Math 140 Final Exam Review

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1 Overview

About a third of the exam will be on material covered after the last midterm; the other 2/3 of the exam will be questions on material you had to know for the midterms.

For studying purposes, I would suggest starting with the most recent material and working backwards to the earlier stuff, since you haven't had any tests yet on this recent material, and there's more to study. However I'll arrange this review sheet chronologically, starting with the beginning of the semester.

2 Material before Exam I

2.1 Domain

A function f takes a number x in the domain to a value $f(x)$ in the range. Functions often have domains that are much smaller than $(-\infty, \infty)$; for instance, maybe a function only takes x between 2 and 5. If 2 is included but 5 is not included, we would write this as $x \in [2, 5)$. If x can be the number 1 in addition to this interval, we include it in our domain using the "union" symbol. We write single numbers in curly braces, and so the domain would be $x \in \{1\} \cup [2, 5)$.

There are only two important things to watch out for when finding the domain:

1) The insides of square roots are always positive. (Note that the insides of a cube root do not have to be positive, since $-3 \cdot -3 \cdot -3 = -27$.)

2) Never divide by zero.

2.1.1 Find the domain for the following expression:

$$f(x) = \frac{\sqrt{x-5}}{2 - \sqrt{25 - x}} \tag{1}$$

2.2 Range and Other Related Things

We write the range in interval notation (the same way we write domain), except instead of looking for all possible values of x, we're now looking for all possible values of $f(x)$.

2.2.1 Find the range for the following expression:

$$f(x) = \frac{1}{\sqrt{x}} \tag{2}$$

Other things that you might need to calculate are things like turning points and the intervals of increase and decrease. For these problems, if you're looking at the graph, always read from left to right, and if the graph is going down, it's decreasing. If the graph is going up, it's increasing. The point where it switches from increasing to decreasing or vice-versa is called the turning point. Find these for the graph $f(x) = |x|$.

2.3 Graphing Cool Things

Now that I mention the graph of $f(x) = |x|$, you should be thinking of all those first graphs that you learned the first few weeks of class. You learned what $y = x$, $y = 1/x$, $y = |x|$, $y = \sqrt{x}$, $y = x^2$, and $y = x^3$ all look like. You should still know these graphs for the final, because more complex graphs that you'll be asked to graph are essentially just shifting, scaling, or reflecting these known graphs.

There are two categories for how a graph is shifted or reflected. The first category are the shifts/reflections that occur when we replace x by something. Replacing x by $-x$ reflects our graph across the y-axis. Replacing x by $x - a$ shifts our graph right by a. Replacing x by $x + a$ shifts our graph left by a.

The second category occurs when we replace $f(x)$ by something. Replacing $f(x)$ by $-f(x)$ reflects us across the x-axis. $f(x) \to f(x) + 1$ shifts us up 1, $f(x) \to f(x) - 1$ shifts us down 1.

2.3.1 Graph the following equation.

$$y = 1 - \sqrt{x+1}. \tag{3}$$

Also, remember that when we're talking about graphs, y is the same thing as saying $f(x)$.

2.4 Rewriting Absolute Values

What do we mean by "absolute value?" You could think of absolute value as a function that takes every negative number x to the number $-x$. If x is already positive, then just $|x| = x$.

What this means is that if we know that the inside of an absolute value is always going to be positive or always going to be negative, we can rewrite $|x|$ as x or $-x$, respectively.

2.4.1 Rewrite without | |'s.

$$|x - 3| - 2|2 - x| \text{ for } x \in [3, 6]. \tag{4}$$

2.5 Plug and Play

Probably the most fun (or easiest) problems to do when you get the hang of them are the problems where you just replace one thing with another.

These problems might give you an equation like $f(x) = 2x^2 + 100x - 29$ and ask you to find $f(2)$. Or they might give you an even more complicated-looking thing with a bunch of $f(x)$'s in it and ask you to solve it for $f(x) = 1/x$.

The great thing about these problems is that they're really simple. The trick is this: every time you see an x, replace it with what x is. Or every time you see an $f(x)$, replace it with what $f(x)$ is.

In the above problem, this means that every time we see an x, we replace it with 2. This means that $f(2) = 2(2)^2 + 100(2) - 29 = 8 + 200 - 29 = 179$.

2.5.1 Simplify the following for $f(x) = 1/(x-1)$.

$$\frac{f(x+h) - f(x)}{h}. \tag{5}$$

2.6 Compositions

For compositions, we do something similar. We just replace things whenever they're equal. If we want to know $f(g(x))$, and we have $f(x) = 1/x$ and $g(x) = x^3$, then $f(g(x)) = f(x^3) = 1/(x^3)$.

We might also have to go the other way. What if we wanted to represent $(x + \sqrt{x})/2$ as the composition of two functions. Well the first thing is to look for the simplest part of the expression (or you could think of it alternatively as the part "closest" to x). I would say the $\sqrt{x}$ is the simplest, so set the inner function $g(x) = \sqrt{x}$. To see the expression as a composition, just take out this part, and then rewrite the equation in such a way that putting $\sqrt{x}$ back in gives us the original expression.

This then looks like $f(g(x)) = [(g(x))^2 + g(x)]/2 = (x + \sqrt{x})/2$ where $g(x) = \sqrt{x}$ and $f(x) = (x^2 + x)/2$.

2.7 Formulas

I'd also recommend you take a look at all the formula sheets for the various midterms on the Math 140 home page. In this section, the formulas like $y = mx + b$, the formulas for circles, and the formulas for the vertex of $y = (x-h)^2 + k$ are important. (The vertex for that parabola is (h, k) by the way.)

3 Material before Exam II

3.1 Inverses

Do you remember what needs to be true about a function for it to have an inverse? It turns out that a function has to be one-to-one, which means that for every value $f(x)$ in it's range, there is only one value x in the domain that maps to it. This idea corresponds to a graph that has at most one point that intersects any given horizontal line.

Once we know that a function even has an inverse, how do we find it? All we have to do is switch the x's and y's, and solve for y. In fact, we do this same process if we're given the inverse $f^{-1}(x)$ and have to find $f(x)$. Again, recall that y is the same thing as $f(x)$.

3.1.1 Find the inverse of

$$f(x) = \sqrt[3]{\frac{x}{1-x}}. \tag{6}$$

Note: If you graph the inverse function, then all you have to do is reflect your original graph across the diagonal $y = x$.

3.2 Word Problems

Word problems are everyone's favorite, because they're so realistic, right? In this section, the word problems mostly have to do with using triangles in clever ways.

For example, if you have a square that's inscribed in a circle, the diameter of that circle is equal to the hypotenuse of the isosceles triangle that comprises 1/2 the square's area.

What if you have a 10 ft polar bear standing on its hind legs in front of a light mounted 20 ft high on the outside of your house. If his shadow is 4 ft long, could you calculate how far from his house he or she is standing? Try it. Warning - you must solve this problem in under 3 minutes, otherwise you're at risk of being devoured.

3.3 Intersections and Asymptotes

Another important part of this section is finding y and x intercepts, and vertical and horizontal asymptotes.

y Intercepts are the points where the graph crosses the y axis. This happens when $x = 0$. To find the y intercepts, just set $x = 0$ and solve for y.

Similarly, x intercepts are the points where the graph crosses the x axis. This happens when $y = 0$. To find the x intercepts, just set $y = 0$ and solve for x.

3.3.1 Find the x and y intercepts of

$$f(x) = \frac{x^2 - 2x}{(x+1)^2}. \tag{7}$$

What about asymptotes? Vertical asymptotes happen whenever we divide by zero. So, the vertical asymptote for the above problem happens at $x = -1$.

Horizontal asymptotes are a little more complicated because there are three possible cases.

1) If the largest degree of the numerator is bigger than the largest degree of the denominator, there is no horizontal asymptote.
2) If the largest degree of the numerator is smaller than the largest degree of the denominator, then the horizontal asymptote is $y = 0$.
3) If the numerator and denominator have the same degree, then the horizontal asymptote is $y =$ the leading coefficient. Ex. If $y = (2x^3 + 1)/3(x - 5)^3$, then the leading coefficient is 2/3.

3.4 Logs and Exponential Functions

In this section we first introduce the awesome logarithmic and exponential functions. They are awesome. If you're not using them at least once a day, something is wrong.

The basic thing we emphasize here is that the natural logarithm $\ln(x)$ (which is the same thing as $\log_e(x)$) is the inverse of the exponential function e^x, and vice-versa. So $\ln(e^x) = x$ and $e^{\ln(x)} = x$. In general, n^x is the inverse of $\log_n(x)$ (and vice-versa). Another cool thing you should remember is $n \log(x) = \log(x^n)$.

Using these facts, solve the following problem:

3.4.1 Solve for t in

$$3^{2t-3} = \sqrt{3}. \tag{8}$$

Oh, and also know the graphs of the exponential and log functions, and how to graph something like $y = 1 - \ln(2 - x)$.

3.5 Formulas

Look at the formula sheet for this exam to make sure you have a good feel for it. Also recall the observation that $2^{10} \approx 10^3$.

4 Material before Exam III

4.1 More Logarithms

Know the rules for adding and subtracting logarithms backwards and forwards. Remember that $\log a + \log b = \log ab$ and $\log a - \log b = \log(a/b)$.

4.1.1 Combine into a single logarithm.

$$2\ln(1/x) - 3\ln(x+1) \tag{9}$$

4.1.2 Expand into a sum/difference/multiple of the simplest logs as possible.

$$\log_2 \frac{\sqrt[5]{x-1}}{(x+1)} \tag{10}$$

Know the change of base formula: $\log_a(x) = \log_b(x)/\log_b(a)$ where b is any natural number you want.

Also be able to use exponential functions and logarithms to solve for x.

4.1.3 Solve for x.

$$\ln(2x) = 1 + \ln(x-1). \tag{11}$$

4.2 Awesome Growth and Decay Word Problems

Although you never tell your friends, every time Discovery Channel's Shark Week comes around, you lock yourself in your room and watch countless hours of shark-based television. You are obsessed with sharks. In fact, that was the main reason why you decided to move from Nebraska and attend the University of Hawaii.

But after one disturbing afternoon researching sharks in the library, you find that the shark population is diminishing rapidly because of illegal shark poaching. It said that in the glory days of shark lovers, there was a population of 1,500,000 sharks that roamed happily and freely across the face (and depths) of the world's oceans. But 100 years later, there were only 500,000 sharks left. If there were 200,000 sharks 5 years ago, how many sharks will your grandchildren be able to enjoy 50 years from now?

As you ask yourself this question, you remember something about solving for a decay constant k in the exponential decay equation $N(t) = N_0 e^{kt}$ that you never thought would ever come in handy. Luckily, because of your TA's exciting word problems, you still remember how to effortlessly solve these kinds of problems.

You should also try a problem involving half-life, and a problem with exponential growth, although essentially all these problems just depend on you knowing that initial equation and solving for what you don't know.

4.3 Trigonometric Functions

These should look really easy to you now that you've been doing much harder problems with trig functions.

4.3.1 Find $\sin\theta$ if

$$\cos\theta = -4/5 \text{ and } \pi < \theta < 3\pi/2. \tag{12}$$

You should have lots of practice on how to simplify expressions involving trig functions too, and should be able to do proofs involving trig functions.

4.3.2 Simplify

$$\frac{\cot^2 t(\sec^2 t - 1)}{\sec^2 t - \tan^2 t + 1}. \tag{13}$$

4.4 Formulas

Remember angular and linear speed equations, and the various trigonometric identities (hint: you can just remember $\sin^2 + \cos^2 = 1$ and derive the others from it).

5 Material before Exam IV

5.1 Double-angle, half-angle, addition, and sum-to-product formulas

Look for sure at the formulas sheet for Exam 4 - everything you knew for Exam IV you should still remember.

5.1.1 If $x = (\sin\theta)/2$ and $\pi/2 < \theta < \pi$,

$$\text{find } \sin 2\theta,\ \sin(\theta/2), \text{ and } \tan(\theta/2). \tag{14}$$

5.1.2 Rewrite the following as a sum or difference of trig functions:

$$\cos(x-1)\sin(x+1). \tag{15}$$

5.2 Inverse Trig Functions

Remember, although $\sin^{-1}(\sin x) = x$, recall that $\sin^{-1}$ and $\tan^{-1}$ (which are the same thing as arcsin and arctan) are only defined for $-\pi/2 \leq \theta \leq \pi/2$, where $\theta = \sin^{-1} x$ or $\theta = \tan^{-1}$. (This is because functions must be 1-1 to have inverses.)

On the other hand, $\cos^{-1}$ is defined from $0 \leq \theta \leq \pi$.

5.3 Graphing Trig Functions

Remember how to find and graph the amplitude, phase shift, and period of the various trig functions. Review your exam 4 notes.

5.4 Solving Trig Functions

Many times solving for x when x is in an expression involving trig functions leads to many solutions. If we are finding all solutions, we need to account for every time we complete a full revolution around the unit circle.

5.4.1 Find all solutions.

$$2\cos^2 x - 5\cos x = -2. \tag{16}$$

6 Material after Exam IV

6.1 Sine and Cosine Laws

Know how and when to use these equations depending on what sides or angles you are given.

Sine Law.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \tag{17}$$

Cosine Law.

$$a^2 = b^2 + c^2 - 2bc\cos A. \tag{18}$$

$$b^2 = a^2 + c^2 - 2ac\cos B. \tag{19}$$

$$c^2 = a^2 + b^2 - 2ab\cos C. \tag{20}$$

6.2 Multiple Triangles

When do you need to watch out for multiple possible triangles? Whenever you don't know two sides and the included angle, or two angles and the included side, you will need to solve for two possible angle measurements (see next problem).

6.3 Triangle Problems

These problems essentially just vary the sides and angles you know, and require your skillful use of the sine or cosine laws. Also remember that the sum of three angles in a triangle is 180 degrees. Try the following problem.

6.3.1 For a certain triangle, the sides $a = 3$, $b = 6$, and $c = 8$. Find the area.

For more problems, check out practice exam 5 on the Math 140 website.

6.4 Rectangular vs. Polar Coordinates: Can x and y hold their own against r and all the trig functions?

Rectangular coordinates are of the form (x, y), whereas polar coordinates are (r, θ).

Rectangular equations only have x and y in them, whereas polar coordinates only have r and the trig functions in them.

You basically need to know that $r = \sqrt{x^2 + y^2}$, $x = r\cos\theta$, and $y = r\sin\theta$. Just replace the different terms in the equations depending on who you want to see win at the end.

6.4.1 Convert

$$r^2 = \cos(2\theta) \text{ to rectangular coordinates.} \tag{21}$$

6.5 Ellipses

Horizontal ellipses are of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{22}$$

where a is always larger than b.

If y^2 has the larger denominator, than we switch the order and it becomes a vertical ellipse:

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1. \tag{23}$$

We call a the major radius and b the minor radius, and they have the relation $a^2 = b^2 + c^2$ where c is the focal radius. Note that this is different for hyperbolas (see below).

Also, remember the shifting rules for ellipses and hyperbolas (warning: they might seem counterintuitive).

6.6 Hyperbolas

In hyperbolas, a does not need to be bigger than b. The important thing is what's being subtracted from what. If y^2 has the negative coefficient, then it's a horizontal hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{24}$$

Otherwise, it's vetical:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1. \tag{25}$$

Where a is the major radius and b is the minor radius with relationship $c^2 = a^2 + b^2$, where c is the focal radius.