CSC_5RO01_TA

Hybrid Optimal Control

Lecture 1: System Modelling & Simulation

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3A/Master Course, 2 ETCS, 2025/26

Learning Goals

Questions:

1. What are hybrid systems? Why do we need them?
2. What is a hybrid automaton (HA)?
3. How do we model hybrid systems?
4. How do we control their modal dynamics?

At the end of this lecture, you will ...

1. know about the basic building blocks of HAs,
2. understand how to interpret a HA,
3. have gained insights on modelling and simulating a HA,
4. have learned how to analyse and control the modal behaviour of a HA.

Why do we need hybrid systems?

Case Study: Human-Robot Collaboration

Geriatric care:

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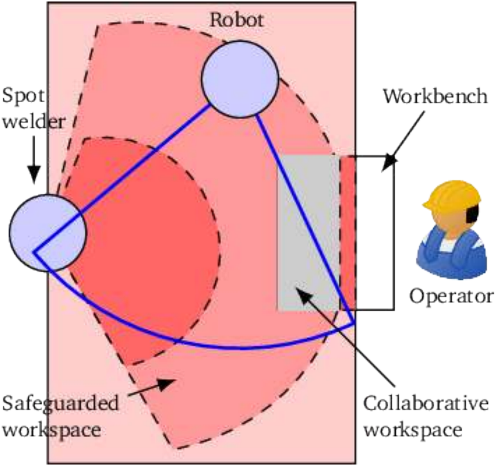
Multi-stage manufacturing:



(Gleirscher et al. 2022)

Case Study: Multi-Vehicle Applications

Industrial processes:

e.g., farming, mining

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Autonomous transportation:

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Goals: Increase control

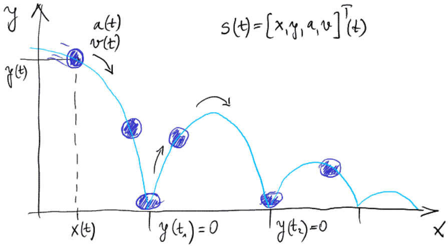
(i) performance (precision, stability),

(ii) dependability (safety, reliability)

Cyber-Physical Systems: Classical Examples

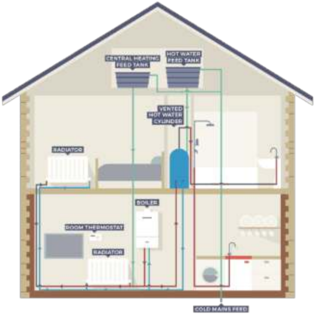
Kinematics:

e.g., bouncing ball



Thermal/chemical processes:

e.g., central heating, coffee machine



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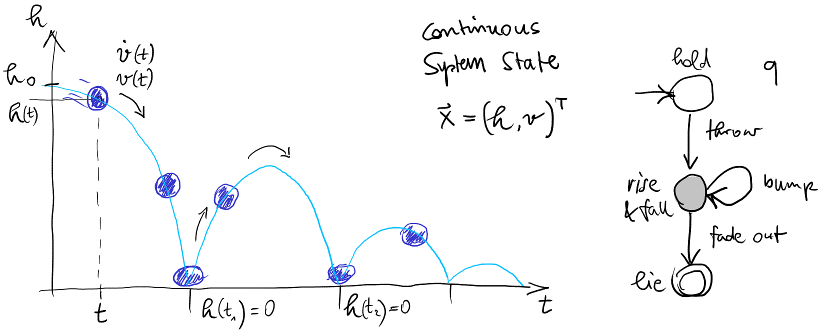
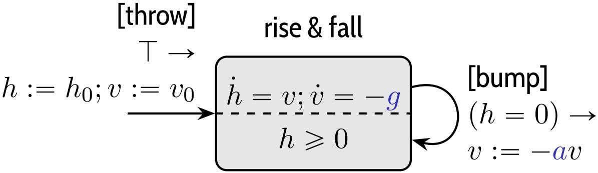
What are hybrid automata?
How do we use them to model complex dynamics?

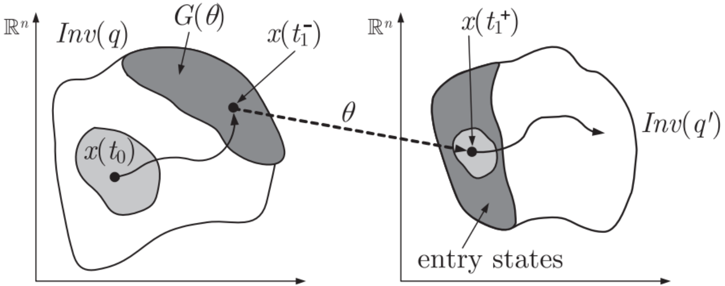
Hybrid Automata: Bouncing Ball



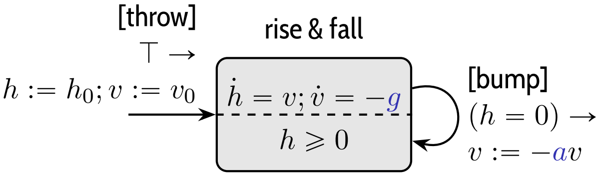


(Alur 2015, p. 382)

Hybrid Automata: Trajectory Semantics

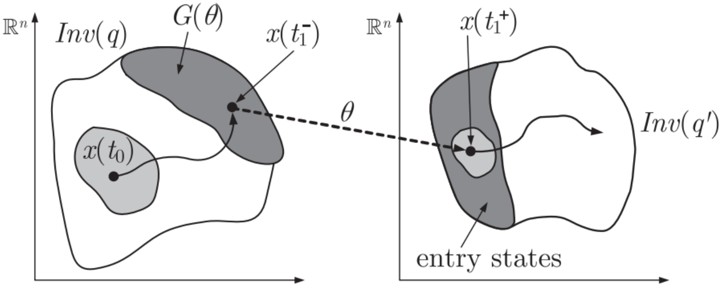


(Lunze and Lamnabhi-Lagarrigue 2009, p. 60)

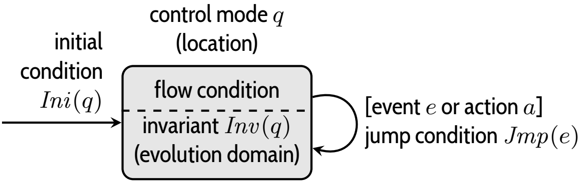


(Alur 2015, p. 382)

Hybrid Automata: Informal Definition



(Lunze and Lamnabhi-Lagarrigue 2009, p. 60)



Reasoning Problems in Hybrid Systems

Given an initialisation, does a (specific mode of a) hybrid system H

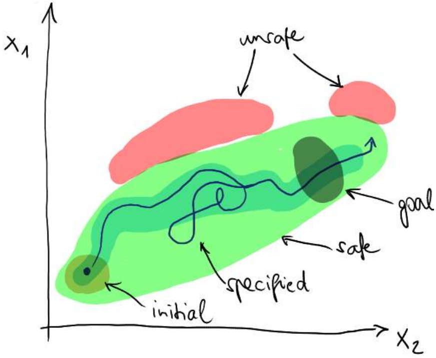
...remain in the safe (or never enter the unsafe) region? ► Safety

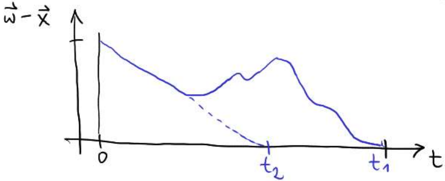
...remain in the specified region? ► Invariance

...eventually reach a specified goal region? ► Reachability

...yield an expected gain within a resource bound? ► Quantification

...show the desired step response (under uncertainty)? ► Stability





Reasoning Problems in Hybrid Systems

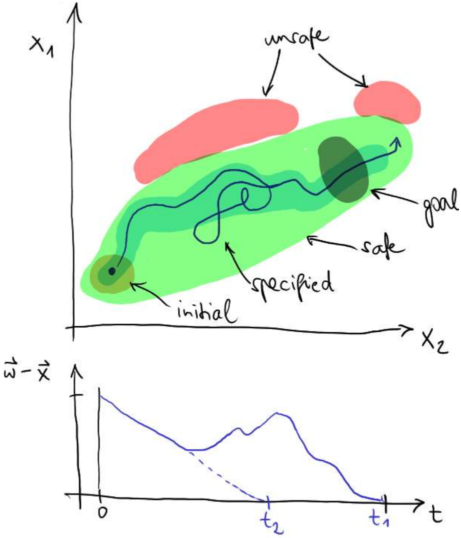
Satisfaction:
Does H fulfil a desired property φ?

Refinement:
Does H_2 fulfil at least the invariants of H_1?

Equivalence: Does H_2 have the same observable behaviour as H_1?

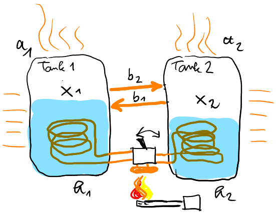
Existence:
Are there any controllers $\mathbf{u}$ for H that satisfy a given constraint φ?

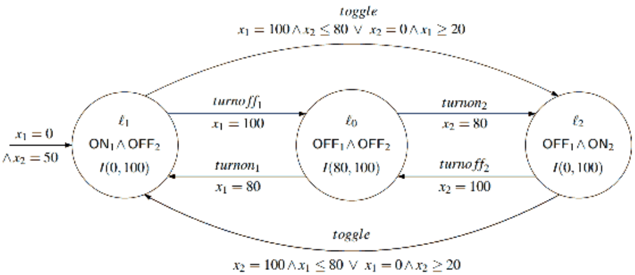
Optimality: Which controller $\mathbf{u}$ optimises a given performance criterion J?



Another Example: Two Heated Tanks

(Doyen et al. 2018, p. 8)

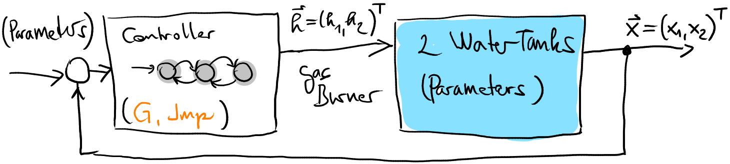




with implicit updates
$x_i^+ = x_i$ for $i \in 1, 2$

$\mathsf{ON}_1 = \dot{x}_1 = h_1 - a_1 x_1 + b_1 x_2$
$\mathsf{OFF}_1 = \dot{x}_1 = -a_1 x_1 + b_1 x_2$
$I(a,b) = a \leq x_1 \leq b \wedge a \leq x_2 \leq b$

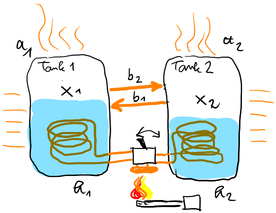
$\mathsf{ON}_2 \equiv \dot{x}_2 = h_2 - a_2 x_2 + b_2 x_1$
$\mathsf{OFF}_2 \equiv \dot{x}_2 = -a_2 x_2 + b_2 x_1$



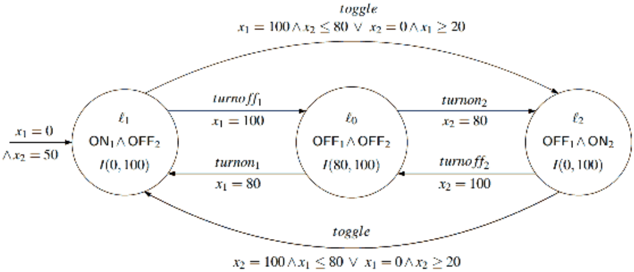
Note: The HA models the closed-loop system.

Another Example: Two Heated Tanks

(Doyen et al. 2018, p. 8)



with implicit updates
$x_i^+ = x_i$ for $i \in 1, 2$



$\mathrm{ON}_1 \equiv \dot{x}_1 = h_1 - a_1 x_1 + b_1 x_2$
$\mathrm{OFF}_1 \equiv \dot{x}_1 = -a_1 x_1 + b_1 x_2$
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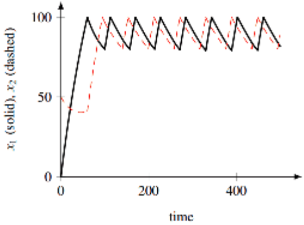
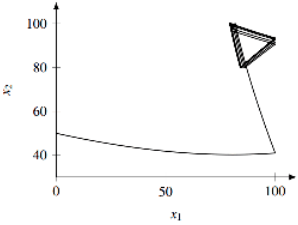
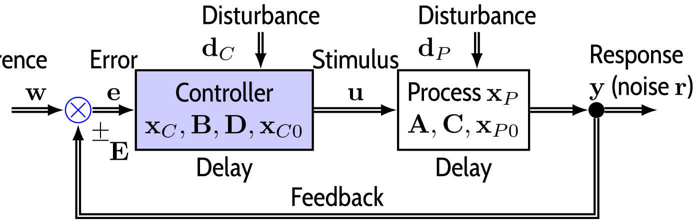




How do we control the modal dynamics of a hybrid automaton?

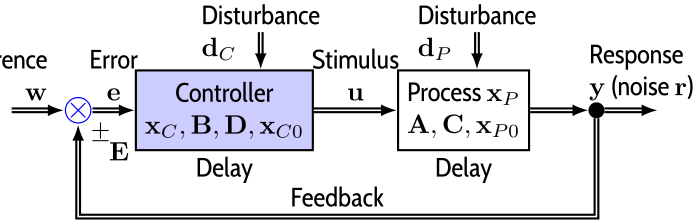
Control (State-based Multi-Parameter Linear Setting)



$$\mathbf{y} = \text{Process}(\mathbf{x}_P, \mathbf{u}, \mathbf{d}_P)$$
$$\mathbf{u} = \text{Controller}(\mathbf{x}_C, \mathbf{e}, \mathbf{d}_C)$$
$$\mathbf{e} = \text{Error}(\mathbf{w}, \mathbf{y})$$

$\mathbf{y}$ often defined via process dynamics $\dot{\mathbf{y}} = f(\mathbf{u})$ or state-space model $\dot{\mathbf{x}}_P = f(\mathbf{x}_P, \mathbf{u})$.
Variables are time-dependent and continuous or discrete. (Lunze 2020, p. 349)

Control (State-based Multi-Parameter Linear Setting)



$$\dot{\mathbf{x}}_P = \mathbf{A}\mathbf{x}_P + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{d}_P \qquad \text{(disturbed state change)}$$

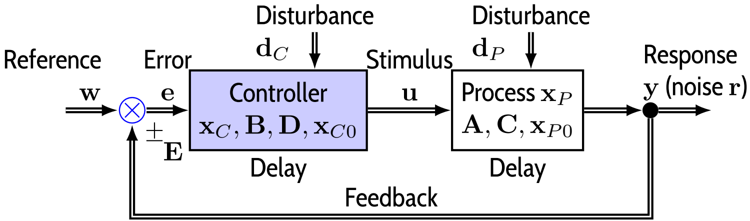
$$\mathbf{y}_m = \mathbf{C}\mathbf{x}_P + \mathbf{D}\mathbf{u} + \mathbf{r} \qquad \text{(noisy output)}$$

$$\mathbf{x}_P(0) = \mathbf{x}_{P0} \qquad \text{(initialisation)}$$

State-space formulation (linear (1,1,1)-parameter case):

$$x(t) = \underbrace{\mathrm{e}^{at} x_0}_{\text{natural}} + \underbrace{\int_0^t \mathrm{e}^{a(t-\tau)} b u(\tau)\, \mathrm{d}\tau}_{\text{forced}} \qquad \text{(motion)}$$

Control (State-based Multi-Parameter Linear Setting)



Equivalent input/output formulation ((1,n,1)-case):

$$y(t) = \mathbf{c}^\intercal \mathrm{e}^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t \mathbf{c}^\intercal \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)\,\mathrm{d}\tau + du(t) \qquad \text{(system response)}$$

$$y_f(t) = \int_0^t \mathbf{c}^\intercal \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)\,\mathrm{d}\tau + du(t) \qquad \text{(forced response)}$$

Stimuli, Transfer, and Response

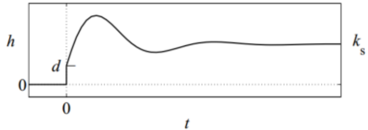
(Lunze 2020, pp. 168-175)

Process P's step response h for a step stimulus $u = u_0\sigma$ with Heaviside function

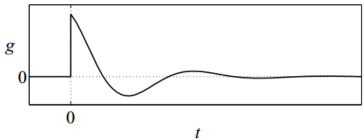
$$\sigma(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geqslant 0 \end{cases} \qquad \text{is}$$

$$h(t) = \int_0^t \mathbf{c}^\mathsf{T} \mathrm{e}^{\mathbf{A}\tau} \mathbf{b} \, \mathrm{d}\tau + d$$

with asymptotic stabilisation at $h(\infty) = k_s$.



Example: Accelerating crane with hanging load

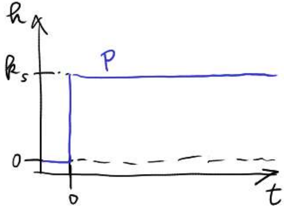


Impulse (or Dirac) stimulus $u = \delta$ and response g: e.g., pushing gas pedal for 3 sec.

Output via stimulus & impulse response: $y(t) = (g * u)(t) = \int_0^t g(t-\tau)u(\tau)\,\mathrm{d}\tau$

Basic Transfer Elements

Approximation of a system using template elements:



Describe that "output is proportional to input".

Definition (P-Element, proportional transfer)

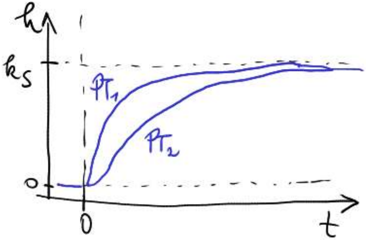
$$y = k_s u \quad \text{(output)}$$

$$h(t) = k_s \quad \text{(step response, gain)}$$

Basic Transfer Elements

Model that “after a delay, output is proportional to input”.

Example: Absorption of medication



Definition (PT_1-Element, P with 1st-order delay T)

$$T\dot{y} + y = k_s u \qquad y(0) = y_0 \qquad \text{(dynamics)}$$

$$\dot{x} = -\frac{1}{T}x + \frac{1}{T}u \qquad x(0) = \frac{1}{k_s}y_0 \qquad \text{(state change)}$$

$$y = k_s x \qquad \text{(output)}$$

$$h(t) = k_s(1 - \mathrm{e}^{-t/T}) \qquad g(t) = (k_s/T)\mathrm{e}^{-t/T} \qquad \text{(step \& impulse)}$$

$$y_s = k_s\sigma \qquad y_{tr}(t) = -k_s\mathrm{e}^{-t/T} \qquad \text{(stationary \& transient)}$$

Open- & Closed-Loop Systems (Stationary/Trans. Resp.)

For a P-controller (as an example)

$$u = k_\mathsf{P}(\underbrace{w - y_\mathsf{m}}_{\text{error } e})$$

the closed-loop system ($(1, n, 1)$-parameter case) results in

$$\dot{\mathbf{x}} = \underbrace{(\underbrace{\mathbf{A} - \mathbf{b}k_\mathsf{P}\mathbf{c}^\mathsf{T}}_{\text{system matrix}})\mathbf{x}}_{\text{controlled system}} + \underbrace{\mathbf{e}d}_{\text{disturb.}} + \underbrace{\mathbf{b}k_\mathsf{P}w}_{\text{command}} + \underbrace{\mathbf{b}k_\mathsf{P}r}_{\text{measurem. noise}} \qquad \text{(state change)}$$

$$y = \mathbf{c}^\mathsf{T}\mathbf{x} \qquad \text{(output)}$$

(Lunze 2020, p. 356)

Theorem (Asymptotic Regulation & Disturbance Rejection)

For a closed-loop system to regulate ($e(\infty) = 0$) and reject step-shaped command and disturbance signals, a stable open-loop system must incorporate an I-component.

Lecture 1: System Modelling & Simulation Summary

- Hybrid systems (HSs) theory combines the theories of finite automata and continuous control.
- HSs enable the modelling, simulation, and reasoning about systems with complex non-linear dynamics or differing modes of operation.
- The evolution of the state or an observed output of a dynamical system can be described by differential equations.
- These equations can be given in the state-space and the input/output formulations. These formulations can be used to simulate and investigate the system.
- The step and impulse forms of control stimuli and responses can be used to investigate basic characteristics of a system.
- One of the simplest elements to construct a closed-loop system with feedback control is the proportional (or P-) controller.

References I

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