CSC_5RO01_TA

# Hybrid Optimal Control

## Lecture 2: Control, Stability & Optimality

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# Learning Goals

Questions:

1. How to perform asymptotic regulation?
2. How to specify and assess control performance?
3. How to design optimal controllers for linear systems?

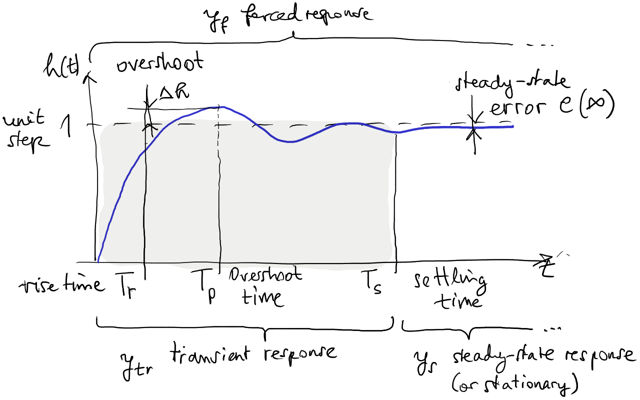
At the end of this lecture, you will ...

1. understand how proportional-integral-derivative (PID) control laws can achieve asymptotic regulation and
2. have learned how linear-quadratic (LQ) regulators (LQRs) provide more flexibility and precision.

How can we perform asymptotic regulation?

# Requirements for Controller and Closed-loop System

- a Stability (of natural and transient responses): $\lim_{t\to\infty} y_{\mathsf{na}}(t) = 0, \lim_{t\to\infty} y_{\mathsf{tr}}(t) = 0$
- b Asymptotic regulation: $\lim_{t\to\infty} \big(w(t) - y(t)\big) = 0, e(\infty) = 0$
- c Error (= stationary response): $e = w - y = y_{\mathsf{s}}$
- d Robustness: a-c under model uncertainties, disturbance, or set-point change




(Nise 2019, pp. 2, 166)

# Design of PID Controllers ((1, n, 1)-parameter case)

Goal: Controller efficiently stabilises closed-loop system under step- and impulse-shaped command and disturbance signals.

## Definition (Basic PID Control Law and Transfer)

$$u(t) = \underbrace{k_{\mathsf{P}} e(t)}_{\text{P-component}} + \underbrace{\frac{k_{\mathsf{P}}}{T_{\mathsf{I}}} \int_0^t e(\tau)\, \mathrm{d}\tau}_{\text{I-component}} + \underbrace{\frac{k_{\mathsf{P}} T_{\mathsf{D}}}{T} \mathrm{e}^{-\frac{t}{T}} \frac{\mathrm{d}e(t)}{\mathrm{d}t}}_{\text{delayed D-component}}$$

$$k_{\mathsf{PID}}(s) = k_{\mathsf{P}}\left(1 + \frac{1}{T_{\mathsf{I}} s} + \frac{T_{\mathsf{D}} s}{T s + 1}\right) \qquad \text{(transfer)}$$

$$h_{\mathsf{PID}}(t) = k_{\mathsf{P}} + \frac{k_{\mathsf{P}}}{T_{\mathsf{I}}} t + \frac{k_{\mathsf{P}} T_{\mathsf{D}}}{T} \mathrm{e}^{-\frac{t}{T}} \delta(t) \qquad \text{(step response, } e = \sigma\text{)}$$

with Dirac impulse $\delta$, proportional factor $k_{\mathsf{P}}$, adjustment and lead times $T_{\mathsf{I}}, T_{\mathsf{D}}$, and very small delay $T \ll T_{\mathsf{D}}$. (Lunze 2020a, p. 412)

Is there a flexible way to assess control performance?

# Asymptotic Stability

(Lin et al. 2022, Defs. 4.1, 4.2, p. 158/161)

For system $\dot{\mathbf{x}} = f(\mathbf{x})$ with solution $\xi(t; \mathbf{x}_0)$ and $\mathbf{x}(0) = \mathbf{x}_0$,
$\mathbf{x}_e = \mathbf{0}$ is an (isolated) equilibrium point if $f(\mathbf{x}_e) = \mathbf{0}$ for $t \geqslant 0$.

$$\mathbf{x}_e \text{ stable} \Leftrightarrow \forall \epsilon > 0 \, \exists \delta\colon |\mathbf{x}_0| < \delta \Rightarrow \forall t \geqslant 0\colon |\xi(t; \mathbf{x}_0)| < \epsilon$$

(Lyapunov stability)

$$\mathbf{x}_e \text{ attractive} \Leftrightarrow \exists \delta > 0\colon |\mathbf{x}_0| < \delta \Rightarrow \lim_{t \to \infty} |\xi(t; \mathbf{x}_0)| = 0$$

(Attractivity)

## Definition (Asymptotic Stability)

An equilibrium point $\mathbf{x}_e$ is said to be asymptotically stable if it is both Lyapunov-stable and attractive.

# Lyapunov's Direct Method

(Lin et al. 2022, Defs. 4.1, 4.2, p. 158/161)

Let $\mathcal{X} \subseteq \mathbb{R}^n$ be convex and closed, and $\mathfrak{C}^1(D)$ be the class of continuous differentiable functions of type $D \to \mathbb{R}$.

**Definition (Strengthened Lyapunov Stability Theorem)**

$\mathbf{x}_e$ asympotically stable $\Leftrightarrow \exists V \in \mathfrak{C}^1(\mathcal{X})\, \forall \mathbf{x} \in \mathcal{X} \backslash \{\mathbf{0}\}:$

$$V(\mathbf{x}) > 0 \wedge \dot{V} = \frac{d}{dt} V(\mathbf{x}) = \frac{\partial V}{\partial \mathbf{x}} f(\mathbf{x}) < 0$$

$V$ is called a Lyapunov function.

Engineer's task: Find $V$ for the control problem at hand.

How can we use stability theory to design optimal control laws?

# The Special Case of LQ Regulators

(Lunze 2020b, p. 306)

We assume to have a linear time-invariant (LTI) system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

with an infinite-horizon quadratic cost functional

$$J(\mathbf{u}) = \int_0^\infty \big( \underbrace{\mathbf{x}^\intercal \mathbf{Q} \mathbf{x}}_{\text{quadratic form}} + \mathbf{u}^\intercal \mathbf{R} \mathbf{u} \big)\, \mathrm{d}t$$

with positive (semi-)definite $\mathbf{Q} = \bar{\mathbf{Q}}^\intercal \bar{\mathbf{Q}}$ and $\mathbf{R}$, and observable $(\mathbf{A}, \bar{\mathbf{Q}})$.

Considering state-feedback controllers of the form $\mathbf{u}^* = -\mathbf{K}^* \cdot \mathbf{x}$ with $\mathbf{K}^* = \mathbf{R}^{-1}\mathbf{B}^\intercal\mathbf{P}$, the solution to $\min_{\mathbf{K}} J$ is given by the symmetric positive definite solution $\mathbf{P}$ of the matrix Riccati equation

$$\mathbf{A}^\intercal\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\intercal\mathbf{P} + \mathbf{Q} = \mathbf{0}\,.$$

# The Special Case of LQ Regulators

(Lunze 2020b, p. 306)

The LQR framework with the Riccati equation forms a special case of Pontryagin's necessary conditions for linear dynamics $f$, free final time $t_f$ (infinite horizon) and final state $x_f(t_f)$, and a quadratic cost function $J$.

A solution to the Riccati equation

- provides a time-invariant state-feedback optimal control law $\mathbf{u}^*(\mathbf{x})$, and
- is rather straightforward with tools, such as the Python and GNU/Octave control packages, for example, via the `lqr` function.

# Observability and Controllability

(Lunze 2020b, pp. 66, 94)

Let a set of state variables $X$ with $|X| = n, \mathbb{R}^X = \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, $t_\mathrm{f} \in \mathbb{R}_{\geqslant 0}$, and $\mathbf{x}_{[a,b]} \colon [a, b] \to \mathbb{R}^X$.

A system $f(q)$: $\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} = \mathbf{C}\mathbf{x} \end{cases}$ with $\mathbf{x}(0) = \mathbf{x}_0$ is

- controllable if $\forall \mathbf{x}_0, \mathbf{x}_\mathrm{f} \in \mathbb{R}^X$ :
  $$\exists t_\mathrm{f} \in \mathbb{R}_{\geqslant 0}, \mathbf{u}_{[0,t_\mathrm{f}]} \in \overline{\mathcal{U}} \colon \mathbf{x}_\mathrm{f} = \mathrm{e}^{\mathbf{A}t_\mathrm{f}}\mathbf{x}_0 + \int_0^{t_\mathrm{f}} \mathrm{e}^{\mathbf{A}(t_\mathrm{f}-\tau)}\mathbf{x}_0\mathbf{B}\mathbf{u}(\tau)\,\mathrm{d}\tau$$
- observable if $\forall \mathcal{I} = [0, t_\mathrm{f}] \subset \mathbb{R}_{\geqslant 0}, \mathbf{u}_\mathcal{I} \in \overline{\mathcal{U}}, \mathbf{y}_\mathcal{I}, \mathbf{y}_{na,\mathcal{I}} \in \overline{\mathcal{Y}}, t \in \mathcal{I}$:
  $$\mathbf{y}_\mathcal{I} = \mathbf{y}_{na,\mathcal{I}} + \mathbf{y}_{f,\mathcal{I}}(\mathbf{u}_\mathcal{I}) \Rightarrow \exists_1 \mathbf{x}_0 \in \mathbb{R}^X \colon \mathbf{y}_{na,\mathcal{I}}(t) = \mathbf{C}\mathrm{e}^{\mathbf{A}t}\mathbf{x}_0$$

Sufficient criterion after Kalman: $rank\,\mathbf{S}_C = rank\,\mathbf{S}_O = n$ with $\mathbf{S}_C = [\mathbf{B}, \mathbf{A}\mathbf{B}, \mathbf{A}^2\mathbf{B} \dots \mathbf{A}^{n-1}\mathbf{B}]$ and $\mathbf{S}_O = [\mathbf{C}, \mathbf{C}\mathbf{A}, \mathbf{C}\mathbf{A}^2 \dots \mathbf{C}\mathbf{A}^{n-1}]^\intercal$.

# Summary

- PID laws are expressed in terms of proportionality factors and lead times to be identified, whereas LQ regulators are expressed in terms of putting costs on state and input vectors.
- Observability implies the existence and uniqueness of a solutions to control problems, for example, a solution to a Riccati equation yielding an LQR law.
- Most controller design (or synthesis) techniques allow one to achieve a particular system specification (e.g., safety constraints, performance criteria) during the design process.
- This reduces the burden on post-hoc verification and validation (V&V) (e.g., simulation and testing).

# References I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