Chapter 20

THE STRUCTURE OF URBAN EQUILIBRIA: A UNIFIED TREATMENT OF THE MUTH–MILLS MODEL*

JAN K. BRUECKNER

University of Illinois at Urbana-Champaign

1. Introduction

A principal challenge facing the urban economist is the formulation of a rigorous economic explanation for a variety of observed regularities in the spatial structures of real-world cities. The most obvious among these is the dramatic spatial variation in the intensity of urban land-use. Buildings are tall near the centers of most cities, while suburban structures embody much lower ratios of capital to land. Providing a precise explanation of this pattern is an important goal of urban economic analysis. Among other obvious regularities requiring explanation is building height variation among (as opposed to within) cities. Buildings near the centers of large urban areas appear to be much taller than those near the centers of small cities, and a successful economic model must be able to isolate the causes of this observed difference.

Urban economics has met the challenge of scientific explanation with considerable success. The last twenty years have seen the emergence and refinement of a simple yet powerful model of urban spatial structure that successfully explains the principal regularities observed in the urban landscape, including those mentioned above. This model, which derives from the work of Alonso (1964), Mills (1967, 1972b), and Muth (1969), is built around the key observation that commuting cost differences within an urban area must be balanced by differences in the price of living space. This compensating price variation, which reconciles suburban residents to long and costly commuting trips, has far-reaching implications for the spatial structure of the city. While Alonso explored these implications in a framework where individuals consume land directly, Muth and Mills analysed a more realistic model where land is an intermediate input in the production of housing, which is the final consumption good.

The purpose of the present chapter is to provide a unified treatment of the

*I wish to thank Mahlon Straszheim, Takahiro Miyao, and Edwin Mills for comments. Any errors are mine.

Handbook of Regional and Urban Economics, Volume II, Edited by E.S. Mills

Muth–Mills version of the urban model, deriving the well-known results on the internal features of cities in a framework which is then used for comparative static analysis. This unified approach offers clear insight into the structure of the urban equilibrium. We begin by deriving the model's implications regarding the intracity spatial variation of the important urban variables (the central city–suburban building height differential noted above is, for example, shown to be an implication of the model). While the approach is somewhat different, the conclusions of the analysis are familiar from Muth (1969). Next, we offer a comparative static analysis of the urban equilibrium, deriving results that are useful in comparing the spatial structures of different cities (the large city–small city building height differential noted above is derived from the model). This discussion generalizes Wheaton's (1974) comparative static analysis of the Alonso model to an urban economy with housing production (many mathematical details are relegated to an appendix). It should be noted that while the method of analysis (and many of the results) are familiar from Wheaton, comparative static analysis of the Muth–Mills model has not previously appeared in the literature.[1] Finally, the chapter concludes with a short survey of papers that attempt to modify in various interesting and realistic ways the basic assumptions of the model.

2. Intracity analysis

In the stylized city represented by the model, each urban resident commutes to a job in the central business district (CBD) along a dense radial road network. Commuting cost per round-trip mile equals t, so that commuting cost from a residence x radial miles from the CBD is tx per period (the CBD is a point at $x=0$).[2] All consumers earn the same income y per period at the CBD, and tastes are assumed to be identical for all individuals. The common strictly quasi-concave utility function is $v(c,q)$, where c is consumption of a composite non-housing good and q is consumption of housing, measured in square feet of floor space. Note that although real-world dwellings are characterized by a vector of attributes, the analysis ignores this fact and focuses on a single important attribute: interior living space. While the price of the composite good c is assumed to be the same everywhere in the city (the price is taken to be unity for simplicity), the rental price per square foot of housing floor space, denoted p, varies with location.

Since consumers are identical, the urban equilibrium must yield identical utility

[1]While Mills (1967, 1972b) was the first to analyse the overall equilibrium of an urban economy, his analysis lacked generality.

[2]All the results of the analysis can be derived for a general commuting cost function $T(x)$, provided the function satisfies a rather weak requirement ($T''<0$ guarantees satisfaction of this requirement).

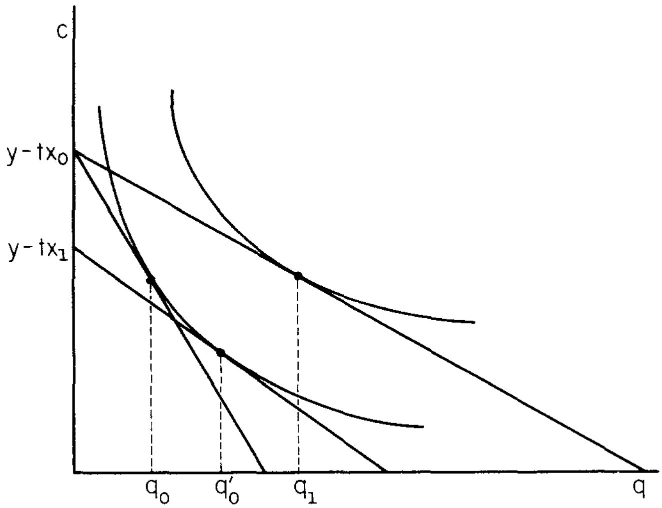

Figure 1.

levels for all individuals. Spatial variation in p provides the key to achieving equal utilities throughout the city. In particular, the price per square foot of housing will vary over space so that the highest utility level attainable at each location equals some constant u. Substituting for c in the utility function using the budget constraint $c+pq=y-tx$, the requirement that the maximized utility level equals u can be written

$$\max_{\{q\}} v(y-tx-pq, q)=u. \tag{1}$$

Eq. (1) reduces to two separate statements. First, since consumers choose q optimally conditional on p, the first-order condition

$$\frac{v_2(y-tx-pq, q)}{v_1(y-tx-pq, q)}=p, \tag{2}$$

must hold (subscripts denote partial derivatives). The key additional requirement is that the resulting consumption bundle must afford utility u, so that

$$v(y-tx-pq, q)=u. \tag{3}$$

The simultaneous system composed of (2) and (3) yields solutions for the unknowns p and q. The solution values depend on the parameters of the equation system: x, y, t, and u.

Figure 1 illustrates various solutions to (2) and (3). The indifference curve with the given utility level is plotted first. Then a budget line with c-intercept equal to

$y-tx$ is drawn so that it is tangent to the indifference curve. The absolute slope of the resulting line equals p, and q is read off from the tangency point. Note that this procedure is the reverse of normal consumer optimization, although the diagram is identical. Utility is fixed, then a price is determined, rather than vice versa. The determination of the urban utility level will be discussed below.

The nature of the dependencies of p and q on the parameters x, y, t, and u can be derived mathematically by totally differentiating (2) and (3). The results can also be inferred diagrammatically from Figure 1. For present purposes, the most important relationships are those between p and x and q and x. These relationships indicate the spatial behavior of housing prices (per square foot) and housing consumption within the city. Totally differentiating (3) with respect to x yields

$$-v_1\left(t+\frac{\partial p}{\partial x}q+p\frac{\partial q}{\partial x}\right)+v_2\frac{\partial q}{\partial x}=0. \tag{4}$$

Since $v_2=pv_1$ by (2), (4) yields

$$\frac{\partial p}{\partial x}=\frac{-t}{q}<0. \tag{5}$$

Thus, the price per square foot of housing is a decreasing function of distance x to the CBD. This result, which is of fundamental importance, can be seen directly in Figure 1. An increase in x from x_0 to x_1 reduces the c-intercept of the budget line, as shown in the figure, so that reestablishing the tangency requires a counter-clockwise rotation of the line around the new intercept. Since the budget line's absolute slope decreases, it follows that p declines as a result of the x increase.

As can be seen from Figure 1, the associated change in q (from q_0 to q_0') is positive, establishing that q is an increasing function of x. Concretely, this means that dwelling sizes increase moving away from the center of the city, a prediction which appears to be confirmed in the real world. Note that since utility is constant, the increase in q corresponds exactly to the substitution effect of the housing price decrease. Formally, it follows that

$$\frac{\partial q}{\partial x}=\eta\frac{\partial p}{\partial x}>0, \tag{6}$$

where $\eta<0$ is the slope of the appropriate income-compensated (constant-utility) demand curve.[3]

The intuitive explanation behind the spatial behavior of p and q is straightforward. Consumers living far from the CBD must be compensated in some fashion for their long and costly commutes (otherwise, no one would live voluntarily at great distances). Compensation takes the form of a lower price per square foot of

[3]Note that $\eta=\partial \mathrm{MRS}/\partial q|^{-1}_{\text{utility}=u}$, a negative expression given the convexity of indifference curves ($\mathrm{MRS}\equiv v_2/v_1$).

housing relative to close-in locations. The resulting decline with x in the price of housing causes consumers to substitute in its favor, leading to larger dwellings at greater distances.[4]

The influences of the parameters y, t, and u on p and q tell us nothing immediate about the internal structure of the city. However, since the various partial derivatives play a crucial role in the comparative static analysis presented below, derivation of their signs is helpful at this point. The discussion will make use of Figure 1. Since an increase in y has the same effect as a decrease in x on the c-intercept of the budget line, it follows that a rotation opposite to that discussed above is needed to restore a tangency. By the above arguments, it then follows that[5]

$$\frac{\partial p}{\partial y}>0, \quad \frac{\partial q}{\partial y}<0. \tag{7}$$

Similarly, since an increase in t has the same effect on the budget line's c-intercept as an increase in x, it follows that

$$\frac{\partial p}{\partial t}<0, \quad \frac{\partial q}{\partial t}>0. \tag{8}$$

Finally, an increase in u holding x, y, and t fixed raises the level of the indifference curve but leaves the c-intercept of the budget line unchanged. A counterclockwise rotation of the line around its fixed intercept is therefore required to restore the tangency, reducing p. The effect on q of the increase in u depends on whether housing is a normal good. If housing is normal, then rotation of the budget line leads to an increase in consumption, as shown in Figure 1 (housing consumption rises from q_0 to q_1). Therefore, when housing is a normal good, it follows that[6]

[4]Note that (6) together with (5) implies that $\partial^2 p/\partial x^2>0$, or that p is a convex function of x. The same conclusion would hold with a general commuting cost function $T(x)$ as long as $T''<0$. Also, note that while the price per square foot of housing declines with x, dwelling rent, which equals pq, may either rise or fall. This follows because $\partial(pq)/\partial x=(1+\eta p/q)q\partial p/\partial x\equiv(1+\sigma)q\partial p/\partial x$, where σ is the income-compensated price elasticity of demand. Note that $\partial(pq)/\partial x \gtreqless 0$ as $\sigma \lesseqgtr -1$.

[5]Note that $\partial q/\partial y$ bears no relation to the regular income effect since utility is held fixed.

[6]The results in (7), (8), and (9) are derived mathematically by total differentiation of (2) and (3) with respect to y, t, and u. This yields

$$\frac{\partial p}{\partial y}=\frac{1}{q}>0, \quad \frac{\partial p}{\partial t}=\frac{-x}{q}<0, \quad \frac{\partial p}{\partial u}=\frac{-1}{qv_1}<0$$

$$\frac{\partial q}{\partial y}=\eta\frac{\partial p}{\partial y}<0, \quad \frac{\partial q}{\partial t}=\eta\frac{\partial p}{\partial t}>0, \quad \frac{\partial q}{\partial u}=\left[\frac{\partial p}{\partial u}-\frac{\partial \mathrm{MRS}}{\partial c}\frac{1}{v_1}\right]\eta>0.$$

The sign of $\partial q/\partial u$ is derived using the inequality $\partial\mathrm{MRS}/\partial c>0$, which must hold for q to be a normal good. The inequality states that indifference curves become steeper moving vertically in Figure 1. Together with convexity, this property implies that a parallel upward shift in the budget line moves the tangency point to the right.

$$\frac{\partial p}{\partial u}<0, \quad \frac{\partial q}{\partial u}>0. \tag{9}$$

Turning now to the supply side of the housing market, it is assumed that housing square footage is produced with inputs of land l and capital N according to the concave constant returns function $H(N, l)$. This function gives the number of square feet of floor space contained in a building with the specified inputs.[7] Concavity of H means among other things that $H_{11}<0$ (capital's marginal productivity diminishes), reflecting the fact that as buildings become taller, capital is increasingly consumed in non-productive uses such as stairways, elevators, and foundations.

An important feature of the model is that the issue of the durability of structures is avoided via the implicit assumption that housing capital is perfectly malleable. In effect, the analysis portrays producers as able to costlessly adjust both their capital and land inputs from period to period. Accordingly, producers are viewed as renting the inputs rather than purchasing them outright, an assumption which may appear particularly unrealistic for the capital input. It should be realized that the assumption of malleable capital is invoked to achieve analytical tractability. Models in which the durability of structures is explicitly recognized are more realistic than the present one but considerably more complex.[8]

Recalling that floor space is rented to consumers at price p, it follows that the revenue from a building is $pH(N, l)$. Note that the building is implicitly being divided up into dwellings (apartments) of the size demanded by consumers. Letting r denote land rent per acre (an endogenous variable) and i denote the spatially-invariant rental price per unit of capital, it follows that producer profit is $pH(N, l)-iN-rl$. Since H exhibits constant returns, profit may be rewritten as $l(pH(N/l, 1)-iN/l-r)$. To simplify notation, let S denote the capital-land ratio N/l, which is an index of the height of buildings (S will be referred to as structural density). Substituting S, profit can be rewritten as

$$l(ph(S)-iS-r), \tag{10}$$

where $h(S)\equiv H(S, 1)$ gives floor space per acre of land. The function h satisfies $h'(S)\equiv H_1(S, 1)>0$ and $h''(S)\equiv H_{11}(S, 1)<0$.

For fixed l, the producer chooses S to maximize profit per acre of land (the expression in parentheses in (10)), and land rent r adjusts so that profit per acre is zero. Since total profit is then zero regardless of the value of l, the scale of the

[7]It is easy to see that in order for this production function to be well-defined, the fraction of the land area covered by the capital must be specified in advance. For a model where the open space surrounding the structure gives utility to the consumer and consequently becomes a choice variable of the producer, see Brueckner (1983).

[8]For a survey of such papers, see the chapter by Miyao in this volume.

producer's building (represented by l) is indeterminate. From (10), the first-order condition for choice of S and the zero-profit condition are

$$ph'(S)=i, \tag{11}$$

$$ph(S)-iS=r. \tag{12}$$

Recalling that p is already a function of x, t, y and u from the solution to the consumer problem, it follows that (11) and (12) determine S and r as functions of these same variables and i. Totally differentiating (11) and (12) with respect to x, t, y, and u yields

$$\frac{\partial p}{\partial \phi}h'+ph''\frac{\partial S}{\partial \phi}=0, \tag{13}$$

$$(ph'-i)\frac{\partial S}{\partial \phi}+\frac{\partial p}{\partial \phi}h=\frac{\partial r}{\partial \phi}, \quad \phi=x,t,y,u. \tag{14}$$

Recalling (11), (14) and (13) yield

$$\frac{\partial r}{\partial \phi}=h\frac{\partial p}{\partial \phi}. \tag{15}$$

$$\frac{\partial S}{\partial \phi}=-\frac{h'}{ph''}\frac{\partial p}{\partial \phi}, \quad \phi=x,t,y,u. \tag{16}$$

The effect of a change in the capital cost parameter i is not considered in the analysis.

Since $h''<0$, (16) implies that $\partial S/\partial \phi$ has the same sign as $\partial p/\partial \phi$, while $\partial r/\partial \phi$ and $\partial p/\partial \phi$ also have the same sign by (15). Recalling (5), the important results

$$\frac{\partial r}{\partial x}<0, \quad \frac{\partial S}{\partial x}<0, \tag{17}$$

are then immediate. Thus, land rent and structural density are both decreasing functions of x, so that land is cheaper and buildings are shorter farther from the CBD. The latter result shows that the model successfully predicts the decline in building heights over distance that is observed in real-world cities. The intuitive explanation for the results in (17) is that lower land rents are required at greater distances to compensate producers for the lower price per square foot of housing. The resulting decline with distance in the relative price of land causes producer substitution in its favor, leading to lower structural densities.

An additional variable of interest is population density, denoted D. Assuming without loss of generality that households each contain one person, D is given by $h(S)/q$, which equals square feet of floor space per acre divided by square feet of floor space per dwelling, or dwellings (persons) per acre. Since $\partial q/\partial x>0$ and

$\partial S/\partial x < 0$, it follows immediately that $\partial D/\partial x < 0$; population density is a decreasing function of distance. The intuitive reason is that since buildings are shorter and the individual dwellings contained within them are larger at greater distances, fewer dwellings and hence fewer people fit on each acre of land. Note that the spatial behavior of population density is a joint result of consumer and producer decisions; consumer substitution in favor of housing and producer substitution in favor of land as x increases are together responsible for the decline of density.

Summing up, the analysis so far has yielded results on the internal structure of cities that appear to recapitulate reality. In particular, the model has predicted that the price per square foot of housing, land rent per acre, and structural and population density are all decreasing functions of distance to the CBD, with dwelling size an increasing function of distance. While these conclusions appear broadly consistent with the results of casual empiricism, systematic empirical tests of the model's predictions have focused mainly on the population density variable. A wealth of evidence has accumulated confirming the negative association between density and distance predicted by the model.[9]

3. Comparative static analysis

While we have seen that the Muth–Mills model does a good job of predicting observed regularities in the internal structures of cities, an equally important goal of the model is to explain intercity differences in spatial structures. For example, the model should be able to explain the building height differential between large and small cities noted in the introduction.

Intercity analysis requires development of the two conditions that characterize the overall equilibrium of the urban area. The first equilibrium condition requires that housing producers outbid agricultural users for all the land used in housing production. Letting $\bar{x}$ denote the distance to the urban–rural boundary, this condition translates into the requirement that urban land rent equals the agricultural rent r_A at $\bar{x}$. Since $\partial r/\partial x < 0$, urban rent will exceed r_A inside $\bar{x}$, as required, and fall short of r_A beyond $\bar{x}$. Recalling that r depends on y, t, and u in addition to x, the first equilibrium condition may be written[10]

$$r(\bar{x}, y, t, u) = r_A. \tag{18}$$

[9]See, for example, Muth (1969). For an analysis of the conditions under which the commonly-fitted negative exponential density function will be the correct specification, see Brueckner (1982).

[10]Recall that while r depends on the capital cost parameter i, this variable is not of interest in the present analysis.

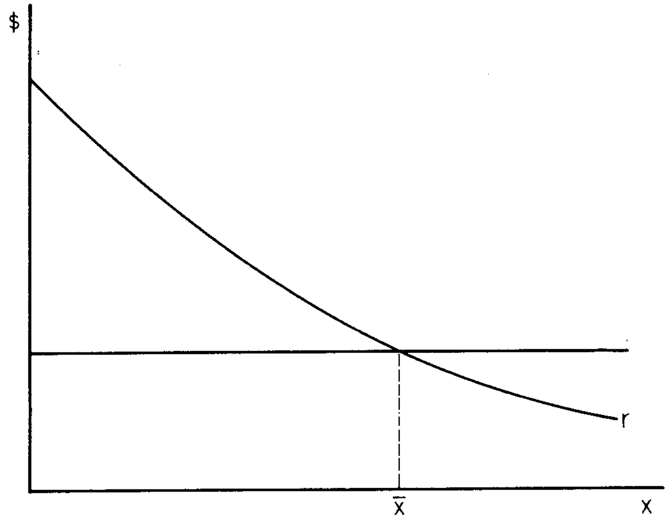

Figure 2.

Figure 2 presents a graphical representation of condition (18) (note that the level of the r function will depend on y, t, and u).[11]

The second equilibrium condition requires that the urban population exactly fit inside $\bar{x}$. To formalize this condition, let θ equal the number of radians of land available for housing at each x, with $0<\theta\leq 2\pi$ (the remaining land will be consumed by the transportation network and topographical irregularities). Then, note that the population of a narrow ring with inner radius x and width $\mathrm{d}x$ will approximately equal $\theta x D(x, y, t, u)\,\mathrm{d}x$, where the functional dependence of population density has been made explicit. The condition that the urban population L fit inside $\bar{x}$ may then be written

$$\int_0^{\bar{x}} \theta x D(x, t, y, u)\,\mathrm{d}x = L. \tag{19}$$

The interpretation of the urban equilibrium conditions depends on whether the city is closed or open to migration. In the "closed-city" case, where migration

[11]The curvature of the land rent contour can be derived by using (15) to compute

$$\frac{\partial^2 r}{\partial x^2} = h' \frac{\partial S}{\partial x}\frac{\partial p}{\partial x} + h\frac{\partial^2 p}{\partial x^2}.$$

Since this expression is positive by previous results, the land rent contour is convex, as shown in Figure 2.

cannot occur, the population variable L is exogenous and the urban utility u is determined along with $\bar{x}$ by balancing of the supply and demand for housing, as expressed in conditions (18) and (19). These conditions constitute two simultaneous equations that determine equilibrium values for u and $\bar{x}$ as functions of the parameters L, r_A, y, and t. In the "open-city" case, costless migration ensures that the urban residents are neither better off nor worse off than consumers in the rest of the economy. In this case, the urban utility level is fixed exogenously, and population L becomes endogenous, adjusting to whatever value is consistent with the prevailing utility level. The boundary distance $\bar{x}$ remains endogenous, and r_A, y, and t remain as parameters. Note that in the open-city case, the system (18)–(19) is recursive instead of fully simultaneous. Eq. (18) determines $\bar{x}$ directly in terms of the parameters, and (19) then gives L.

With the urban equilibrium conditions established, the stage is now set for comparative static analysis.[12] The closed-city case is discussed first, with attention then turning to the open-city case.

3.1. *The closed-city case*

The goal of the analysis is to deduce the impacts of changes in the parameters L, r_A, y, and t on the spatial size of the city ($\bar{x}$), housing prices (p), land rents (r), dwelling sizes (q), and building heights (S). The first step in the analysis is total differentiation of the equation system (18) and (19), which yields the comparative-static derivatives $\partial u/\partial\psi$ and $\partial\bar{x}/\partial\psi$, $\psi = L, r_A, y, t$. While this calculation shows the impact of parameter changes on $\bar{x}$, more work is required to derive the effects on p, q, r, and S. Recalling that at a given x, each of these variables depends on y, t, and u, the sources of change are clear. When y or t increases, there is both a direct effect and an indirect effect operating through the induced change in u. When L or r_A increases, the indirect effect alone is felt since p, q, r, and S do not depend directly on population and agricultural rent. Since the analysis outlined above is quite complex, details are relegated to an appendix.

Before proceeding, it is important to realize that the discussion will focus on the impact of parameter changes on the equilibrium of a single city. Once the differences between the pre-change and post-change cities are known, the conclusions can be used to make intercity predictions (separate cities with parameter levels corresponding to the pre- and post-change values can be compared).

[12]A final point regarding the urban equilibrium concerns the disposition of the rent earned by urban land. Since urban residents subsist on wage income alone, it is clear that the analysis implicitly assumes that rent is paid to absentee landlords living outside the urban boundary. See Pines and Sadka (1986) for comparative static analysis of a model where land is internally owned.

3.1.1. *The effects of an increase in L*

It is shown in the appendix that $\bar{x}$ and u are respectively increasing and decreasing functions of L, or that

$$\frac{\partial \bar{x}}{\partial L} > 0, \quad \frac{\partial u}{\partial L} < 0. \tag{20}$$

A population increase thus causes the city to expand spatially and leads to a lower urban utility level. The population increase will also lead to changes in p, q, r, and S at each location. Since y and t are fixed, the impact of the increase in L is felt entirely through the induced change in u, as explained above. The impact on p is given by

$$\frac{\mathrm{d}p}{\mathrm{d}L} = \frac{\partial p}{\partial u}\frac{\partial u}{\partial L} > 0, \tag{21}$$

where the inequality follows from (20) and the fact that $\partial p/\partial u$ is negative (see (9)). Eq. (21) indicates that the price per square foot of housing rises at all locations as a result of the population increase.

The total derivatives of q, S, and r with respect to L are given by expressions analogous to (21), with p replaced by the appropriate variable. Since the sign of $\partial q/\partial u$ is positive by (9) and since $\partial r/\partial u$ and $\partial S/\partial u$ share the negative sign of $\partial p/\partial u$ by (15) and (16), it follows using (20) that

$$\frac{\mathrm{d}q}{\mathrm{d}L} < 0, \quad \frac{\mathrm{d}r}{\mathrm{d}L} > 0, \quad \frac{\mathrm{d}S}{\mathrm{d}L} > 0. \tag{22}$$

Thus, an increase in L leads to smaller dwellings, higher land rents, and higher structural densities (taller buildings) at all locations. Since q falls and S rises, it follows that population density rises everywhere ($\mathrm{d}D/\mathrm{d}L > 0$).

It is helpful to trace through the effects of the population increase using a heuristic approach. When the city starts in equilibrium and population increases, excess demand for housing is created at the old prices: the urban population no longer fits inside the old $\bar{x}$. As a result, housing prices are bid up throughout the city. On the consumption side of the market, this increase in prices leads to a decline in dwelling sizes at all locations. On the production side, the price increase causes land rents to be bid up everywhere, and higher land rents in turn lead producers to substitute away from land, resulting in higher structural densities. Since buildings are taller and dwellings smaller, population density rises everywhere, so that more people fit inside any given $\bar{x}$. Finally, the rise in the level of the land rent function leads to an increase in the value of $\bar{x}$ that satisfies (18) (Figure 3 shows the upward shift in the land rent contour (from r_0 to r_1) together with the increase in $\bar{x}$ (from $\bar{x}_a$ to $\bar{x}_b$)). The resulting spatial expansion of the city,

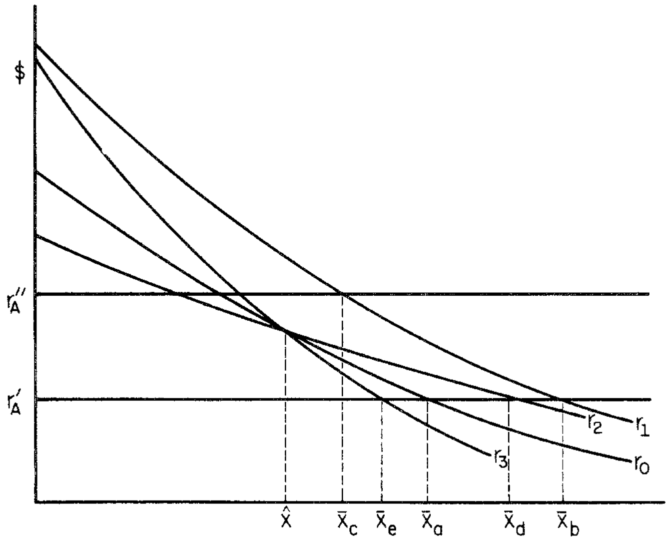

Figure 3.

together with the increase in population densities, tends to eliminate the excess demand for housing, restoring equilibrium.

By describing what appear to be instantaneous changes in the structure of the city as a result of the population increase, the above discussion ignores the fact that buildings are not easily or quickly replaceable. This, of course, is a reflection of the assumption that housing capital is perfectly malleable. Realistically, one would expect the adjustments described above to unfold over a long time period as buildings are torn down and replaced. Thus, the comparative static results are best viewed as predicting the long-run effect of a population increase.[13]

Although the appropriate time horizon must be considered in predicting changes in the spatial structure of a particular city, this issue does not arise when the comparative static results are used to predict intercity differences at a given point in time. The reason is that in a stationary or gradually changing world, the spatial structures of different cities will reflect equilibrium (or approximate equilibrium) outcomes, so that the comparative static results will give correct predictions in intercity comparisons. In the case of population differences, we would expect (holding r_A, y, and t constant) that larger cities would have bigger spatial areas. Moreover, at any given distance from the center, the larger city will have taller buildings and smaller dwellings, and thus a higher population density.

[13]A problem with this view is that the model is being interpreted in a dynamic sense even though producer decision-making has been modeled in a static context.

In addition, the price per square foot of housing and land rent per acre will be higher at a given distance from the center in a larger city. These predictions appear to be consistent with the observed features of cities in the real world.[14]

3.1.2. The effects of an increase in r_A

The effects on the closed-city equilibrium of an increase in the agricultural rent r_A are similar to those of an increase in population. The appendix establishes that

$$\frac{\partial \bar{x}}{\partial r_A}<0, \quad \frac{\partial u}{\partial r_A}<0, \tag{23}$$

indicating that an increase in agricultural rent reduces the spatial size of the city and lowers the urban utility level. Since r_A is not a direct argument of p, q, r, and S, the derivation of the impacts of the agricultural rent change on these variables proceeds as in the case of a population increase, with each total derivative equal to $\partial u/\partial r_A$ times the partial derivative of the relevant variable with respect to u (for example, $\mathrm{d}p/\mathrm{d}r_A=(\partial p/\partial u)(\partial u/\partial r_A)$). Using (9), (15), and (16) together with (23), the signs of the total derivatives are

$$\frac{\mathrm{d}p}{\mathrm{d}r_A}>0, \quad \frac{\mathrm{d}q}{\mathrm{d}r_A}<0, \quad \frac{\mathrm{d}r}{\mathrm{d}r_A}>0, \quad \frac{\mathrm{d}S}{\mathrm{d}r_A}>0. \tag{24}$$

Note that the directions of change in (24) are the same as in the case of a population increase. In particular, the price per square foot of housing, land rent, and structural density rise at each location, while dwelling sizes fall. Note that the movements of q and S raise population density everywhere, so that $\mathrm{d}D/\mathrm{d}r_A>0$.

The intuitive explanation for the above results borrows from the explanation of the impact of a population increase. Proceeding heuristically, the first-round effect (holding u constant) of an increase in r_A is spatial shrinkage of the city, with land near the boundary returned to agricultural use (with u and hence r fixed, an increase in r_A from r'_A to r''_A reduces $\bar{x}$ below its original value of $\bar{x}_a$, as can be seen in Figure 3). This change, however, creates excess demand for housing, so that further adjustments unfold as in the case of a population increase. Note that while $\bar{x}$ expands from its first-round adjustment value (reaching $\bar{x}_c$ in Figure 3 in the case where the final land rent function corresponds to r_1), the variable never rises to its original level (since the city becomes denser, its population fits in a smaller area).

These results can be used to predict differences between otherwise identical cities facing different agricultural rents. For given values of L, y, and t, a city in a

[14]For empirical confirmation of the comparative static predictions regarding the spatial sizes of cities, see Brueckner and Fansler (1983).

region of low agricultural rent (a desert, for instance) will have a larger area than a city located amidst productive farmland. In addition, at a given distance from the center the low-rent city will have shorter buildings and larger dwellings, and thus a lower population density. Also, the price per square foot of housing and land rent per acre will be lower in the low-rent city at a given distance from the center. Real-world observation appears to confirm these predictions.

3.1.3. The effects of an increase in y

An increase in the urban income level raises the demand for housing, so that the city grows spatially. In addition, the utility level of urban residents rises. These results are established in the appendix, where it is shown that

$$\frac{\partial \bar{x}}{\partial y}>0, \quad \frac{\partial u}{\partial y}>0. \tag{25}$$

Deriving the impacts of a higher y on p, q, S, and r is more difficult than the analogous earlier calculations since the direct effect of y must be considered along with the indirect effect that operates through u. The total derivative of p with respect to y is given by

$$\frac{\mathrm{d}p}{\mathrm{d}y}=\frac{\partial p}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial p}{\partial y}. \tag{26}$$

Using (25), (9), and (7), it is clear that the first term in (26) is negative while the second term is positive, so that the sign of the expression is not immediately apparent. However, calculations in the appendix show that

$$\frac{\mathrm{d}p}{\mathrm{d}y} \gtreqless 0 \text{ as } x \gtreqless \hat{x}, \quad \text{where } 0<\hat{x}<\bar{x}. \tag{27}$$

That is, at locations inside some $\hat{x}<\bar{x}$, p falls as y increases, while p rises at locations beyond $\hat{x}$. Thus, an increase in y causes a counterclockwise rotation of the p contour. Moreover, since

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}y}&=\frac{\partial r}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial r}{\partial y}\\ &=h\frac{\partial p}{\partial u}\frac{\partial u}{\partial y}+h\frac{\partial p}{\partial y}\\ &=h\frac{\mathrm{d}p}{\mathrm{d}y},\end{aligned} \tag{28}$$

by (16), it follows that the land rent contour rotates counterclockwise in step with the p contour (the point of rotation is the same $\hat{x}$). Figure 3 illustrates this outcome (r rotates from r_0 to r_2 and $\bar{x}$ increases from $\bar{x}_a$ to $\bar{x}_d$). Since

$$\frac{\mathrm{d}S}{\mathrm{d}y} = \frac{-h'}{ph''}\frac{\mathrm{d}p}{\mathrm{d}y}, \tag{29}$$

by (15), it follows (recalling $h''<0$) that the S contour rotates in the same fashion as the p and r contours. Thus, an increase in y lowers the price per square foot of housing, land rent, and structural density at central locations while increasing the levels of these variables at more distant points. These conclusions, as well as the results in (25), can be used as before to predict differences in the features of otherwise identical cities with different values of y.

It is easy to establish that q rises in response to the increase in y at any location where p falls. This follows because the new consumption bundle must lie on a higher indifference curve [recall (25)] at a point where the MRS is lower (the absolute slope of the budget line, p, has fallen by assumption). Such a point must lie to the right of the original bundle in Figure 1 given that housing is a normal good and indifference curves are convex. Thus, since p falls inside $\hat{x}$, dwelling sizes rise at central locations in response to the increase in income. Since S falls inside $\hat{x}$ from above, it follows that population density also falls at central locations.

By referring to Figure 1, it is easy to see that at locations where p rises, q may either rise or fall. Thus, at locations between $\hat{x}$ and the old $\bar{x}$, the change in q in response to the increase in y is ambiguous. Since it may be shown that the value of p at the new urban boundary is the same as at the old,[15] it follows that the new boundary value of q must be higher than the old boundary value. It is easy to see, however, that this conclusion is not inconsistent with a decline in q at some intermediate location.

To gain an intuitive understanding of the rotation of the p contour, consider the change in locational incentives caused by an increase in income. When income rises, desired housing consumption increases, and since housing is cheaper at greater distances, consumers have an incentive to move to less central locations. This desire to relocate drives up houses prices at distant locations and depresses prices in the now less-attractive central part of the city. These changes lead to sympathetic movements in r and S.

[15]To see this, note that r in (12) must be replaced by r_A when (11) and (12) are evaluated at $\bar{x}$. The two equations then serve to determine boundary values of p and S. Since $\bar{x}$ does not appear explicitly, it follows that the boundary values of both these variables are independent of $\bar{x}$. Note that this argument implies that the boundary values of S and p will also be invariant to changes in L and t (they will change with r_A, however).

3.1.4. The effects of an increase in t

When the commuting cost parameter t increases, commute trips of any given length become more expensive, and the city shrinks spatially in response. The urban utility level declines. These results are proved in the appendix, where it is shown that

$$\frac{\partial \bar{x}}{\partial t}<0, \quad \frac{\partial u}{\partial t}<0. \tag{30}$$

As in the case of an increase in income, the impacts of higher commuting cost on p, q, r, and S are complex. The appendix establishes that

$$\frac{\mathrm{d}p}{\mathrm{d}t}=\frac{\partial p}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial p}{\partial t}\gtrless 0 \quad \text{as } x \lesseqgtr x^*, \quad \text{where } 0<x^*<\bar{x}. \tag{31}$$

Thus, the housing price contour rotates in a clockwise direction, with p rising inside some $x^*<\bar{x}$ and falling beyond x^*. As in the earlier discussion, the r and S contours rotate in exactly the same fashion as the p contour (in this case, clockwise). An increase in t therefore raises the price per square foot of housing, land rent, and structural density at central locations while lowering the values of these variables at more distant points. These impacts, of course, are just the reverse of those generated by an increase in y. The rotation of the land rent contour from r_0 to r_3 and the resulting decline of $\bar{x}$ from $\bar{x}_a$ to $\bar{x}_e$ are shown in Figure 3 (for simplicity, the Figure assumes $x^*=\hat{x}$).

Applying the same argument as before, it follows that q falls at any location where p rises (recall that utility declines in the present case). Thus, dwelling sizes decrease in the central part of the city, although they may rise beyond x^*. Recalling that S rises inside x^*, it follows that central population densities rise.

An intuitive understanding of the rotation of the p contour comes from noting that the increase in t makes close-in locations more attractive given the original pattern of housing prices. The resulting desire of consumers to move toward the CBD bids up central prices and reduces prices at more distant locations, causing a clockwise rotation of the contour.

3.2. The open-city case

For the predictions of the closed-city model to be valid, urban populations must be captive, ruling out utility-equalizing migration flows. When such flows occur, the urban utility level is no longer determined internally, and the open-city model is appropriate. The discussion now turns to a comparative static analysis of this model.

As noted above, the exogenous parameters in the open-city model are u, r_A, y, and t. Holding u fixed, the goal of the analysis is to derive the impact of changes in the observable parameters r_A, y, and t on $\bar{x}$, L, p, q, r, and S (recall that population is now endogenous). The analysis is considerably simpler than in the closed-city case. First, since u is now a parameter, the impact of a changes in r_A, y or t on $\bar{x}$ follows immediately from (18). With the $\bar{x}$ impact thus determined, the effect of a parameter change on L can be read off directly from (19) (the system (18)–(19) is now recursive rather than simultaneous). In addition, the exogeneity of u means that indirect effects on p, q, r, and S (which figured prominently in the closed-city analysis) are absent. Direct effects are the sole sources of change.

An increase in the agricultural rent level has an especially simple impact on the open-city equilibrium. Since y, t, and u are fixed, the land rent function is unchanged as r_A increases (both direct and indirect effects are absent in this case). As a result, $\bar{x}$ must fall as r_A increases, as can be seen in Figure 3. Since p, q, and S are (like r) unchanged at a given x, it follows that the effect of the increase in r_A is simply to truncate the city at a smaller $\bar{x}$, reducing its population but not altering the structure of its remaining area. The model therefore predicts that an open city in a high-r_A region will be smaller spatially and have a lower population than an open city in a low-r_A region. At a given distance from the CBD, however, the cities will be identical.

An increase in y leads to more extensive changes in the open-city equilibrium since the increase in income alters p, q, r, and S at every location, with the impacts (direct effects) given by the partial derivatives in (7), (15), and (16). The signs of these derivatives indicate that the price per square foot of housing, land rent, and structural density rise at all locations in response to the increase in y, while dwelling sizes fall. The upward shift in r leads to an increase in $\bar{x}$, and since population density increases everywhere, L from (19) also increases. These results indicate that in an open urban system, a high-income city will have a higher population, a larger area, and will be denser and more expensive to live in than a low-income city. Note that the model predicts the positive correlation between income and city size noted in various empirical studies.[16]

The changes in the open-city equilibrium following from an increase in t are just the reverse of the impacts of higher income. From (8), (15), and (16), p, r, and S fall at all locations as t increases, while dwelling sizes increase everywhere. Since r falls, $\bar{x}$ declines in Figure 3. Lower population densities at all locations together with a smaller $\bar{x}$ lead to a smaller L by (19). Thus, a high-t city in an open system will have cheaper housing, a smaller area and population, and will be less dense than a low-t city.

To appreciate the connection between these comparative static results and

[16]See, for example, Hoch (1972).

those for a closed city, it is helpful to decompose the open-city changes into two parts. First, let the given parameter (r_A, y, or t) increase holding L fixed, and predict impacts using the closed-city analysis. Then, adjust population to cancel the utility change generated by the parameter increase, again inferring impacts using the closed-city model. The net impact of the two changes corresponds to the open-city effect of the given parameter change. Consider first the case of an increase in income. Holding L fixed, a higher y increases $\bar{x}$ and causes counter-clockwise rotations of the p, r, and S contours (recall Section 3.1.3). Since utility rises and since $\partial u/\partial L<0$, it follows that an increase in population is required to restore the original utility level. The required increase in L raises $\bar{x}$ further and shifts the rotated p, r, and S contours upward while shifting the q contour downward. From above, the net changes relative to the starting point turn out to be an upward shift in each of the p, q, and S contours, and a downward shift in the q contour. Note that the changes holding L fixed could be interpreted as short-run adjustments, with the remaining impacts unfolding after migrants have begun to enter the city in response to the higher utility level.

While the decomposition of the effect of an increase in t parallels the above, an increase in r_A yields a somewhat different series of changes. Holding L fixed, a higher r_A lowers $\bar{x}$, raises the p, r, and S contours, and lowers the q contour. Since utility falls as a result of the higher r_A, a decline in population is required to restore the original utility level. The required decrease in L lowers the p, r, and S contours and raises the q contour, restoring their original positions. The decline in r drives $\bar{x}$ further below its original level. Again, the first-round impacts can be viewed as occuring before migration begins, with the second-round impacts capturing the effects of the outward migration flow induced by the decline in the city's standard of living.

In concluding this section, it is interesting to consider the question of whether cities in a national economy are best viewed according to the open- or closed-city model. On the one hand, the costs of migration (both pecuniary and psychic) are often high, so that utility differences between cities may persist over long periods. On the other hand, migration flows must ultimately eliminate intercity differences in standards of living, especially over a time horizon as long as one or two generations. Casual empiricism suggests that the predictions of both the open- and closed-city models are partly borne out in reality. For example, the real world appears to exhibit the positive correlation between income and city population predicted by the open-city model. Conversely, the low-density character of desert cities is consistent with predictions of the closed-city model but at variance with those of the open-city model. This kind of evidence suggests that utility levels in certain cities may diverge appreciably from the national norm at any point in time. Such differences, of course, are always in the process of being eliminated as consumers migrate.

4. Modifications of the model

Having gained an understanding of the properties of the Muth–Mills model, it is important to remember that its portrayal of the urban economy is highly stylized. While the good predictive performance of the model suggests that its simplifications are artfully chosen, capturing the essential features of real-world cities, it is nevertheless instructive to list the ways in which the model is unrealistic and note the attempts of various authors to add greater realism.

The assumption that the city is monocentric is perhaps the most obvious source of difficulty. While the assumption will be reasonably accurate for many urban areas, many cities have important secondary employment centers outside the CBD. By demonstrating that land-use patterns around such centers follow the predictions of the basic model, Muth (1969) showed that the lessons of the analysis are largely unchanged in a polycentric setting. In a different vein, White (1976) analyzed the forces leading to decentralization of employment by exploring the incentives that might lead a CBD firm to seek a suburban location. In more ambitious studies, Mills (1972a) and Fujita and Ogawa (1982) constructed models where the location of all employment within the city is endogenous and potentially decentralized.

The assumption that all urban residents earn the same income is also unrealistic. The effect of relaxing this assumption has been discussed by Mills (1972b) and extensively analyzed by Muth (1969), who offers a complete treatment of the effect of income differences on household location. In addition, Hartwick et al. (1976) and Wheaton (1976) present comparative static analyses of the equilibrium of a city with multiple income groups. The results of these studies show that many of the key properties of the model are unaffected by income heterogeneity.

While the Muth–Mills approach essentially ignores the urban transportation system by assuming an exogenous commuting cost function, the fact that urban traffic congestion (and hence the cost of travel) is endogenous has been stressed in a number of studies. Although the endogeneity of commuting costs has received most attention in normative models of city structure [see, for example, Dixit (1973)], early positive analyses recognizing the importance of investment in the transportation network were provided by Mills (1967, 1972a).

Another unrealistic feature of the model is its treatment of the housing commodity. As was noted in Section 2, the model's focus on a single housing attribute (floor space) is inconsistent with the fact that real-world dwellings are characterized by a vector of attributes. Although awareness of this fact gave birth to the empirical hedonic price literature in the early 1970s, incorporation of multiple housing attributes in urban spatial models has been more recent [see Büttler (1981) and Brueckner (1983)]. Interestingly, this modification leaves most of the important predictions of the model unchanged.

Modification of the Muth–Mills assumption that housing capital is perfectly malleable has been the goal of a growing new literature in urban economics. The resulting models, which stress the importance of spatial variation in the age of buildings, are considerably more complex than the basic malleable-capital framework (see Miyao, Chapter 22 in this volume, for a survey). The models do, however, generate the kind of spatial irregularities that are observed on a micro level in real-world cities (erratic local building height patterns, for example) but are not successfully explained by the Muth–Mills model.

Finally, a number of studies have introduced local public goods (which are absent in the Muth–Mills framework) into urban spatial analysis. While most studies assume that public consumption is spatially uniform, Schuler (1974) and Yang and Fujita (1983) add a new spatial element to the analysis by focusing on models where the public good level varies with location.

Appendix

This appendix derives the comparative static results cited in Section 3. While the analysis largely parallels that of Wheaton (1974), the derivations do not rely on Wheaton's assumption that the non-housing good is normal (more extensive substitutions in several expressions made avoidance of the assumption possible).

The first step is computation of the partial derivatives of r with respect to x, y, t, and u. Substituting for $\partial p/\partial \phi$ in (15) using (5) and footnote 6, it follows that

$$\frac{\partial r}{\partial x}=\frac{-th}{q}<0,\quad \frac{\partial r}{\partial y}=\frac{h}{q}>0,\quad \frac{\partial r}{\partial t}=\frac{-xh}{q}<0,\quad \frac{\partial r}{\partial u}=\frac{-h}{qv_1}<0. \tag{1a}$$

Then, noting that $D\equiv h/q=-(\partial r/\partial x)/t$, (19) may be rewritten as

$$-\int_0^{\bar{x}} x\frac{\partial r}{\partial x}\,\mathrm{d}x=tL/\theta. \tag{2a}$$

Integrating (2a) by parts then yields

$$-r_A\bar{x}+\int_0^{\bar{x}} r\,\mathrm{d}x=tL/\theta, \tag{3a}$$

where (18) has been used. Letting λ denote any one of the parameters (L, r_A, y, t), total differentiation of (3a) gives

$$\int_0^{\bar{x}}\left(\frac{\partial r}{\partial \lambda}+\frac{\partial r}{\partial u}\frac{\partial u}{\partial \lambda}\right)\mathrm{d}x=\frac{1}{\theta}\left(t\frac{\partial L}{\partial \lambda}+\frac{\partial t}{\partial \lambda}L\right)+\frac{\partial r_A}{\partial \lambda}\bar{x}, \tag{4a}$$

(note that the terms involving $\partial\bar{x}/\partial\lambda$ cancel and that since u is an endogenous variable, the effect of λ on the u argument of r must be taken into account). Since

$\partial u/\partial\lambda$ does not depend on x, this term may be brought outside the integral in (4a), yielding

$$\frac{\partial u}{\partial\lambda}=\frac{\frac{1}{\theta}\left(t\frac{\partial L}{\partial\lambda}+\frac{\partial t}{\partial\lambda}L\right)+\frac{\partial r_A}{\partial\lambda}\bar{x}-\int_0^{\bar{x}}\frac{\partial r}{\partial\lambda}\mathrm{d}x}{\int_0^{\bar{x}}\frac{\partial r}{\partial u}\mathrm{d}x}. \tag{5a}$$

Noting that $\partial L/\partial\lambda$ equals one for $\lambda=L$ and equals zero otherwise, and similarly for $\partial t/\partial\lambda$, and recalling the results of (1a), as well as $\partial r/\partial r_A=\partial r/\partial L=0$, the following inequalities emerge simply from inspection of (5a):

$$\frac{\partial u}{\partial L}<0,\quad \frac{\partial u}{\partial r_A}<0,\quad \frac{\partial u}{\partial y}>0,\quad \frac{\partial u}{\partial t}<0. \tag{6a}$$

Computation of the derivatives of $\bar{x}$ makes use of (5a). Differentiating (18) yields

$$\frac{\partial\bar{r}}{\partial x}\frac{\partial\bar{x}}{\partial\lambda}+\frac{\partial\bar{r}}{\partial u}\frac{\partial u}{\partial\lambda}+\frac{\partial\bar{r}}{\partial\lambda}=\frac{\partial r_A}{\partial\lambda}, \tag{7a}$$

where the bar over r indicates that the function is evaluated at $\bar{x}$. Since $\partial\bar{r}/\partial x<0$, the sign of $\partial\bar{x}/\partial\lambda$ is the opposite of the sign of

$$\frac{\partial r_A}{\partial\lambda}-\frac{\partial\bar{r}}{\partial u}\frac{\partial u}{\partial\lambda}-\frac{\partial\bar{r}}{\partial\lambda}. \tag{8a}$$

Since $\partial r_A/\partial L=\partial\bar{r}/\partial L=0$ and $\partial\bar{r}/\partial u<0$, and since $\partial u/\partial L<0$ from (6a), it follows that (8a) is negative for $\lambda=L$, implying

$$\frac{\partial\bar{x}}{\partial L}>0. \tag{9a}$$

Setting $\lambda=r_A$, (8a) becomes $1-(\partial\bar{r}/\partial u)(\partial u/\partial r_A)$, which, using (5a) with $\lambda=r_A$, becomes

$$\frac{\int_0^{\bar{x}}\frac{\partial r}{\partial u}\mathrm{d}x-\bar{x}\frac{\partial\bar{r}}{\partial u}}{\int_0^{\bar{x}}\frac{\partial r}{\partial u}\mathrm{d}x}. \tag{10a}$$

The sign of (10a) is the opposite of that of its numerator, which, integrating by parts, becomes

$$\left[\bar{x}\frac{\partial\bar{r}}{\partial u}-\int_0^{\bar{x}}x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial r}{\partial u}\right)\mathrm{d}x\right]-\bar{x}\frac{\partial\bar{r}}{\partial u}=\int_0^{\bar{x}}x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{h}{qv_1}\right)\mathrm{d}x, \tag{11a}$$

using (1a). Now $\partial(h/q)/\partial x \equiv \partial D/\partial x < 0$. Also, it may be shown that $\mathrm{d}v_1/\mathrm{d}x > 0$ holds as long as q is a normal good. These facts imply $\mathrm{d}(h/qv_1)/\mathrm{d}x < 0$ and yield positive signs for (10a) and (8a), giving

$$\frac{\partial \bar{x}}{\partial r_A} < 0. \tag{12a}$$

The positive sign of $\mathrm{d}v_1/\mathrm{d}x = v_{11}\partial c/\partial x + v_{12}\partial q/\partial x = (v_{12} - v_2 v_{11}/v_1)\partial q/\partial x$ is established by noting that the term multiplying the positive expression $\partial q/\partial x$ will itself be positive when q is a normal good. Note that $\partial c/\partial x = -(v_2/v_1)\partial q/\partial x$ since utility is constant over x.

When $\lambda = y$, (8a) becomes $-((\partial \bar{r}/\partial u)(\partial u/\partial y) + \partial \bar{r}/\partial y)$, which, substituting (5a), has the sign of

$$\frac{\partial \bar{r}}{\partial y}\int_0^{\bar{x}} \frac{\partial r}{\partial u}\,\mathrm{d}x - \frac{\partial \bar{r}}{\partial u}\int_0^{\bar{x}} \frac{\partial r}{\partial y}\,\mathrm{d}x. \tag{13a}$$

Gathering all terms under the same integral sign and substituting from (1a), (13a) becomes

$$\int_0^{\bar{x}} \frac{\bar{h}h}{\bar{q}q}\left(\frac{1}{\bar{v}_1} - \frac{1}{v_1}\right)\mathrm{d}x, \tag{14a}$$

where the bar again indicates that the variable is evaluated at $\bar{x}$. Since $\mathrm{d}v_1/\mathrm{d}x > 0$, it follows that the integrand in (14a) is negative over the range of integration, making (13a) and (8a) positive and yielding

$$\frac{\partial \bar{x}}{\partial y} > 0. \tag{15a}$$

Computation of $\partial \bar{x}/\partial t$ is more difficult and proceeds in a reverse manner to the above. Eq. (19) is differentiated directly and then $\partial u/\partial t$ is eliminated using (7a). Differentiating (19) with respect to t yields

$$\overline{xD}\,\frac{\partial \bar{x}}{\partial t} + \int_0^{\bar{x}} x\left(\frac{\partial D}{\partial t} + \frac{\partial D}{\partial u}\frac{\partial u}{\partial t}\right)\mathrm{d}x = 0. \tag{16a}$$

Rearranging (7a) to solve for $\partial u/\partial t$ and substituting in (16a) yields, after more rearrangement,

$$\frac{\partial \bar{x}}{\partial t}\left[\overline{xD}\,\frac{\partial \bar{r}}{\partial u} - \int_0^{\bar{x}} x\frac{\partial D}{\partial u}\frac{\partial \bar{r}}{\partial x}\,\mathrm{d}x\right] = \int_0^{\bar{x}} x\left(\frac{\partial D}{\partial u}\frac{\partial \bar{r}}{\partial t} - \frac{\partial D}{\partial t}\frac{\partial \bar{r}}{\partial u}\right)\mathrm{d}x. \tag{17a}$$

Using the definition of D,

$$\frac{\partial D}{\partial t} = \frac{h'}{q}\frac{\partial S}{\partial t} - \frac{h}{q^2}\frac{\partial q}{\partial t}. \tag{18a}$$

Substituting for $\partial S/\partial t$ and $\partial q/\partial t$ using (16) and footnote 6, (18a) becomes

$$\frac{\partial D}{\partial t} = -\left[\frac{(h')^2}{ph''q} + \frac{h}{q^2}\eta\right]\frac{\partial p}{\partial t} \equiv \Gamma\frac{\partial p}{\partial t}, \tag{19a}$$

with $\Gamma > 0$. To evaluate $\partial D/\partial u$, $\partial S/\partial u$ and $\partial q/\partial u$ from (16) and footnote 6 are substituted into an expression analogous to (18a). Since

$$\partial q/\partial u = (\partial p/\partial u - (\partial\,\mathrm{MRS}/\partial c)(1/v_1))\eta,$$

the result is

$$\frac{\partial D}{\partial u} = \Gamma\frac{\partial p}{\partial u} + \frac{h}{q^2}\frac{\partial \mathrm{MRS}}{\partial c}\frac{\eta}{v_1} \equiv \Gamma\frac{\partial p}{\partial u} + \Lambda, \tag{20a}$$

with $\Lambda \equiv (h/q^2)(\partial\mathrm{MRS}/\partial c)(\eta/v_1) < 0$ (see footnote 6). Since $\partial p/\partial u < 0$, it follows that $\partial D/\partial u$ is negative, which implies that the integral on the LHS of (17a) is positive, making the entire term multiplying $\partial\bar{x}/\partial t$ negative. Next, substituting (19a) and (20a) to evaluate the RHS of (17a), the expression reduces to

$$\int_0^{\bar{x}}\left[x\Gamma\left(\frac{\partial p}{\partial u}\frac{\partial\bar{r}}{\partial t} - \frac{\partial p}{\partial t}\frac{\partial\bar{r}}{\partial u}\right) + x\Lambda\frac{\partial\bar{r}}{\partial t}\right]\mathrm{d}x =$$

$$\int_0^{\bar{x}}\left[x\Gamma\frac{\bar{h}}{q\bar{q}}\left(\frac{\bar{x}}{v_1} - \frac{x}{\bar{v}_1}\right) + x\Lambda\frac{\partial\bar{r}}{\partial t}\right]\mathrm{d}x. \tag{21a}$$

The second term in the integrand of (21a) is positive while the first term is also positive since $\bar{x}/v_1 > x/\bar{v}_1$ holds over the range of integration by virtue of $\mathrm{d}v_1/\mathrm{d}x > 0$. Thus the RHS of (17a) is positive, and it follows that

$$\frac{\partial\bar{x}}{\partial t} < 0. \tag{22a}$$

The final results to be derived are the effects of an increase in y or t on p. Substituting in (26) using footnote 6 yields

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}y} = \frac{1}{\tilde{q}}\left(1 + \frac{1}{\tilde{v}_1}\frac{\partial u}{\partial y}\right), \tag{23a}$$

where the ˜ indicates that the variable is evaluated at some $\tilde{x}$ between 0 and $\bar{x}$. Substituting for $\partial u/\partial y$ from (5a) and factoring out $\int_0^{\bar{x}}\partial r/\partial u\,\mathrm{d}x$, (23a) has the sign of

$$-\left[\int_0^{\bar{x}}\frac{\partial r}{\partial u}\mathrm{d}x + \frac{1}{\tilde{v}_1}\int_0^{\bar{x}}\frac{\partial r}{\partial y}\mathrm{d}x\right] \tag{24a}$$

$$= -\int_0^{\tilde{x}} \frac{h}{q}\left(\frac{1}{\tilde{v}_1} - \frac{1}{v_1}\right) \mathrm{d}x, \tag{25a}$$

using (1a). When $\tilde{x}=\bar{x}$, so that $\tilde{v}_1=\bar{v}_1$, (25a) is positive given $\mathrm{d}v_1/\mathrm{d}x>0$. Conversely, when $\tilde{x}=0$, so that $\tilde{v}_1=v_1^0$, (25a) is negative. Furthermore, since (25a) is increasing in $\tilde{x}$, it follows that the expression changes sign just once, establishing that $\mathrm{d}\tilde{p}/\mathrm{d}y$ is negative for $\tilde{x}$ less than some $\hat{x}$ and positive beyond $\hat{x}$.

Substituting in (31) using footnote 6 gives

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}t} = -\frac{1}{\tilde{q}}\left(\tilde{x} + \frac{1}{\tilde{v}_1}\frac{\partial u}{\partial t}\right). \tag{26a}$$

Substituting for $\partial u/\partial t$ using (5a), (26a) has the sign of

$$\tilde{x}\int_0^{\bar{x}} \frac{\partial r}{\partial u}\mathrm{d}x + \frac{1}{\tilde{v}_1}\frac{L}{\theta} - \frac{1}{\tilde{v}_1}\int_0^{\tilde{x}} \frac{\partial r}{\partial t}\mathrm{d}x. \tag{27a}$$

Using (2a) to eliminate L/θ and replacing $\partial r/\partial x$ by $(t/x)(\partial r/\partial t)$ using (1a), (27a) reduces to

$$\int_0^{\bar{x}} \frac{h}{q}\left[\frac{2x}{\tilde{v}_1} - \frac{\tilde{x}}{v_1}\right]\mathrm{d}x, \tag{28a}$$

making further substitutions from (1a). For $\tilde{x}=0$, (28a) is clearly positive. Although (28a) is ambiguous in sign for $\tilde{x}=\bar{x}$, the facts that the boundary value of p is invariant with t (see footnote 12) while $\partial\bar{x}/\partial t<0$ together imply that $\mathrm{d}\bar{p}/\mathrm{d}t<0$ (recall $\partial p/\partial x<0$). Thus (28a) must be negative for $\tilde{x}=\bar{x}$. The fact that its derivative with respect to $\tilde{x}$ is negative then means that (28a) changes sign just once between $\tilde{x}=0$ and $\tilde{x}=\bar{x}$, implying that $\mathrm{d}\tilde{p}/\mathrm{d}t$ is positive inside some x^* and negative beyond x^*.

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