 = \frac{1}{4}y^2$

$$E(S) = E\left(\tfrac{1}{4}y^2\right) = \int_6^{10} \frac{y^2}{4}\,\frac{1}{(10-6)}\,dy = \frac{1}{16}\cdot\frac{y^3}{3}\Big]_6^{10} = \frac{1}{16}\times\frac{10^3}{3} - \frac{1}{16}\times\frac{6^3}{3}$$
$$= 16.33$$

4. a) $E(x) = \sum x\cdot p(x=k) = 1\times\frac{1}{3} + 2\times\frac{4}{15} + 3\times\frac{1}{5} + 4\times\frac{2}{15} + 5\times\frac{1}{15} = 2.33$

b) $Var(x) = \sum (x-\mu)^2\cdot p(x) = (1-2.33)^2\times\frac{1}{3} + \cdots + (5-2.33)^2\times\frac{1}{15} = 1.56$

c) $\mu_3^0 = \sum (x-\mu)^3\cdot p(x) = (1-2.33)^3\times\frac{1}{3} + \cdots + (5-2.33)^3\times\frac{1}{15} = 1.16$

$r_1 = \frac{\mu_3^0}{\sigma^3} = \frac{1.16}{1.95} = 0.595 \Rightarrow \mu_3^0 > 0$ & $r_1 > 0$

positively skewed.