Problem Set for Dynamical Systems

April 15, 2015

Problem 1

The velocity of a free-falling skydiver is given by the following nonlinear ODE:

$$m\frac{dv}{dt} = mg - kv^2$$

In this case, m is the mass of the sky-diver, g is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

A) Qualitatively analyze this ODE. That is, plot $\frac{dv}{dt}$ vs v, look for fixed points, and analyze their stability. Find an expression for the terminal velocity.

Problem 2

Suppose a vat contains 100 gallons of liquid, along with an impressive stir bar that keeps the solution inside well-mixed. Now, suppose that there are two pipes (pipe A and pipe B) that pump liquid into this vat, and one pipe (pipe C) that drains liquid from the vat. Sugar water at a concentration of 30 grams/liter enters the vat at 2 liters/min through pipe A, and sugar water at a concentration of 45 grams/liter enters the vat at 1 liter/min through pipe B. Finally, sugar water leaves pipe C at a rate of 3 liters/min.

A) Write the differential equation describing the rate of change of sugar ($\frac{dS}{dt}$, where S is the amount of sugar in grams) in this vat over time.

B) Solve this differential equation. Find a particular solution using $S(0) = 0$ as your initial condition. What happens to the amount of sugar in the vat over time?

Problem 3

The quadratic integrate-and-fire model is given by the following differential equation:

$$\tau\frac{dv}{dt} = \alpha(v - v_{rest})(v - v_{crit}) + RI$$

where $\alpha > 0$, $v_{rest} < v_{crit}$, and I corresponds to the amount of applied current. Similar to the leaky integrate-and-fire model neuron we discussed in class, a threshold is placed on the voltage such that if v reaches v_{thresh}, v is reset back to v_{reset}.

A) Assume (for now) that I = 0. Qualitatively analyze this differential equation. What happens if $v = v_{rest}$? Or if $v = v_{crit}$? Or if $v_{rest} < v < v_{crit}$? How does this compare to the leaky integrate and fire model neuron?

B) How do the fixed points (specifically, the number of fixed points and their stability) change with the amount of applied current? In the dynamical systems field, a qualitative change in the behavior of a system due to a change in a parameters are called a bifurcation.

Problem 4

Let's consider the following 2-dimensional system:

$$\frac{dx}{dt} = x + y$$

$$\frac{dy}{dt} = 4x - 2y$$

A) Write the corresponding matrix equation (in the form $\dot{\mathbf{z}} = A\mathbf{z}$).

B) Confirm that the only fixed point is the origin (x = 0, y = 0). Find the eigenvalues of A, and associated eigenvectors.

C) On a graph with x on the x-axis and y on the y-axis, draw the line spanned by the first eigenvector and the line spanned by the second eigenvector. Solutions along these lines (also called manifolds) will either travel towards or away from the fixed point, depending on the sign of the associated eigenvalue. Solutions on stable manifolds will travel towards the fixed point, and solutions on unstable manifolds will travel away from the fixed point. Place the appropriate arrows on the manifolds to indicate which way solutions travel.

(D) Remember that, in this simple case, every solution is a linear combination of the manifolds plotted in (C). Using this, try to determine the qualitative behavior for a solution starting with any given (x_0, y_0).