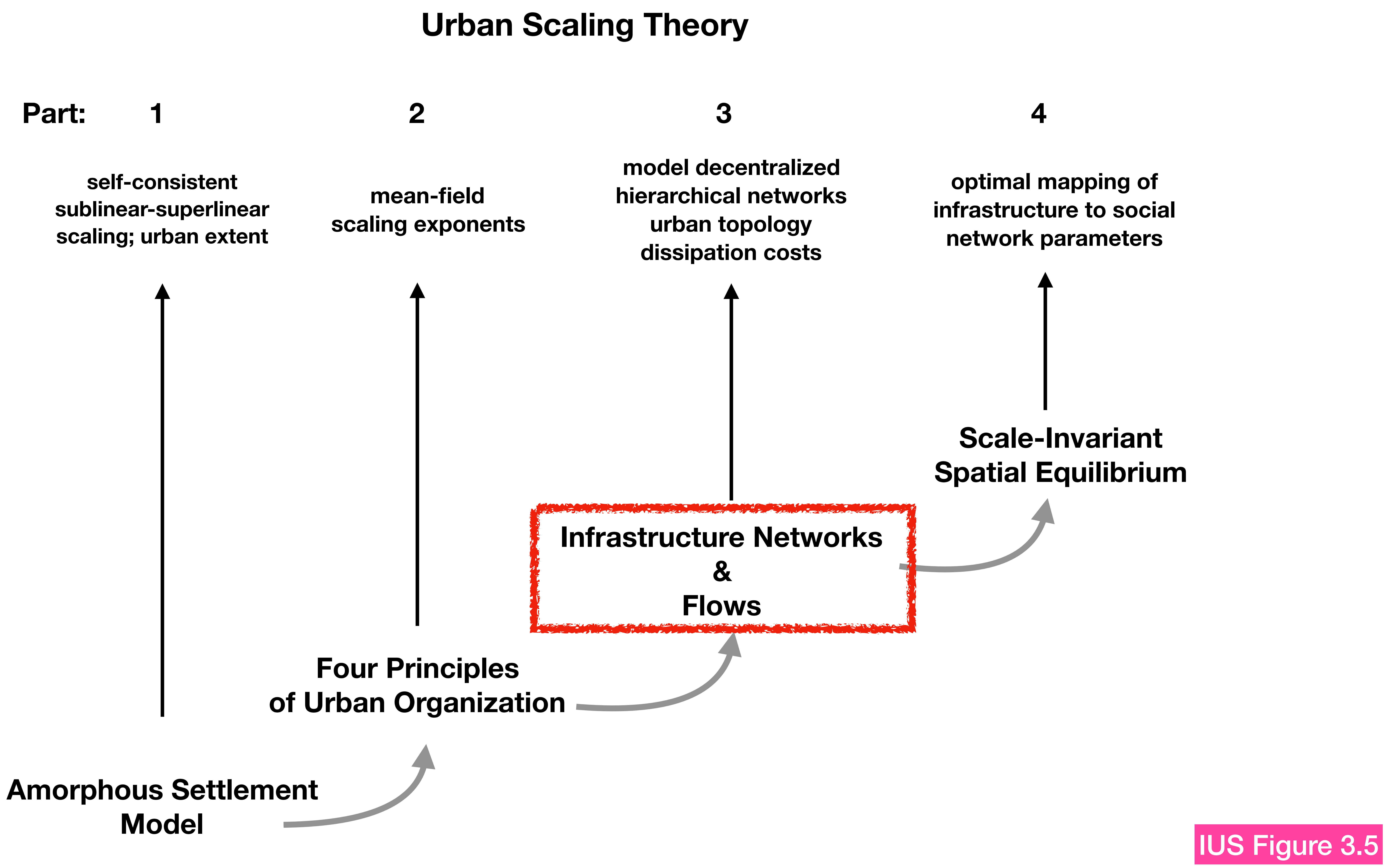
Lecture 7

Network Models of Cities

7.3 Urban Infrastructure, Energy Costs of Movement, Spatial Equilibrium

IUS 3.2.2 pp 82-89


Urban Scaling Theory
Part: 1
2
3
4
self-consistent sublinear-superlinear scaling; urban extent
mean-field scaling exponents
model decentralized hierarchical networks urban topology dissipation costs
optimal mapping of infrastructure to social network parameters
Scale-Invariant Spatial Equilibrium
Infrastructure Networks & Flows
Four Principles of Urban Organization
Amorphous Settlement Model
IUS Figure 3.5

To get closer to the right answer need:

To understand fundamental constraints on human interactions

To understand the general characteristics of urban spaces

A better model of social interactions over built space

To better compute costs of transportation and land rents

The Scale-Independence of City Size

in the presence of increasing returns and transportation costs

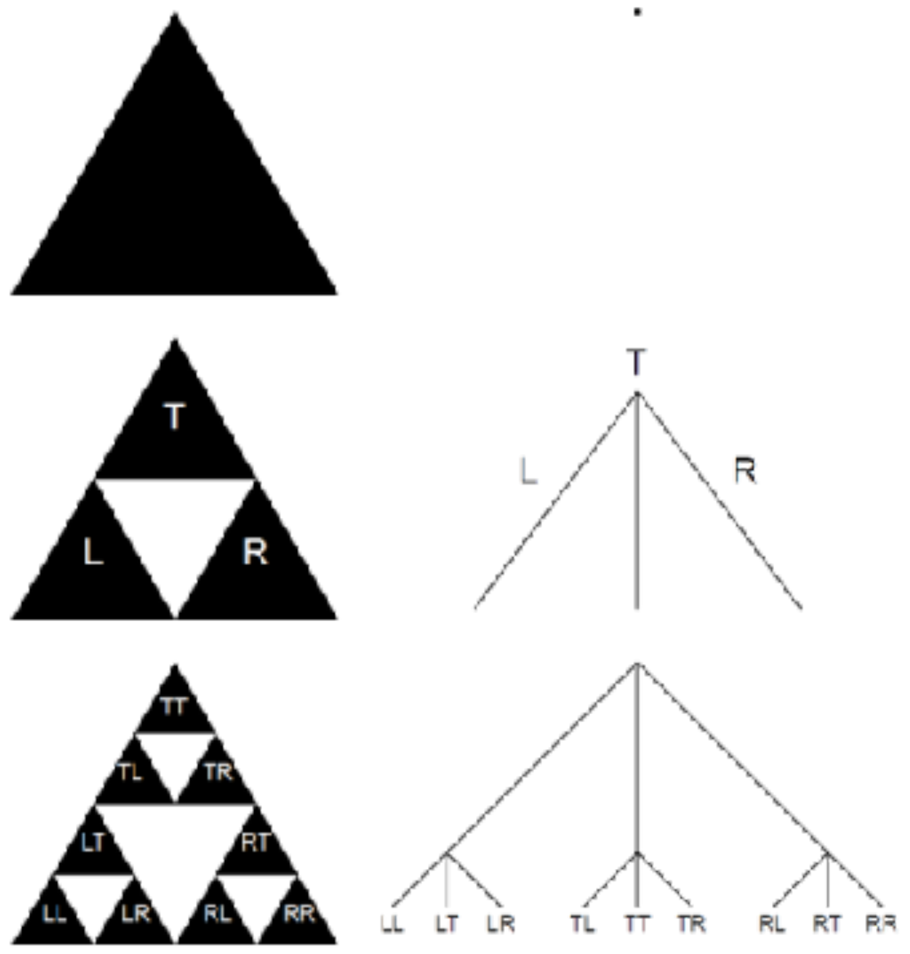
1. Detailed Model of **Urban Infrastructure**

2. General Model of **Cost of Transportation in Cities**

3. The Properties of **Scale-Independent Equilibrium**

Seoul, S. Korea

Infrastructure Networks in the City have a Hierarchy



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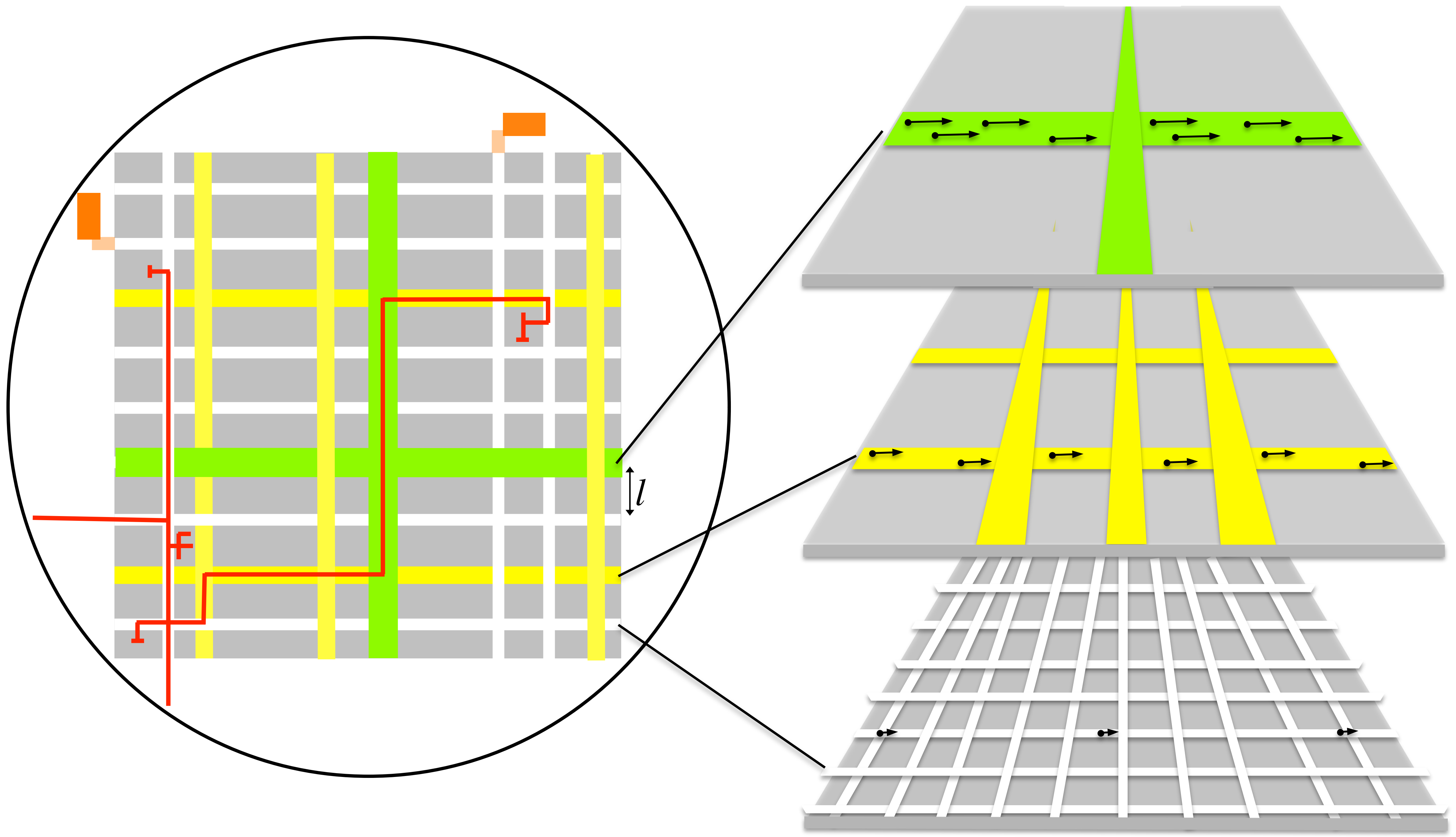


$i = 0$ **Highways: wider + faster**

infrastructure
hierarchy
levels

$N_i = b^i$ **Main Roads**

units of infrastructure at level i

$i = h$ $h = \dfrac{\ln N}{\ln b}$ **Local Roads**

$N = b^h$

$$s_i = s_* b^{(1-\delta)(h-i)}$$ **width segments**

$$s_0 = s_* b^{(1-\delta)h} \gg s_h = s_*$$

width highways — keeps increasing with city size (and individual flows)

width doorways — same everywhere

$$a_i = a b^{(\alpha-1)i}$$ **land area segments**

$$a_h = a b^{(\alpha-1)h} = a N^{\alpha-1}$$

land area per person

$$l_i = \frac{a_i}{l}$$ **infrastructure length segments**

$$l_h = \frac{a}{l} N^{\alpha-1}$$ **minimal distance between people**

We will need this math trick to sum over levels of hierarchies:

Sum of geometric series:

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\left(\frac{1-r^n}{1-r}\right),$$

hierarchical networks (exponentials)

number of nodes at level 0 (root)

branching ratio: b=2

$N_0 = 2^0 = 1$

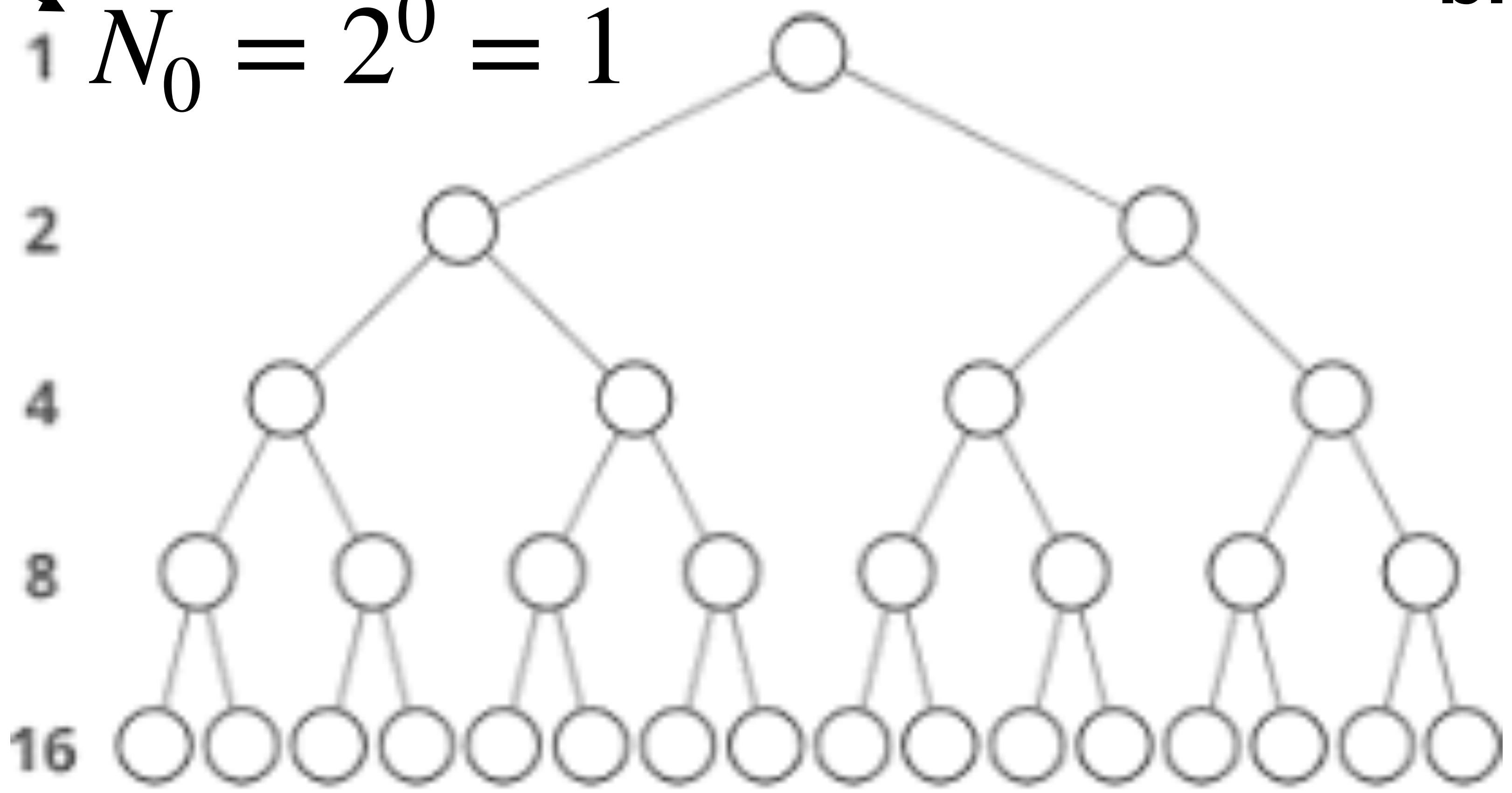
i=0

i=1

i=2

i=3

i=h=4



This is a form of self-similarity (~fractals)

$N_h = b^h = 2^4 = 16$

number of nodes at level h (leaves)

Sum of geometric series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\left(\frac{1-r^n}{1-r}\right),$$

Total Length and Area of Infrastructure Networks

$$L_n = \sum_{i=0}^{h} l_i N_i = \frac{a}{l}\sum_{i=0}^{h} b^{\alpha i} = \frac{a}{l}\frac{b^{\alpha(h+1)}-1}{b^{\alpha}-1} \simeq L_0 N^{\alpha}, \quad L_0 = a/l,$$

Length is area filling
L ~ A

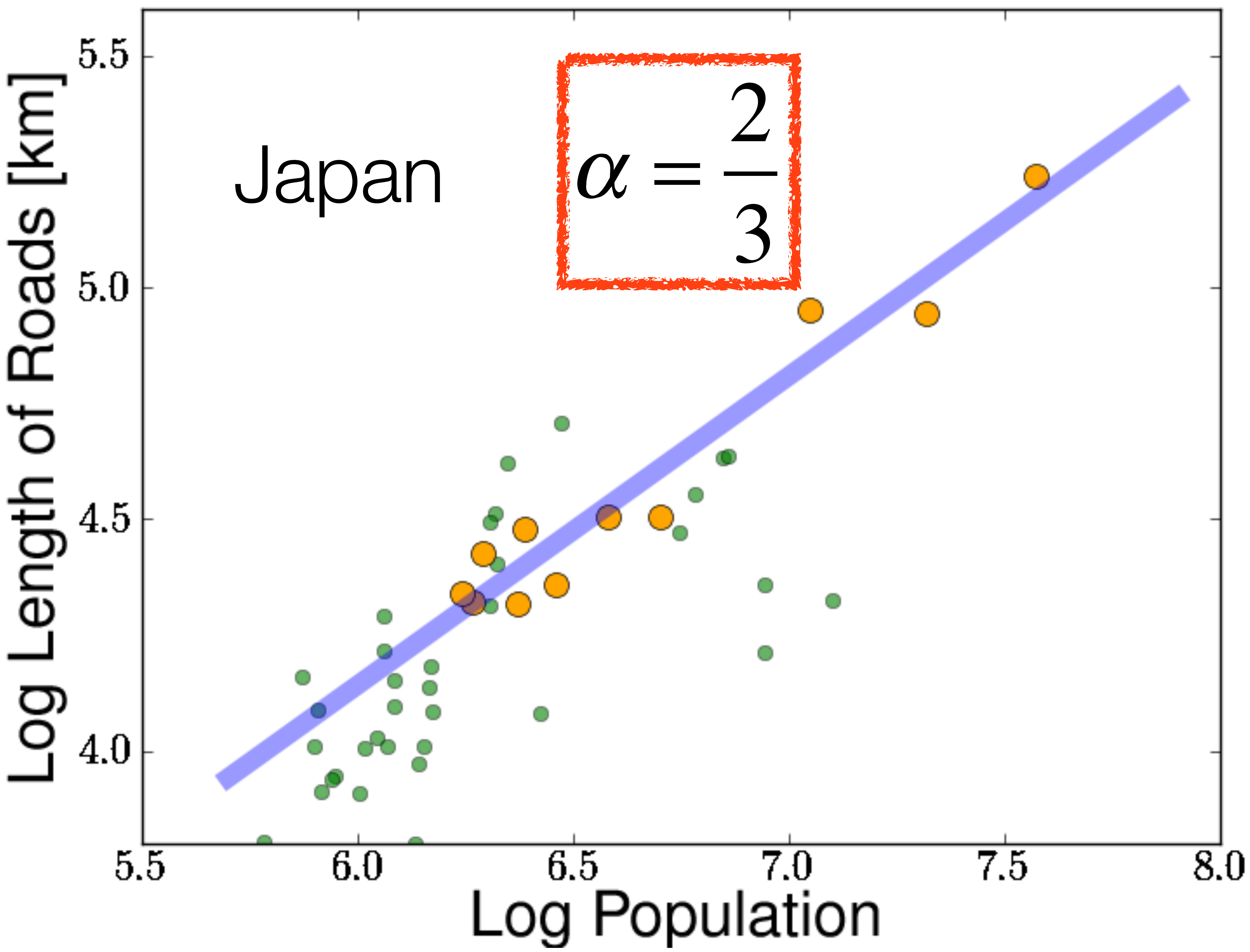
$$A_n = \sum_{i=0}^{h} s_i l_i N_i = s_* \frac{a}{l} b^{(1-\delta)h} \sum_{i=0}^{h} b^{(\alpha+\delta-1)i} \simeq A_0 N^{1-\delta}, \quad A_0 = \frac{s_* a}{l(1-b^{\alpha+\delta-1})},$$

width of network

This gives us back our empirical observations with

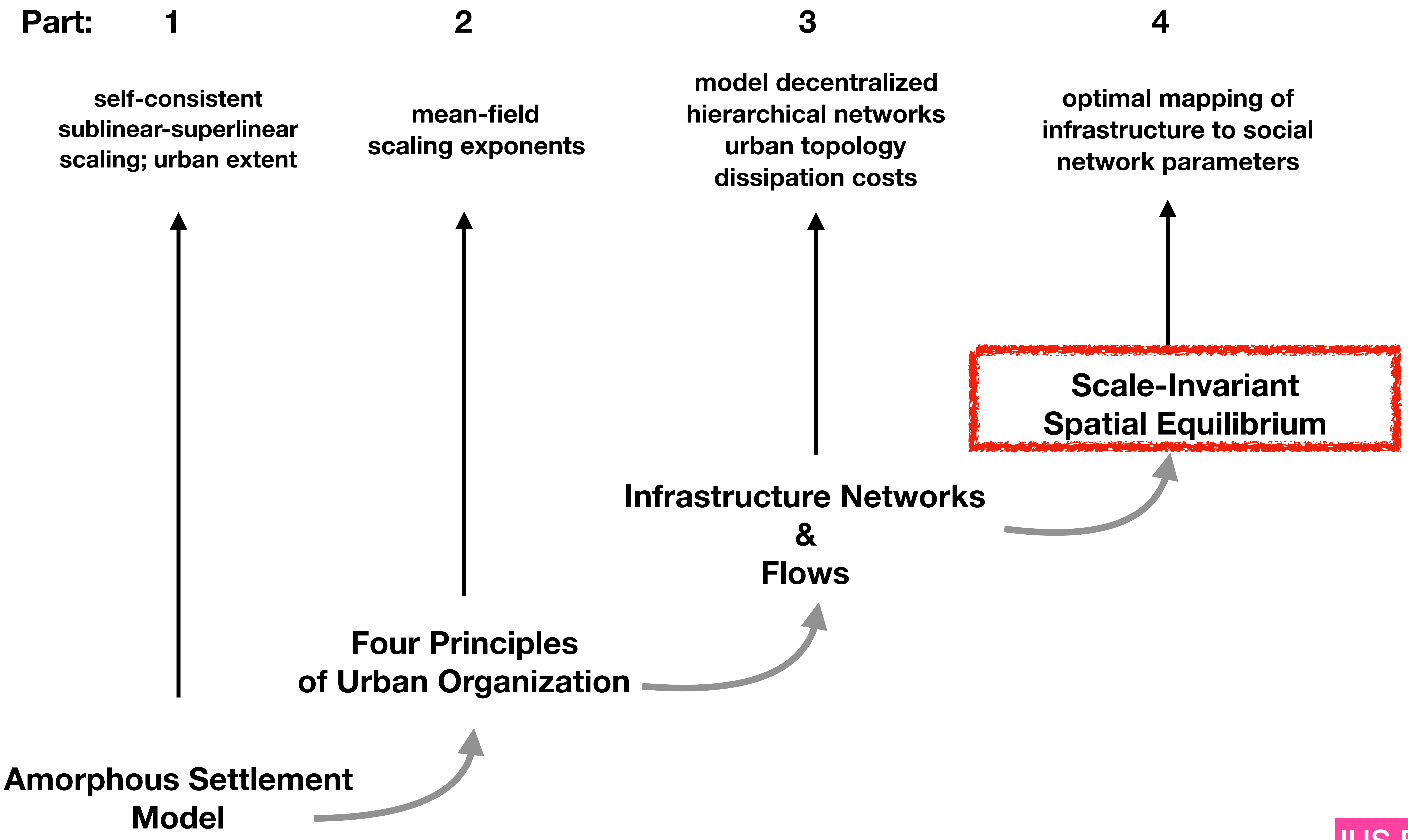
$$\alpha = \frac{2}{3}; \quad \delta = \frac{1}{6}$$

but now also gives us a theory of the entire infrastructure of cities



Japan
$\alpha = \frac{2}{3}$
Log Length of Roads [km]
Log Population
5.5
5.0
4.5
4.0
5.5
6.0
6.5
7.0
7.5
8.0

Urban Scaling Theory



IUS Figure 3.5

The Cost of Socializing in the City

Conservation of Current across infrastructural levels

$$J_i = s_i \rho_i v_i N_i = s_{i-1} \rho_{i-1} v_{i-1} N_{i-1} = J_{i-1}$$

$$\rho_0 v_0 \gg \rho_h v_h$$

highways faster and more densely packed

doorways the same everywhere

$$\rho_i v_i = b^{\delta(h-i)} \rho_* v_*$$

flow per unit area

$$J_i = J = J_0 N, \text{ with } J_0 = s_* \rho_* v_*$$

Resistance accounts for Cost of Movement:

$$r_i = r \frac{l_i}{s_i} \qquad R_i = \frac{r_i}{N_i} = \frac{ar}{ls_*} b^{-(1-\alpha+\delta)i-(1-\delta)h}$$

Parallel resistance because flow can take alternate routes (decentralized networks)

$$\boxed{W = J^2 \sum_{i=1}^{h} R_i = J^2 \frac{ar}{ls_*} b^{-(1-\delta)h} \frac{1-b^{-(1-\alpha+\delta)(h+1)}}{1-b^{-1+\alpha-\delta}} \simeq W_0 N^{1+\delta}, \quad W_0 = \frac{arJ_0^2}{ls_*(1-b^{-1+\alpha-\delta})},}$$

Cost of Transportation scales super linearly
like Social Benefits -> Spatial Equilibrium independent of City population size

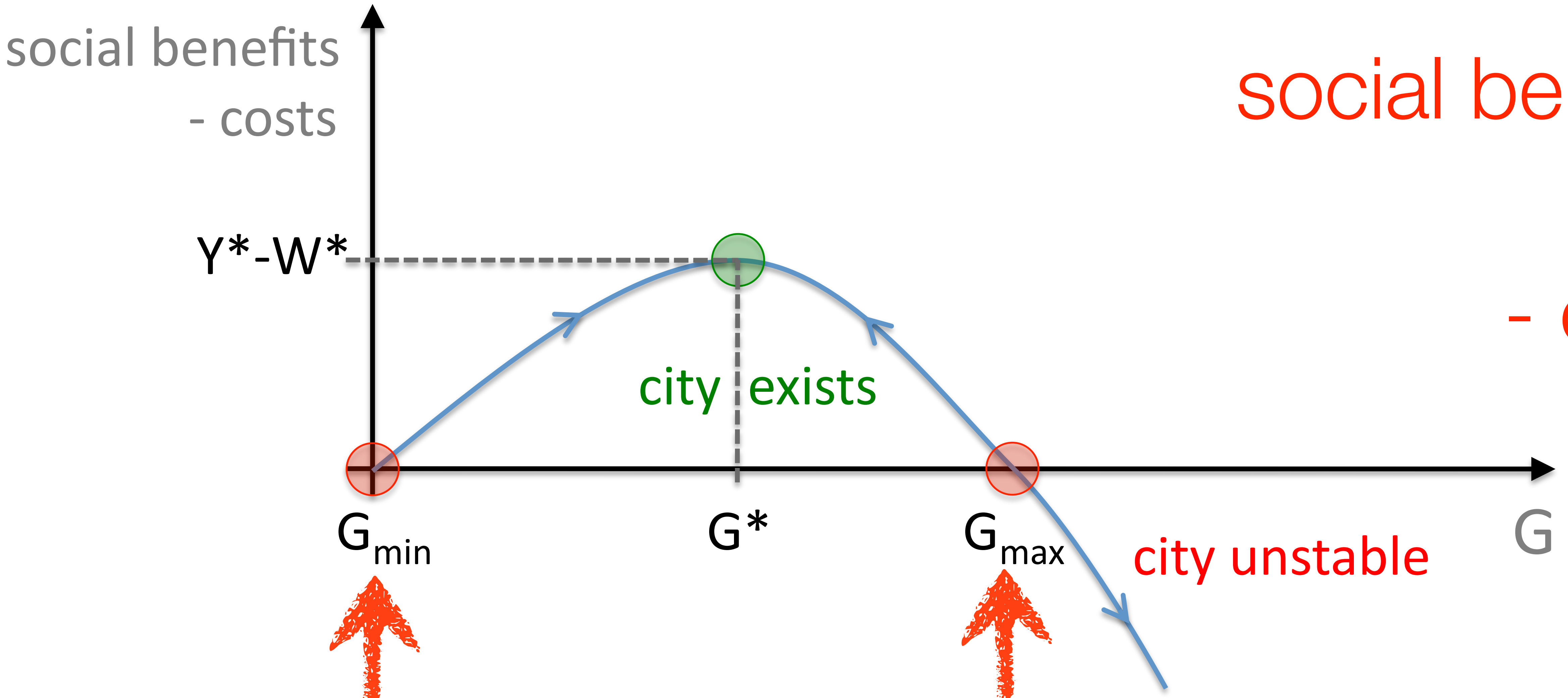
New (scale-invariant) Spatial Equilibrium

Spatial equilibrium between networked social benefits and costs

Scale invariance: cities can exist at “any” population size

Different from Urban Economics: V Henderson

Real “Income”: $Y(N) - W(N)$



social benefits: $Y(N) \sim G \dfrac{N^2}{A_n(N)} \sim N^{7/6}$

- costs: $W(N) \sim W_0 \dfrac{I^2}{A_n(N)} \sim N^{7/6}$

$\boxed{Y_0 \sim G^{1-\alpha} \text{ and } W_0 \sim G^{\alpha}}$

calculation in IUS pp 88-89

Congestion/Danger

High Costs/Dispersion

~Lagos

~”Florida”

Poor, congested, intense, dangerous

Diffuse, sprawling, uneventful

Human Effort is conserved: **Estimating G**

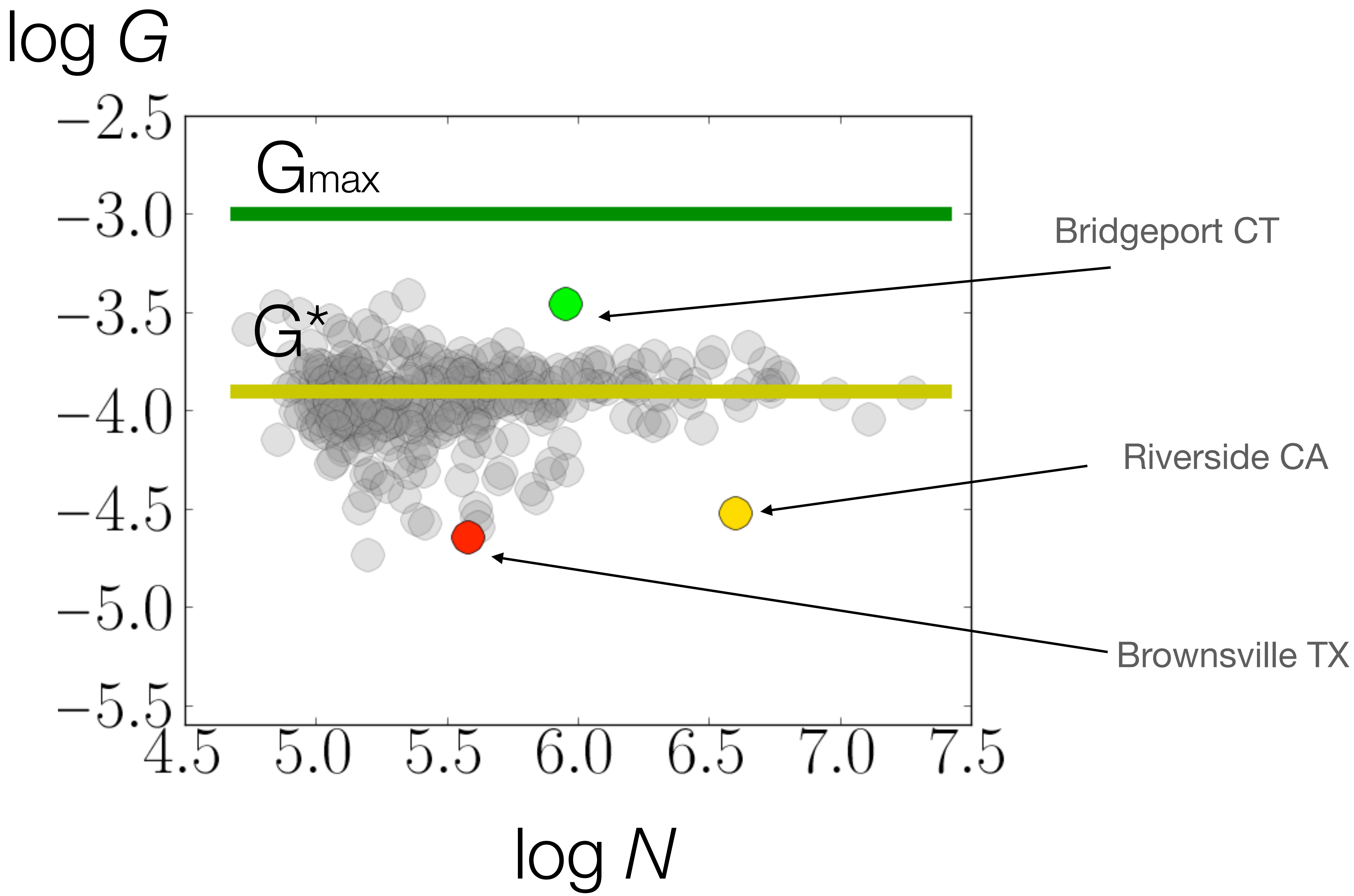
The parameter G measures the strength of social interactions over a life path.

For a city it is:

$$\boxed{G = \left(\frac{Y}{N}\right)\left(\frac{A_n}{N}\right) = a_0 \ell \bar{g}}$$

average income x average area per capita

Total "freedom" = social x spatial freedom



Tradeoff between more Social Space versus more Physical Space

larger cities smaller towns

Many quantitative predictions + general consequences

Urban scaling relation	Exponent prediction $D = 2, H_m = 1$	Exponent prediction General D, H_m
Land area $A = aN^{\alpha}$	$\alpha = 2/3$	$\alpha = \frac{D}{D + H_m}$
Network volume $A_n = A_0 N^{\nu}$	$\nu = 5/6$	$\nu = 1 - \delta$
Network Length $L_n = L_0 N^{\lambda}$	$\lambda = 2/3$	$\lambda = \alpha$
Interactions/capita $k = k_0 N^{\delta}$	$\delta = 1/6$	$\delta = \frac{H}{D(D + H_m)}$
Social outputs $Y = Y_0 N^{\beta}$	$\beta = 7/6$	$\beta = 1 + \delta$
Power dissipation $W = W_0 N^{\omega}$	$\omega = 7/6$	$\omega = 1 + \delta$
Land rents ($\$/m^2$) $P_L = P_0 N^{\beta_L}$	$\beta_L = 4/3$	$\beta_L = 1 + 2\,\delta$

Summary of Urban Scaling relations and exponent predictions for various important quantities. Note that agglomeration effects vanish when $H_m \to 0$ because then people remain spatially separated social networks fail to emerge (we will look at internet quantities later).