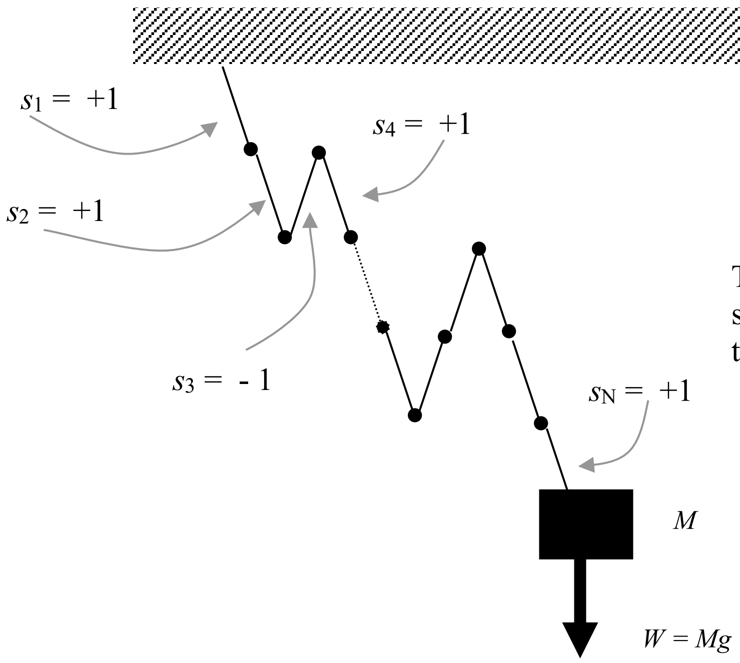
6304NSC HONOURS STATISTICAL & CONDENSED MATTER PHYSICS
STATISTICAL MECHANICS PROBLEM SHEET 1
(Not assessed; answers will be discussed in a tutorial.)

1. Say system 1 has these energy eigenvalues (in some suitable, very small, unit): 0, 1, 1, 2.
(that is, the 1-unit energy state is doubly degenerate etc. and the maximum possible energy is 2.)
Say system 2 has these energy eigenvalues:
0, 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, …

a) Say there are 5 units of energy shared between the two systems. What is $g(5)$? That is, how many microstates have energy 5? What is the distribution of energy in system 1 under the "equal *a priori* probability" assumption? What is the average energy of system 1?

b) Now consider a canonical ensemble for system 1 with the same mean energy. What is its temperature? What is the distribution of energy in system 1? How well does this agree to the answer in part (a). Comment.

2. An elastic band has the unusual property that, if a weight is hung on it, it shrinks when heated. Since rubber is made out of polymer strands, a highly simplified model is that of a single polymer strand consisting of short links of length a. Each link (say the i^{th} from the top) can point either straight down ($s_i = +1$) or straight up ($s_i = -1$).



Typical portion of chain, slightly displaced sideways to make viewing clearer

Treat a chain of N links, with a mass M hanging off the bottom. Assume the links have negligible mass, so that they make no contribution to the potential or kinetic energy. Thus the total energy is the gravitational potential energy of the mass M, which depends on the length of the (multiply folded) chain.

(a) First, treating the entire chain by the **microcanonical** ensemble:

(i) Find the number of ways the chain can fold to achieve net length $L = ma$.

(ii) Find the entropy and temperature as functions of L.

(iii) Show that the chain shrinks when heated

(iv) In what temperature regime does L agree with Hooke's law of elasticity?

(b) Next, we will use the **canonical** ensemble to treat the same chain.

(i) First show that the partition function for any collection of N noninteracting, distinguishable but identically-constructed systems is

$$Z_N = \left(Z_1\right)^N$$

where Z_1 is the partition function of any one system.

(ii) Hence calculate the mean length $\langle L \rangle$ as a function of T.

(iii) Also calculate the RMS deviation (i.e. the standard deviation) in L, and hence the relative deviation,

$$\Delta L_{\text{frac,RMS}} = \left(\frac{\left\langle \left(L - \langle L \rangle \right)^2 \right\rangle}{\langle L \rangle^2} \right)^{1/2}$$

(iv) Are the links in the chain really independent systems, given the (highly idealised) set up shown in the Figure?
Hint: what constraints does the Figure imply?

(v) Given your answers to (iv) and (iii), in what parameter regime do you think your answer to (ii) is accurate?

(vi) Why do you not have to worry about this in part (a)?
Hint: in what limit does it make sense to talk about temperature for the microcanonical ensemble?

3. Consider a single spin-s particle in a magnetic field B, in contact with a heat bath at temperature T. That is, the eigenvalues of the spin $\hat{s}$ in the direction of the magnetic field are $-s, -s+1, \dots, +s$. The Hamiltonian is $\hat{H} = -\mu_s \hat{s} B$. For ease of comparison, assume that the maximum possible magnetisation M is the same for all s, by setting $\mu_s = M/s$.

(a) Calculate and plot the magnetization m as a function of scaled temperature, $k_\text{B}T/MB$, for the cases $s = \frac{1}{2}$ and $s = 1$. Comment on the difference.

(b) Calculate and plot the RMS-deviation in m for these two cases. Comment on the difference. What do you think will happen as s increases?

(c) What about if there are N independent spins. How does the RMS-deviation in the total magnetisation scale with N? How about the *relative* RMS-deviation?