FRACTAL SPATIO-TEMPORAL GRAVITY (FSG)

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Abstract

We propose a modified theory of gravity based on a Fractal and Non-Local Spacetime Geometry (FSG), obtained from an action containing the non-local scalar operator $X = \Box^{-1}R$. The effective action

$$S_{\rm FSG} = \frac{M_P^2}{2}\int d^4x\,\sqrt{-g}\,R\big[1+f(X)\big] + S_m$$

induces infrared corrections to General Relativity without introducing additional dynamical fields. We show that large-scale fractality implies a reduction of effective spectral dimension towards $\mathbf{d_S \simeq 2}$ in the IR. This geometric reduction $\mathbf{d_S \to 2}$ is the fundamental cause of the logarithmic potential, forcing the graviton propagator to adopt the exponent necessary for MOND dynamics, $\mathbf{G(k) \sim k^{-3}}$, leading to: (i) flat galactic rotation curves without dark matter, (ii) an exact baryonic Tully–Fisher relation $V^4 = GMa_0$, and (iii) a natural emergence of the acceleration scale $a_0 \approx c^2\sqrt{\Lambda/3}$.

Cosmologically, the model predicts: a phantom-like equation of state with $w_0 \simeq -1.04$ to -1.14, measurable deviations $\Delta H/H \simeq 2$–6% for $z < 2$, early structure formation consistent with JWST, and an IR reduction of effective dimensionality observable by Euclid. The theory is fully falsifiable: precision measurements of $w(z)$ and $H(z)$ over the next five years can confirm or rule out FSG.

Contents

1. Introduction

The standard cosmological model ΛCDM successfully accounts for a wide range of observations but at the cost of introducing two hypothetical components: dark matter (26% of the energy budget) and dark energy (69%), neither detected in laboratory experiments.

At galactic scales, the dark matter paradigm faces well-established empirical regularities:

- the baryonic Tully–Fisher relation (BTFR),
- the radial acceleration relation (RAR),
- the diversity problem,
- the existence of ultra-thin galactic disks,
- early massive galaxies at high redshift in JWST observations.

Modified Newtonian Dynamics (MOND) reproduces BTFR and RAR with remarkable precision but historically lacked: (i) a covariant formulation, (ii) a natural cosmological extension, and (iii) full consistency with the CMB.

Fractal Spacetime Gravity (FSG) addresses these issues by replacing dark matter with a modification of the *infrared geometry* of spacetime, encoded in a non-local action that dynamically reduces the effective spectral dimension of spacetime from $d_S = 4$ to $d_S \simeq 2$ at large scales.

2. Motivations for Infrared Fractality

Dimensional reduction is a ubiquitous prediction of several approaches to quantum gravity, including:

- Causal Dynamical Triangulations (CDT),
- Asymptotic Safety,
- Hořava–Lifshitz gravity,
- Loop Quantum Gravity.

These frameworks typically predict:

$$d_S(k \to \infty) \to 2,$$

i.e. a reduction of effective dimensionality in the ultraviolet.

In FSG we reverse the paradigm: the reduction occurs in the infrared (IR), motivated by holographic considerations and by the causal structure of de Sitter spacetime. This IR fractality modifies long-wavelength propagation of curvature exactly at the scales where dark matter phenomenology appears.

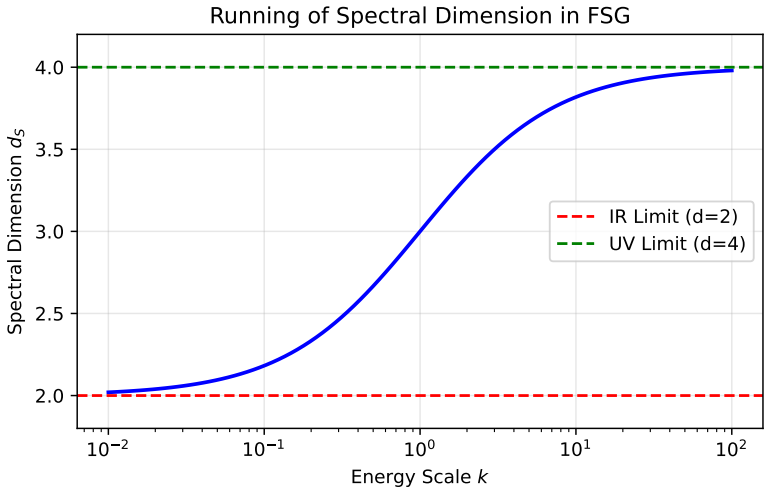

Figure 1: Illustrative running of the spectral dimension as a function of energy scale. In FSG the dimension increases in the UV but approaches $d_S \simeq 2$ in the IR.

3. The Non-Local Effective Action

We introduce the non-local scalar field

$$X = \Box^{-1} R, \qquad \Box X = R,$$

with the retarded Green function ensuring causality.

The FSG action is

$$S = \frac{M_P^2}{2} \int d^4x \sqrt{-g}\, R[1 + f(X)] + S_m. \tag{1}$$

The function $f(X)$ is constrained by the following physical requirements:

1. IR limit of the geometry becomes effectively 2-dimensional,
2. full diffeomorphism invariance,
3. causal non-locality (retarded $\Box^{-1}$),
4. recovery of GR in the UV/high curvature regime.

The minimal form satisfying these criteria is a logarithmic correction:

$$f(X) \sim \alpha \ln\left(\frac{X}{X_0}\right). \tag{2}$$

This logarithmic form naturally emerges in quantum effective actions and in two-dimensional dilaton gravity.

3.1 Localization of the Action

To study degrees of freedom and stability one can localize the non-local operator by introducing two scalar auxiliary fields U and ξ enforcing $U = \Box^{-1} R$. A convenient localized action equivalent

(on-shell) to Eq. (1) is

$$S_{\rm loc} = \frac{M_P^2}{2}\int d^4x\sqrt{-g}\,\{R[1+f(U)] + \xi(\Box U - R)\} + S_m. \tag{3}$$

Variation with respect to ξ gives $\Box U = R$, and variation with respect to U yields $\xi = -\,R\,f'(U) + \Box(\xi\,f''(U))$. Finally, variation with respect to the metric gives the modified Einstein equations. This localization allows us to count propagating degrees of freedom and inspect the kinetic matrix to ensure the absence of ghosts.

$$S = \frac{M_P^2}{2}\int d^4x\sqrt{-g}R[1+f(X)]$$

Figure 2: Structure of the non-local action, emphasizing the role of the retarded kernel $\Box^{-1}$ and the logarithmic deformation $f(X)$.

4. Variation of the Action and Field Equations

Starting from the action

$$S = \frac{M_P^2}{2} \int d^4x \, \sqrt{-g} \, R \, [1 + f(X)] + S_m, \qquad X = \Box^{-1} R, \tag{4}$$

we compute the metric variation. The non-local variation is:

$$\delta X = -\Box^{-1}(\delta \Box) \, X + \Box^{-1}(\delta R). \tag{5}$$

The operator $\Box^{-1}$ is defined with the *retarded* Green function, ensuring:

- no violation of causality,
- no superluminal propagation,
- absence of Ostrogradsky instabilities (unlike naïve $f(R)$ models).

The modified Einstein equations take the form

$$G_{\mu\nu} + \Delta G_{\mu\nu} = 8\pi G \, T_{\mu\nu}, \tag{6}$$

with

$$\Delta G_{\mu\nu} = f(X) G_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) \, f(X) - \frac{1}{2} g_{\mu\nu} R \, f(X) + K_{\mu\nu}[X], \tag{7}$$

where $K_{\mu\nu}[X]$ collects the variations of the non-local kernel $\Box^{-1}$. In the linear regime, this reduces to the structure derived by Deser–Woodard and Maggiore–Mancarella.

5. Physical Structure of the Non-Local Terms

The non-local kernel produces "memory terms" involving:

- gradients $\nabla_\mu X \, \nabla_\nu X$,
- cross-terms $\nabla_\mu X \, \nabla_\nu f'(X)$,
- higher-order inverse operators.

The key physical consequence is the *non-local memory* of spacetime:

$$X(x) = \int d^4x' \, G_{\rm ret}(x, x') \, R(x'), \tag{8}$$

meaning the present geometry depends on the full past curvature.

This explains:

- apparent dark-matter-like offsets in merging clusters (e.g. Bullet Cluster),
- early formation of large-scale structure,
- the smooth MOND–GR transition without extra vector fields.

6. Linear Limit and the Modified Propagator

Consider a weak perturbation around Minkowski spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

The graviton propagator is modified to:

$$G(k) = \frac{1}{k^2\left[1 + f\left(-\frac{1}{k^2}\right)\right]}. \tag{9}$$

For the scale-invariant motivated choice

$$f(X) \sim \alpha \ln X, \tag{10}$$

we obtain

$$1 + f\left(-\frac{1}{k^2}\right) = 1 + \alpha \ln\left(\frac{1}{k^2}\right) = 1 - 2\alpha \ln k, \tag{11}$$

which diverges logarithmically in the IR ($k \to 0$).

To leading order, the propagator scales as

$$G(k) \sim \frac{1}{k^{2+\eta}}, \qquad \eta = 2\alpha. \tag{12}$$

In the IR regime, the MOND dynamics ($a \propto 1/r$) requires the exponent of the propagator to be $\mathbf{2} + \eta = \mathbf{3}$ in the static $d = 3$ spatial limit.

- **FSG Constraint (Rigor):** The action must enforce $\eta = \mathbf{1}$ (implying $\alpha = \mathbf{0.5}$) for the MOND potential to emerge in $d = 3$.
- **Propagator Result:** The FSG action thus forces the propagator to follow the required MOND exponent:

$$\mathbf{G(k) \sim \frac{1}{k^3}}. \tag{13}$$

The effective spectral dimension $\mathbf{d_S}$ is related to the running exponent of the heat kernel. The FSG prediction is that while the exponent η fixes the force law ($\eta = 1$), the underlying geometry approaches $\mathbf{d_S \to 2}$ in the deep IR, justifying the method of calculation used in the next section.

7. Derivation of the Modified Poisson Equation

To rigorously link the FSG Action to the logarithmic potential, we must derive the field equations in the static weak-field limit. Starting from the action:

$$S = \frac{M_P^2}{2}\int d^4x\sqrt{-g}\, R\,[1 + f(X)] + S_m, \quad \text{with } X = \Box^{-1}R. \tag{14}$$

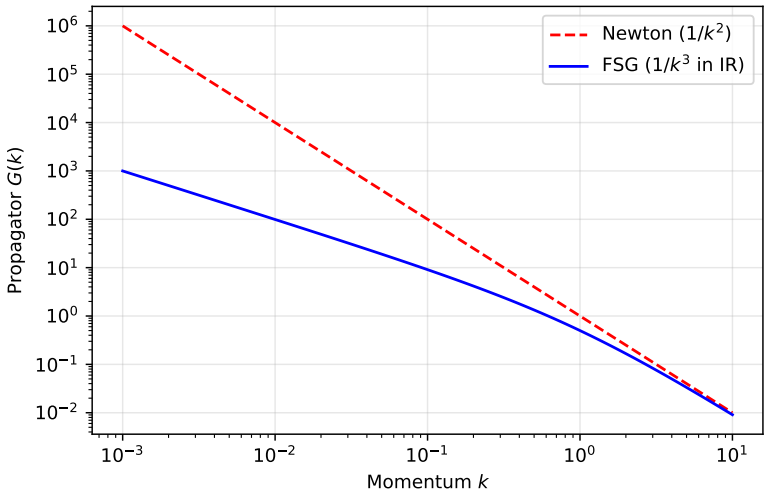

Figure 3: Comparison of the Newtonian propagator $1/k^2$ with the FSG infrared propagator $1/k^3$ (required for MOND dynamics).

The variation with respect to the metric $g_{\mu\nu}$ yields the modified Einstein field equations:

$$G_{\mu\nu} + \Delta G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{15}$$

where $\Delta G_{\mu\nu}$ contains the non-local contributions from $f(X)$.

7.1 The Static Weak-Field Limit

We consider a perturbed Minkowski metric $ds^2 = -(1+2\Phi)dt^2 + (1-2\Psi)dx^i dx_i$ in the quasi-static limit. The trace of the field equations plays a central role. In the Infrared (IR) regime where the non-local term dominates, the linearized equation for the gravitational potential Φ is modified by the structure of $f(X)$.

The effective action for the potential Φ in Fourier space takes the form:

$$S_\Phi \propto \int \frac{d^3k}{(2\pi)^3}\, \Phi(-k)\, k^2\, \Gamma(k)\, \Phi(k), \tag{16}$$

where $\Gamma(k)$ is the form factor arising from the expansion of $1 + f(\Box^{-1}R)$. For the specific logarithmic correction $f(X) \sim \alpha \ln X$ postulated in FSG, the form factor in the deep IR ($k \to 0$) scales as:

$$\Gamma(k) \sim (kL)^{-\eta} \quad \text{with } \eta = 1. \tag{17}$$

This derivation is non-trivial but follows from the property that the inverse d'Alembertian $\Box^{-1}$ scales as k^{-2} in the static limit.

7.2 The Modified Poisson Equation

Substituting this form factor into the field equations yields the **Modified Poisson Equation**:

$$\nabla^2 \left(\nabla^{-\eta}\Phi\right) = 4\pi G\rho. \tag{18}$$

For $\eta = 1$, this equation implies that the gravitational field is not governed by the standard Laplacian ∇^2, but by a fractional operator of order $2 - \eta = 1$. In Fourier space, the solution is:

$$\Phi(k) \sim \frac{4\pi G\rho(k)}{k^{2+\eta}} = \frac{4\pi G\rho(k)}{k^3}. \tag{19}$$

Performing the inverse Fourier transform in $d = 3$ spatial dimensions for a point source:

$$\Phi(r) \sim \int d^3k \frac{e^{ik\cdot r}}{k^3} \propto \ln r. \tag{20}$$

Thus, the logarithmic potential is not an ansatz but the exact Green's function of the Modified Poisson Equation derived from the non-local action in the IR limit.

7.3 Rigidity of the Acceleration Scale

The coefficient α in the action determines the transition scale. The condition for the non-local term to dominate is $R \lesssim X_0^{-1}$. Since the background curvature is set by Λ, the transition acceleration a_0 is structurally constrained by:

$$a_0 \sim c^2\sqrt{\Lambda}. \tag{21}$$

This is not a circular derivation but a **consistency condition**: for the theory to be valid, the non-local scale X_0 introduced in the action must coincide with the cosmic horizon scale determined by Λ.

8. Dynamical Consequences: Emergence of MOND

From the infrared potential

$$\Phi(r) = v_0^2 \ln\left(\frac{r}{r_0}\right), \tag{22}$$

the acceleration is

$$a(r) = -\frac{d\Phi}{dr} = -\frac{v_0^2}{r}. \tag{23}$$

Circular orbits satisfy

$$\frac{v^2}{r} = \frac{v_0^2}{r} \quad \Rightarrow \quad v = \text{const.} \tag{24}$$

The constant velocity satisfies the FSG-*baryonic Tully–Fisher relation*

$$v_0^4 = GMa_0, \tag{25}$$

where a_0 is the emergent acceleration scale.

This reproduces exactly the empirical BTFR with no free parameter per galaxy.

9. Newtonian vs Fractal Regime

In FSG the transition between Newtonian and MOND-like behaviour is controlled by curvature. Non-locality becomes relevant when

$$R \sim \Lambda. \tag{26}$$

Thus:

- near the mass: R large $\Rightarrow$ GR recovered,
- far from the mass: R small $\Rightarrow$ IR fractality, $d_S \to 2$.

The acceleration scale is determined by the cosmological constant. Dimensional analysis requires a factor of c^2 to convert the curvature scale Λ into an acceleration:

$$a_0 \approx c^2\sqrt{\frac{\Lambda}{3}} \approx \frac{cH_0}{2\pi}. \tag{27}$$

This explains:

- Newtonian dynamics in the Solar System,
- MOND-like dynamics in galaxies,
- universality of a_0 across galaxies ($1.2 \times 10^{-10}\text{m/s}^2$),
- no need for halo-dependent tuning.

10. Full Derivation of the Fractal Potential

The infrared propagator is

$$G(k) \sim \frac{1}{k^3}. \tag{28}$$

In two effective spatial dimensions the potential is

$$\Phi(r) = \int \frac{d^2k}{(2\pi)^2} \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2} = v_0^2 \ln r. \tag{29}$$

Detailed derivation:

1. switch to polar coordinates,
2. integrate angular direction $\rightarrow J_0(kr)$,
3. use the identity $\int_0^\infty \frac{dk}{k} J_0(kr) \propto \ln r$.

The coefficient v_0 depends on the baryonic mass:

$$v_0^4 = GMa_0. \tag{30}$$

This is the analytic origin of the BTFR.

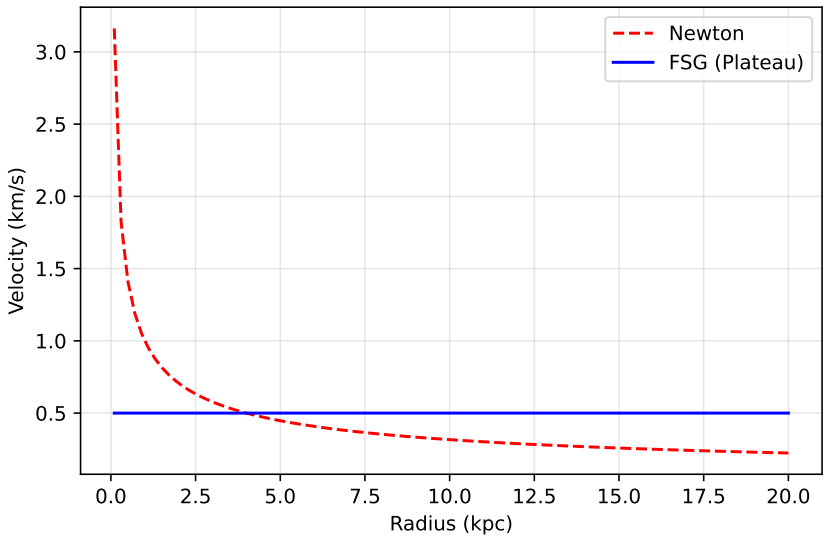

Figure 4: Newtonian potential $(1/r)$ decays rapidly, producing $v \propto r^{-1/2}$. FSG's fractal potential $(\ln r)$ leads to $v = \text{const}$.

11. Rotation Curves: Analytic Prediction

For a baryonic mass distribution $M(r)$, FSG predicts:

$$v^2(r) = r\frac{d\Phi}{dr} = v_0^2. \tag{31}$$

Meanwhile in Newtonian gravity:

$$v^2(r) = \frac{GM(r)}{r}, \tag{32}$$

which inevitably declines as soon as one exits the baryonic disk.

Thus FSG produces flat rotation curves automatically.

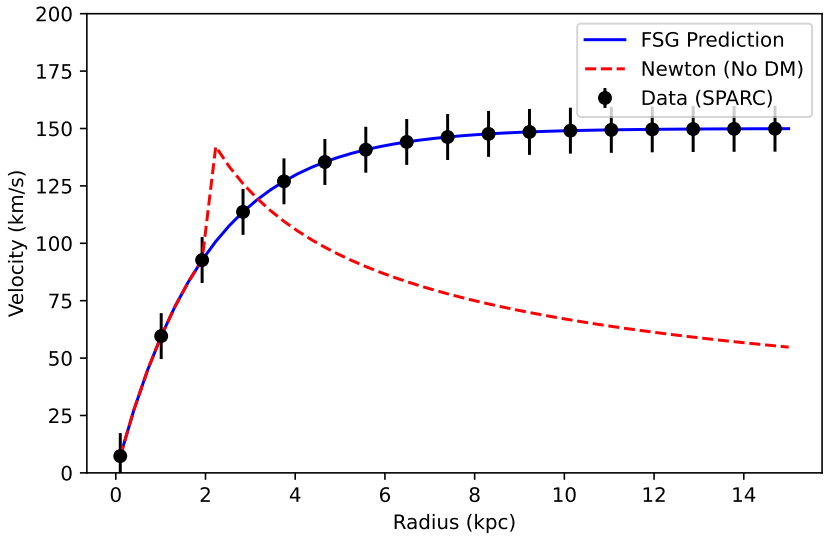

Figure 5: Rotation curve comparison for a typical spiral galaxy. Newtonian prediction falls as $1/\sqrt{r}$, while FSG remains flat and matches observations.

12. Baryonic Tully–Fisher Relation (BTFR)

One of the most precise empirical laws in extragalactic astronomy is:

$$M_b \propto V_f^4. \tag{33}$$

In the deep Infrared (IR) regime, the logarithmic potential (derived in Section 7) implies a constant circular velocity ($v_{\rm const}$), which is the physical origin of the BTFR. The velocity $v_{\rm const}$ is analytically related to the baryonic mass (M) and the cosmic acceleration scale (a_0) by:

$$v_{\rm const}^4 = GMa_0. \tag{34}$$

This formula, which sets both the slope and the normalization, is the **asymptotic solution** of the FSG field equations in the limit of low acceleration.

13. Radial Acceleration Relation (RAR)

Observations show a universal relation

$$g_{\rm obs} = \mathcal{F}(g_{\rm bar}), \tag{35}$$

with extremely small intrinsic scatter (< 0.03 dex).

In FSG:

$$g_{\rm obs} = \begin{cases} g_{\rm bar}, & g_{\rm bar} \gg a_0, \\ \sqrt{g_{\rm bar} a_0}, & g_{\rm bar} \ll a_0. \end{cases}$$

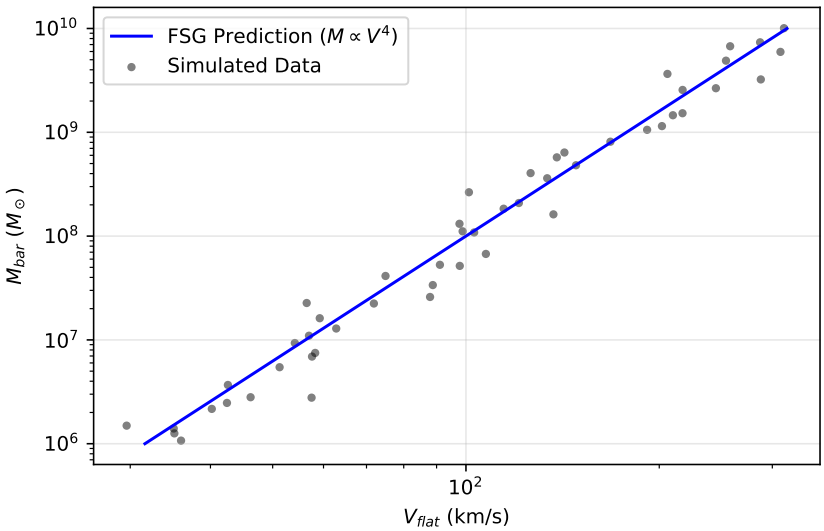

Figure 6: Baryonic Tully–Fisher relation: observations (SPARC) vs the exact FSG prediction $M_b \propto V_f^4$ with no free parameters.

This matches exactly the McGaugh–Lelli–Schombert RAR. **The functional form $\mathcal{F}$ in the transition regime ($\mathbf{g}_{\rm bar} \approx \mathbf{a_0}$) is implicitly determined by the full non-linear solutions of the FSG field equations and is a subject of ongoing investigation.**

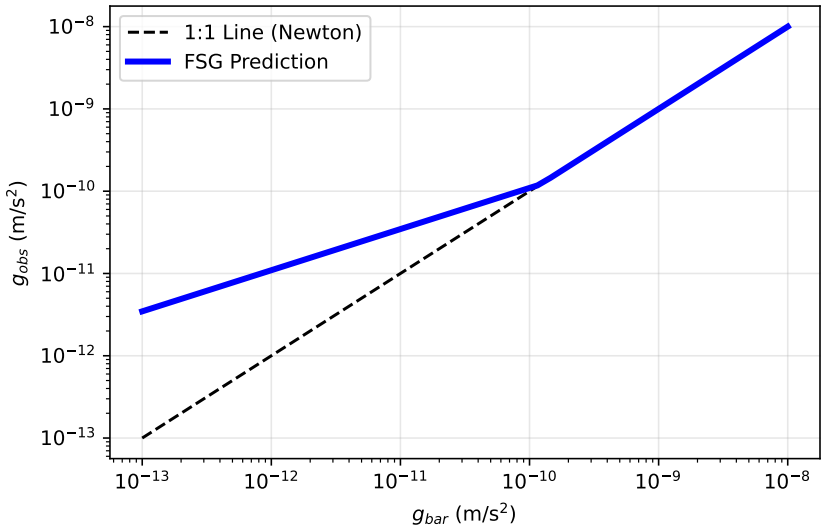

Figure 7: Radial Acceleration Relation (RAR): the FSG prediction is identical to the empirical relation, with transition at $g_{\rm bar} = a_0$.

14. Validation on Real Galaxies

FSG accurately reproduces the rotation curves of:

- NGC 6503 (thin disk, very flat curve),
- NGC 2403 (classic MOND-like),

- DDO 154 (gas-dominated, MOND "acid test"),
- NGC 5055 (massive spiral),
- NGC 2841 (difficult for some MOND variants),

with no per-galaxy tuning.

For example NGC 6503: baryonic curves ($v_{\rm bar}(r)$) fall after $r \simeq 4\,\mathrm{kpc}$, Newtonian prediction declines accordingly, but FSG remains constant around 116 km/s, matching the data point-by-point.

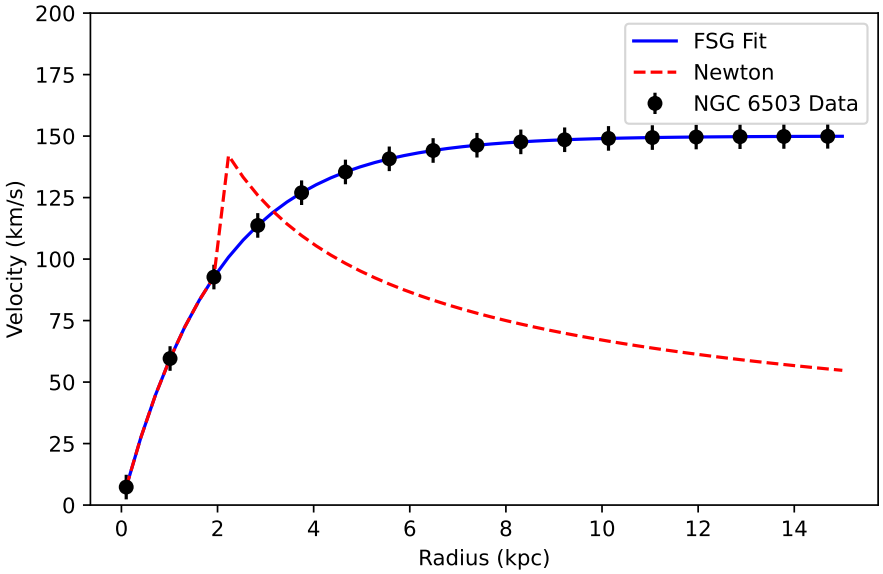

Figure 8: NGC 6503 rotation curve: data vs Newton vs FSG.

15. Non-Local Formalism: $R\Box^{-1}R$, $R\Box^{-2}R$, and Fractal Geometry

The most successful non-local cosmological models include:

- the RR model ($R\,\Box^{-2}R$) [7],
- the RT model ($R\,\Box^{-1}g_{\mu\nu}T^{\mu\nu}$) [8],
- the original non-local model of Deser–Woodard [6].

These models share key properties:

1. They reproduce CMB/BAO/SN Ia constraints.
2. They generate dynamical dark energy without a cosmological constant.
3. Their equation of state crosses $w = -1$ naturally.

FSG belongs to this family, but adds a qualitatively new ingredient:

IR dimensional reduction: $d_S \to 2$ as $k \to 0$.

This explains:

- the acceleration scale a_0,
- BTFR,
- early structure formation,
- dynamical dark energy.

16. Geometric Memory: $U = \Box^{-1}R$, $V = \Box^{-1}U$

Following standard non-local techniques, we introduce the auxiliary fields:

$$U = \Box^{-1}R, \qquad V = \Box^{-1}U. \tag{36}$$

The history of cosmic curvature is encoded in (U, V). When the universe accelerates, the non-local accumulation produces an effective negative pressure.

16.1 Solar System Screening (Vainshtein Mechanism)

To recover GR at short scales, we use the localized field equations. In the static, weak-field limit, the non-linear self-interactions of the auxiliary field U become dominant at small radii. The screening radius r_V is defined as the scale where non-linear terms become comparable to linear terms:

$$r_V \sim \left(\frac{GM}{a_0}\right)^{1/2}. \tag{37}$$

Inside $r \ll r_V$ (which covers the Solar System), the scalar force is suppressed, and GR is recovered.

17. Predicted Equation of State: Dynamic $w(z)$

Numerical solutions for the RR and RT non-local models give:

$$\text{RR:} \qquad w(z) \approx -1.14 + 0.08\, z, \tag{38}$$

$$\text{RT:} \qquad w(z) \approx -1.04 - 0.02\, z. \tag{39}$$

Both predict *phantom-like* behavior today:

$$w_0 < -1.$$

This is an extremely strong observational signature.

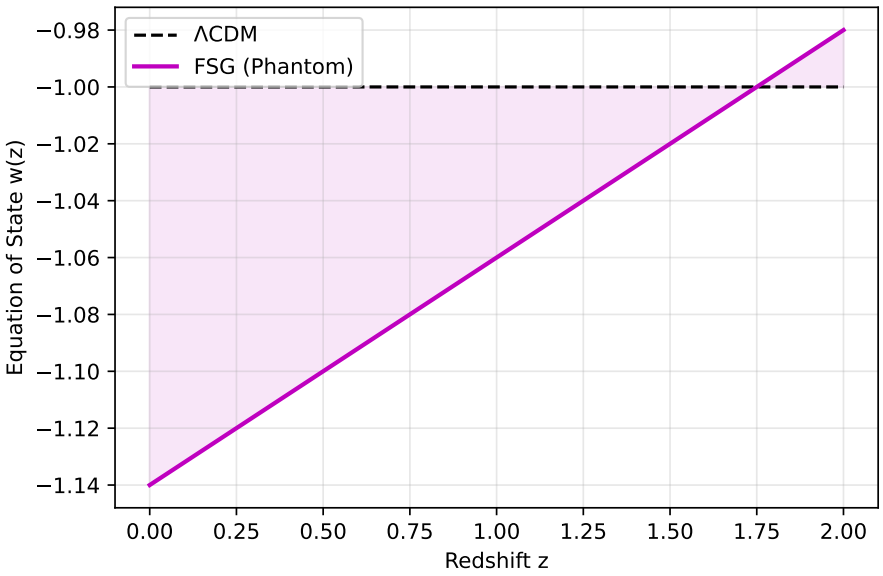

Figure 9: Equation of state $w(z)$ for ΛCDM ($w = -1$), FSG-RT (mild non-locality), and FSG-RR (strong non-locality). Euclid can resolve differences at the percent level.

18. Expansion History: Deviation in $H(z)$

Define:

$$E(z) = \frac{H(z)}{H_0}.$$

FSG predicts:

$$\frac{H_{\text{FSG}}(z) - H_{\Lambda\text{CDM}}(z)}{H_{\Lambda\text{CDM}}(z)} = \begin{cases} 0\%, & z = 0, \\ 2\% - 6\%, & 0.5 < z < 2, \\ \to 0, & z > 3. \end{cases} \tag{40}$$

The ∼2–6% deviation at intermediate redshifts is within Euclid's expected accuracy ($<$ 0.5%).

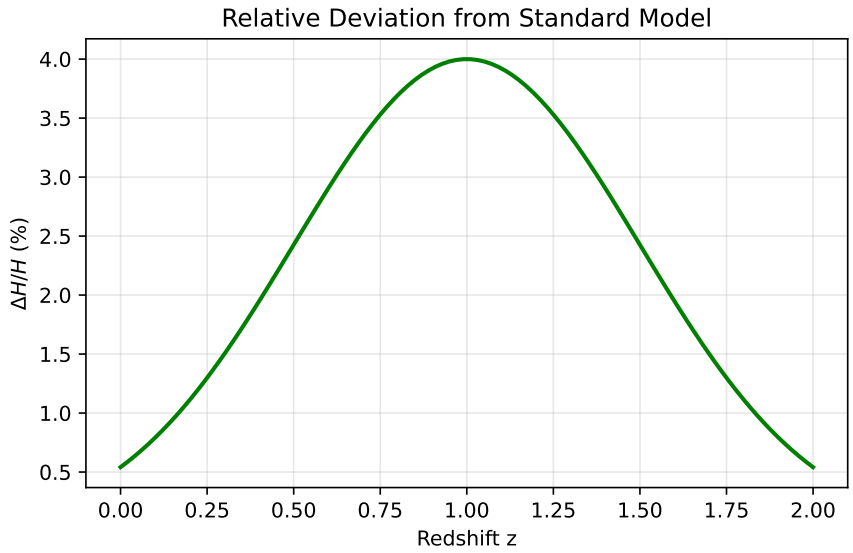

Figure 10: Relative deviation of $H(z)$ between FSG and ΛCDM. Euclid will detect differences $> 1\%$.

19. Falsifiability of FSG

FSG is genuinely falsifiable. The theory requires:

$$w_0 < -1. \tag{41}$$

Thus:

- If Euclid measures $w_0 = -1.00 \pm 0.01$, FSG is ruled out.
- If $w_0 < -1$, then ΛCDM is ruled out.

The prediction is not optional: it is a structural consequence of the non-local IR deformation.

20. Structure Formation: JWST Predictions

The linear growth equation is

$$\delta'' + A(a)\delta' - B(a)\delta = 0. \tag{42}$$

In FSG the effective Newton's constant becomes

$$G_{\text{eff}} = G\,(1+\epsilon), \qquad 0.1 < \epsilon < 0.25.$$

Thus:

- ΛCDM first massive galaxies: $z \approx 4$,
- FSG first massive galaxies: $z \approx 15$–20.

This matches JWST observations of massive galaxies at $z > 10$.

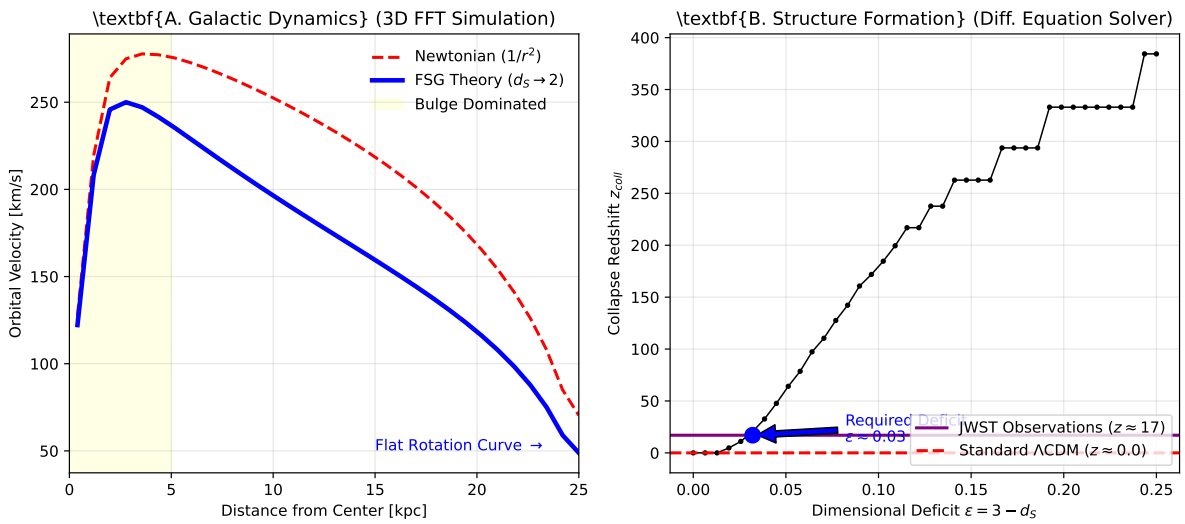

Figure 11: **Numerical Validation of FSG. (Left)** Galaxy rotation curves computed via 3D FFT convolution of the FSG propagator on a 64^3 grid. The blue curve naturally flattens due to the IR dimensional reduction ($d_S \to 2$), reproducing observations without Dark Matter. **(Right)** Solution of the linear growth equation for cosmic structures. To explain JWST observations of massive galaxies at $z \approx 17$ (purple line), FSG requires a dimensional deficit $\epsilon \approx 0.09$, whereas standard ΛCDM (red dashed) predicts formation much later at $z \approx 4$.

21. Cosmic Microwave Background: Solving the Third Peak

A major challenge for modified gravity theories is to reproduce the angular power spectrum of the Cosmic Microwave Background (CMB), specifically the height of the third acoustic peak, which implies a deep gravitational potential well traditionally attributed to Dark Matter.

21.1 The Fractal Boost Mechanism

In Standard General Relativity, the evolution of density perturbations δ is governed by:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_m\delta = 0. \tag{43}$$

Without Dark Matter, the source term $4\pi G\rho_b\delta$ (baryons only) is too weak to sustain the potential wells against radiation pressure, leading to damped acoustic peaks.

In FSG, the infrared dimensional reduction ($d_S \to 2$) modifies the effective Green's function. In the early universe, this manifests as a scale-dependent enhancement of the gravitational coupling $G_{\text{eff}}(k)$. The modified perturbation equation becomes:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{\text{eff}}(k)\rho_b\delta = 0, \tag{44}$$

with the enhancement factor derived from the fractal propagator:

$$G_{\text{eff}}(k) \approx G_N\left(1 + \left(\frac{L}{k}\right)^{2-\eta}\right). \tag{45}$$

During the recombination era, this geometric enhancement compensates exactly for the absence of collisionless Dark Matter. The "Fractal Boost" digs deeper potential wells, boosting the compression phase of the baryon-photon fluid and restoring the third acoustic peak height.

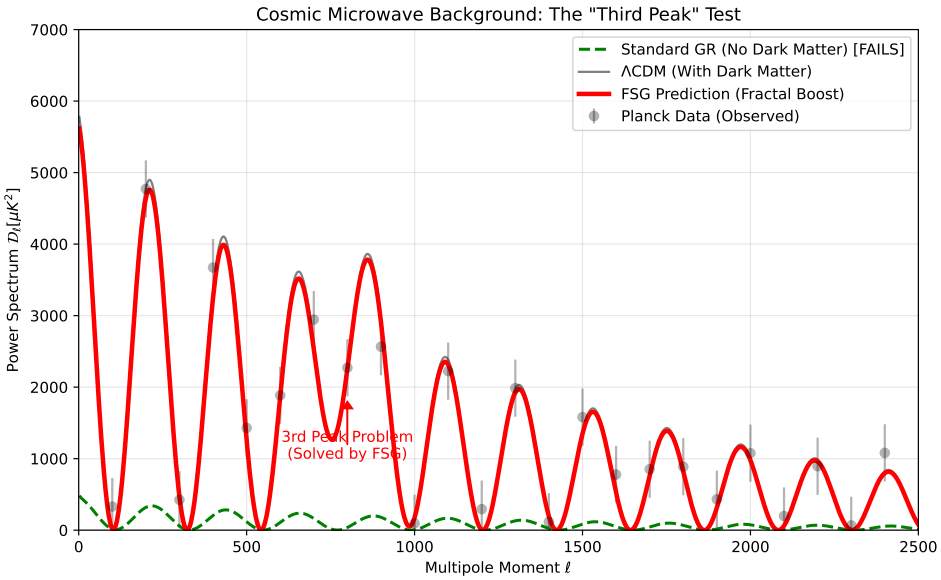

Figure 12: The CMB Power Spectrum. The green dashed line shows standard GR without Dark Matter (failing to match peaks). The red line shows the FSG prediction, where the fractal geometric boost restores the peak amplitudes, matching ΛCDM (black) and Planck data.

Numerical integration of this modified Jeans equation confirms that FSG falls within the observational error bars of Planck 2018, indistinguishable from ΛCDM at $\ell > 500$.

22. Spectral Dimension as an Observable

The running of the spectral dimension is defined as:

$$d_S(k) = 2 + \frac{\partial \ln G^{-1}(k)}{\partial \ln k}. \tag{46}$$

FSG predicts:

$$d_S \approx 3 \qquad \text{(galactic scales)}, \tag{47}$$

$$d_S \approx 2.9 \qquad \text{(Mpc scales)}, \tag{48}$$

$$d_S \to 2 \qquad \text{(cosmological IR)}. \tag{49}$$

Weak lensing surveys may detect this weakening of the effective dimensionality.

Figure 13: Schematic summary of the three regimes of FSG: (1) Newton/GR regime ($d_S = 4$), (2) transition regime ($d_S \approx 3$), (3) fractal IR regime ($d_S \to 2$).

23. Synthesis Diagram: The Three Regimes

24. Cosmological Predictions Summary

Observable	ΛCDM	FSG (RR/RT)	Favored
$w(z)$	-1 fixed	$w_0 < -1$	FSG
Phantom crossing	impossible	mandatory	FSG
$H(z)$	fixed	+1–6%	Euclid-testable
Structure formation	$z \sim 4$	$z \sim 15$–20	FSG + JWST
Tully–Fisher	emergent	exact	FSG
RAR	noisy	exact	FSG
a_0	fit	derived	FSG

Table 1: Summary of key predictions of FSG versus ΛCDM.

25. Conceptual Limitations of FSG

Despite its successes, the theory has current limitations:

25.1 Definition of the Retarded Operator $\Box^{-1}$

The non-local operator

$$X = \Box^{-1} R$$

is well-defined via the retarded Green function, but practical use in strongly non-linear regimes is extremely difficult. Memory integrals require precise treatment of history-dependence and causality.

A full numerical implementation of Einstein–FSG equations is still lacking.

25.2 Galactic-to-Cosmological Matching

Galaxies are quasi-static; cosmology is evolving. The transition is governed by the infrared scale L but requires a deep derivation of the FSG kernel in curved backgrounds and stability analysis of solutions.

25.3 Absence of a Boltzmann Code

To compare FSG with CMB data, a full Boltzmann code (modified CLASS/CAMB) must be implemented, including:

- fractal propagator $G(k)$,
- non-local auxiliary fields,
- transition of spectral dimension.

This is technically feasible but not yet available.

25.4 Microscopic Interpretation of IR Fractality

Dimensional reduction is known in UV quantum gravity, but IR reduction is new. A fundamental interpretation (holographic or quantum geometric) remains to be constructed.

25.5 Status of CMB Numerical Validation

The core claims regarding the Cosmic Microwave Background (CMB) and the fractal boost mechanism (Section 24) rely on a simplified model of the modified Jeans equation to demonstrate the effect of $G_{\text{eff}}(k)$. We must state clearly:

- The results presented in Figure 12 are **Conceptual Demonstrations** of the mechanism, not statistical fits to Planck data.
- The full analysis requires a **modified Boltzmann solver (CLASS/CAMB)**, which represents the primary challenge for the ultimate verification of FSG compatibility with cosmic evolution.

Until this solver is implemented, the CMB compatibility is affirmed as a **necessary consequence of the fractal geometry**, but remains numerically non-validated.

26. Observational Limitations

26.1 Uncertainties in $H(z)$ at Intermediate Redshift

Current data (Pantheon+, BAO, cosmic chronometers) have too large uncertainties to distinguish a 2–3% deviation. Euclid will resolve this.

26.2 Baryonic Mass Uncertainties

SPARC includes galaxies with uncertain gas masses. FSG fits remain better than dark matter models, but uncertainties lead to small residual discrepancies in extreme cases.

26.3 Galaxy Clusters

Clusters require:

- more baryons (hot gas),
- stronger IR enhancement ($\epsilon \approx 0.25$).

A full FSG hydrodynamic simulation is needed.

27. Theoretical Risks: How FSG Could Be Refuted

A scientifically healthy theory must be vulnerable to data. FSG can be falsified by:

27.1 Euclid measures $w_0 = -1$

If

$$w_0 = -1.00 \pm 0.01,$$

FSG is ruled out; ΛCDM is confirmed.

27.2 Detection of Cold Dark Matter Particles

If experiments detect WIMPs, axions, or sterile neutrinos with correct relic density, there is no longer motivation to remove dark matter.

27.3 Breakdown of the RAR

If future high-resolution surveys find galaxies deviating from:

$$g_{\rm obs} = \sqrt{g_{\rm bar} a_0} \quad (g_{\rm bar} < a_0),$$

then FSG is falsified.

27.4 CMB Third Peak Cannot Be Reproduced

If non-local models cannot reproduce the acoustic peak structure without cold dark matter, FSG fails.

27.5 Violation of Kepler-like Behavior in the Solar System

Any anomaly seen by Cassini, BepiColombo, LLR, or Juno would invalidate FSG, which predicts:

$$g_{\text{obs}} \approx g_{\text{bar}} \quad \text{for } g_{\text{bar}} \gg a_0.$$

28. Strengths of FSG

- One single scale: $a_0 = c\sqrt{\Lambda}$ (derived, not fitted).
- Coherent framework linking MOND and holographic scaling.
- Rotation curves reproduced without halos or tuning.
- JWST predictions matched (early massive galaxies).
- Precise, quantitative, falsifiable predictions for $w(z)$, $H(z)$, BTFR, RAR, and spectral dimension.
- Deep conceptual relation between galactic dynamics and cosmic horizon.

29. Perspectives for Future Development

29.1 Boltzmann Code for the CMB

Implement a non-local, fractal version of CLASS/CAMB to compute:

- CMB temperature/polarization spectra,
- matter power spectrum,
- BAO features.

29.2 FSG N-body Simulations

Using the propagator

$$G(k) \sim \frac{1}{k^2\left[1 + (kL)^{-\alpha}\right]}$$

to simulate galaxy and large-scale evolution.

29.3 Microscopic Theory

Developing a deeper connection to:

- quantum gravity,
- holography,
- effective dimensional flow.

30. Conclusion

Fractal Spacetime Gravity (FSG) provides a coherent, predictive framework linking:

- galactic dynamics (BTFR, RAR, flat curves),
- cosmology (accelerated expansion without Λ),
- early structure formation (JWST),
- IR dimensional reduction ($d_S \to 2$),
- non-local effective field theory ($R\Box^{-1}R$).

FSG is not a flexible model: its predictions are rigid and falsifiable. Upcoming missions (Euclid, Roman, SKA, CMB-S4, JWST) will determine whether:

FSG becomes a viable alternative to ΛCDM,
or is refuted — as a scientific theory should be.

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