

# Application of Plasticity Theory to Slope Stability Problems

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The purpose of this paper is to present a closed-form plasticity solution to slope stability problems in homogeneous soils. The paper discusses the basic concepts and limitations of the plasticity solution by comparing it to existing limit equilibrium solutions. Stability factors computed by the plasticity technique assuming straight-line, circular, and logarithmic-spiral failure surfaces are presented. These are compared with stability factors computed from existing limit equilibrium solutions including the Culmann method, the friction circle method, and the Rendulic logarithmic-spiral method. Design charts developed from the plasticity solution are presented for a useful range of friction angles and slope geometries.

•THERE ARE numerous methods currently available for performing slope stability analyses. The majority of these may be categorized as limit equilibrium methods. The basic assumption of the limit equilibrium approach is that Coulomb's failure criterion is satisfied along the failure surface. The failure surface may be assumed to be a straight line, circular arc, logarithmic spiral, or other irregular surface. Analytical, graphical, semigraphical, and numerical procedures for solving limit equilibrium problems are available.

A weakness of the limit equilibrium approach is that it neglects the soil's stress-strain relationship. From the mechanics of solids point of view, a valid solution is obtained if equilibrium, compatibility, and the material's stress-strain relationship are simultaneously satisfied. The limit equilibrium method considers only equilibrium.

In an attempt to take account of the stress-strain relationship in analyzing the stability of slopes, it has been suggested that the theory of plasticity be applied to the problem (8). The purpose of this paper is to utilize plasticity theory to solve homogeneous earth slope stability problems, and to compare the results with conventional limit equilibrium solutions. The results are presented in the form of a design chart of stability factors computed for a useful range of friction angles and slope geometries.

## LIMIT EQUILIBRIUM METHOD

The Culmann method (7) represents a typical limit equilibrium solution and is briefly reviewed here for purposes of illustration and comparison. The method assumes that failure occurs on a plane (straight line) passing through the toe of the earth slope. The Culmann failure mechanism is shown in Figure 1, where  $W$  is the weight of soil in the wedge ABD,  $C$  is the total cohesion along the failure plane AB, and  $\alpha$ ,  $\beta$ , and  $\theta$  are the slope angles.  $P$  is the resultant force necessary to hold wedge ABD in equilibrium,  $H$  is the height of the earth slope, and  $\phi$  is the friction angle for the soil. From the geometrical relationships shown in Figure 1, the weight of soil in the wedge ABD is

$$W = \frac{1}{2} \gamma l H \csc \beta \sin (\beta - \theta) \quad (1)$$

where  $\gamma$  is the unit weight of the soil and  $l$  is the length of the failure plane AB. If c

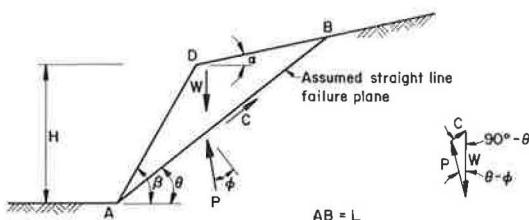


Figure 1. Straight-line limit equilibrium failure plane (Culmann method).

is the unit cohesion, then

$$C = cl \quad (2)$$

Substitution of Eqs. 1 and 2 into the law of sines expressed for the force diagram in Figure 1 yields

$$\frac{\gamma H}{c} = \frac{2 \sin \beta \cos \phi}{\sin(\beta - \theta) \sin(\theta - \phi)} \quad (3)$$

The term  $\gamma H/c$  is a dimensionless expression called the stability factor,  $N_s$ . The critical stability factor (most dangerous plane) may be obtained by minimizing the first derivative of the stability factor with respect to  $\theta$ . This yields

$$N_{s_{crit.}} = \frac{4 \sin \beta \cos \phi}{1 - \cos(\beta - \phi)} \quad (4)$$

For a vertical cut where  $\beta = 90^\circ$  deg =  $\frac{\pi}{2}$ , Eq. 4 becomes

$$N_{s_{crit.}} = 4 \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \quad (5)$$

The critical height of the slope,  $H_c$ , is

$$H_c = N_{s_{crit.}} \frac{c}{\gamma} = \frac{4c}{\gamma} \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \quad (6)$$

The Culmann method has been widely used for slope stability analyses because of its simplicity, even though the assumptions on which the method is based are of questionable validity.

Extensive refinements of the limit equilibrium method have been undertaken by many investigators. These refinements have generally been concerned with defining a more acceptable failure surface or with modifying the method by which the forces acting on the failure surface are handled. A summary of the various limit equilibrium methods currently in use is given in Table 1.

The foregoing description of limit equilibrium methods is intended to provide a background for, and to distinguish these methods from, the plasticity method. Thus only the basic features of the methods have been introduced. The results obtained by the various methods will be compared with plasticity solutions later in the paper.

### PLASTICITY ANALYSES

Plasticity analyses are based on two main theorems for any body or assemblage of bodies of elastic-perfectly plastic material:

1. Lower Bound Theorem—If an equilibrium distribution of stress can be found that balances the applied load and nowhere violates the yield criterion that includes  $c$ , the cohesion, and  $\phi$ , the angle of internal friction, the soil mass will not fail or will be just at the point of failure.
2. Upper Bound Theorem—The soil mass will collapse if there is any compatible pattern of plastic deformation for which the rate of work of the external loads exceeds the rate of internal energy dissipation.

According to the upper bound theorem, it is necessary to find a compatible failure mechanism (velocity field) in order to obtain an upper bound solution. A stress field satisfying all conditions of the lower bound theorem is required for a lower bound

TABLE 1  
CLASSIFICATION OF SLOPE STABILITY ANALYSIS BY LIMIT EQUILIBRIUM METHOD

Type of Failure Plane	Name of Method	Type of Solution	Basic Assumptions	References
Straight line	Culmann method	Analytical	Failure occurs on a plane through the toe of the slope	Culmann (7)
	Method of infinite slope <sup>a</sup>	Analytical	The slope is constant with unlimited extent A vertical column is typical of the entire mass No cohesion may be depended on within the depth to which tension occurs	Résal (20) Frontard (11)
	Wedge method	Semigraphical, analytical	Sliding block mechanism is assumed with lateral earth forces	Culmann (7), Terzaghi and Peck (23)
Circular arc <sup>a</sup>	Slices method <sup>b</sup>	Semigraphical, numerical	The lateral forces are equal on two sides of each slice	Statens Järnvägars Geotekniska Commission, Fellenius (10)
	Bishop's method <sup>b</sup>	Analytical, numerical	Oblique side forces on each slice are considered	Bishop (2)
	Simplified Bishop's method <sup>b</sup>	Analytical, numerical	Vertical component of lateral earth forces are considered to be equal and opposite	Bishop (2), Little and Price (14)
	$\phi$ -circle method <sup>b</sup>	Analytical, graphical	Resultant acting on rupture arc is tangential to a concentric circle with radius = $R \sin \phi_d$	Gilboy (12), Taylor (22), Casagrande (3)
	Modified $\phi$ -circle <sup>b</sup>	Analytical, graphical	The resultant misses tangency to the $\phi_d$ circle by a small amount; radius of $\phi_d$ circle = $KR \sin \phi_d$	Taylor (22)
Logarithmic spiral	Log-spiral method	Analytical	No assumptions required to make the problem statically determinate	Rendulic (19), Taylor (22), Spencer (21)
Irregular	Irregular	Analytical, numerical	General slip surface Forces between slices are considered	Morgenstern and Price (15)

<sup>a</sup>Circular arc failure plane, suggested by Petterson and Hultin (16).

<sup>b</sup>Long-term stability analysis (seepage considered).

solution. If the upper bound and lower bound provided by the velocity field and stress field coincide, the exact value of the collapse load is determined (for an elastic-perfectly plastic material). The following section gives examples demonstrating the application of plasticity theorems to problems of earth slope stability.

#### Straight-Line Failure Plane

A plasticity analysis of the slope stability problem was first performed by Drucker and Prager (8). They assumed a straight-line failure plane and suggested the upper bound failure mechanism shown in Figure 2. The symbols are as previously defined. As the wedge formed by the shear plane slides downward along the discontinuity surface, the rate of work done by the external forces is equal to the vertical component of the velocity multiplied by the weight of the soil wedge. For a vertical cut ( $\beta = 90$  deg) and horizontal backfill ( $\alpha = 0$  deg), the rate of external work is

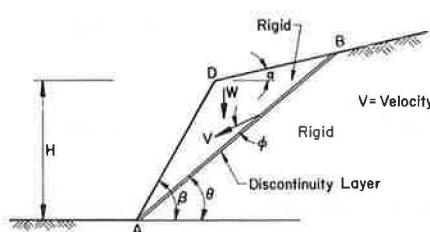


Figure 2. Straight-line plasticity failure mechanism—velocity field (upper bound solution).

$$\text{Rate of Work}_{\text{ext.}} = \frac{1}{2} \gamma H^2 V \cot \theta \sin (\theta - \phi) \quad (7)$$

where  $V$  equals the velocity of the wedge and is inclined at angle  $\phi$  to the discontinuity surface. The rate of internal energy dissipation along the discontinuity surface is

$$\text{Rate of Work}_{\text{int.}} = \frac{cH}{\sin \theta} V \cos \phi \quad (8)$$

Equating the rate of external work (Eq. 7) to the rate of internal energy dissipation (Eq. 8) gives

$$H \leq \frac{2c}{\gamma} \frac{\cos \phi}{\sin (\theta - \phi) \cos \theta} \quad (9)$$

Minimizing the right side of Eq. 9 gives

$$\theta = \frac{\pi}{4} + \frac{\phi}{2} \quad (10)$$

and

$$H_c \leq \frac{4c}{\gamma} \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \quad (11)$$

Thus it may be seen that the upper bound plasticity solution for the critical height,  $H_c$ , of a vertical cut yields the same value as the Culmann limit equilibrium solution.

A lower bound solution for a vertical cut was also developed by Drucker and Prager (8) and later refined by Chen and Scawthorn (4). The suggested stress field is composed of three regions, as shown in Figure 3a.

Region I, in the bank itself, is subjected to uniaxial compression that increases with depth. Region II is under biaxial compression and region III is under hydrostatic pressure ( $\sigma_x = \sigma_y$ ). Figure 3b shows the corresponding Mohr circles for each region. Failure occurs when the circle representing region I meets the yield curve. Therefore,

$$\frac{1}{2} \gamma H = c \cos \phi + \frac{1}{2} \gamma H \sin \phi \quad (12)$$

and

$$H_c \geq \frac{2c}{\gamma} \tan \left( \frac{\pi}{4} + \frac{\phi}{2} \right) \quad (13)$$

Hence, the lower bound solution is only half the value given by the upper bound solution.

#### Logarithmic-Spiral Failure Plane

Model studies have indicated that the shape of the failure plane for earth slopes is frequently close to a logarithmic spiral (13). Drucker and Prager (8) have suggested that the upper bound solution could be improved by choosing a logarithmic spiral as the discontinuity surface rather than a straight line. Recently, a closed-form mathematical solution of the upper bound problem has been

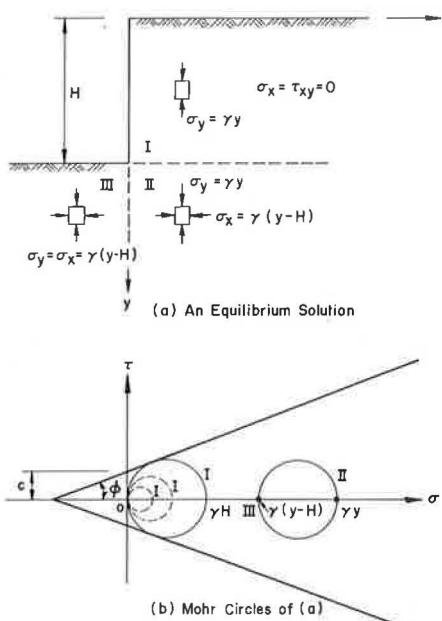


Figure 3. Plasticity stress field for a vertical cut (lower bound solution): (a) an equilibrium condition, and (b) Mohr circles of the equilibrium condition.

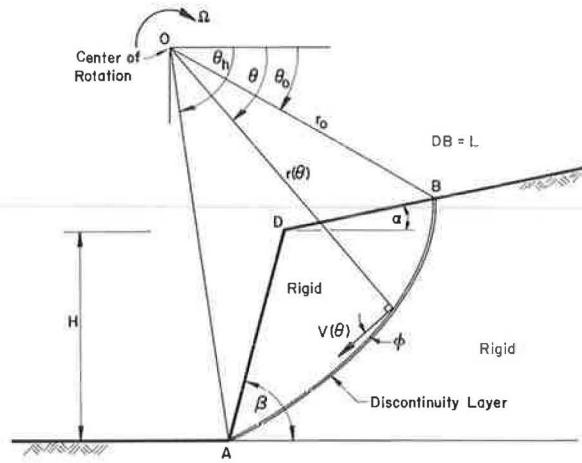


Figure 4. Logarithmic-spiral plasticity failure mechanism—velocity field (upper bound solution).

developed by Chen, Giger, and Fang (6). The assumed failure mechanism is shown in Figure 4. The triangular-shaped region ABD rotates as a rigid body about the center of rotation O, with the materials below the logarithmic surface AB remaining at rest. Thus, the surface AB is a surface of velocity discontinuity. The mechanism can be specified by three variables,  $\theta_o$ ,  $\theta_h$ , and  $H$ , which are the slope angles of the chords OB and OA and the height of the embankment respectively. From the geometrical relations it may be shown that the ratios  $H/r_o$  and  $L/r_o$  can be expressed in terms of the angles  $\theta_o$  and  $\theta_h$  in the forms

$$\frac{H}{r_o} = \frac{\sin \beta}{\sin (\beta - \alpha)} \left\{ \sin (\theta_h + \alpha) \exp [(\theta_h - \theta_o) \tan \phi] - \sin (\theta_o + \alpha) \right\} \quad (14)$$

where  $r_o$  = radius of logarithmic spiral at  $\theta = \theta_o$ . Also, if  $L$  = length of the slope DB (Fig. 4), then

$$\begin{aligned} \frac{L}{r_o} &= \frac{\sin (\theta_h - \theta_o)}{\sin (\theta_h + \alpha)} \\ &- \frac{\sin (\theta_h + \beta)}{\sin (\theta_h + \alpha) \sin (\beta - \alpha)} \left\{ \exp [(\theta_h - \theta_o) \tan \phi] \sin (\theta_h + \alpha) - \sin (\theta_o + \alpha) \right\} \end{aligned} \quad (15)$$

The rate of external work may be obtained by the simple algebraic summation of the rates of work resulting from rotation of the soil weight in the regions OAB, OAD, and ODB:

$$\text{Rate of Work}_{\text{ext.}} = \gamma r_o^3 \Omega (f_1 - f_2 - f_3) \quad (16)$$

where  $\Omega$  is the angular velocity of the region ABD and  $f_1$ ,  $f_2$ , and  $f_3$  are defined as

$$\begin{aligned} f_1 (\theta_h, \theta_o) &= \frac{1}{3(1 + 9 \tan^2 \phi)} \left\{ (3 \tan \phi \cos \theta_h + \sin \theta_h) \exp [3(\theta_h - \theta_o) \tan \phi] \right. \\ &\quad \left. - (3 \tan \phi \cos \theta_o + \sin \theta_o) \right\} \end{aligned} \quad (17)$$

$$f_2(\theta_h, \theta_o) = \frac{1}{6} \frac{L}{r_o} \left( 2 \cos \theta_o - \frac{L}{r_o} \cos \alpha \right) \sin (\theta_o + \alpha) \quad (18)$$

$$f_3(\theta_h, \theta_o) = \frac{1}{6} \exp [(\theta_h - \theta_o) \tan \phi] \left[ \sin (\theta_h - \theta_o) - \frac{L}{r_o} \sin (\theta_h + \alpha) \right]$$

$$\left\{ \cos \theta_o - \frac{L}{r_o} \cos \alpha + \cos \theta_h \exp [(\theta_h - \theta_o) \tan \phi] \right\} \quad (19)$$

The rate of internal dissipation of energy occurs along the discontinuity surface AB and may be found by integration of the differential rate of dissipation of energy along the surface, over the whole surface. This is obtained by multiplying the differential area,  $r d \theta / \cos \phi$ , by the cohesion,  $c$ , times the tangential discontinuity in velocity,  $V \cos \phi$ , and integrating from  $\theta_o$  to  $\theta_h$ :

$$\text{Rate of Work}_{\text{int.}} = \int_{\theta_o}^{\theta_h} c(V \cos \phi) \frac{r d \theta}{\cos \phi}$$

$$= \frac{c r_o^2 \Omega}{2 \tan \phi} \left\{ \exp [2(\theta_h - \theta_o) \tan \phi] - 1 \right\} \quad (20)$$

Equating the external rate of work (Eq. 16) to the rate of internal energy dissipation (Eq. 20) yields

$$H = \frac{c}{\gamma} f(\theta_h, \theta_o) \quad (21)$$

where  $f(\theta_h, \theta_o)$  is defined as

$$f(\theta_h, \theta_o) = \frac{\sin \beta \{ \exp [2(\theta_h - \theta_o) \tan \phi] - 1 \}}{2 \sin (\beta - \alpha) \tan \phi (f_1 - f_2 - f_3)}$$

$$\{ \sin (\theta_h + \alpha) \exp [(\theta_h - \theta_o) \tan \phi] - \sin (\theta_o + \alpha) \} \quad (22)$$

By the upper bound theorem, Eq. 21 gives a least upper bound solution for  $H$  when  $f(\theta_h, \theta_o)$  is a minimum. Such a condition is attained when

$$\frac{\delta f}{\delta \theta_h} = 0 \quad \text{and} \quad \frac{\delta f}{\delta \theta_o} = 0 \quad (23)$$

Solving these equations and substituting the resulting values of  $\theta_h$  and  $\theta_o$  into Eq. 21 yields the critical height,  $H_c$ , of an earth slope

$$H_c \leq \frac{c}{\gamma} N_s \quad (24)$$

where

$$N_s = \min. f(\theta_h, \theta_o) \quad (25)$$

The value  $N_s$  depends on the slope angles,  $\alpha$  and  $\beta$ , and the angle of internal friction,  $\phi$ .

The results of this analysis are shown in Figures 5 and 6 for the special cases  $\alpha = 0$  or  $\beta = 90$  deg and are tabulated numerically in Table 2 for the more general cases.

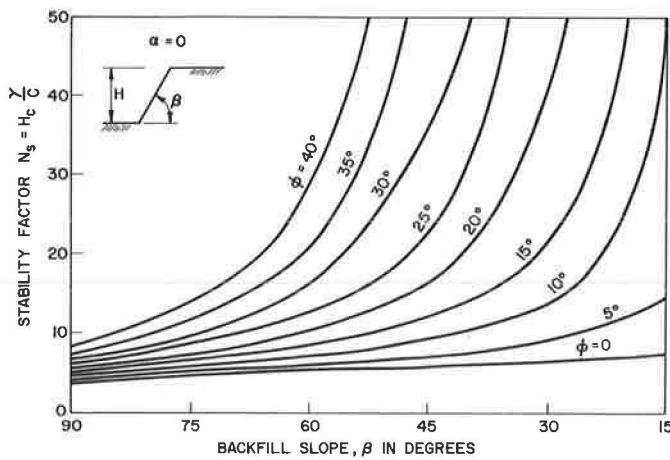


Figure 5. Stability factor versus slope angle,  $\beta$ , for various friction angles.

TABLE 2  
SUMMARY OF STABILITY FACTORS,  $N_s$ , COMPUTED BY PLASTICITY THEORY  
(UPPER BOUND)

Friction Angle, $\phi$ (deg)	Slope Angle, $\alpha$ (deg)	Slope Angle, $\beta$ (deg)					
		90	75	60	45	30	15
0	0	3.83	4.57	5.25	5.86	6.51	7.35
5	0	4.19	5.14	6.17	7.33	9.17	14.80
	5	4.14	5.05	6.03	7.18	8.93	14.62
10	0	4.59	5.80	7.26	9.32	13.53	45.53
	5	4.53	5.72	7.14	9.14	13.26	45.15
	10	4.47	5.61	6.98	8.93	12.97	44.56
15	0	5.02	6.57	8.64	12.05	21.71	
	5	4.97	6.49	8.52	11.91	21.50	
	10	4.90	6.39	8.38	11.73	21.14	
	15	4.83	6.28	8.18	11.42	20.59	
20	0	5.51	7.48	10.39	16.18	41.27	
	5	5.46	7.40	10.30	16.04	41.06	
	10	5.40	7.31	10.15	15.87	40.73	
	15	5.33	7.20	9.98	15.59	40.16	
	20	5.24	7.04	9.78	15.17	39.19	
25	0	6.06	8.59	12.75	22.92	120.0	
	5	6.01	8.52	12.65	22.78	119.8	
	10	5.96	8.41	12.54	22.60	119.5	
	15	5.89	8.30	12.40	22.37	118.7	
	20	5.81	8.16	12.17	21.98	117.4	
	25	5.71	7.97	11.80	21.35	115.5	
30	0	6.69	9.96	16.11	35.63		
	5	6.63	9.87	16.00	35.44		
	10	6.58	9.79	15.87	35.25		
	15	6.53	9.67	15.69	34.99		
	20	6.44	9.54	15.48	34.64		
	25	6.34	9.37	15.21	34.12		
	30	6.22	9.15	14.81	33.08		
35	0	7.43	11.68	20.94	65.53		
	5	7.38	11.60	20.84	65.39		
	10	7.32	11.51	20.71	65.22		
	15	7.26	11.41	20.55	65.03		
	20	7.18	11.28	20.36	64.74		
	25	7.11	11.12	20.07	64.18		
	30	6.99	10.93	19.73	63.00		
	35	6.84	10.66	19.21	60.80		
40	0	8.30	14.00	28.99	185.6		
	5	8.26	13.94	28.84	185.5		
	10	8.21	13.85	28.69	185.3		
	15	8.15	13.72	28.54	185.0		
	20	8.06	13.57	28.39	184.6		
	25	7.98	13.42	28.16	184.0		
	30	7.87	13.21	27.88	183.2		
	35	7.76	12.95	27.49	182.3		
	40	7.61	12.63	26.91	181.1		

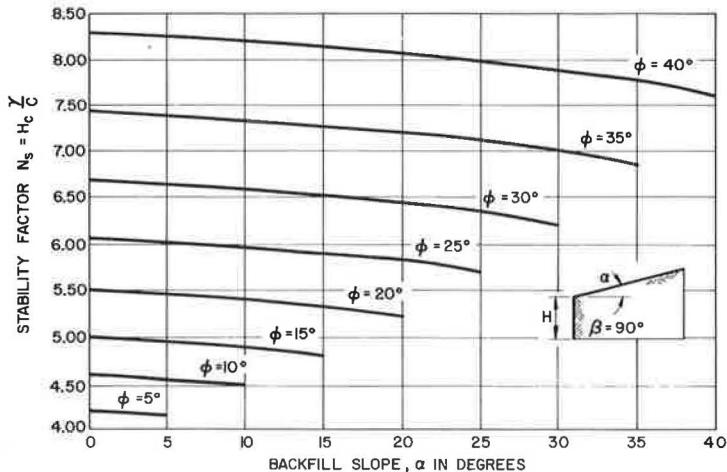


Figure 6. Stability factor versus slope angle,  $\alpha$ , for various friction angles.

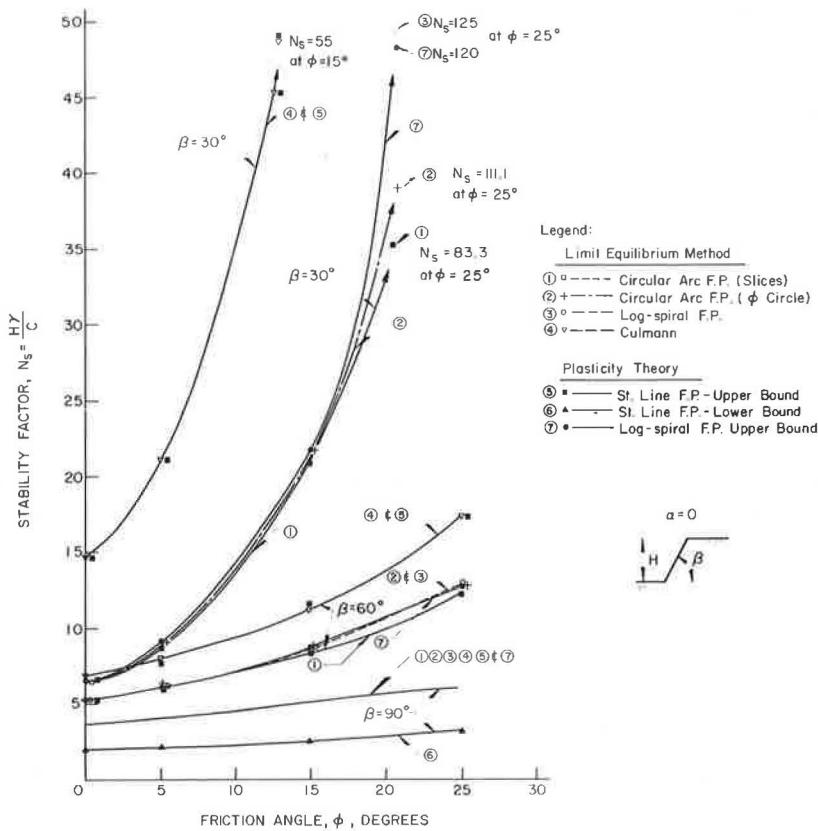


Figure 7. Comparison of stability factor values computed by various methods for  $\beta$  of 30, 60, and 90 deg.

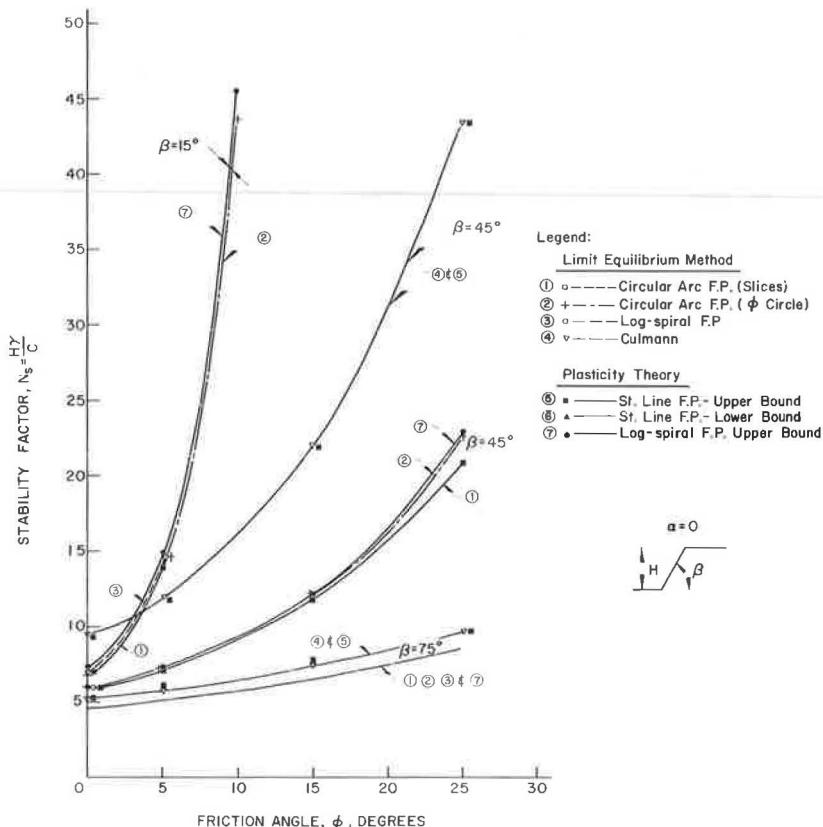


Figure 8. Comparison of stability factor values computed by various methods for  $\beta$  of 15, 45, and 75 deg.

### COMPARISON OF LIMIT EQUILIBRIUM AND PLASTICITY RESULTS

Figures 7 and 8 show the stability factor,  $N_s$ , versus friction angle,  $\phi$ , for various angles,  $\beta$ , with angle  $\alpha$  constant and equal to zero. Both limit equilibrium and plasticity solutions are shown.

It may be noted that in the case of steep embankments, limit equilibrium methods and upper bound plasticity methods yield similar results. The discrepancies between the various methods increase with increasing friction angle and with decreasing slope angle,  $\beta$ .

### SUMMARY AND CONCLUSIONS

Additional insight into many soil mechanics problems such as bearing capacity (17), earth pressure (1, 18), and slope stability (8) may be obtained by considering soil as a perfectly plastic material. The limitations of plasticity theory applied to soil mechanics problems have been discussed by Drucker (9) and more recently by Chen (5).

In applying plasticity theory to problems of earth slope stability, the following conclusions may be considered applicable within the limitations of the assumptions used:

- For a straight-line failure plane, both limit equilibrium and upper bound plasticity methods yield the same results, suggesting that the conventional limit equilibrium method provides an upper bound solution.

2. For a logarithmic-spiral failure plane, the upper bound plasticity method provides a closed-form mathematical solution. Currently, only numerical procedures are available for solving the limit equilibrium equations. The results of the upper bound plasticity solution are presented in the form of a design chart giving stability factors for a wide range of soil properties and slope geometries (Table 2).

3. The most significant advantage to using the plasticity method in analyzing the stability of earth slopes is that the upper and lower bound theorems provide a mechanism for bounding the solution. On the other hand, limit equilibrium methods do not provide any indication of their own validity. This is not meant to imply that stability analyses should be performed solely by plasticity methods. Soil is not a perfectly plastic material although it does exhibit certain plastic characteristics. What the upper and lower bound theorems do provide are additional guidelines to aid the designer in establishing a rational factor of safety for the problem under consideration.

#### ACKNOWLEDGMENTS

The work described in this paper was conducted in the Geotechnical Engineering Division, Fritz Engineering Laboratory, Lehigh University, as part of the research program on soil plasticity. This particular study is sponsored by the Envirotronics Corporation.

The authors express their appreciation to Professor W. F. Chen for his constructive criticism and review of the manuscript.

#### REFERENCES

1. Bell, A. L. The Lateral Pressure and Resistance of Clay, and the Supporting Power of Clay Foundations. Minutes of Proc. of Institution of Civil Engineers, Paper No. 4131, London, 1915.
2. Bishop, A. W. The Use of the Slip Circle in the Stability Analysis of Slopes. *Geotechnique*, Vol. 5, 1955, pp. 7-17.
3. Casagrande, A. The Shearing Resistance of Soils. *Jour. Boston Society of Civil Engineers*, Vol. 21, No. 3, 1934.
4. Chen, W. F., and Scawthorn, C. R. Limit Analysis and Limit Equilibrium Solutions in Soil Mechanics. Lehigh Univ., Bethlehem, Penn., Fritz Engineering Lab. Rept. 355.3, 1968.
5. Chen, W. F. Soil Mechanics and the Theorems of Limit Analysis. *Jour. Soil Mech. and Found. Div.*, ASCE, Vol. 95, No. SM2, 1969.
6. Chen, W. F., Giger, M. W., and Fang, H. Y. On the Limit Analysis of Stability of Slopes. *Soils and Foundations*, Vol. 9, No. 4, Dec. 1969.
7. Culmann, K. *Die Graphische Statik*. Berlin, 1866.
8. Drucker, D. C., and Prager, W. Soil Mechanics and Plastic or Limit Design. *Quarterly of Applied Mathematics*, Vol. 10, 1952, pp. 157-165.
9. Drucker, D. C. Limit Analysis of Two and Three Dimensional Soil Mechanics Problems. *Jour. Mechanics and Physics of Solids*, Vol. 1, No. 4, 1953, pp. 217-226.
10. Fellenius, W. *Erdstatische Berechnungen* (Calculation of Stability of Slopes), Revised Edition. W. Ernst und Sohn, Berlin, 1939.
11. Frontard, M. *Cycloides de Glissement des Terres*. Comptes Rendues, Paris, 1922.
12. Gilboy, G. Soil Mechanics Research. *Trans. ASCE*, Vol. 98, 1933.
13. Jumikis, A. R. Stability Analysis of Soil-Foundation Systems: A Design Manual. Rutgers, The State University, Engineering Research Publ. No. 44, 1965.
14. Little, A. L., and Price, V. E. The Use of an Electronic Computer for Slope Stability Analysis. *Geotechnique*, Vol. 8, 1958, pp. 113-120.
15. Morgenstern, N. R., and Price, V. E. The Analysis of the Stability of General Slip Surfaces. *Geotechnique*, Vol. 15, 1965, pp. 79-93.
16. Pettersson, K. E., and Hultin, S. *Kajraseti Göteborg des 5 re Mars 1916*. *Teknisk Tidskrift*, Vol. 46, 1916, pp. 281-294.

17. Prandtl, L. Ueber die Harte plastischer Körper. Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Berlin, 1920, pp. 74-85.
18. Rankine, W. J. M. A Manual of Applied Mechanics. Charles Griffin and Co., London, 1885, pp. 219-220.
19. Rendulic, L. Ein Beitrag zur Bestimmung der Gleitsicherheit. Der Bauingenieur, No. 19/20, 1935.
20. Résal, J. Poussée des Terres. Paris, 1910.
21. Spencer, E. Circular and Logarithmic Spiral Slip Surfaces. Jour. Soil Mech. and Found. Div., ASCE, Vol. 95, No. SM1, 1969, pp. 227-234.
22. Taylor, D. W. Stability of Earth Slopes. Jour. Boston Society of Civil Engineers, Vol. 24, 1937, pp. 197-246.
23. Terzaghi, K., and Peck, R. B. Soil Mechanics in Engineering Practice. John Wiley and Sons, 1948.

### *Discussion*

LESLIE L. KARAFIATH, Grumman Aerospace Corporation—The application of plasticity theory to slope stability problems requires the finding of solutions for the basic differential equations of plastic equilibrium. These equations combine the differential equations of static equilibrium with the Mohr-Coulomb failure criterion characterized by the effective strength parameters  $c$  and  $\phi$ ; the development of the differential equations in a form suitable for numerical integration is due to Sokolovski. The solution of these equations is a family of slip lines that form zones of plastic equilibrium; each point within these zones is in failure condition. The mode of failure of granular materials that follow that Mohr-Coulomb criterion is, indeed, zone failure, as earth pressure experiments and sand amply demonstrate.

The strength properties of soils that generate pore pressures may be described by the undrained strength parameters  $c_u$  and  $\phi_u$ . Formal solutions of the differential equations of plasticity can be obtained by substituting these parameters in the Mohr-Coulomb failure criterion. These solutions, however, are invalid in the physical sense, because the inclination of the slip lines to the direction of principal stresses is  $45 \text{ deg} \pm \phi_u/2$  in the formal solutions, whereas theory and experiments equally indicate that the inclination of the failure surface is governed by the effective angle of friction and its angle is  $45 \text{ deg} \pm \phi/2$ . Thus plasticity theory does not apply to cases where the use of undrained strength parameters is appropriate; indeed, the soil in such cases fails along a narrow band that may be called a single failure surface. The development of such a single failure surface is often the result of pore water migration toward the most stressed portion of the soil mass and other progressive failure phenomena. This process will be discussed in more detail elsewhere.

The authors do not present zone solutions for the basic differential equations of plastic equilibrium but assume a surface of "velocity discontinuity" in their analyses and apply the limit theorems of plasticity theory to determine upper bounds for the height of a stable slope. The assumption of a discontinuity surface is essentially equivalent to assuming a single failure surface in the conventional ("limit equilibrium" in the authors' terminology) methods; it is a reasonable assumption for short-term slope stability problems controlled by the undrained shear strength of soil. By skillful selection of the variables, the authors arrive at an expression for the stable slope height (Eq. 21) that is amenable to minimizing; the authors call the minimum value of the stable slope height "least upper bound." Although more elegant, this procedure corresponds to the search for the critical failure surface in conventional analyses. It appears that the same equation could be obtained by considering moment equilibrium about point O, and, therefore, it is not surprising that the stability factors for the same failure surface geometry the authors show in Figures 7 and 8 for the limit equilibrium (3) and plasticity methods (7) are the same except for a slight difference

at  $\phi = 25$  deg for  $\beta = 30$  deg, which may be an inaccuracy in the calculation. The authors also show that, for plane failure surfaces, the limit equilibrium method and the upper bound theorem yield identical results and conclude that the application of limit theorems developed in plasticity theory provides a mechanism for bounding the solution and contributes thereby additional information on the validity of the solution.

There are several reasons why the limit theorems fail in this respect when applied to soil mechanics, particularly slope stability problems. First, if the upper and lower bound theorems, originally applied in soil mechanics to zone solutions of the basic differential equations of plasticity, are considered valid for any surface discontinuity, then limit equilibrium solutions are statically admissible and so, therefore, are lower bound solutions. Second, the lower and upper bound theorems may provide a mechanism for bounding the solution, as the authors state, but this bounding would be valid only for the particular discontinuity geometry chosen. To obtain true brackets of the stability factors for the slope stability problem, bounds for all potential failure surface geometries would have to be determined. If a zone solution for the problem had been found, the boundary discontinuity would be unique and this problem would not arise. Third, the whole philosophy of the upper bound theorem is objectionable from the soil mechanics point of view. The validity of a solution is judged on the basis of a velocity field that would obtain after failure if the soil obeyed the prescribed plastic strain rate. Strains and volume changes that the soil undergoes prior to failure in the process of mobilizing its ultimate strength are completely disregarded. These strains and volume changes are appreciable; their compatibility with the stress field is the criterion that should determine the validity of solutions. Admissibility of velocity fields may be important in analyses of plastic flow, as for example in extrusion problems, but preoccupation with velocity fields in slope stability, earth pressure, or bearing capacity problems diverts attention from the analysis of prefailure volume changes and strains.

H. Y. FANG and T. J. HIRST, Closure—The authors wish to thank Karafiath for his valuable comments. They agree that analyses of prefailure volume change and deformation are desirable. However, until improved soil stress-strain characterizations are available, practical solutions to soil engineering problems will continue to be based on simplified idealizations of soil behavior. Limit equilibrium and plasticity approaches are two such idealizations.

Karafiath notes that, in the case of analyses involving the undrained strength parameters,  $c_u$  and  $\phi_u$ , plasticity solutions are invalid in the physical sense because the principal stresses are inclined at  $45 \text{ deg} \pm \phi/2$  to the slip lines and not at  $45 \text{ deg} \pm \phi_u/2$ . Such an observation may be equally applied to limit equilibrium solutions. This incompatibility of theory with real behavior is further complicated by the dilatant nature of soils. Volume changes accompanying the application of shear stress can magnify inhomogeneities initially present in all soils and can lead to the development of preferred planes of failure that are unrelated to the theoretically predicted failure surfaces. In the case of both limit equilibrium and plasticity approaches, such inhomogeneity renders all solutions rigorously unacceptable. The authors never intended to suggest that, in terms of stress-strain behavior, the theory of plasticity is more rigorous than limit equilibrium. Certainly both methods have very real limitations.

That the plasticity solutions (7) are in agreement with the limit equilibrium solutions (3) is not sufficient to imply that the plasticity solutions correspond to conventional analyses. The assumptions on which the plasticity method is based were outlined by Chen and Scawthorn (4) and others, and are fundamentally different from those of the conventional limit equilibrium approach. The fact that the solutions are the same attests to little else than an apparent insensitivity of the stability factors to the assumed material behavior. In this regard, it should be noted that the discrepancy between the limit equilibrium solution (3) and the plasticity solution (7) for  $\phi = 25$  deg and  $\beta = 30$  deg is believed to be due to rounding errors in the original limit equilibrium hand calculation.

Application of plasticity theory to slope stability problems will provide upper and lower bounds for the critical height. The nearness of the bounds to the true critical height of an embankment depends on two considerations: (a) How well does the theory of plasticity model the real behavior of the soil in the embankment, and (b) how close are the chosen stress and plastic deformation fields to those that would obtain in a perfectly plastic embankment? The bounding of the solution is valid for any stress or deformation field that satisfies the conditions of the limit theorems. It is not necessary to determine all potential stress and deformation fields (or failure surface geometries) in order to establish true bounds to the solution. The bounds may frequently be improved (i.e., brought closer to the true solution), however, by searching for additional acceptable stress and deformation fields.

If conventional limit equilibrium solutions are statically admissible, then, by definition, they represent lower bound solutions. Conventional limit equilibrium solutions consider conditions along an assumed failure surface only. This, in itself, does not guarantee that the yield criterion is not violated elsewhere, and therefore such solutions are not lower bounds.

Karafiath's objection to the overall philosophy of the upper bound approach cannot be accepted by the authors. The stress-strain relationship used in the upper bound solutions is based on the idealized flow rule applied at the moment of incipient failure, not after failure. Limit equilibrium methods, on the other hand, do not consider the stress-strain relationship of the soil at any stage prior to, at, or after failure.

In summary, a useful and acceptable method of analysis must satisfy two conditions: (a) It must yield realistic answers, and (b) it must be simple to apply. The plasticity theory approach is simple because it provides a closed-form solution to the problem. It has been shown to yield reasonable answers when compared with limit equilibrium solutions whose validity has been established on the basis of practical experience. Finally, application of plasticity theory to slope stability problems provides an opportunity for bounding the solution. Rational interpretation of all of the approximate solutions available to an engineer will continue to be the most practical and attractive approach to slope stability problems until material behavior in complex stress states, such as embankments, is better understood.