Lecture 5

Network Models of Cities

5.3 Scale Invariance: Examples

IUS 3.1, 3.2

Network Effects

The value of a network is proportional to the number of its connections

Connections grow faster than proportionally to the number of nodes

$$Y \sim N^2$$

Metcalfe's Law

Can this happen for cities?

Scaling Effects

“The laws of complex systems”
(log world)

How a quantity in a system depends on another (scale)

$$Y = f(N)$$

Scale Invariance

$$Y = f(\lambda N) = \lambda^{\beta} f(N) \quad \textbf{?}$$

“homogeneous of degree β”

“power-law"

solution:

$$f(N) = aN^{\beta}$$

Note that there is no single special N: that is scale invariance

Properties of the Logarithm

The living world is a “log world”: of relative growth .

$$\log_{10}(e^x) = \frac{\log_e(e^x)}{\log_e(10)}$$

change of base: just divides by a constant

we will commit the base for simplicity
plots often in base 10; equations in base e

1. Inverse of exponential: $$e^{\log(x)} = x, \qquad \log(e^x) = x$$

2. Turns multiplication into addition: $$\log(x\,.\,y) = \log(x) + \log(y)$$

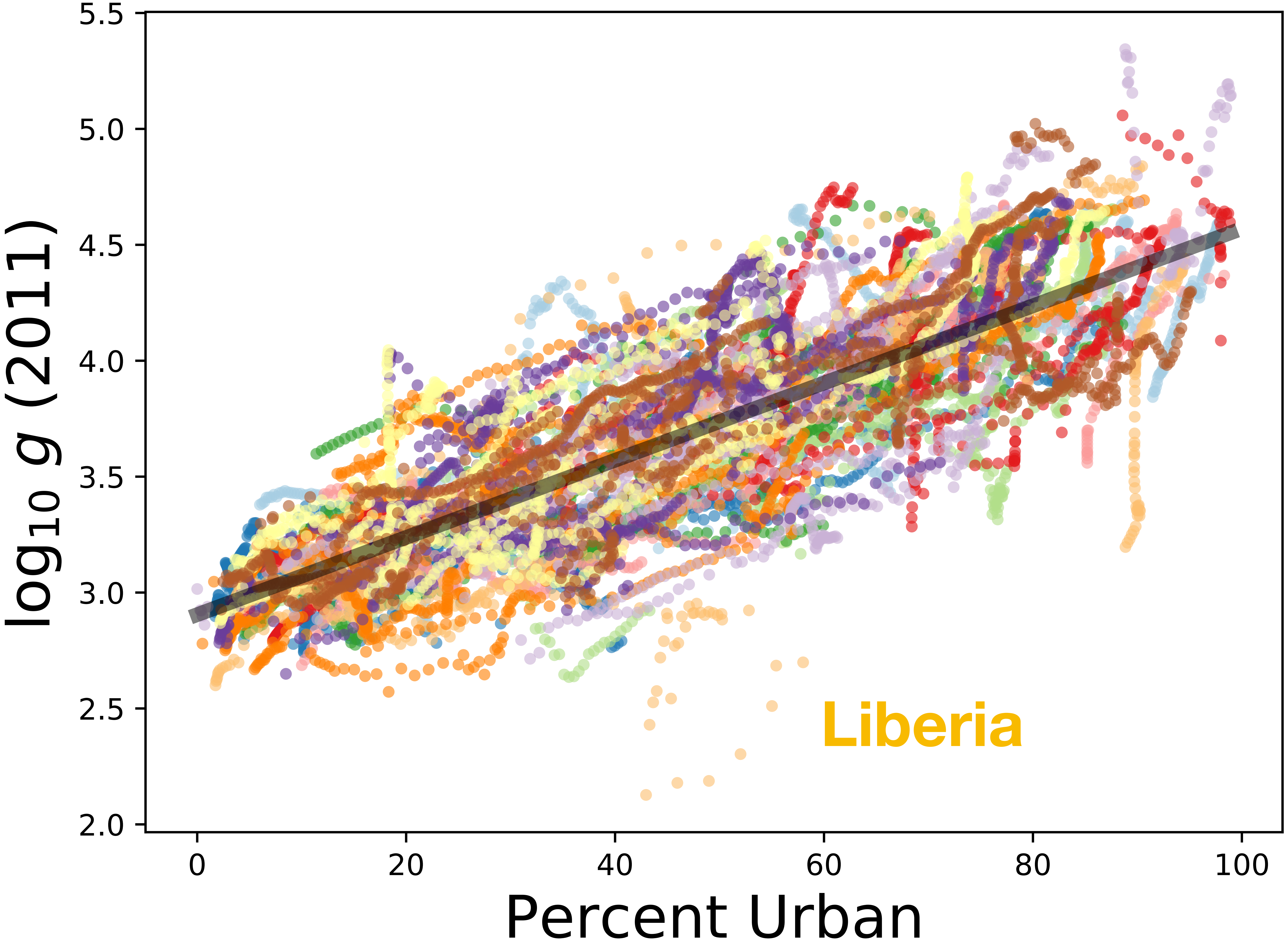
3. Exponents come out and multiply: $$\log(A^x) = x\,.\log(A),$$

4. Derivatives are “percent” changes: $$\gamma(t) = \frac{d\log(A)}{dt} = \frac{\frac{dA}{dt}}{A} \longrightarrow \log A(t) = \int dt\,\frac{d\log A}{dt} = \bar{\gamma}t$$

growth rate

3. Economic Growth and Urbanization

Remember? what does it mean?



National GDP per capita increases 4-5% with each percent increase in the percent of people living in cities

Scaling Effects

$$f(N) = aN^{\beta}$$

	more than doubles	$\beta > 1$	superlinear
When you double N **(**$\lambda = 2$**),** Y	**also doubles**	$\beta = 1$	linear
	less than doubles	$\beta < 1$	sublinear

Properties of logs

Can write this as:

$$\ln f(N) = \ln a + \beta \ln N$$

logarithms transform products into sums

recall that:

$$\ln e^{\beta} = \beta, \quad N^{\beta} = e^{\beta \ln N}$$

$$\beta = \frac{d \ln f(N)}{d \ln N} = \frac{\frac{df}{f}}{\frac{dN}{N}}$$

exponent

"elasticity"

"an x percent change in N, leads to a β x percent change in f"

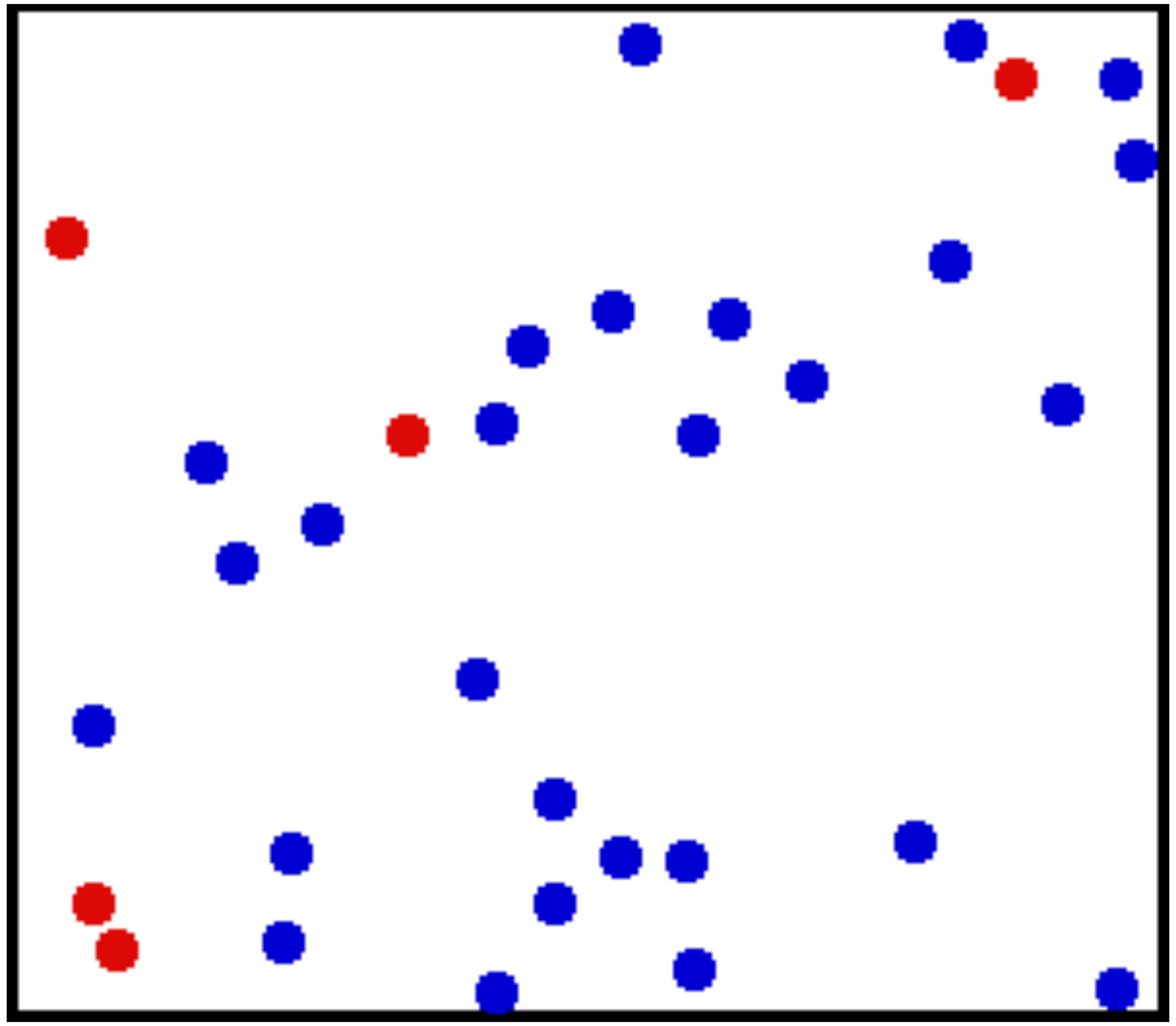
Metcalfe's law says that the value of a network is **superlinear** on the number of its nodes

Examples of scaling relations

Example: the air in this room

IUS 3.1.2

Scaling Effects and Systems



Nothing much happens with size:

a bottle of gas has the same properties as the entire Earth's atmosphere

"Ideal gas" : no interactions

energy/ball

$$E = eN$$

total energy

number of balls

$$A = nN$$

area

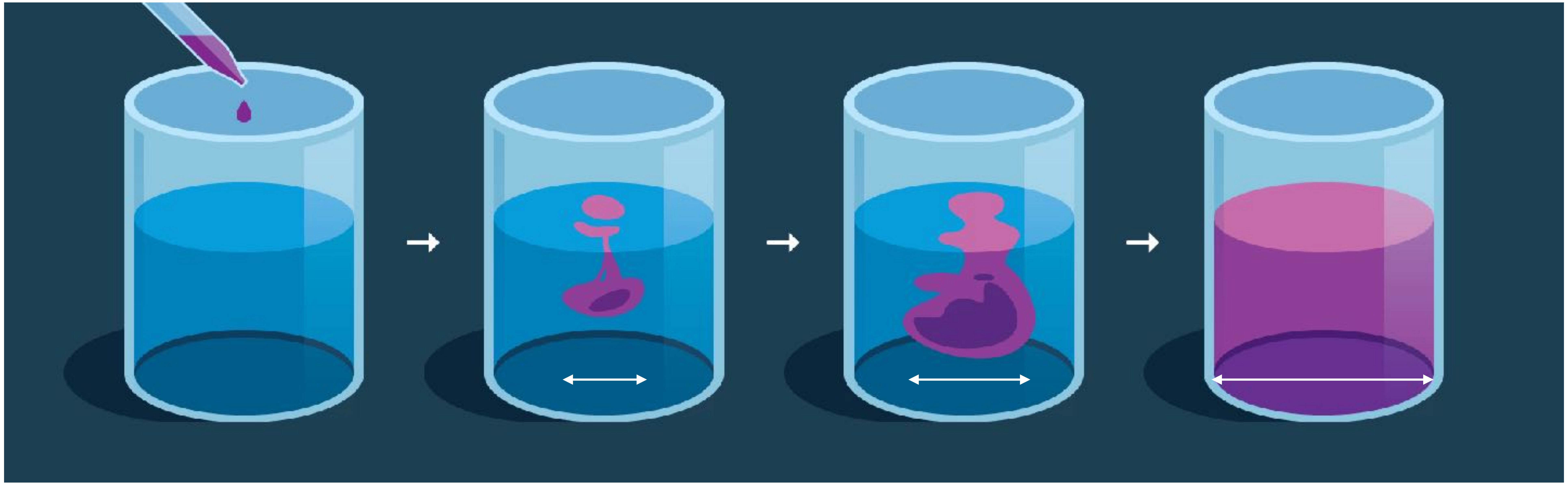
density

linear

Scaling Effects and Systems (examples)

diffusion

the milk in your coffee



$$\left|\Delta x\right|$$

Time and space become intertwined …

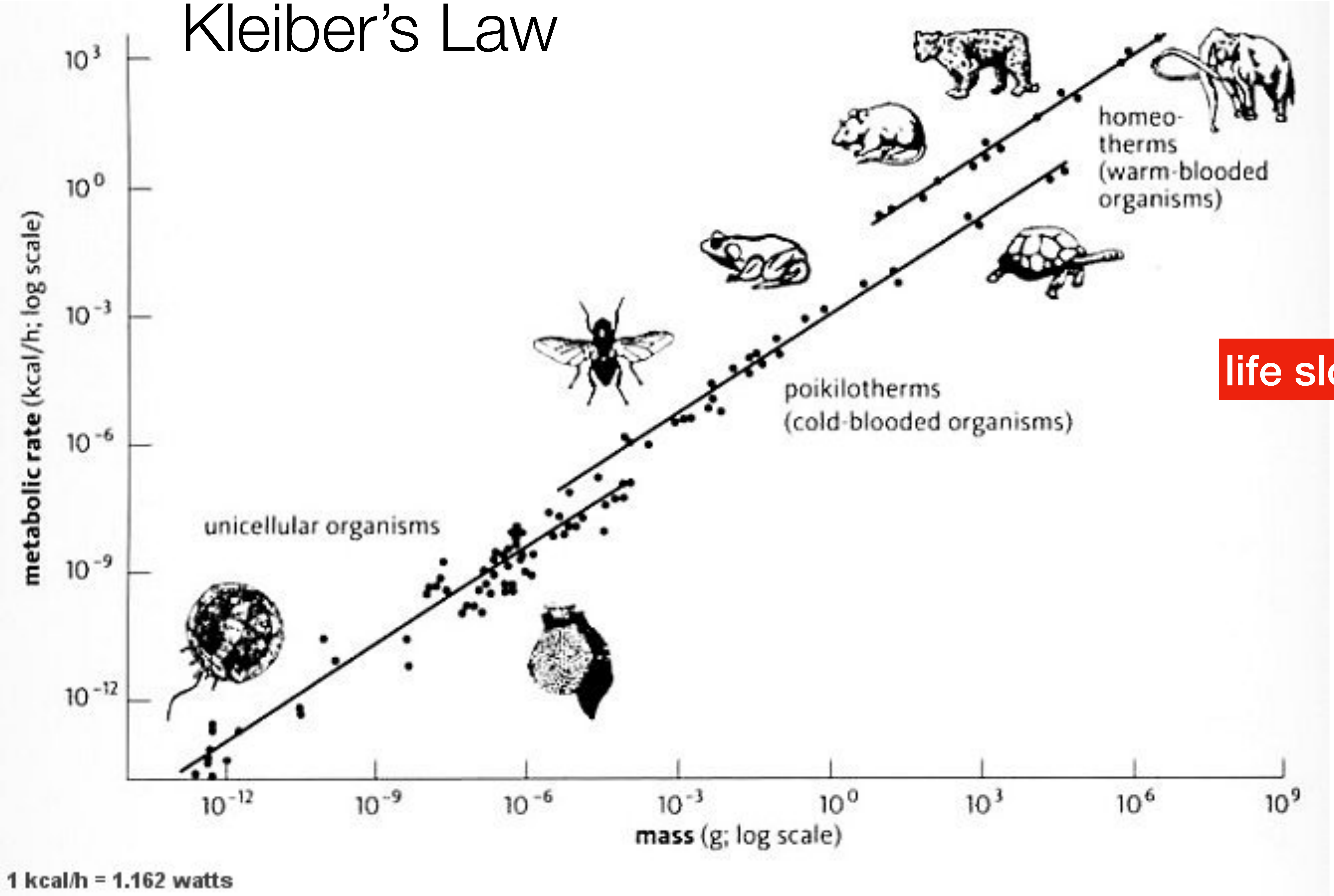
$$\Delta x = \sqrt{2D\Delta t}$$

sublinear: $\Delta x \sim t^{1/2}$

Spatial extent of ink

Time elapsed

Kleiber's Law



life slows down with size

biological metabolism=Energy/time ~ Mass $^{3/4}$

West, Brown, Enquist 1997, …

Star’s Luminosity versus its Mass

Mass–luminosity relation

From Wikipedia, the free encyclopedia

In astrophysics, the **mass–luminosity relation** is an equation giving the relationship between a star's mass and its luminosity, first noted by Jakob Karl Ernst Halm.[1] The relationship is represented by the equation:

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^a$$

where $L_\odot$ and $M_\odot$ are the luminosity and mass of the Sun and $1 < a < 6$.[2] The value $a = 3.5$ is commonly used for main-sequence stars.[3] This equation and the usual value of $a = 3.5$ only applies to main-sequence stars with masses $2M_\odot < M < 55M_\odot$ and does not apply to red giants or white dwarfs.

credit : wikipedia

Stars burn brighter, live faster the more massive they are

Different complex systems have different scaling relations

these express properties of structures that result from attractive and repulsive forces

What kind of scaling relations will cities have?