Solution to problems related to random walks.

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1 Problems and solutions

An ant leaves its anthill in order to forage for food. It moves with the speed of 10 cm per second, but it doesn't know where to go, therefore every second it moves randomly 10 cm directly north, south, east or west with equal probability.

Problem 1. *If the food is located on east-west lines* 20 *cm to the north and* 20 *cm to the south, as well as on north-south lines* 20 *cm to the east and* 20 *cm to the west from the anthill, how long will it take the ant to reach it on average?*

Solution 1. *The answer is* ***4.5 seconds****. We may model the problem as an absorbing Markov chain, such that the absorbing states are the ones in which the food is located, the others are transient and the transition probability are given by hypothesis. For any absorbing Markov chain there exists a permutation of the nodes such that the transition matrix P associated to the Markov chain has the following canonical form*

$$P = \begin{pmatrix} Q & R \\ \mathbf{0} & I_r \end{pmatrix},$$

where Q is a $t \times t$, R is a nonzero $t \times t$ matrix, 0 is an $r \times r$ zero matrix, andI_r is the $r \times r$ identity matrix. It is a standard fact that the Resolvent Matrix *(also called fundamental matrix) $N = (I_t - Q)^{-1}$, associated to the absorbing Markov Chain incapsulate many properties of the chain. Indeed the sum of the $i - th$ row of N represents the expected number of steps before being absorbed, starting from node i.*

In Problem1.jl*, a simple algorithm which compute the steps above is presented.*

Problem 2. *What is the average time the ant will reach food if it is located only on a diagonal line passing through (10cm, 0cm) and (0cm, 10cm) points.*

Solution 2. *The answer is* ***infinite****. Indeed, this comes from the fact that a random walk in a semiplane is a composition of two 1D random walk on an half line, plus the fact that the expected hitting time to hit state 0 from all starting states $k \geq 1$ in a balanced one 1D random walk is infinite. Let us precise that in a 2D balanced random walk on a square lattice, the probability to hit* every *state is* 1 *(binomial distribution + stirling), but, intuitively, once we hit a state we need to wait more time than the last time to hit it again; hence we obtain an infinite average hitting time. Formally, we will prove the following:*

Lemma 1. *Let (X_n) be a Markov chain defined on $\{0, 1, 2, \cdots\}$, with transition probabilities $p_{i,i-1} = p_{i,i+1} = 1/2$. Then the expected hitting time to hit state 0 from all states $k \geq 1$ is infinite.*

This lemma, combined with the fact that a 2D random walk can be decomposed in two 1D random walk, proves the results. With a change of coordinates we can set the initial position of the ant in any $(-k, 0)$, with $k > 0$, and the food diagonal *to be the y-axis. In this way the application of the lemma should be straightforward. Finally let us prove the lemma.*

Proof. Let $y_n = \mathbb{E}[X_n]$. The transition probabilities imply the following linear recurrence relation

$$y_{n+1} - 2y_n + y_{n-1} = -2.$$

We also have one boundary condition, which is $y_0 = 0$; to solve the system we need another boundary condition, which we artificially set to $y_N = 0$. In this way we are searching for the expected time to hit 0 or N starting from inside. Asymptotically, we are guaranteed to hit 0 first. Now this is a standard analysis problem, we search for a homogenous solution and a particular one. If we impose a quadratic function to be a particular solution we obtain

$$y_n = -n^2.$$

Then we search for a general homogenous solution of the form $y_n = c_1 + c_2 n$; hence

$$y_n = c_1 + c_2 n = n^2.$$

Plugging in the boundary conditions to found constants c_1, c_2 we obtain $c_1 = 0$ and $c_2 = N$. Thus

$$y_n = Nn - n^2.$$

We end the proof by observing that for any fixed $n > 0$, $lim_{N \to \infty} y_n = \infty$. $\square$

In Problem2.jl *we may found an algorithm which compute the average hitting time, when the food is located on a diagonal line, but the ant can move only inside a triangle, i.e. the finite approximation of the discussed problem 2.*

Problem 3. *Can you write a program that comes up with an estimate of average time to find food for any closed boundary around the anthill? What would be the answer if food is located outside and defined by $((x\text{–}2.5cm)/30cm)^2 + ((y\text{–}2.5cm)/40cm)^2 < 1$ in coordinate system where the anthill is located at $(x = 0cm, y = 0cm)$? Provide us with a solution rounded to the nearest integer.*

Solution 3. *The answer is **14 seconds**. In the presence of a (let us say piecewise continuous) closed boundary, we just need to compute the right absorbing Markov chain, then we can compute the answer as explained in* **??**.

1. *In Problem3.jl, the boundary is defined by a quadratic form (such as an ellipse)*

2. *In Problem3bis.jl, the boundary is defined by the graph of four arbitrary piecewise continuous functions (one for each "side" of the boundary). For instance solution to problem1, is an application of the algorithm in Problem3bis.jl, with a boundary defined by the four functions $x = \pm 20$ and $y = \pm 20$.*