CSC_5RO01_TA

Hybrid Optimal Control

Lecture 6: Switched Optimal Control

Mario Gleirscher (v0.1, 9/1/26)

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Learning Goals

Questions:

1. How can we solve combined problems of optimal modal control and optimal switching times?

At the end of this lecture, you will ...

1. understand the problem formulation and solution of switched linear-quadratic (LQ) regulator (LQR) problems.

Switched LQR

(Lin et al. 2022, p. 284)

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{A}_1\mathbf{x} + \mathbf{B}_1\mathbf{u} & t_0 \leqslant t < t_1 \\ \mathbf{A}_2\mathbf{x} + \mathbf{B}_2\mathbf{u} & t_1 \leqslant t < t_f \end{cases}$$

(original system)

$$J(t_1, \mathbf{u}) = \mathbf{x}^T(t_\mathrm{f})\mathbf{Q}_\mathrm{f}\mathbf{x}(t_\mathrm{f}) + \int_{t_0}^{t_\mathrm{f}} (\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,\mathrm{d}t$$

(cost functional)

$$t = \begin{cases} t_0 + \overbrace{(x_2 - t_0)}^{d_1}\,\tau & 0 \leqslant \tau < 1 \\ x_2 + \underbrace{(t_\mathrm{f} - x_2)}_{d_2}(\tau - 1) & 1 \leqslant \tau < 2 \end{cases}$$

(state variable x_2 for time scaling)

$$\mathrm{d}\mathbf{x}/\mathrm{d}\tau = \begin{cases} d_1(\mathbf{A}_1\mathbf{x} + \mathbf{B}_1\mathbf{u}) & \tau \in [0, 1) \\ d_2(\mathbf{A}_2\mathbf{x} + \mathbf{B}_2\mathbf{u}) & \tau \in [1, 2) \end{cases}, \quad \mathrm{d}x_2/\mathrm{d}\tau = 0$$

(augmented system)

Switched LQR

(Lin et al. 2022, p. 284)

$$J(t_1, \mathbf{u}) = \mathbf{x}^T(2)\mathbf{Q}_\mathsf{f}\mathbf{x}(2) + \int_0^1 d_1(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,\mathrm{d}t$$
$$+ \int_1^2 d_2(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u})\,\mathrm{d}t \quad \text{(switched cost functional)}$$

Obtain an optimal controller $\mathbf{u}_i^* = -\mathbf{K}_i^*(\tau, x_2)\mathbf{x}(\tau, x_2)$ via $\min_{\mathbf{K}_i^*} J$, or via $\mathbf{K}_i^*(\tau, x_2) = \frac{1}{2}\mathbf{R}^{-1}\mathbf{B}_i^T\mathbf{P}^*(\tau, x_2)$ where $\mathbf{P}_i^*$ is the solution of two parameterised matrix Riccati equations

$$-\frac{\partial \mathbf{P}}{\partial \tau} = 0 = d_i(\mathbf{Q} + \mathbf{P}\mathbf{A}_i + \mathbf{A}_i^T\mathbf{P} - \mathbf{P}\mathbf{B}_i\mathbf{R}^{-1}\mathbf{B}_i^T\mathbf{P}) \quad \text{for} \quad \tau \in [i-1, i)$$

where $i = 1, 2$, such that optimal switching time can be obtained via

$$\min_{x_2} J(x_2, \mathbf{u}^*) \quad \text{via} \quad \frac{\partial J}{\partial x_2}(x_2) = 0 = \frac{V(\mathbf{x}_0, 0, x_2)}{\partial x_2} = \frac{1}{2}\mathbf{x}_0^T \frac{\mathbf{P}(0, x_2)}{\partial x_2}\mathbf{x}_0 \,.$$

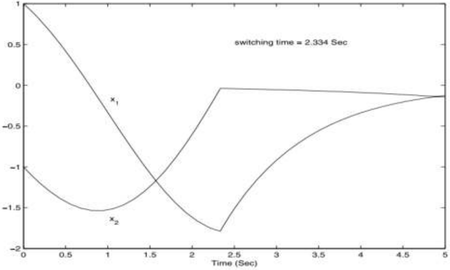
A more detailed derivation can be found in Lin et al. (2022, p. 284).

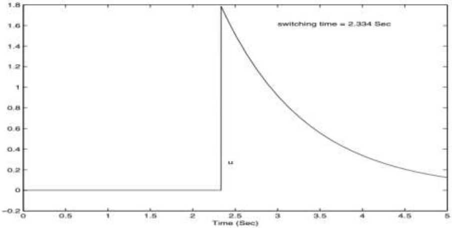
Example: Switched System

$\mathbf{A}_1 = [0,0;0,0.5], \mathbf{B}_1 = [1;0], \mathbf{A}_2 = [.1,1;-1,.1], \mathbf{B}_2 = [0;0]$

$$J(\mathbf{x}_0, \mathbf{u}, \sigma) = \mathbf{x}^T(t_\mathrm{f})\mathbf{x}(t_\mathrm{f}) + \int_0^{t_\mathrm{f}} \mathbf{x}^T(t)\mathbf{x}(t) + \mathbf{u}^2(t)\,\mathrm{d}t$$

Optimal switching sequence: $(0,2), (2.334,1)$ at optimal cost $J(\mathbf{x}_0) \approx 9.1808$





Summary

- For linear hybrid systems with controlled switches and a quadratic cost function, the hybrid optimal control problem can be reduced to a two-staged LQR-style optimal control problem based on system augmented with variables for time scaling.

Project: Hybrid Optimal Control of Heated Tanks

1. Submit Worksheet 6 as a group.
2. Final evaluation in-class based on the submitted PDF/ipynb:
   - 3 min: Solution pitch (explain hybrid automaton trace)
   - 3 min: Oral exam (1 Q&A per student)

Please, submit timely. Your final result will be put on the beamer.

References I

Lin, Hai and Panos J. Antsaklis (2022). Hybrid Dynamical Systems. Springer.