Inverse Geometric model (IGM)

Usefulness of the inverse kinematic models for manipulator : $q = f^{-1}(X)$

- Finding the joint coordinates q needed to bring the robot tool in a desired position and orientation X ;
- Transforming the coordinates for computer control algorithms from the desired task coordinates to references to joints coordinates.

Tricky problem : a general approach for finding its solution does not exist !

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- Type and number of equations raising the problem of the *existence* and *multiplicity* of the solution(s), in the general case :
  - No solution (ex. specification of a targeted position X out of the robot workspace) ;
  - A finite set of solutions ;
  - Infinite numbers of solutions.

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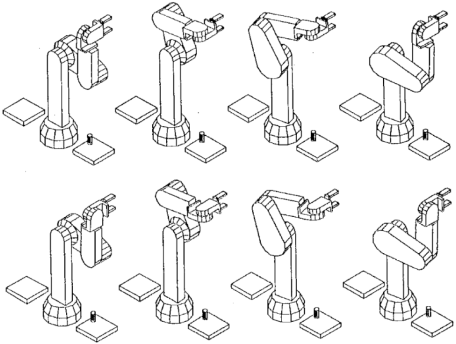
  - No solution (ex. specification of a targeted position X out of the robot workspace) ;
  - A finite set of solutions ;
  - Infinite numbers of solutions.
- Two main classes of approaches for solving the IGM :
  1. explicit solutions (true for decoupled six-dof robot, e.g. 6-dof robots with 3-dof spherical wrist mounted on a 3-dof arm [25], or robots with relatively simple geometry that have many zero distances and parallel or perpendicular joint axes [23])
  2. iterative numerical methods when no explicit form exists (mainly exploiting the inverse differential model).

Admissible solutions for the case $N = M$

Set of admissible solutions to the IGM problem : illustration for the case of a serial 6R robot (case where $M = N = 6$)



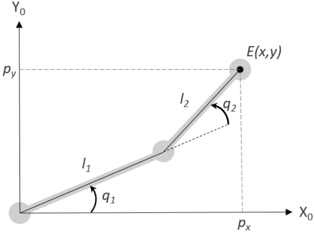
- 4 solutions out of singularities (for the only positioning of the wrist center) ;
- 8 solutions when considering the complete pose of the end-effector (spherical wrist : 2 alternative solutions for the last 3 joints).

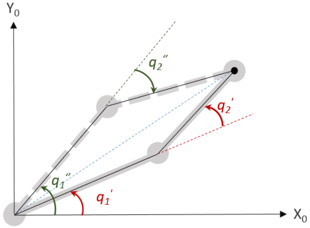
Admissible solutions for the case $N = M$

Explicit solution : example of the 2R robot ($M = N = 2$)
Direct Geometric model :

$$\begin{cases} p_x & = l_1 cos\left(q_1\right) + l_2 cos\left(q_1 + q_2\right) \\ p_y & = l_1 sin\left(q_1\right) + l_2 sin\left(q_1 + q_2\right) \end{cases}$$

In the regular case, the IGM problem has two pairs of solutions : $\left(q_1^{'}, q_2^{'}\right)$ and $\left(q_1^{''}, q_2^{''}\right)$.





Example

Algorithms for numerical computation of IGM

Newton-Raphson method (for $M = N$)

- Usefulness when an analytical solution to the problem $X_d = f(q)$ does not exist or is diffcult to obtain ;
- When considering a first-order *Taylor* series approximation of the function f giving the DGM,

$$X_d = f(q_k) + \underbrace{\frac{\partial f(q_k)}{\partial q}}_{J(q_k)} (q - q_k) + o\left((q - q_k)^2\right)$$

we propose the iteration of joint variables at next step as follows :

$$q_{k+1} = q_k + J^{-1}(q_k)\left[X_d - f(q_k)\right]$$

- convergence if q^0 (initial conditions) *relatively* close to the solution $q^* : X_d = f(q^*)$;
- quadratic convergence rate in the neighbourhood of the solution ;
- problems *close to* singularities of the Jacobian matrix $J(q)$ in the redundant case $(M < N)$.

Algorithms for numerical computation of IGM

Gradient-based method

- When considering the minimisation of the following objective-function,

$$H(q) = \frac{1}{2}\left\|X_d - f(q)\right\|^2 = \frac{1}{2}\left(X_d - f(q)\right)^t\left(X_d - f(q)\right)$$

the iteration joint variables at next step is made in the opposite direction of the gradient, so as to decrease the function H :

$$q_{k+1} = q_k - \alpha\nabla H(q_k) = q_k + \alpha J^t\left(q_k\right)\left[X_d - f(q_k)\right]$$

  - simpler on the computational point (transpose of the Jacobian matrix, and not its inverse) ;
  - direct usefulness for the case of task-redundant robot ;
  - searching for the amplification gain step α to guaranty decreasing of the error function H at each iteration (*linesearch* technique) ;
  - linear convergence rate ;

Algorithms for numerical computation of IGM

Gradient-based method

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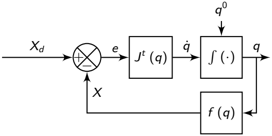
$$q_{k+1} = q_k - \alpha \nabla H(q_k) = q_k + \alpha J^t(q_k)[X_d - f(q_k)]$$

  - simpler on the computational point (transpose of the Jacobian matrix, and not its inverse) ;
  - direct usefulness for the case of task-redundant robot ;
  - searching for the amplification gain step α to guaranty decreasing of the error function H at each iteration (*linesearch* technique) ;
  - linear convergence rate ;
  - algorithm revisited as a feedback scheme ($\alpha = 1$).

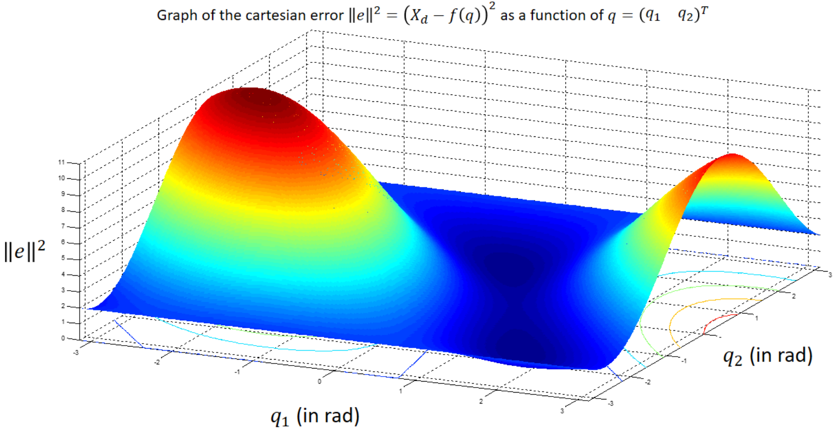


Demonstration for asymptotic stability of the algorithm

Demonstration

Algorithms for numerical computation of IGM

Case study of robot 2R when considering $X_d = (1, 1)$ and $l_1 = l_2 = 1$.



Algorithms for numerical computation of IGM

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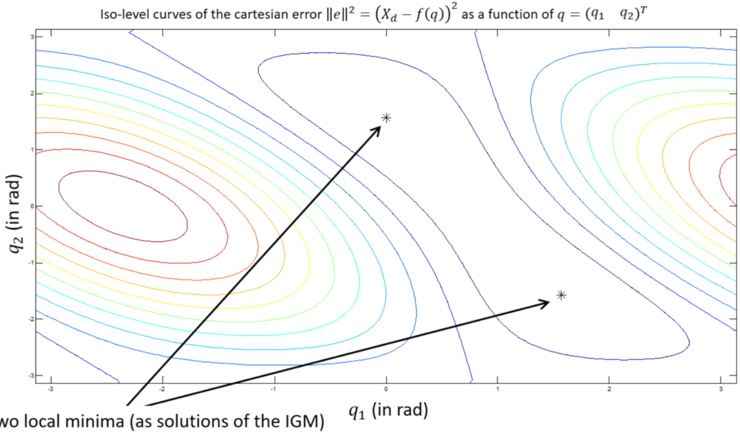

Pair of solutions coming from the explicit formulation : $\left(q_1', q_2'\right) = \left(0, \frac{\pi}{2}\right)$ and $\left(q_1'', q_2''\right) = \left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$

Algorithms for numerical computation of IGM

- Iterative optimisation procedure

$k \leftarrow 0$
while $\left\| J^t\left(q_k\right) \right\| > \epsilon$ **do**
  $k \leftarrow k+1$
  ▷ *Case of the Gradient-based method*
  $q_k \leftarrow q_{k-1} + \alpha J^t\left(q_{k-1}\right)\left[X_d - f(q_{k-1})\right]$
  ▷ *Case of the Newton-Raphson-based method*
  $q_k \leftarrow q_{k-1} + J^{-1}\left(q_{k-1}\right)\left[X_d - f(q_{k-1})\right]$
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- Difficulties coming from these methods
  - Lack of convergence when the error $e = X_d - f(q_{k-1})$ is in the null of J^t or in the singular configuration cases ;
  - Multiple initialisations with different q_0 to avoid local minima ;
  - Search for an adaptive step to choose the best α at each iteration ;
  - Consideration of joint limits only when the algorithm is ended.

Algorithms for numerical computation of IGM

- Iterative optimisation procedure

```
k ← 0
while ||J^t(q_k)|| > ε do
    k ← k + 1
                ▷ Case of the Gradient-based method
    q_k ← q_{k-1} + α J^t(q_{k-1}) [X_d − f(q_{k-1})]
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    q_k ← q_{k-1} + J^{-1}(q_{k-1}) [X_d − f(q_{k-1})]
end while
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return (q*)
```

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  - Search for an adaptive step to choose the best α at each iteration ;
  - Consideration of joint limits only when the algorithm is ended.

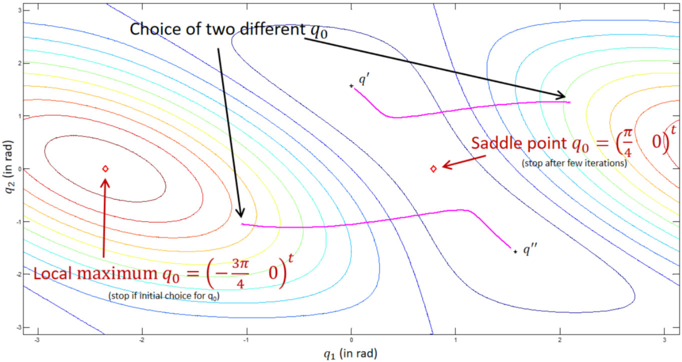
- Remarks
  - Possibilities to combine Gradient-based method for the first iterations (guaranteed convergence but with low convergence rate) and the *Newton-Raphson*-based method for the last iterations (quadratic convergence rate) ;
  - Other possible stop criteria ;
    - cartesian error $\|X_d - f(q_k)\| < \epsilon_x$
    - joint error $\|q_k - q_{k-1}\| < \epsilon_q$

Algorithms for numerical computation of IGM

Case study of robot 2R when considering $X_d = (1, 1)$ and $l_1 = l_2 = 1$.

- Analysis of the algorithm convergence

Iso-level curves of the cartesian error $\|e\|^2 = \left(X_d - f(q)\right)^2$ as a function of $q = (q_1 \quad q_2)^T$



- Case of no convergence when the Jacobian matrix $J(q)$ becomes singular in q and the error e belongs to the null of $J^t(q)$.

Example

Inversion of the kinematic model

Usefulness of the inverse kinematic model

1. Search for a joint velocity $\dot{q}$ enabling to achieve the cartesian velocity $\dot{X}$ according to $\dot{X} = J(q)\dot{q}$: search for inversion technique of $J \in \mathbb{R}^{m \times n}$ in the general case.
2. Inverse differential kinematic model also used for kinematic control along a continuous time scaling end-effector trajectory $X_d(t)$ in task space :
   - Tracking the trajectory $X_d\,(t)$ of the robot end-effector consisting in providing to the robot controller a succession of values X_{d_k} (index k corresponding to the sampling time $t_k = kT_e$) ;
   - Coordinates transformation needed to provide to the axes controllers the series of reference values q_{d_k} corresponding to X_{d_k} ;
     - execution of the previous iterative algorithm at each sample $t_0, \ldots, t_k, \ldots, t_f$: $q \leftarrow q + J^{-1}(q)\left(X_{d_k} - f\,(q)\right)$ (in general, 1 or 2 iterations being sufficient) ;
     - "reasonable" choice from q_{0_k} at t_k being the solution to the previous problem at t_{k-1} ;
   - Eventual problems requiring the search for robust inversion techniques :
     - crossing a singular configuration (case where $J(q)$ non-invertible) ;
     - redundant or under-determined robots (case where $J(q)$ not square).

Inversion of the kinematic model in the redundant case $(M < N)$

- Infinity of solutions to the inverse kinematic problem, from which one can choose the one that is *the closest possible* of a particular or preferred configuration $\dot{q}_0$
- Use of this redundancy for :
  - determining a motion out of singularity to avoid the previous case ;
  - avoiding obstacles or increasing dexterity.
- Inverse kinematics as a convex quadratic optimization problem with equality constraint :

$$\min_{\dot{q}} \frac{1}{2} (\dot{q} - \dot{q}_0)^t \, W \, (\dot{q} - \dot{q}_0) \, ,$$
$$\text{s.c.} \quad J(q) \, \dot{q} = \dot{X}_d$$

  - Positive-definite matrix $W \in \mathbb{R}^{n \times n}$ used for norm weighting purpose $\|\dot{q} - \dot{q}_0\|_W$:
    - to give *more or less importance* to the position or the orientation ;
    - to optimize the involved kinetic energy during the motion : $\frac{1}{2} \dot{q}^t A(q) \, \dot{q}$, etc.
  - Let note that in this problem the kinematic constraints are totally satisfied, which means that the desired pose in the task space is strictly reached (provided that $\dot{X}_d \in \mathscr{R}(J)$).

Inversion of the kinematic model in the redundant case $(M < N)$

Solution of the quadratic optimisation

The general solution to the problem of inverse kinematics in the redundant case is written as follows

$$\dot{q}^* = J_W^\sharp \dot{X}_d + \left(I_n - J_W^\sharp J\right) \dot{q}_0$$

- Particular solution to the problem $J_W^\sharp \dot{X}_d$, that of minimum weighted norm (for $\dot{q}_0 = 0$), with :

$$J_W^\sharp = W^{-1} J^t \left(J W^{-1} J^t\right)^{-1}$$

- Set of homogeneous solutions $(J\dot{q} = 0)$ being determined through the orthogonal projection of $\dot{q}_0$ in the null of J given by $\left(I_n - J_W^\sharp J\right)$

Demonstration

Inversion of the kinematic model in the redundant case ($M < N$)

Particular case where $W = I_n$:

- General solution of the non-weighted norm problem :

$$\dot{q}^* = \underbrace{J^\sharp \dot{X}_d}_{\substack{\text{Particular solution} \\ \text{(here the pseudo-inverse)}}} + \underbrace{\left(I_n - J^\sharp J\right) \dot{q}_0}_{\substack{\text{Orthogonal projection} \\ \text{of } \dot{q}_0 \text{ on } \mathcal{N}(J)}}$$

- right pseudo-inverse matrix defined by $J^\sharp = J^t \left(JJ^t\right)^{-1}$:
  - *Moore-Penrose* inverse of $\dot{q}^* = J^\sharp \dot{X}_d$ (*pinv()* function of *Matlab*^{TM}) defined as the unique solution of

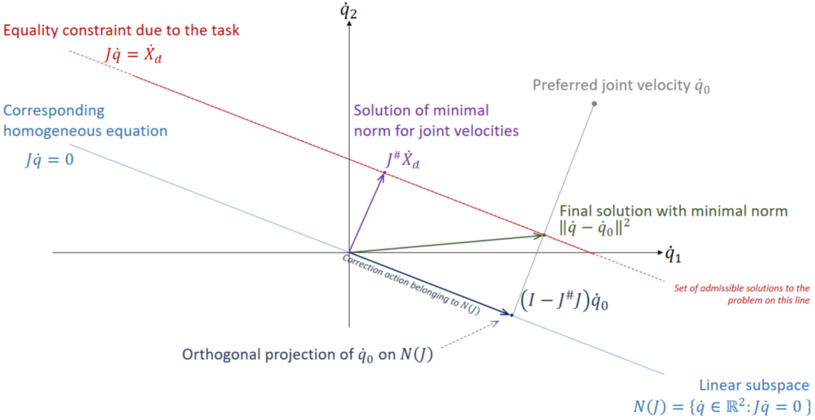
    $$\min_{\dot{q}} \frac{1}{2} \left\|\dot{q}\right\|^2$$

    for desired $\dot{X}_d$ (if $Rank\,(J) = M$, i.e. full row rank).
  - $J^\sharp$ said *pseudo-inverse* of J in the sense that it is not a real inverse (indeed, $JJ^\sharp = I_m$ - *right pseudo-inverse* - but $J^\sharp J \neq I_n$) ;
- Properties of the projector $N_J = \left(I_n - J^\sharp J\right)$:
  1. Symmetry : $\left(I_n - J^\sharp J\right)^t = \left(I_n - J^\sharp J\right)$;
  2. Idempotent : $\left(I_n - J^\sharp J\right)^2 = \left(I_n - J^\sharp J\right)$;
  3. Orthogonality between $J^\sharp \dot{X}_d$ and $\left(I_n - J^\sharp J\right) \dot{q}_0$.
  4. Invariance through pseudo-inversion : $\left(I_n - J^\sharp J\right)^\sharp = \left(I_n - J^\sharp J\right)$.

Inversion of the kinematic model in the redundant case $(M < N)$

Graphical illustration of joint velocity for the case where $N = 2$ and $M = 1$ in a given configuration $\overline{q} = (\overline{q}_1, \overline{q}_2)$:

$$J(\overline{q})\,\dot{q} = \dot{X}_d \Leftrightarrow \begin{bmatrix} J_1 & J_2 \end{bmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \dot{X}_d$$



Inversion of the kinematic model in the redundant case $(M < N)$

Secondary task in joint space

- Term $\left(I_n - J^\sharp J\right)\dot{q}_0$ belonging to the null space of J :
  - No influence on the value of $\dot{X}_d$;
  - Physical description of some internal motions of the robot ;
  - Use for satisfying additional optimisation constraints by projecting a secondary task on the null space of the Jacobian matrix.
- Search for preferred configurations using the *projected gradient* technique
  - Choice of a differentiable, scalar and positive-definite objective function H :

$$\dot{q}_0 = -\alpha \nabla_q H(q) = -\alpha \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_n} \end{pmatrix}$$

    where the term $\alpha > 0$ denotes the tradeoff between the minimisation objectives of $\frac{1}{2}\left\|\dot{q}\right\|^2$ and $H(q)$.
  - Decrease of the values taken by $H(q)$ at each iteration during the execution of the task $\dot{X}_d(t)$.

Inversion of the kinematic model in the redundant case $(M < N)$

Some usual cases for the choice of $H(q)$:

- Avoiding joint limits ($q_i \in [q_{min}, q_{max}]$) [7] :

$$H_{lim.}(q) = \sum_{i=1}^{n} \left(\frac{q_i - \overline{q}_i}{q_{max} - q_{min}} \right)^2 \quad where \quad \overline{q}_i = \frac{q_{min} + q_{max}}{2}$$

- Increasing manipulability (recall the velocity ellipsoids) :

$$H_{man.}(q) = \sqrt{\det\left(J(q)\,J^t(q)\right)}$$

  (maximisation of the distance to singularities)

- Avoiding or searching for particular joint configurations through some force fields deriving from attractive or repulsive potential functions [15]
- Avoiding obstacles through the maximisation of the minimal caretsian distance of the robot to obstacles :

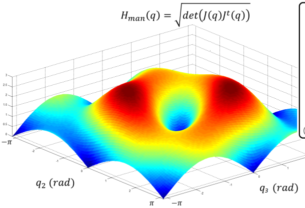
$$H_{obs.}(q) = \min_{\substack{\text{a : robot} \\ \text{b : obstacles}}} \|a(q) - b\|_2$$

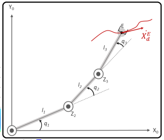
  (difficulties arising from the potential non-differentiability of the function)

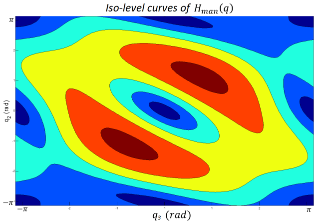
Inversion of the kinematic model in the redundant case ($M < N$)

Study of the manipulability : positioning of the end-effector E with the planar robot RRR (body lengths chosen to be unitary)

$$H_{man.}(q) = \sqrt{\det\left(J(q)\,J^t(q)\right)}$$



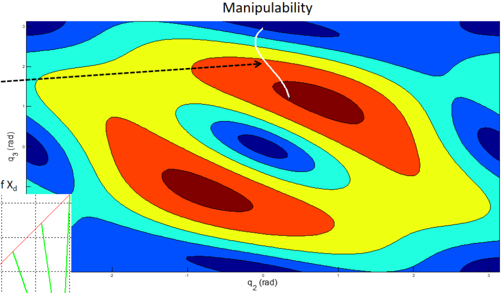


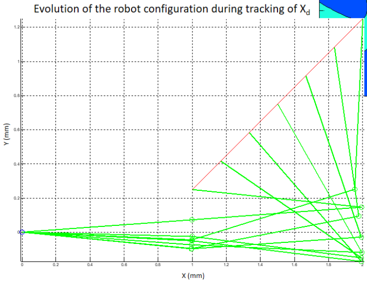


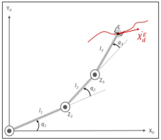
- Redundancy of order 1 ($M = 2$ and $N = 3$) ;
- Potential function independent of q_1 ;
- Minima of H_{man} for q_2 and q_3 belonging to $\{-\pi; 0; \pi\}$.

Inversion of the kinematic model in the redundant case ($M < N$)

Study of the manipulability : positionning of the end-effector E with the planar robot RRR (body lengths chosen to be unitary)







Planar RRR robot with tracking of the desired cartesian velocity

$$^0\dot{X}_d^E = [1 \quad 1]^t$$

Inversion of the kinematic model in the redundant case ($M < N$)

Secondary task in the operational space
Taking into account priorities through projections in the subspaces of the successive null spaces of the different tasks :

- Description of the tasks to be realized according to their order of priority :
  - First-priority subtask $\dot{X}_{d_1}$ corresponding to $J_1(q)$:

$$\dot{X}_{d_1} = J_1(q)\dot{q}$$

  - Second-priority subtask $\dot{X}_{d_2}$ corresponding to $J_2(q)$:

$$\dot{X}_{d_2} = J_2(q)\dot{q}$$

- General formulation of the solution :

$$\boxed{\dot{q}^* = J_1^\sharp \dot{X}_{d_1} + (J_2 N_{J_1})^\sharp \left(\dot{X}_{d_2} - J_2 J_1^\sharp \dot{X}_{d_1}\right)}$$

where we use the contracted notation N_{J_1} to denote the projector in the null space of J_1 : $N_{J_1} = \left(I - J_1^\sharp J_1\right)$.

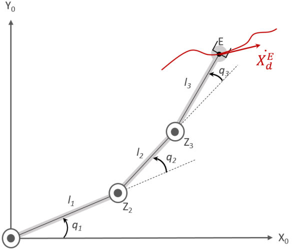
Demonstration

Inversion of the kinematic model in the redundant case ($M < N$)

Secondary task in the operational space

Case of the planar RRR robot with specification of the linear velocity of the end-effector ($M = 2$) :

Coming back to example 10 :



(unitary body lengths)

1. Given a desired velocity of the end-effector ${}^0\dot{X}_d^E(t) = (1,1)^t$, searching for the joint velocity $\dot{q}_1^*(t)$ (numerical evaluation when the robot is in configuration $\overline{q} = \left(\left(\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{3}\right)^t\right).$;
2. Searching for $\dot{q}_2^*(t)$ respecting the previous objective while adding $\dot{q}_1$ and $\dot{q}_2$ to be null (evaluation in $\overline{q}$).

Example

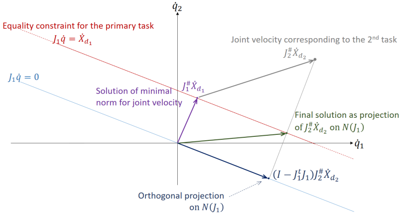
Inversion of the kinematic model in the redundant case ($M < N$)

Secondary task in the operational space

The previous method may lead to an algorithmic singularity when the null spaces of J_1 and J_2 are neighbours (inducing too high joint velocities).

- To avoid algorithmic singularities, an alternative formulation leads to determine the solution for the secondary task and then to project it on the null space of J_1 :

$$\dot{q}^* = \underbrace{J_1^\sharp \dot{X}_{d_1}}_{\text{Primary task}} + N_{J_1} \underbrace{\left(J_2^\sharp \dot{X}_{d_2}\right)}_{\text{Secondary task}}$$



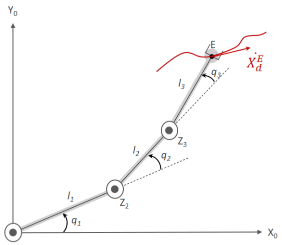
- For highly redundant system, possibility to put in series several tasks according to :

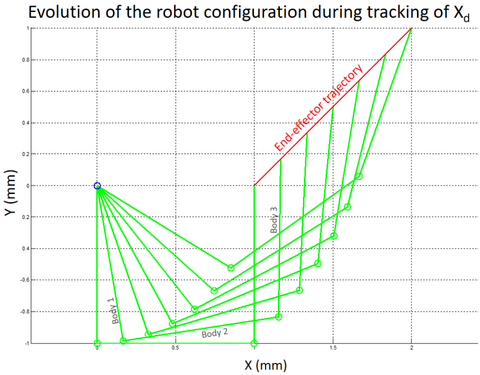
$$\dot{q}^* = J_1^\sharp \dot{X}_{d_1} + N_{J_1} \left(J_2^\sharp \dot{X}_{d_2} + \left(N_{J_1} \cap N_{J_2}\right) \left(J_3^\sharp \dot{X}_{d_3} + \ldots \right) \right)$$

the vector $J_3^\sharp \dot{X}_{d_3}$ being projected on the intersection of null spaces of both J_1 and J_2 so as to avoid perturbing both tasks.

Inversion of the kinematic model in the redundant case ($M < N$)

Study for avoiding obstacles : trajectory tracking of the end-effector with imposed linear velocity $\dot{X}_d^E(t) = (1,1)^t$ using a planar RRR robot





Inversion of the kinematic model in the redundant case ($M < N$)

Study for avoiding obstacles : trajectory tracking of the end-effector with imposed linear velocity $\dot{X}_d^E(t) = (1,1)^t$ using a planar RRR robot

- Searching for $\dot{q}_2$ for avoiding ponctual obstacles, as solution of

$$\min_{\dot{q}_2} \left\| \dot{X}_2(t) - J_2 \dot{q}_2 \right\|_2$$

  where
  - $\dot{X}_2(t)$ follows the repulsive law defined by :

$$\dot{X}_2(t) = \alpha(t)\, v_0 n_0(t)$$

    with :
    - the repulsion coefficient α given by :

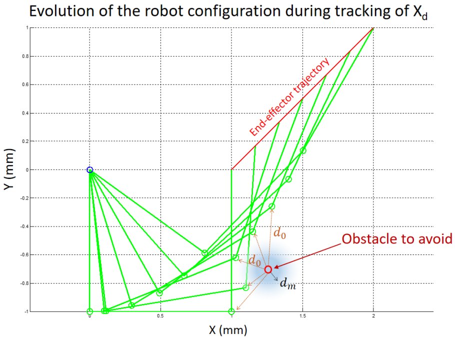
$$\alpha(t) = \begin{cases} \left(\frac{d_m}{\|d_0\|}\right)^2 - 1 & \text{if} \quad \|d_0\| < d_m \\ 0 & \text{if} \quad \|d_0\| \geq d_m \end{cases}$$

    - v_0 a nominal velocity arbitrary chosen ;
    - d_0 denotes the distance between the obstacle point and the extremity of the 2^{nd} body according to the unitary vector n_0 ;
    - d_m the distance corresponding to the influence radius ;
  - J_2 the Jacobian matrix associated to the extremity of the 2^{nd} segment.
- Final joint configuration as follows :

$$\dot{q}^* = J^\sharp \dot{X}_d^E(t) + \left(I_n - J^\sharp J\right) J_2^\sharp \dot{X}_2$$

Inversion of the kinematic model in the redundant case ($M < N$)

Study for avoiding obstacles : trajectory tracking of the end-effector with imposed linear velocity $\dot{X}_d^E(t) = (1,1)^t$ using a planar RRR robot



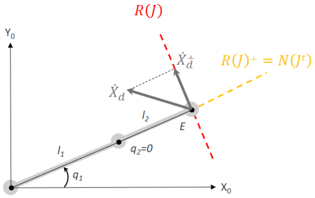
Inversion of the kinematic model in the singular case ($M = N$)

Pseudo-inverse method : $\dot{q}^* = J^\sharp \dot{X}_d$

- Singular configuration q limiting the capacity to generate arbitrary motions, and formalized by a loss of rank for $J(q)$:

$$Rank(J) = r < M$$

- Several possible cases in a singular configuration :
  1. If $\dot{X}_d \in \mathscr{R}(J)$ exclusively, then the constraint $J(q)\,\dot{q} = \dot{X}_d$ is satisfied (the velocity is achievable even if the inverse J^{-1} does not exist) ;
  2. Otherwise, the constraint $J(q)\,\dot{q} = \dot{X}_d$ is not fulfilled : $J\dot{q}^* = \dot{X}_d^\perp$ where $\dot{X}_d^\perp$ is the orthogonal projection of $\dot{X}_d$ on $Im(J)$ (subspace of achievable velocities), so that the error $\left\| J(q)\,\dot{q}^* - \dot{X}_d \right\|$ is minimum.
     - Let note that, in the particular case where $\dot{X}_d \in Im(J)^\perp$ exclusively (i.e. $\dot{X}_d \in N(J^t)$), then the solution according to the pseudo-inverse method returns $\dot{q} = 0$ (unachievable velocity, since $\dot{X}_d^\perp = 0$).



$\dot{q} = J^\# \dot{X}_d$
Joint velocity vector of minimal norm generating $\dot{X}_d^\perp$

Inversion of the kinematic model in the singular case $(M = N)$

Pseudo-inverse method : $\dot{q}^* = J^\sharp \dot{X}_d$

- To sum-up, insights in the returned *pseudo-inverse* solution for kinematic inverse problem
  - Regular case : *exact and unique solution out of singular configurations* (since $J^\sharp = J^{-1}$ in this case), but solution not acceptable in the neighbourhood of singular configurations (use of the inverse kinematic model with a bad conditioning of J which can lead to high joint velocities, incompatible with the actuators capabilities) ;
  - Singular case : approximated solution *on a singular configuration* allowing nevertheless to calculate the joint velocity $\dot{q}$ with minimal norm, while minimizing the error $\left\| J(q)\,\dot{q}^* - \dot{X}_d \right\|$ (constraint fully satisfied if $\dot{X}_d \in \mathscr{R}(J)$).
- Discontinuous behavior when passing from *regular* to *singular* case :

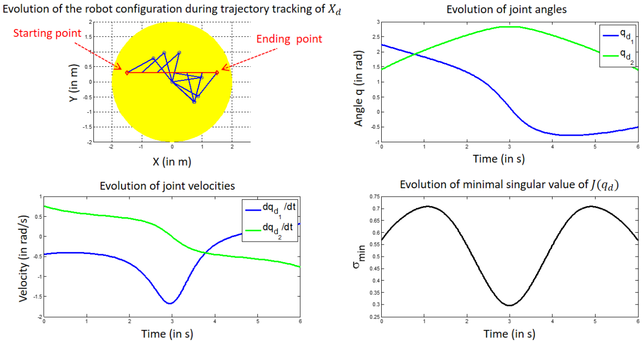
$$\dot{q}^* = \sum_{i=1}^{m} \frac{1}{\sigma_i} V_i U_i^t \dot{X}_d$$

When coming close to a singularity $\sigma_{min} \to 0$ ($\|\dot{q}\|$ high), then $\sigma_{min} = 0$ (sum being stopped at $m-1$) involving a discontinuity for crossing the singularity.

Inversion of the kinematic model in the regular case ($M = N$)

Case study of robot 2R

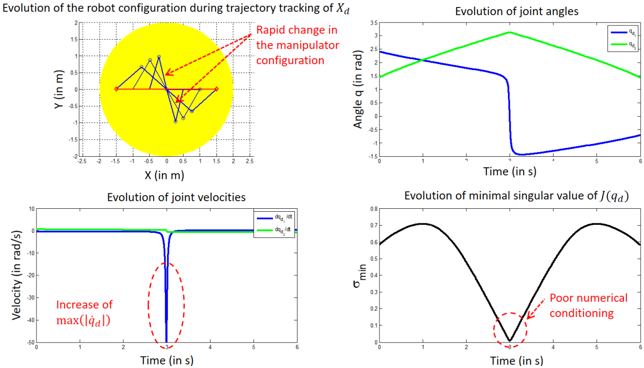
1. Behaviour analysis of $\dot{q} = J^{-1}(q)\dot{X}_d$ in the regular case out of singularities
   - Tracking a straight path $X_d(t)$ at constant speed $v = 0.5 m.s^{-1}$ during $T = 6$s.



Inversion of the kinematic model in the quasi-singular case $(M = N)$

Case study of robot 2R

2. Behaviour analysis of $\dot{q} = J^{-1}(q)\dot{X}_d$ in the regular case but close to singularity
   - new trajectory reference $X_d(t)$ close to singular case $(\min_{t_k} \{\sigma_{min}(J(q_d))\} \approx 0)$;
   - increase of $max\,|\dot{q}_i|$ in the neighbourhood of singular configurations.



Inversion of the kinematic model in the singular case $(M = N)$

Inversion using the damped least-squares method

To decrease the excessive joint velocities close to singularities, we can think about tolerating an *error on the trajectory tracking*, replacing the inversion problem $\dot{x}_d = J(q_d)\,\dot{q}$ by the following minimisation problem [29] :

$$\min_{\dot{q}} \left\{ \frac{1}{2} \left\| J(q)\,\dot{q} - \dot{X}_d \right\|^2 + \frac{\lambda^2}{2} \|\dot{q}\|^2 \right\}, \quad \lambda \geq 0.$$

- Inverse kinematic seen as an optimisation problem :
  - 1^{st} term of the objective function representative of the norm of the trajectory error ;
  - 2^{nd} term of the objective function representative of the norm of the joint velocity ;
- Role of the coefficient λ :
  - weighting coefficient named as *damping ratio* ;
  - choice of $\lambda = 0$ when far away from singular configurations, then $\lambda > 0$ when $\sigma_{min}(J(q))$ close to 0) ;
  - when $\lambda > 0$, decrease (*damping*) of the amplitude of the joint velocity $max\,|\dot{q}_i|$ obtained to the detrimental to velocity trajectory error $\epsilon_{\dot{X}} = \lambda^2 \left(\lambda^2 I_m + JJ^t\right)^{-1}$

Inversion of the kinematic model in the singular case ($M = N$)

Inversion using the damped least-squares method

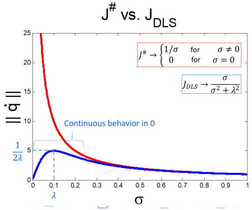
- Solution to the inversion problem using the damped least-squares method :

$$\boxed{\dot{q}^* = \left(\lambda^2 I_n + J^t J\right)^{-1} J^t \dot{X}_d = \underbrace{J^t \left(\lambda^2 I_m + J J^t\right)^{-1}}_{J_{DLS}(q)} \dot{X}_d}$$

- Possible use of the Jacobian matrix J_{DLS} both for the case $m = n$ as well as for the redundant case $m < n$ (in this case, we will prefer the expression $J^t \left(\lambda^2 I_m + J J^t\right)^{-1}$ for computing J_{DLS}).
- Rewriting J_{DLS} from its decomposition in singular values

$$\boxed{\dot{q}^* = \sum_{i=1}^{m} \frac{\sigma_i}{\sigma_i^2 + \lambda^2} V_i U_i^t \dot{X}_d}$$

- damping ratio limiting the maximal amplitude in the detrimental to the precision when $\sigma_i << \lambda$;
- damping ratio with few impact when $\sigma_i >> \lambda$.

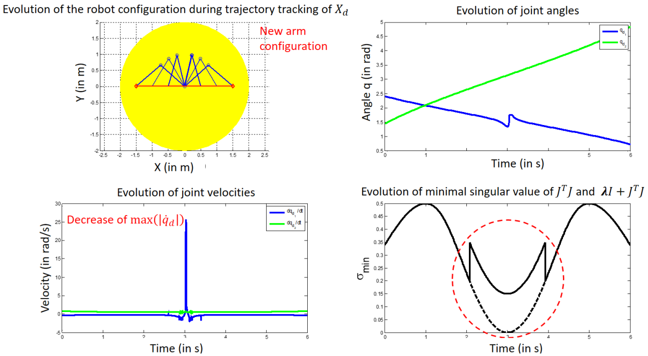


Demonstration

Inversion of the kinematic model in the quasi-singular case ($M = N$)

Case study of robot 2R

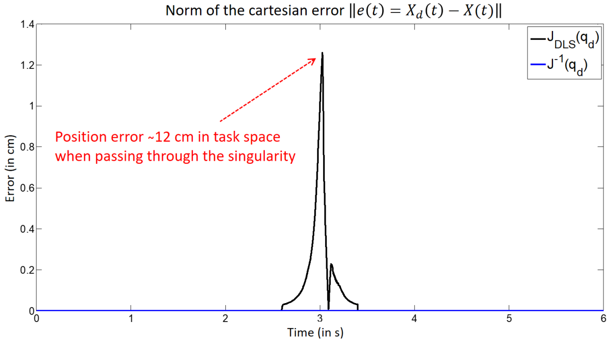
2. Behaviour analysis of $\dot{q} = J_{DLS}(q)\dot{X}_d$ in the regular case but close to singularity
   - Physically, high-dynamic reconfiguration of the arm avoided ;
   - Use of the damped pseudo-inverse matrix when $\sigma_{min}(J)$ becomes close to 0 ;
   - Damping velocities $\dot{q}_i$.



Inversion of the kinematic model in the quasi-singular case ($M = N$)

Case study of robot 2R

- Comparing the behaviour of $\dot{q} = J_{DLS}(q)\dot{X}_d$ and $\dot{q} = J^{-1}(q)\dot{X}_d$ on the cartesian error in the case close to singularity.



Inversion of the kinematic model in the over-determined case ($M > N$)

Constrained system : no exact solution to the inverse kinematic problem (system said *incompatible*)

- Situation corresponding to the case where the desired velocity cannot be exactly obtained ;
- Search for an approximated solution that minimize the Euclidian norm of the cartesian error :

$$\min_{\dot{q}} \frac{1}{2} \left\| J(q)\,\dot{q} - \dot{X}_d \right\|$$

where, if $Rank(J) = N$ (full column rank), the solution, said *the least-squares solution*, equals $\dot{q}^* = J^\sharp(q)\,\dot{x}_d$ with

$$J^\sharp(q) = \left(J^t J\right)^{-1} J^t$$

- Property of left-inversion : $J(q)^\sharp J = I_n$ (but $JJ(q)^\sharp \neq I_m$) ;

Inversion of kinematic model

Sum-up about *Moore-Penrose* pseudo-inverse Jacobian matrix :

1. If $Rank\,(J) = r = M < N$ (full row rank), then $J^\sharp = J^t\left(JJ^t\right)^{-1}$ (right pseudo-inverse - redundant case) ;
2. If $Rank\,(J) = r = M = N$, then $J^\sharp = J^{-1}$ (regular case) ;
3. If $Rank\,(J) = r = N < M$ (full column rank), then $J^\sharp = \left(J^tJ\right)^{-1}J^t$ (left pseudo-inverse - over-determined case).

Properties of the pseudo-inverse matrix (valid irrespective of the conditions on the rank of J) :

- $JJ^\sharp J = J$, $J^\sharp JJ^\sharp = J^\sharp$, $\left(JJ^\sharp\right)^t = JJ^\sharp$ and $\left(J^\sharp J\right)^t = J^\sharp J$ (Hermitian matrices) ;
- Algorithmically, the pseudo-inverse always exists and is obtained from the decomposition into singular values of $J = U\Sigma V^t$:

$$J^\sharp = V\Sigma^+U^t \Rightarrow \dot{q} = \sum_{i=1}^{r}\frac{1}{\sigma_i}V_iU_i^t\dot{X}_d$$

where $\Sigma^\sharp$, pseudo-inverse of the diagonal matrix Σ, is a diagonal matrix whose non-zero elements are obtained by inverting the non-zero elements (of the diagonal) of Σ :

$$\Sigma^+ = \left[\begin{array}{cccc} \frac{1}{\sigma_1} & & & \\ & \ddots & & \\ & & \frac{1}{\sigma_r} & \\ & & & 0 \\ \hline \multicolumn{4}{c}{0_{(N-M)\times(N)}} \end{array}\right]$$