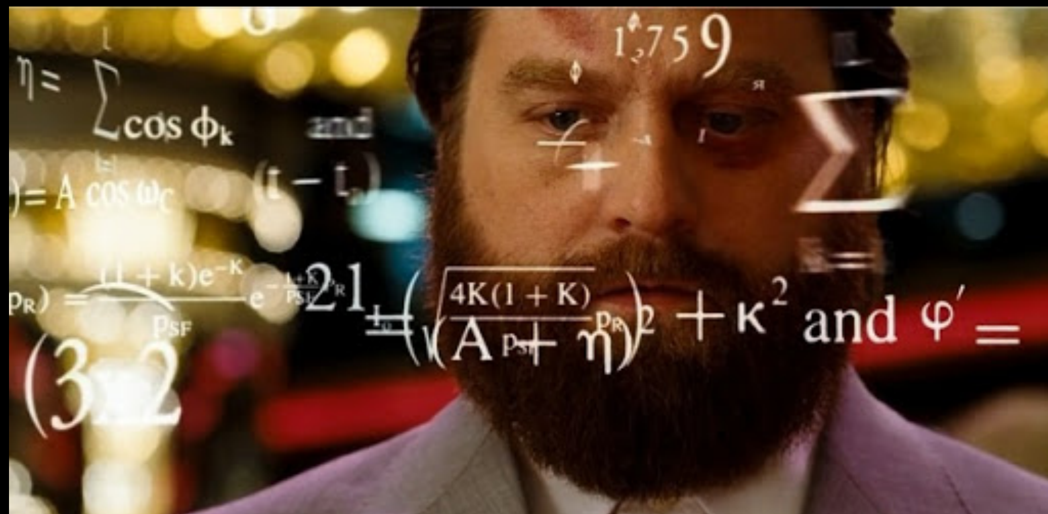
NBIO 228: Math tools for neuroscientists



“What is an eigenvector?”

“What exactly is PCA doing?”

“How do I know when I can model
my data with a linear classifier?

What *really* is a Fourier transform?

Winter 2017 Thursdays 12:30-2:30pm
Fairchild D202 nbio228.stanford.edu

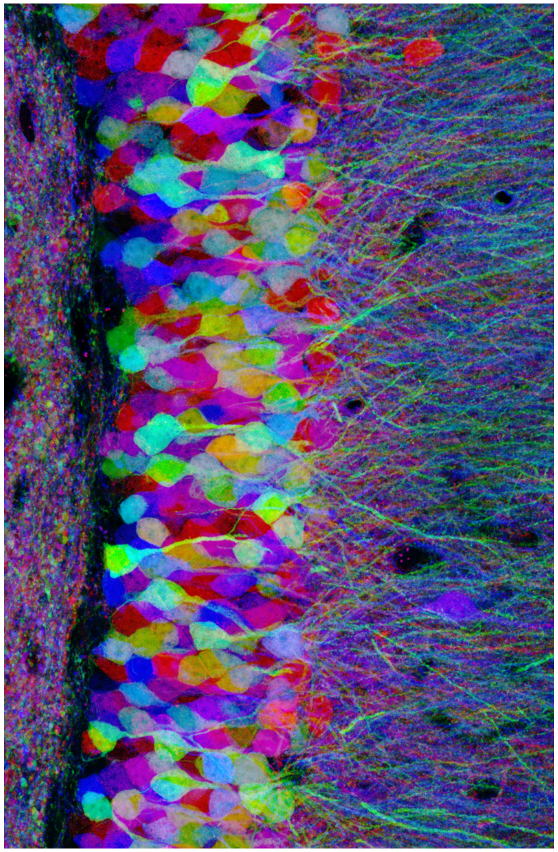
Lecture 1:
Introduction & Linear Algebra

January 12th, 2017

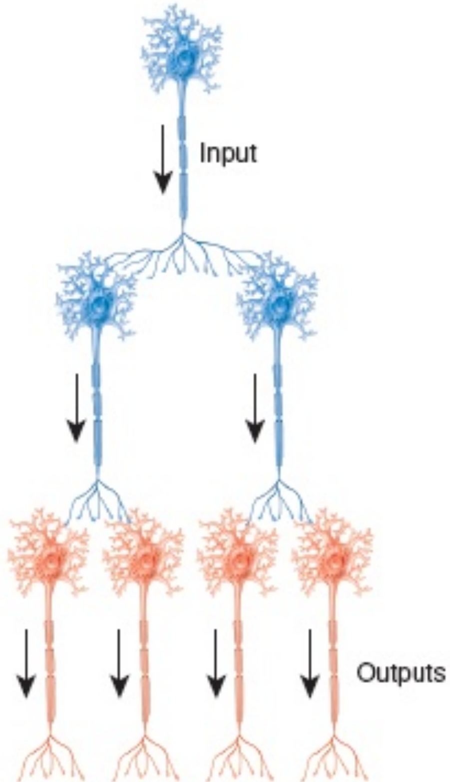
Lane McIntosh & Kiah Hardcastle

Math Tools for Neuroscience

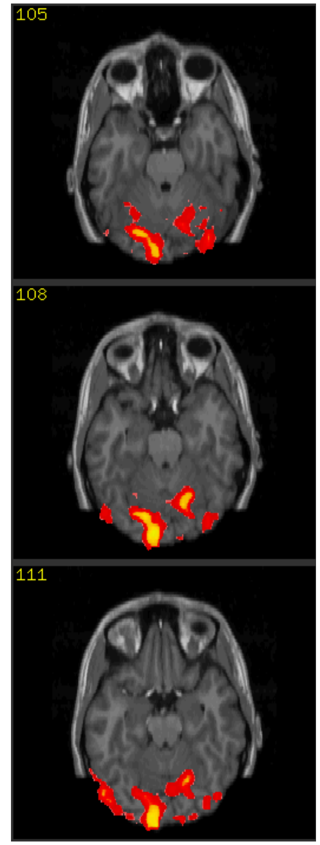
Welcome to NBIO 228!



cellular & molecular



circuits & systems

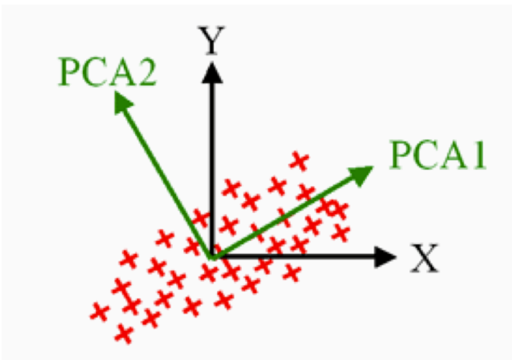


cognitive & behavioral

translational

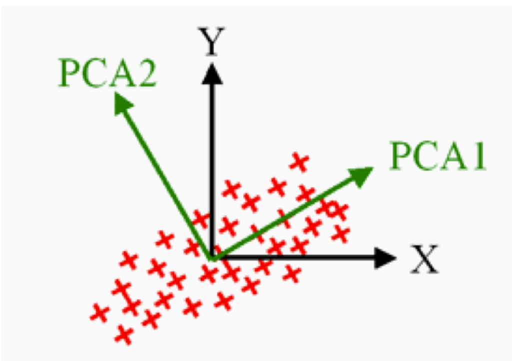
Who is this class designed for?

Topics we will cover this quarter

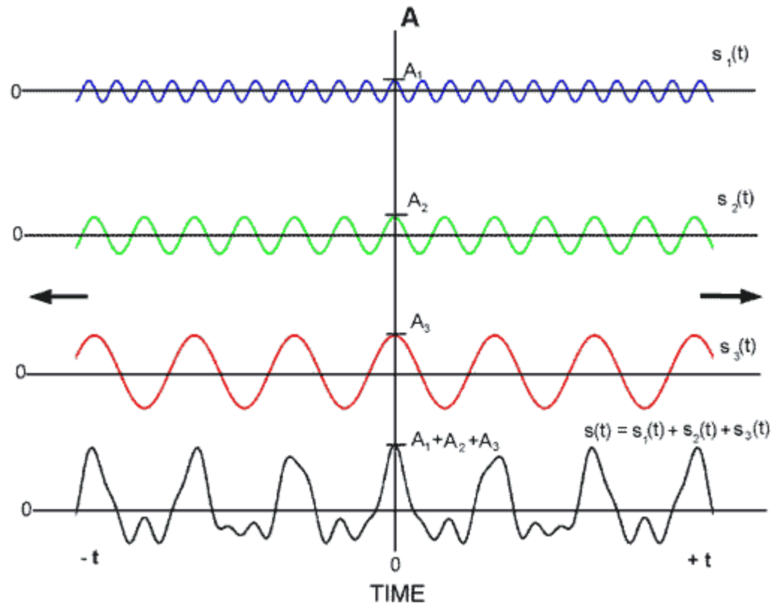


dimensionality
reduction

Topics we will cover this quarter

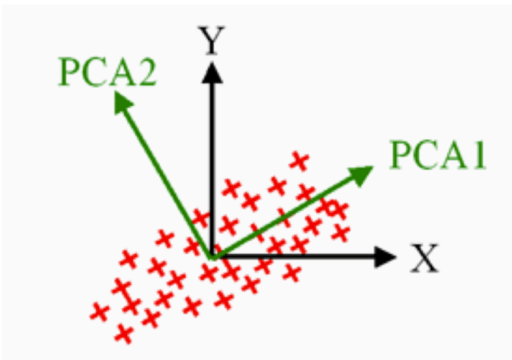


dimensionality reduction



Fourier transforms, convolutions, and filtering out noise

Topics we will cover this quarter



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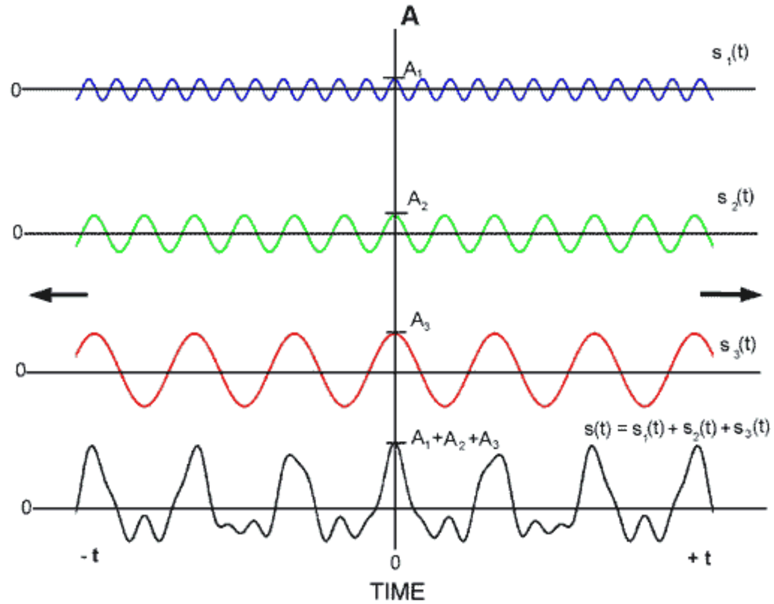


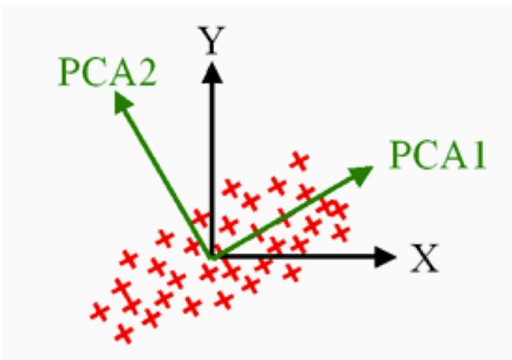
Fourier transforms, convolutions, and filtering out noise

$$C_m \frac{dV}{dt} = -g_L(V - E_L) + g_L \Delta_T e^{\frac{V-V_T}{\Delta_T}} - u + I$$

$$\tau_w \frac{du}{dt} = a(V - E_L) - u$$

differential equations and modeling

Topics we will cover this quarter



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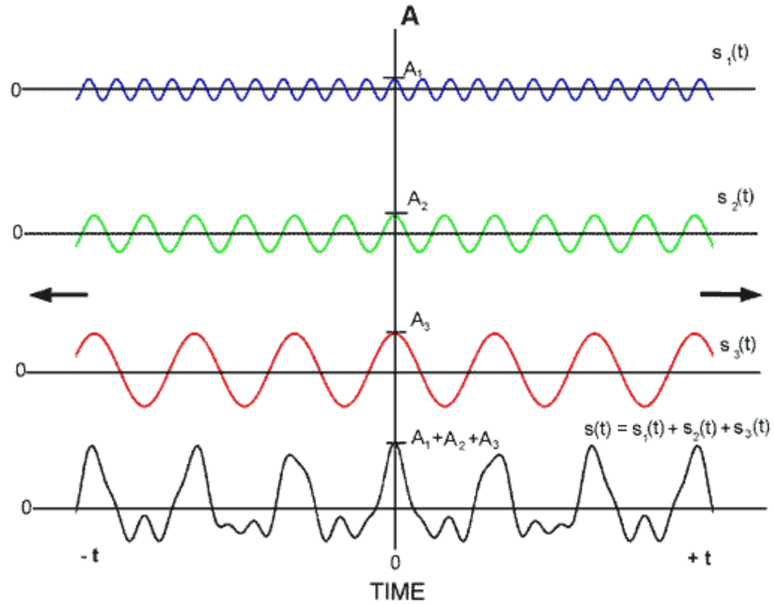

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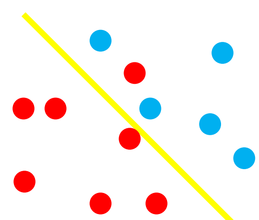
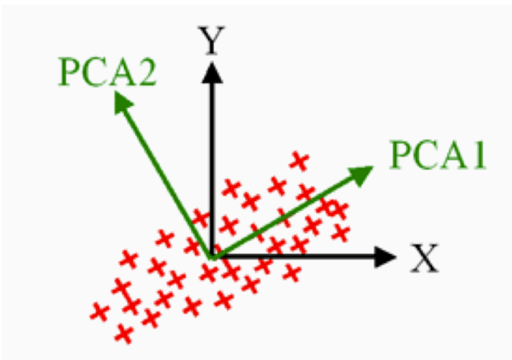
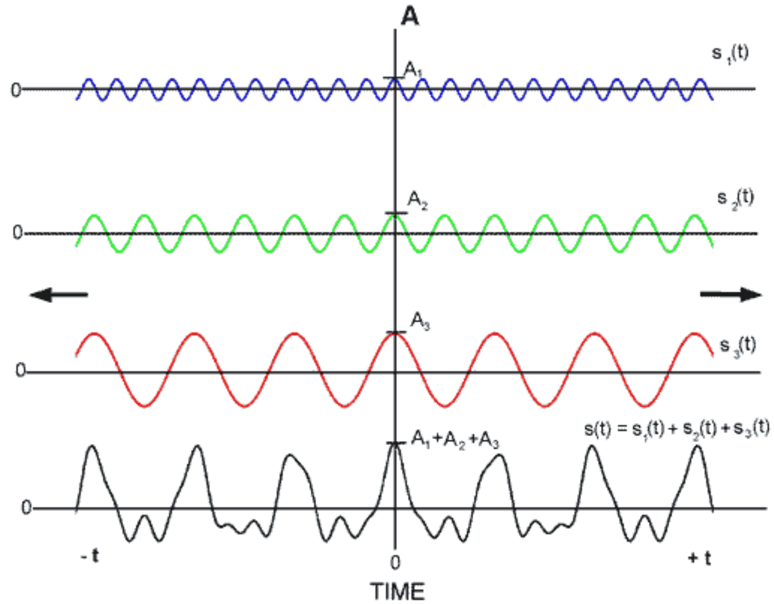


statistical models

Topics we will cover this quarter



dimensionality reduction



Fourier transforms, convolutions, and filtering out noise

$$C_m \frac{dV}{dt} = -g_L(V - E_L) + g_L \Delta_T e^{\frac{V - V_T}{\Delta_T}} - u + I$$

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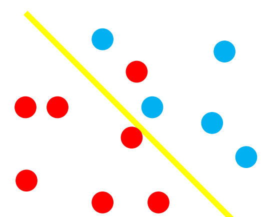
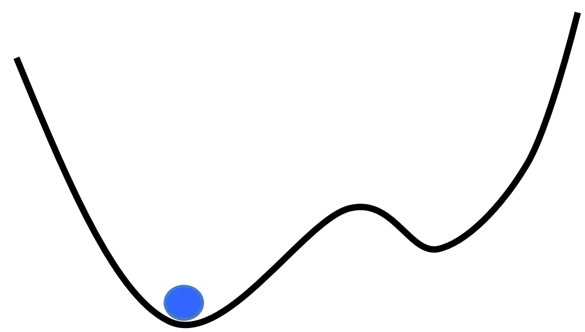
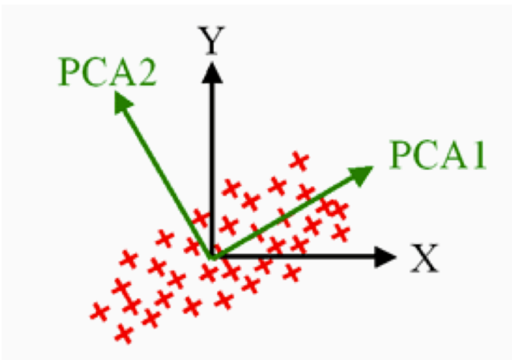


statistical models



optimization

Topics we will cover this quarter



dimensionality
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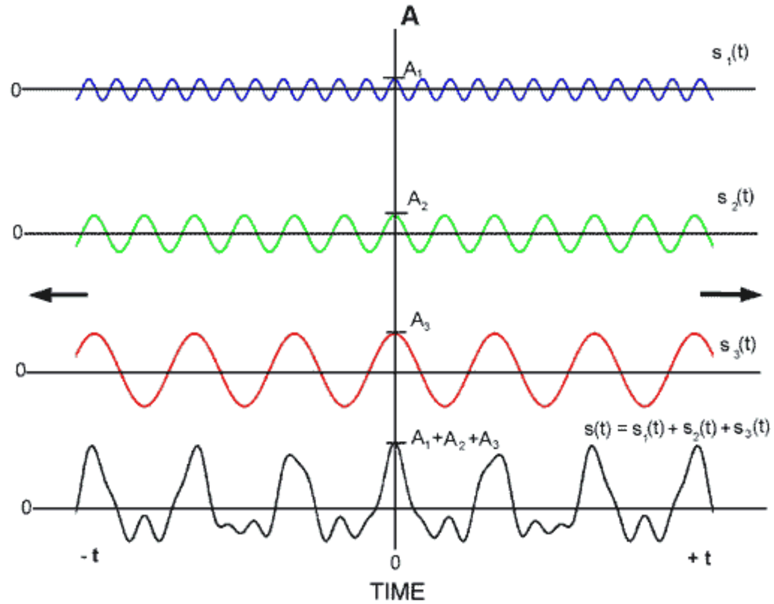
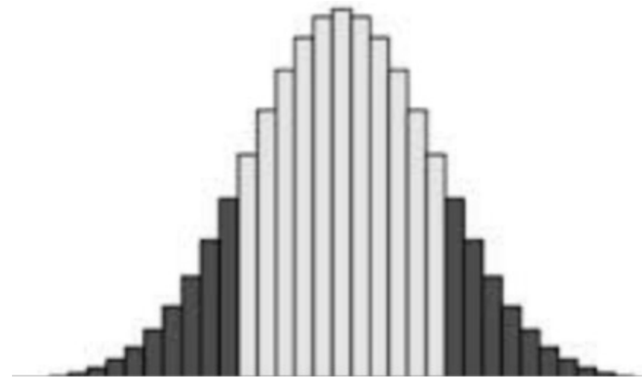


Fourier transforms, convolutions,
and filtering out noise

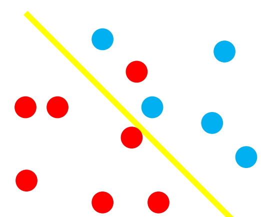


statistics, Bayesian
probability, and
information theory

$$C_m \frac{dV}{dt} = -g_L(V - E_L) + g_L \Delta_T e^{\frac{V-V_T}{\Delta_T}} - u + I$$

$$\tau_w \frac{du}{dt} = a(V - E_L) - u$$

differential equations and modeling



statistical
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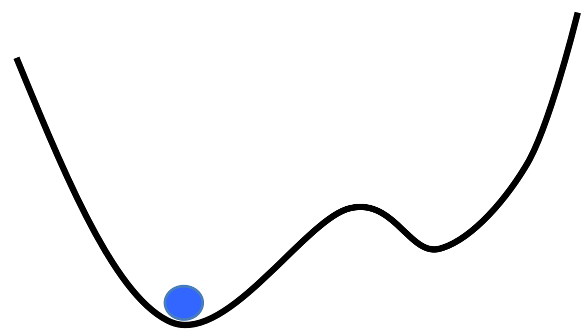

optimization

Today's lecture

NBIO 228 overview

Linear algebra

Who we are

Instructors	Contact
Kiah Hardcastle	khardcas@stanford.edu
Lane McIntosh	lmcintosh@stanford.edu

For announcements, lecture slides, syllabus, problem sets:

nbio228.stanford.edu

Projects

- Workshop your own data

- Present a paper that uses a mathematical tool you want to understand better

Grading policy

- Problem Sets: 30%
  - states = {0, check, checkplus}
  - 2 allowed late homeworks up to a week each
- Participation: 10%
- Final Project – the main emphasis of this course!
  - ½ page project proposal: 5%
  - 10 minute final presentation: 25%
  - 1-2 page write-up: 30%

Today's lecture

NBIO 228 overview

Linear algebra

Today's lecture

NBIO 228 overview

Linear algebra

- When and why is linear algebra useful?
- Vectors and their operations
- Matrices and their operations
- Special matrices
- Determinants

Why linear algebra?

Why linear algebra?

1.63	5.20	7.66	8.12	3.22
4.98	5.90	8.21	9.29	20.10
10.10	8.57	5.73	8.17	2.22
0.02	0.21	0.14	0.93	1.40
9.27	10.27	13.12	8.90	9.01
7.44	6.98	5.62	8.20	7.21
100.10	8.22	7.54	60.10	1.69
40.20	29.21	12.45	10.41	8.90
32.33	21.59	10.21	4.99	2.62
2.99	1.67	1.01	0.80	0.07

Datasets are matrices

time →

neuron 1	1.63	5.20	7.66	8.12	3.22
neuron 2	4.98	5.90	8.21	9.29	20.10
neuron 3	10.10	8.57	5.73	8.17	2.22
neuron 4	0.02	0.21	0.14	0.93	1.40
neuron 5	9.27	10.27	13.12	8.90	9.01
neuron 6	7.44	6.98	5.62	8.20	7.21
neuron 7	100.10	8.22	7.54	60.10	1.69
neuron 8	40.20	29.21	12.45	10.41	8.90
neuron 9	32.33	21.59	10.21	4.99	2.62
neuron 10	2.99	1.67	1.01	0.80	0.07

Datasets are matrices

time →

voxel 1	1.63	5.20	7.66	8.12	3.22
voxel 2	4.98	5.90	8.21	9.29	20.10
voxel 3	10.10	8.57	5.73	8.17	2.22
voxel 4	0.02	0.21	0.14	0.93	1.40
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voxel 9	32.33	21.59	10.21	4.99	2.62
voxel 10	2.99	1.67	1.01	0.80	0.07

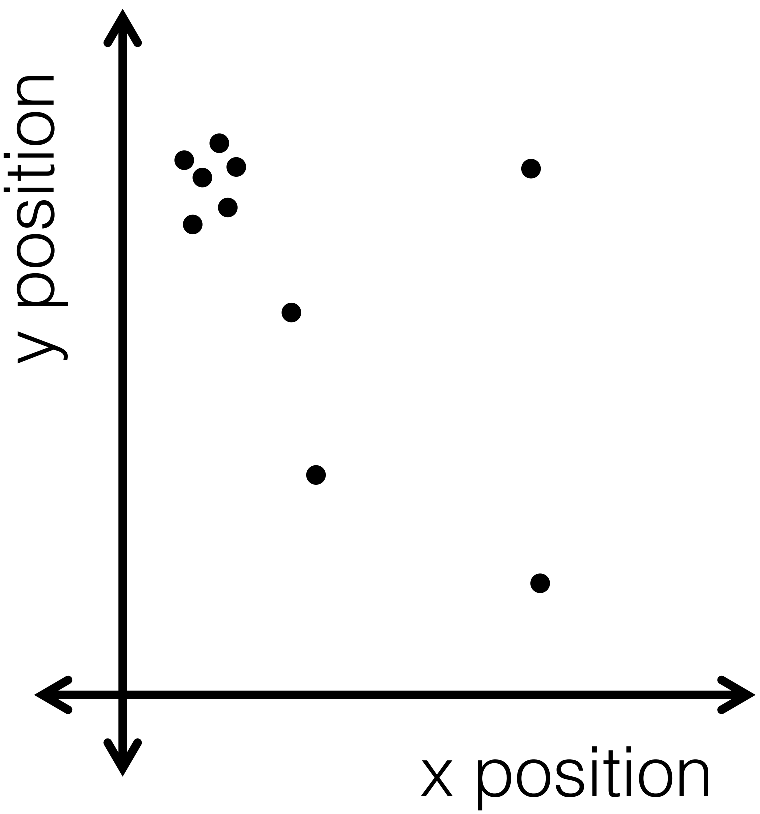
Datasets are matrices

patient ⟶

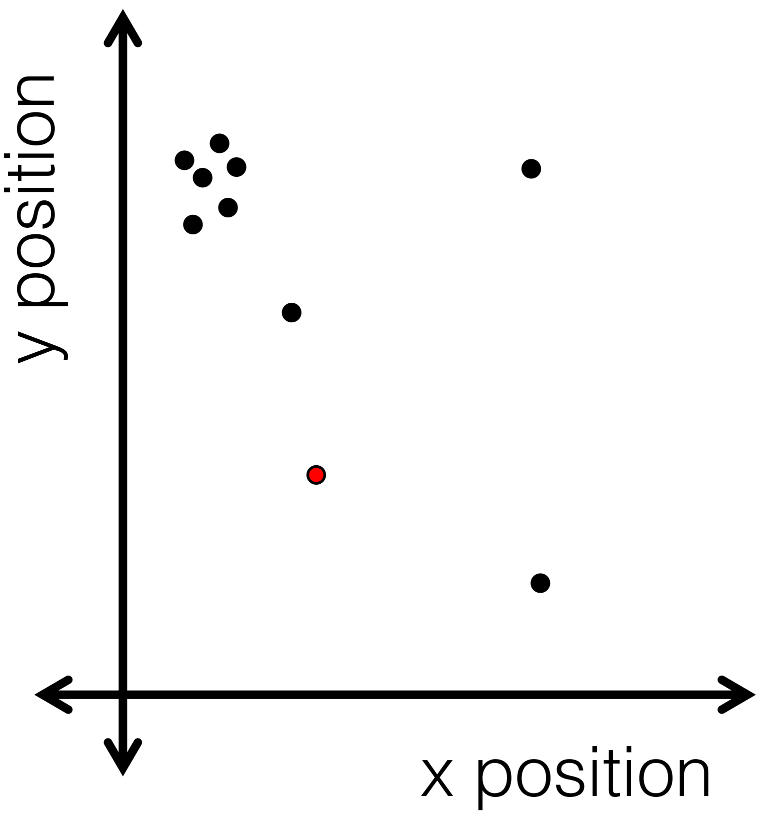
gene 1	1.63	5.20	7.66	8.12	3.22
gene 2	4.98	5.90	8.21	9.29	20.10
gene 3	10.10	8.57	5.73	8.17	2.22
gene 4	0.02	0.21	0.14	0.93	1.40
gene 5	9.27	10.27	13.12	8.90	9.01
gene 6	7.44	6.98	5.62	8.20	7.21
gene 7	100.10	8.22	7.54	60.10	1.69
gene 8	40.20	29.21	12.45	10.41	8.90
gene 9	32.33	21.59	10.21	4.99	2.62
gene 10	2.99	1.67	1.01	0.80	0.07

Part 1: Matrix Arithmetic
(w/ applications to an experiment)

Measure a mouse's location



Measure a mouse's location



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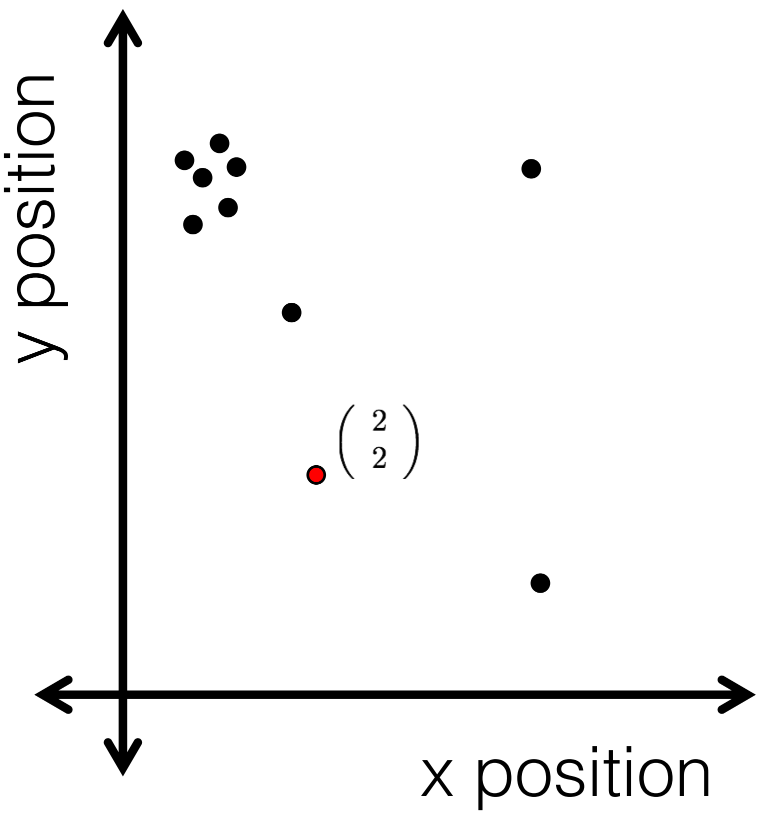

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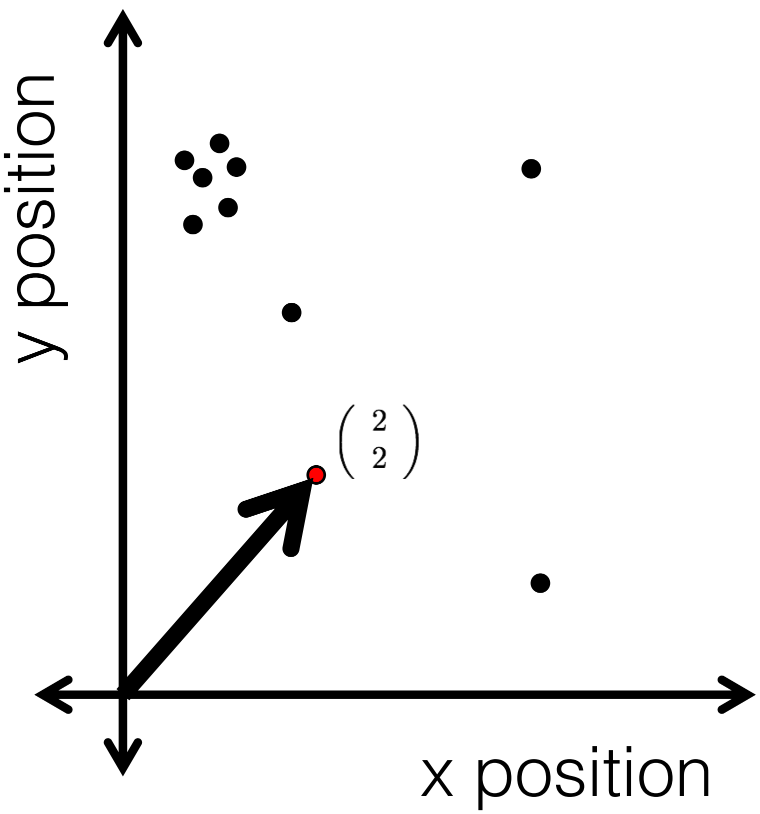
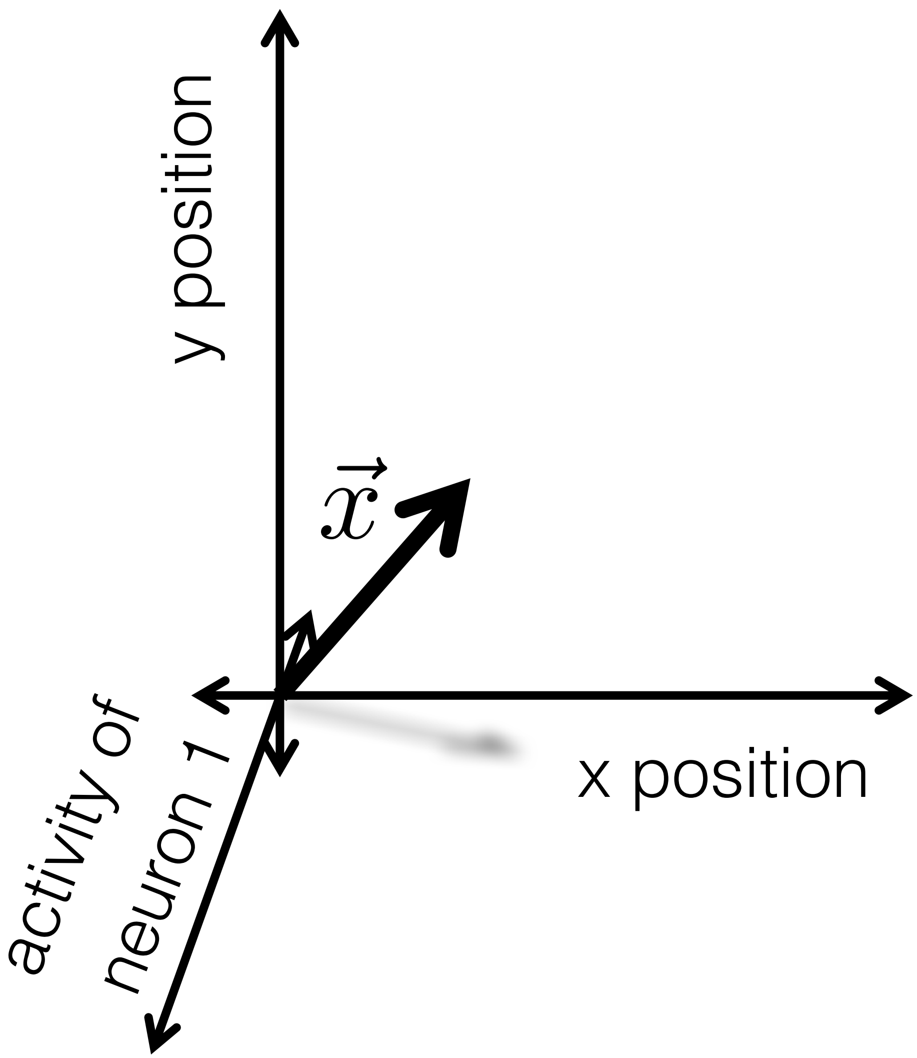
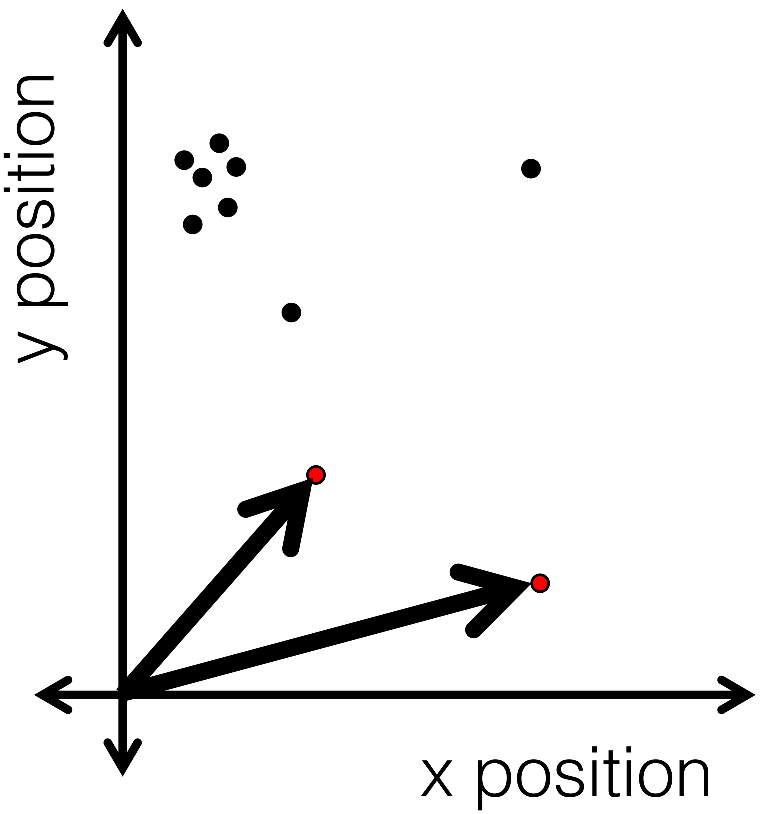


Measure a mouse's location and 10 of its neurons

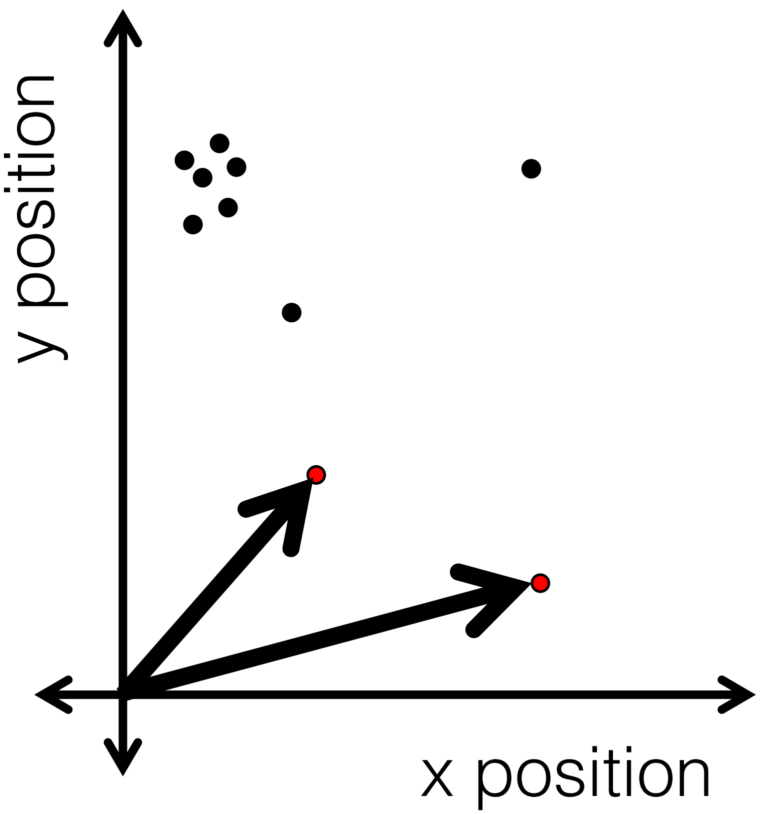


Matrix addition and subtraction



$$
\begin{array}{lcc}
 & \text{measurement 1} & \text{measurement 2} \\
\text{feature 1} & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix}$}} \\
\text{feature 2} & &
\end{array}
$$

Matrix addition and subtraction

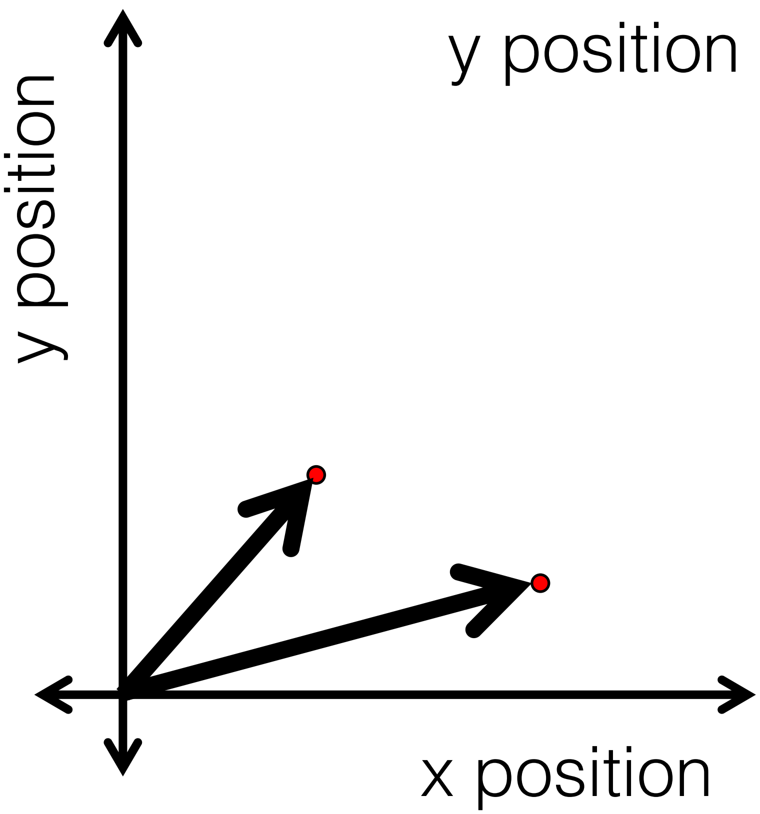


$$\begin{array}{lc} & \begin{array}{cc} \text{measurement 1} & \text{measurement 2} \end{array} \\ \begin{array}{l} \text{x position} \\ \text{y position} \end{array} & \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} \end{array}$$

Matrix addition and subtraction

$$\begin{array}{l} \text{x position} \\ \text{y position} \end{array} \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}$$

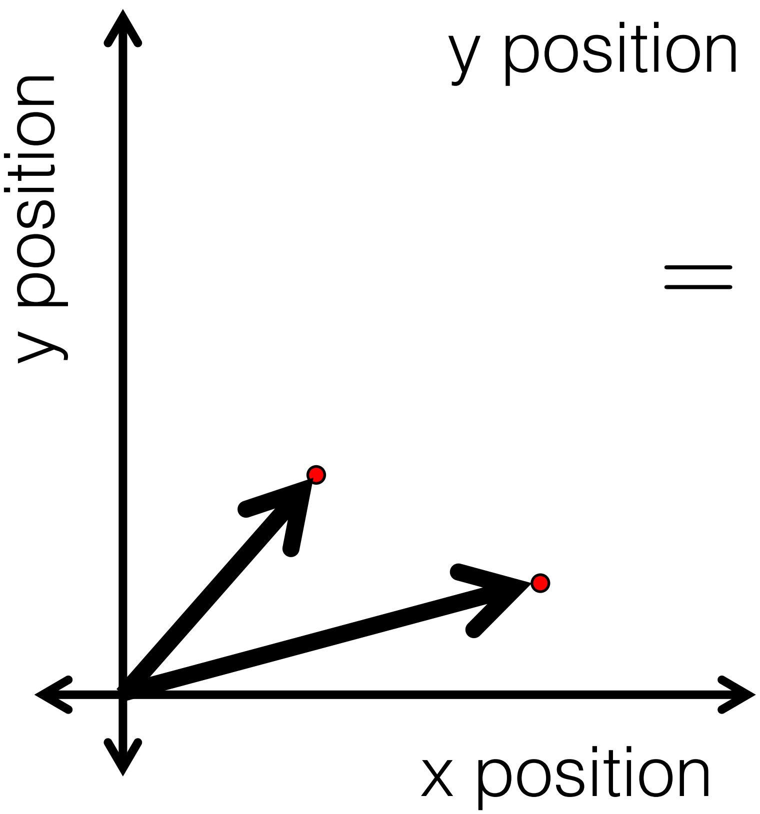
MATLAB: `mean(A, 2)`



Matrix addition and subtraction

$$\begin{array}{l} \text{x position} \\ \text{y position} \end{array} \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 3 \\ 1.5 & 1.5 \end{pmatrix}$$



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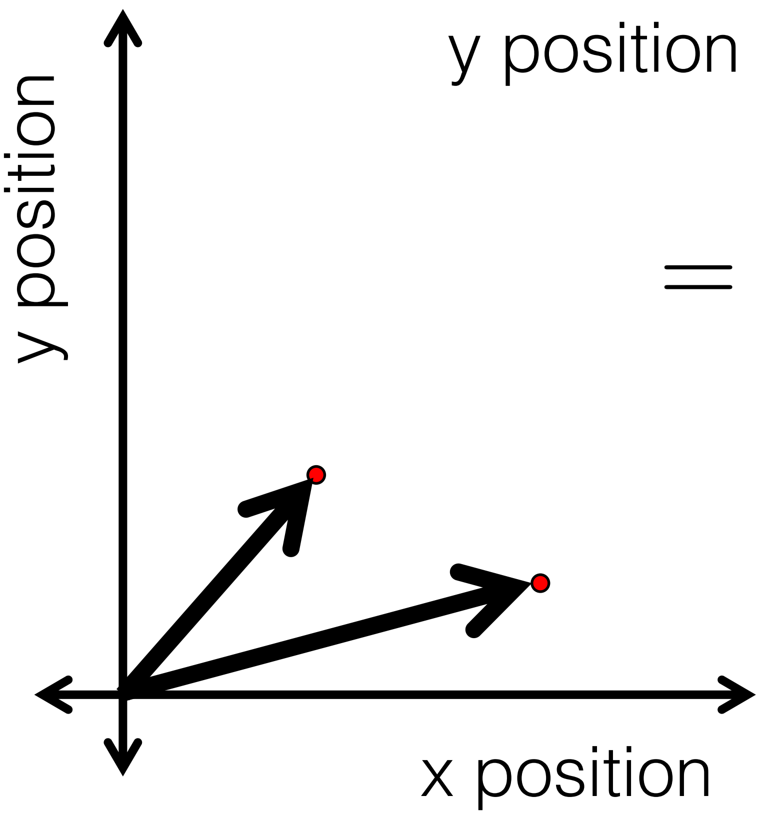

$$\begin{array}{l} \text{x position} \\ \text{y position} \end{array} \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 3 \\ 1.5 & 1.5 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 0.5 & -0.5 \end{pmatrix}$$

Matrix addition and subtraction

$$
\begin{array}{l} \text{x position} \\ \text{y position} \end{array}
\begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}
$$

$$
= \begin{pmatrix} 2 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 3 \\ 1.5 & 1.5 \end{pmatrix}
$$

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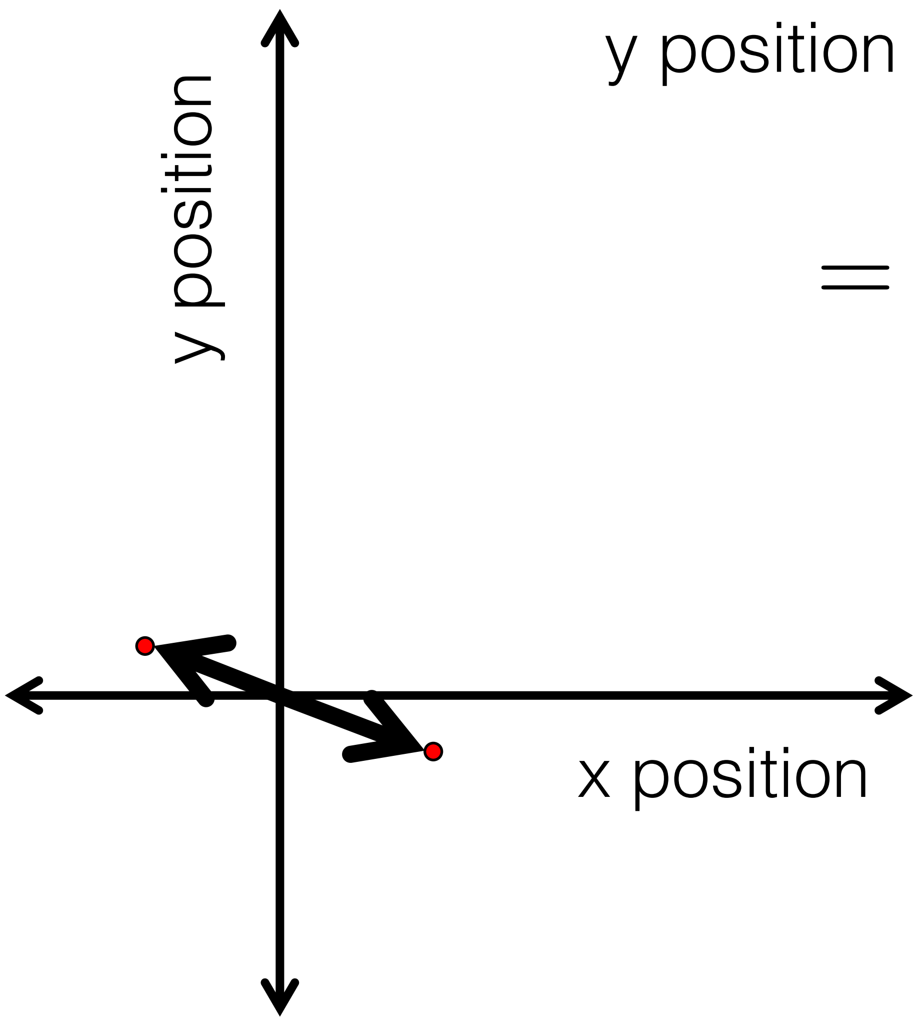
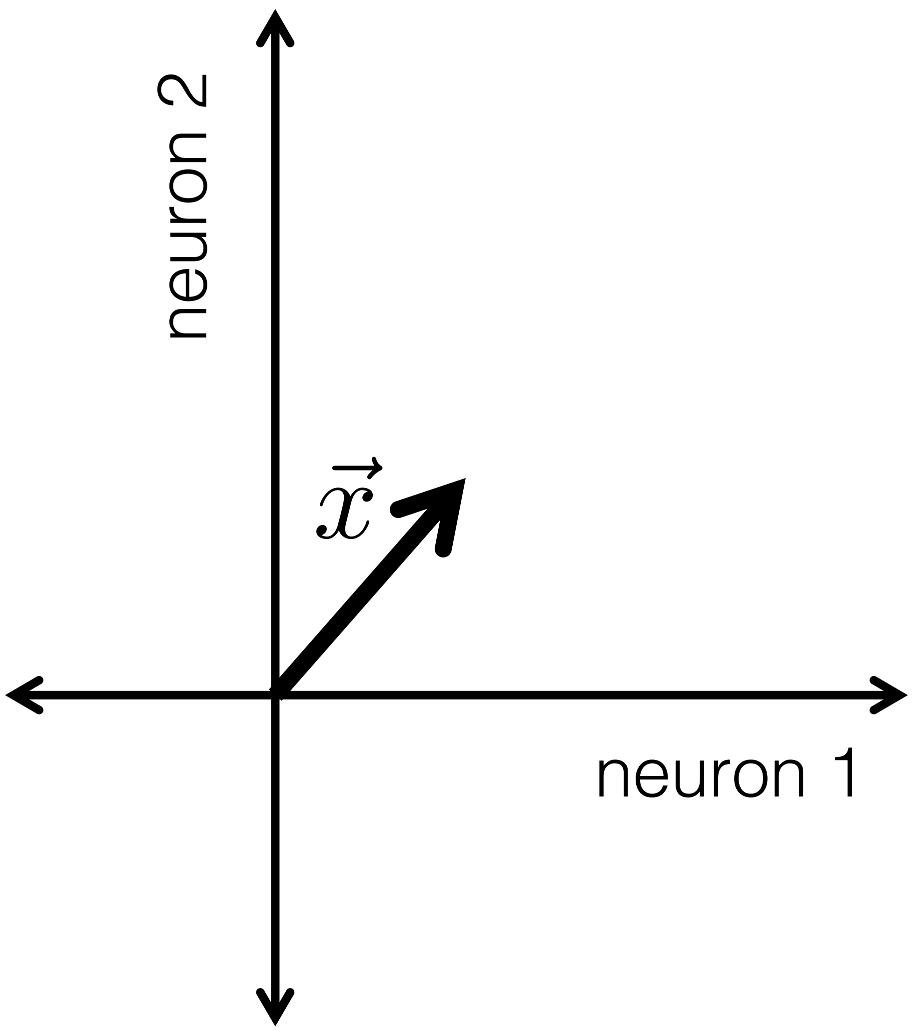
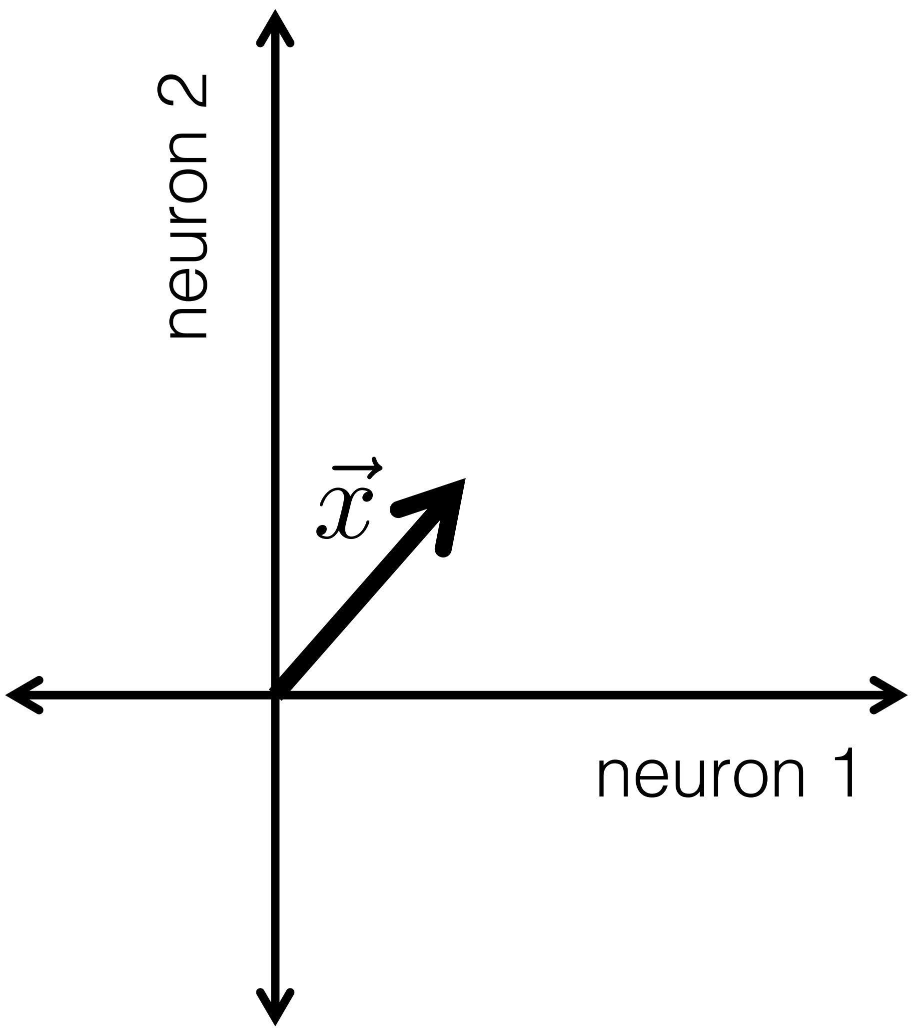


Scalar times vector



Measure 10 neurons at 100 HZ

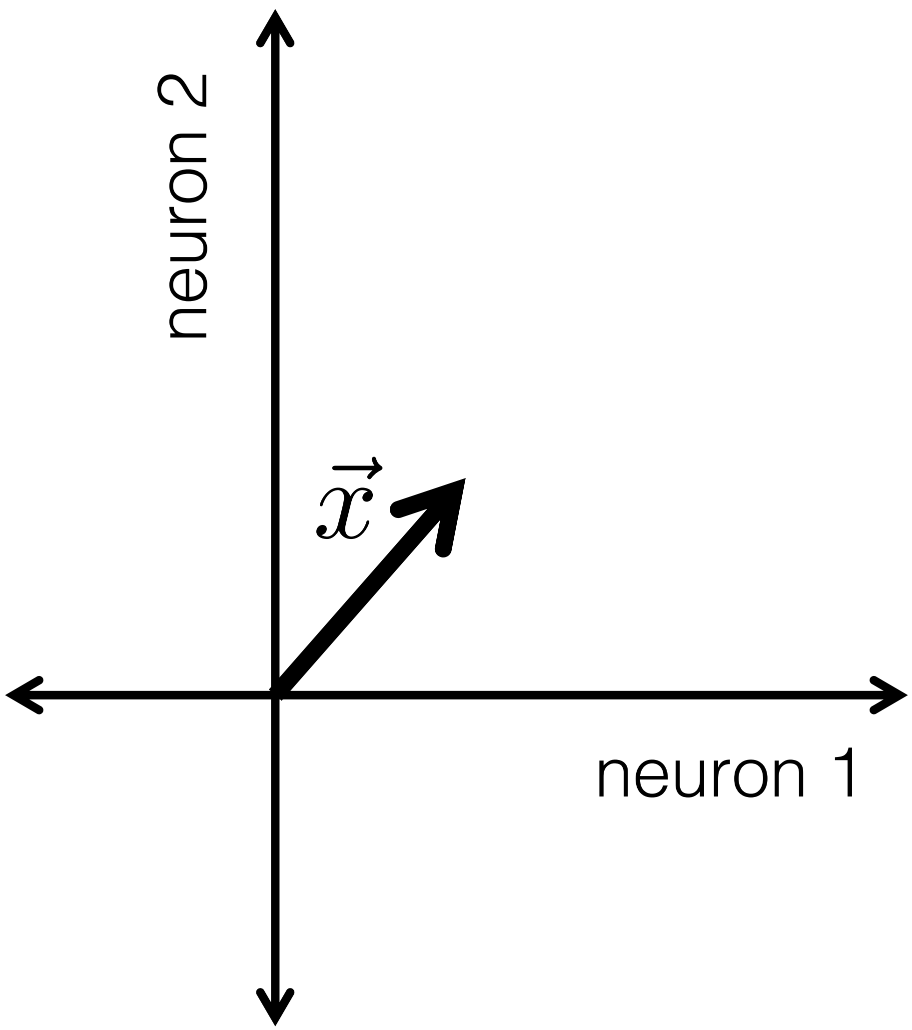
Scalar times vector



Measure 10 neurons at 100 HZ

Number of spikes per 10ms bin

Scalar times vector

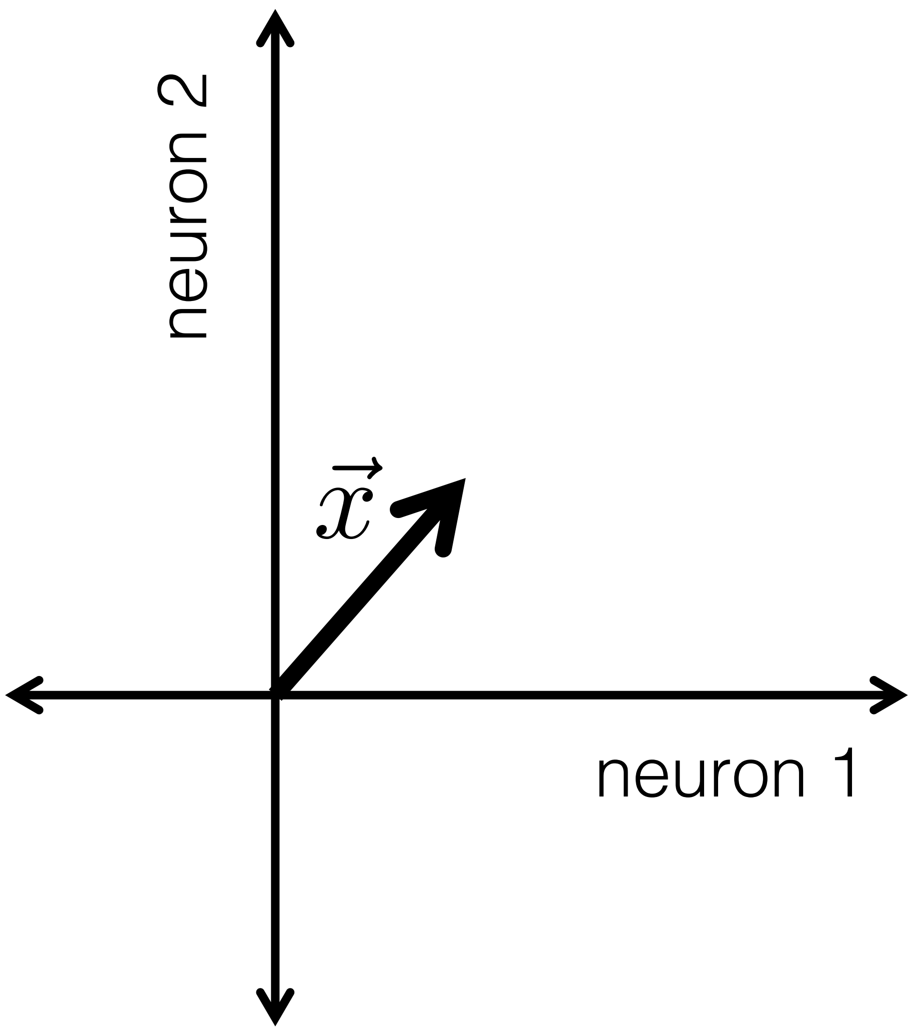


Measure 10 neurons at 100 HZ

Number of spikes per 10ms bin

How to convert to a firing rate in spikes/sec?

Scalar times vector



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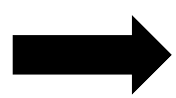
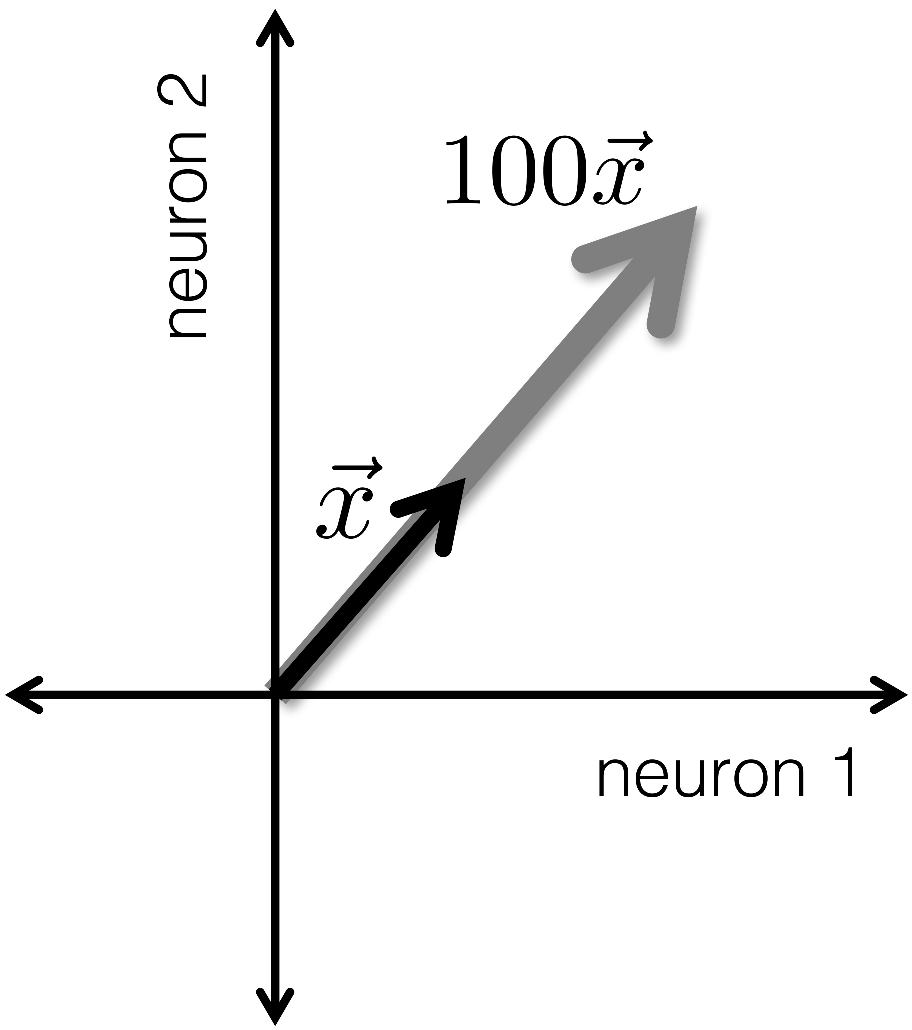


Multiply each measurement by 100

Scalar times vector



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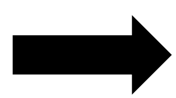

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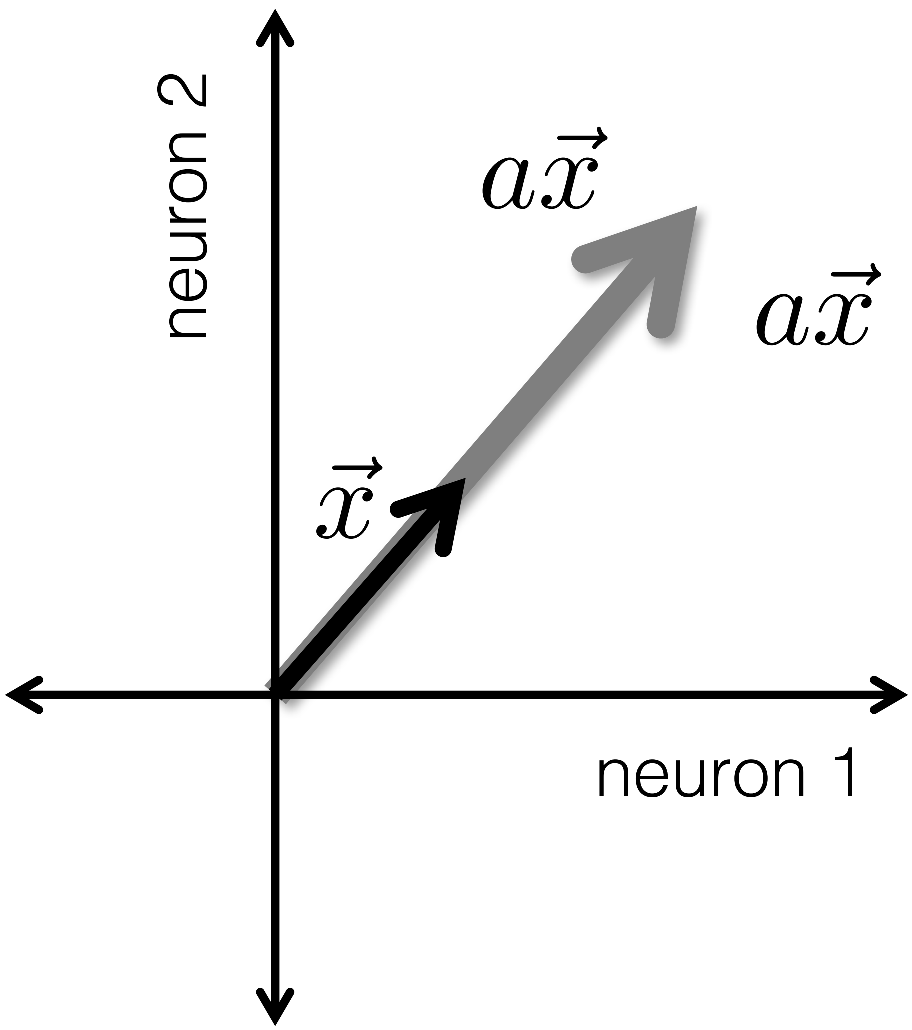


$$a\vec{x} = a \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_N \end{pmatrix}$$

Product of Two Vectors

Three ways to multiply

- Element-by-element
- Inner product
- Outer product

Element-by-element product

(Hadamard product)

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} .* \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}$$

- Element-wise multiplication (`.*` in MATLAB)

Element-by-element product

(Hadamard product)

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} .* \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} .^* \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_2 b_2 \\ a_3 b_3 & a_4 b_4 \end{pmatrix}$$

- Element-wise multiplication (`.*` in MATLAB)

Element-by-element product

Example

$$\begin{array}{l} \text{neuron 1} \\ \text{neuron 2} \end{array} \begin{pmatrix} 102 & 34 \\ 2 & 3 \end{pmatrix}$$

Element-by-element product

Example

$$\begin{array}{l} \text{neuron 1} \\ \text{neuron 2} \end{array} \begin{pmatrix} 102 & 34 \\ 2 & 3 \end{pmatrix} \begin{array}{l} \text{high standard deviation (\textasciitilde 48)} \\ \text{low standard deviation (\textasciitilde 0.7)} \end{array}$$

Element-by-element product

Example

$$\begin{array}{l} \text{neuron 1} \\ \text{neuron 2} \end{array} \begin{pmatrix} 102 & 34 \\ 2 & 3 \end{pmatrix} \begin{array}{l} \text{high standard deviation (\textasciitilde 48)} \\ \text{low standard deviation (\textasciitilde 0.7)} \end{array}$$

$$\begin{pmatrix} 102 & 34 \\ 2 & 3 \end{pmatrix} ./ \begin{pmatrix} \sigma_1 & \sigma_1 \\ \sigma_2 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 102/\sigma_1 & 34/\sigma_1 \\ 2/\sigma_2 & 3/\sigma_2 \end{pmatrix}$$

- Element-wise division (`./` in MATLAB)

Element-by-element product

Example

$$\begin{array}{l} \text{neuron 1} \\ \text{neuron 2} \end{array} \begin{pmatrix} 102 & 34 \\ 2 & 3 \end{pmatrix} \begin{array}{l} \text{high standard deviation (\textasciitilde48)} \\ \text{low standard deviation (\textasciitilde0.7)} \end{array}$$

$$\begin{pmatrix} 102 & 34 \\ 2 & 3 \end{pmatrix} ./ \begin{pmatrix} \sigma_1 & \sigma_1 \\ \sigma_2 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 102/\sigma_1 & 34/\sigma_1 \\ 2/\sigma_2 & 3/\sigma_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2.1 & 0.7 \\ 2.9 & 4.2 \end{pmatrix}$$

- Element-wise division (`./` in MATLAB)

Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

Multiplication:

Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1$$

Multiplication:

Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2$$

Multiplication:
Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

Multiplication:

Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

$$= \sum_{i=1}^{N} x_i y_i$$

Multiplication:

Dot product (inner product)

$$\vec{x} \cdot \vec{y} =$$

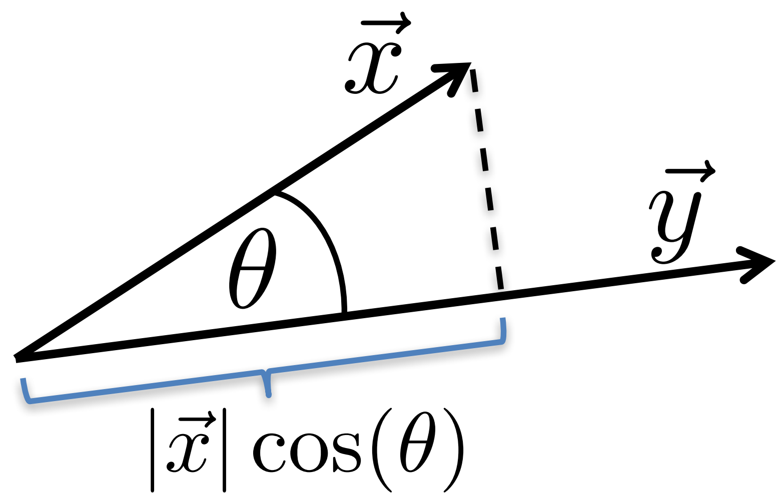
$$\begin{pmatrix} x_1 & x_2 & \cdots & x_N \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_N y_N$$

1 X N   N X 1   1 X 1

- MATLAB: 'inner matrix dimensions must agree'

Outer dimensions give size of resulting matrix

Dot product geometric intuition: "Overlap" of 2 vectors

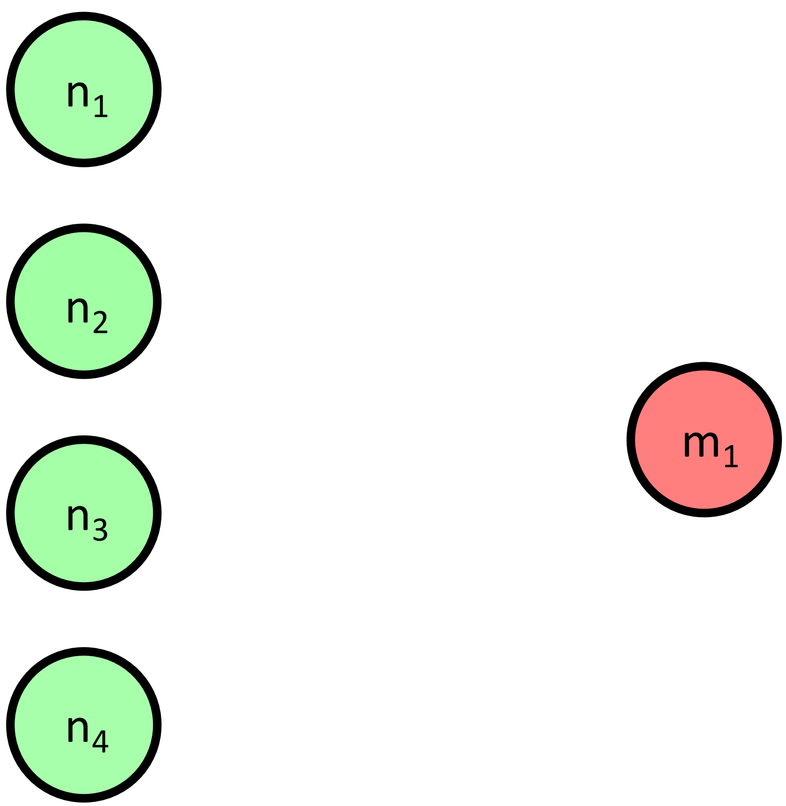


$$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos(\theta)$$

Multiplication:

Dot product (inner product) Example 1

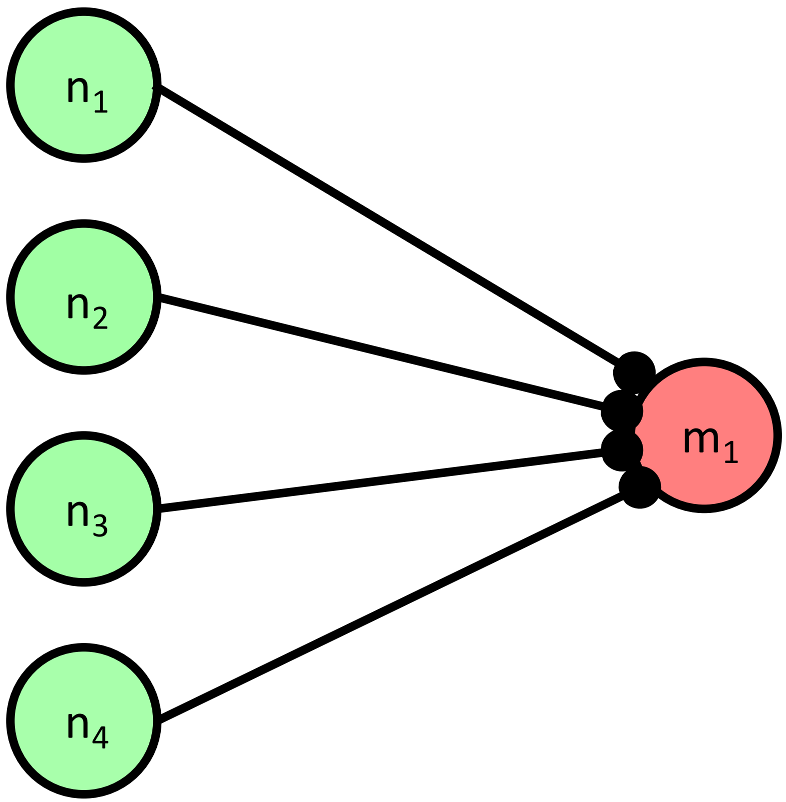
weighted average



Multiplication:

Dot product (inner product) Example 1

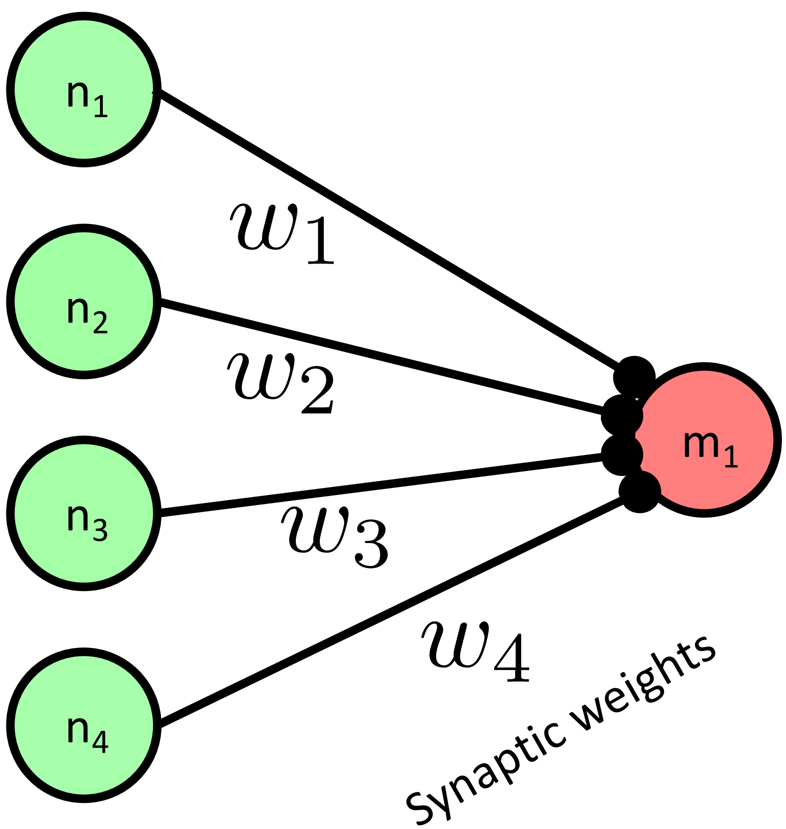
weighted average



Multiplication:

Dot product (inner product) Example 1

weighted average

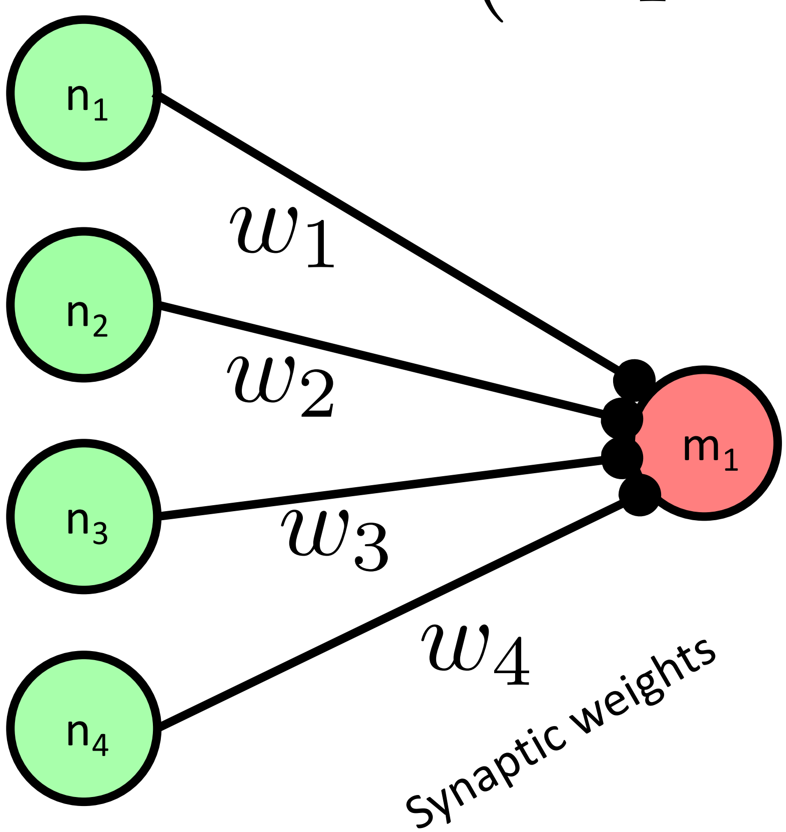


Multiplication:

Dot product (inner product) Example 1

weighted average

$$m_1 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}$$

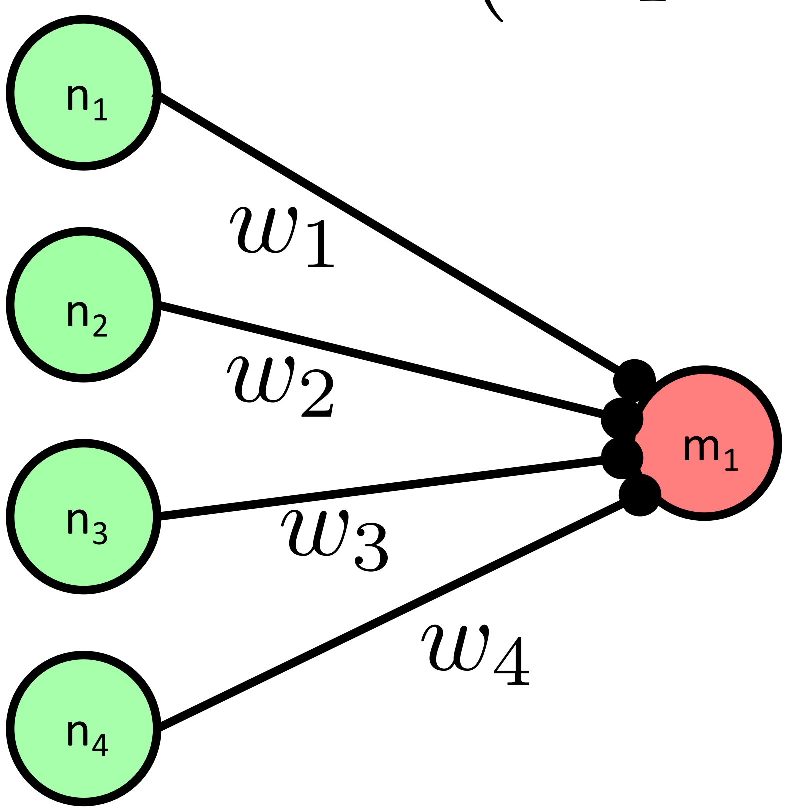


Multiplication:

Dot product (inner product) Example 1

weighted average

$$m_1 = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix}$$



$$= w_1 n_1 + w_2 n_2 + w_3 n_3 + w_4 n_4$$

$$= w^T n$$ (`w'*n` in MATLAB)

Multiplication:
Dot product (inner product) Example 2

linear regression

patient ⟶

gene 1	1.63	5.20	7.66	8.12	3.22
gene 2	4.98	5.90	8.21	9.29	20.10
gene 3	10.10	8.57	5.73	8.17	2.22
gene 4	0.02	0.21	0.14	0.93	1.40
gene 5	9.27	10.27	13.12	8.90	9.01
gene 6	7.44	6.98	5.62	8.20	7.21

Multiplication:
Dot product (inner product) Example 2

linear regression

patient $\longrightarrow$

gene 1	1.63	5.20	7.66	8.12	3.22
gene 2	4.98	5.90	8.21	9.29	20.10
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gene 5	9.27	10.27	13.12	8.90	9.01
gene 6	7.44	6.98	5.62	8.20	7.21

For each patient, you also measure their Asperger's disorder quotient

Multiplication:

Dot product (inner product) Example 2

linear regression

patient ⟶

gene 1	1.63	5.20	7.66	8.12	3.22
gene 2	4.98	5.90	8.21	9.29	20.10
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gene 6	7.44	6.98	5.62	8.20	7.21
score	2	0	9	44	48

Multiplication:

Dot product (inner product) Example 2

linear regression

patient $\longrightarrow$

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gene 6	7.44	6.98	5.62	8.20	7.21
score	2	0	9	44	48

$$\text{score} = w_1 \text{gene}_1 + w_2 \text{gene}_2 + \cdots + w_6 \text{gene}_6$$

Multiplication:

Dot product (inner product) Example 2

linear regression

$$
\begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{pmatrix}
\begin{pmatrix} 8.12 \\ 9.29 \\ 8.17 \\ 0.93 \\ 8.90 \\ 8.20 \end{pmatrix}
\begin{array}{l} \text{gene 1} \\ \text{gene 2} \\ \text{gene 3} \\ \text{gene 4} \\ \text{gene 5} \\ \text{gene 6} \end{array}
$$

score 44

$$44 = w_1 8.12 + w_2 9.29 + \cdots + w_6 8.20$$

Multiplication:

Dot product (inner product) Example 2

linear regression

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{pmatrix} \begin{array}{cc} \begin{pmatrix} 8.12 \\ 9.29 \\ 8.17 \\ 0.93 \\ 8.90 \\ 8.20 \end{pmatrix} & \begin{array}{l} \text{gene 1} \\ \text{gene 2} \\ \text{gene 3} \\ \text{gene 4} \\ \text{gene 5} \\ \text{gene 6} \end{array} \end{array}$$

$$\text{score} = w^T \vec{\text{genes}}$$

Multiplication:
Dot product (inner product) Example 2

linear regression

patient $\longrightarrow$

gene 1	1.63	5.20	7.66	8.12	3.22
gene 2	4.98	5.90	8.21	9.29	20.10
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gene 6	7.44	6.98	5.62	8.20	7.21
score	2	0	9	44	48

$$\text{score} = w^T \vec{\text{genes}}$$

(`regress(score',X')` in MATLAB to get w's, where `X` is the full genes by patients matrix)

Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix}$$

N X 1 1 X M

Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\ \vdots & \vdots & \ddots & \ldots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_M \end{pmatrix}$$

N X 1 1 X M N X M

Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\ \vdots & \vdots & \ddots & \ldots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_M \end{pmatrix}$$

Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_M \\ x_2y_1 & x_2y_2 & \cdots & x_2y_M \\ \vdots & \vdots & \ddots & \ldots \\ x_Ny_1 & x_Ny_2 & \cdots & x_Ny_M \end{pmatrix}$$

Multiplication: **Outer product**

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Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\ \vdots & \vdots & \ddots & \vdots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_M \end{pmatrix}$$

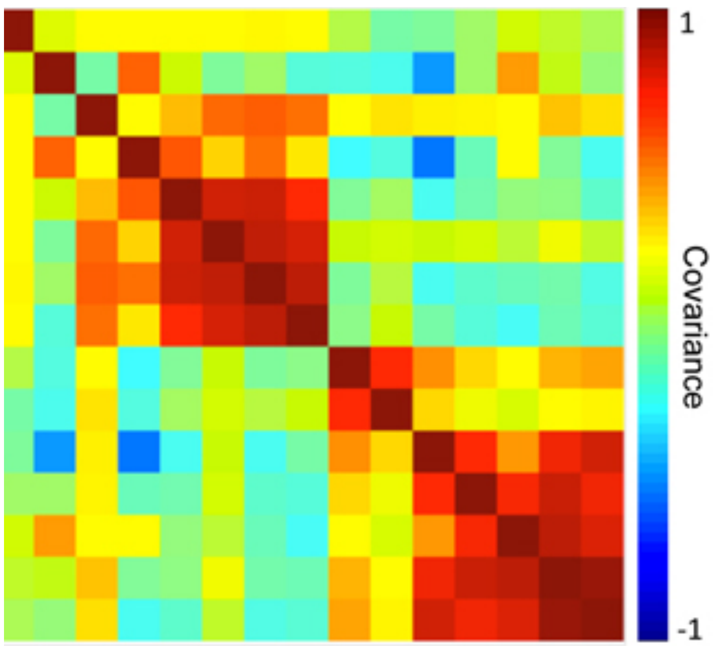
Multiplication: **Outer product**

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_M \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_M \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_M \\ \vdots & \vdots & \ddots & \vdots \\ x_N y_1 & x_N y_2 & \cdots & x_N y_M \end{pmatrix}$$

- Note: each column or each row is a multiple of the others

Multiplication: **Outer product**

Example: Covariance Matrices



$$= \begin{pmatrix} x_1 x_1 & x_1 x_2 & \dots x_1 x_N \\ x_2 x_1 & x_2 x_2 & \\ \vdots & & \ddots \\ x_N x_1 & & x_N x_N \end{pmatrix}$$

- When $\vec{x} = \vec{y}$ and $\vec{x}$ has an average of zero, this outer product is called the covariance matrix

Matrix times vector

$$\overrightarrow{y} = \overleftrightarrow{W}\,\overrightarrow{x}$$

Matrix times vector

$$\overrightarrow{y} = \overleftrightarrow{W}\,\overrightarrow{x}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

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M X 1   M X N   N X 1

Matrix times vector: **inner product interpretation**

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{i1} & W_{i2} & \cdots & W_{iN} \\ \vdots & \vdots & \ddots & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

- Rule: the i^{th} element of **y** is the dot product of the i^{th} row of W with **x**

Matrix times vector:
inner product interpretation

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Matrix times vector:
inner product interpretation

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- Rule: the i^{th} element of **y** is the dot product of the i^{th} row of W with **x**

Matrix times vector: **outer product interpretation**

$$\begin{array}{l} \quad \overrightarrow{W}^{(1)} \\ \quad\ \downarrow \\ \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} =$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

Matrix times vector:
outer product interpretation

$$\begin{array}{c} \begin{array}{llll} \vec{W}^{(1)} & \phantom{W_{12}} & & \phantom{W_{1N}} \end{array} \\ \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = x_1 \vec{W}^{(1)} +$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

Matrix times vector:
outer product interpretation

$$
\begin{array}{c}
\overrightarrow{W}^{(1)} \\
\downarrow \\
\end{array}
$$

$$
\begin{pmatrix}
W_{11} & W_{12} & \cdots & W_{1N} \\
W_{21} & W_{22} & \cdots & W_{2N} \\
\vdots & \vdots & & \vdots \\
W_{M1} & W_{M2} & \cdots & W_{MN}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ \vdots \\ x_N
\end{pmatrix}
= x_1 \overrightarrow{W}^{(1)} + x_2 \overrightarrow{W}^{(2)}
$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

Matrix times vector:
outer product interpretation

$$\begin{array}{c} \begin{array}{cccc} \vec{W}^{(1)} & & & \\ \downarrow & & & \end{array} \\ \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \end{array} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = x_1 \vec{W}^{(1)} + x_2 \vec{W}^{(2)} + \cdots + x_N \vec{W}^{(N)}$$

- The product is a weighted sum of the columns of W, weighted by the entries of x

Example of the outer product method

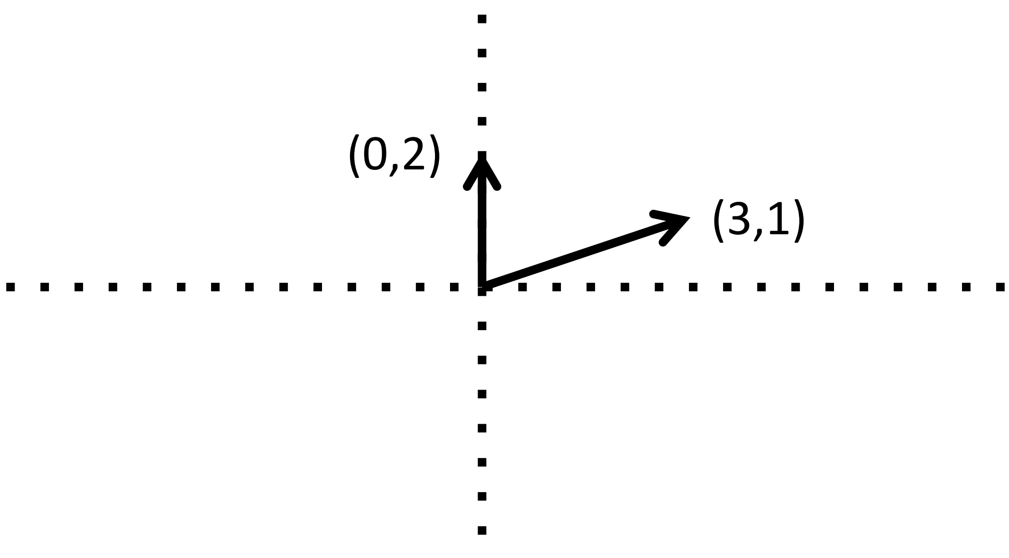
$$\overleftrightarrow{M}$$

$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

Example of the outer product method

$\overleftrightarrow{M}$

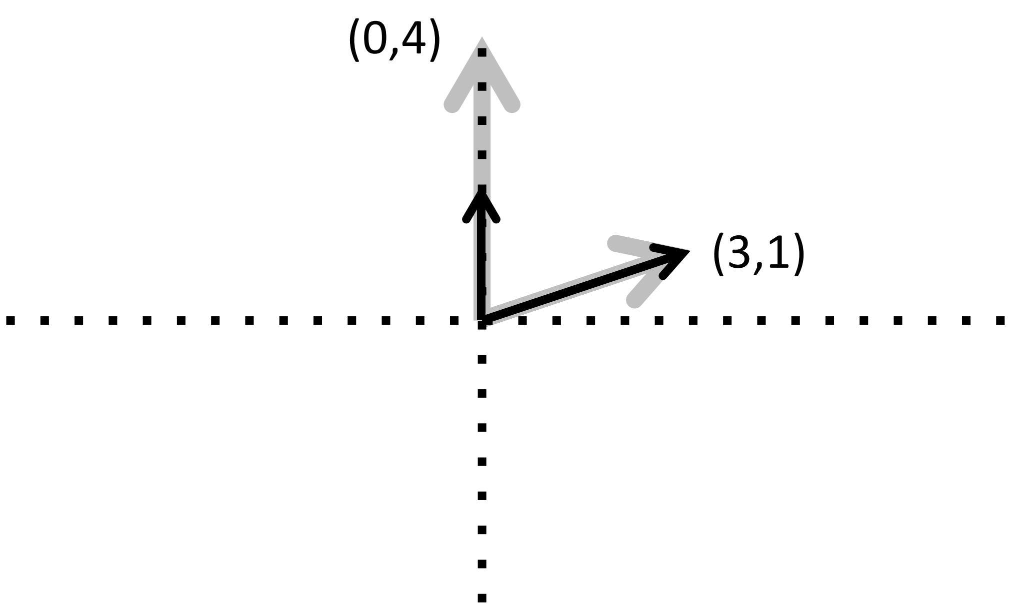
$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



Example of the outer product method

$\overleftrightarrow{M}$

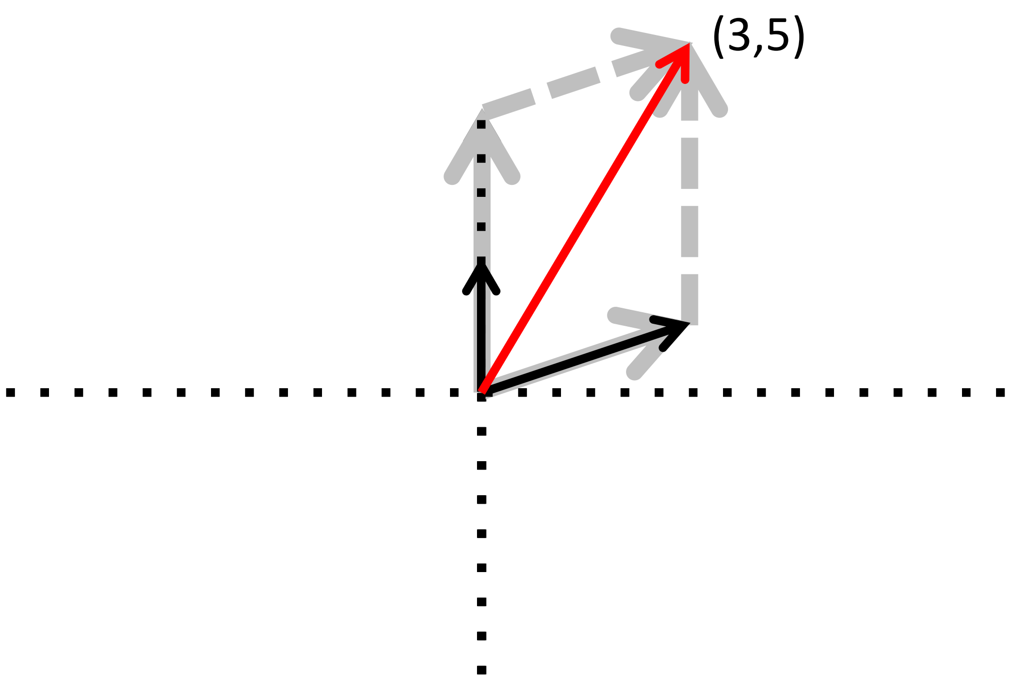
$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



Example of the outer product method

$$\overleftrightarrow{M}$$

$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$



- Note: different combinations of the columns of **M** can give you any vector in the plane

(we say the columns of **M** "span" the plane)

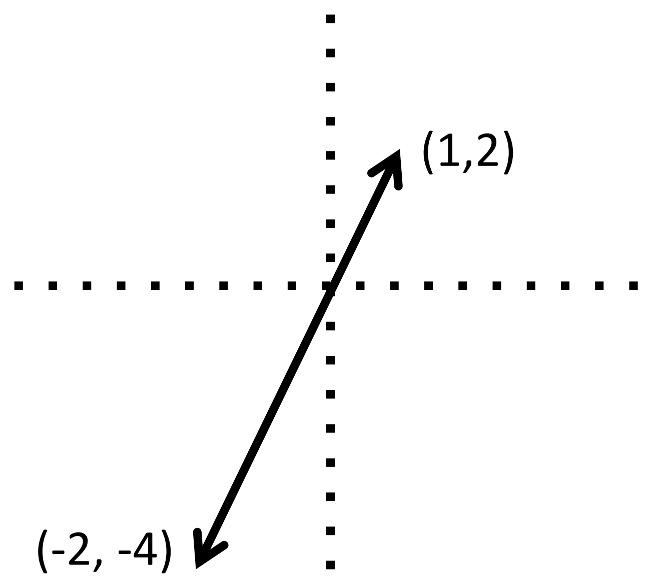
Rank of a Matrix

- Are there special matrices whose columns don’t span the full plane?

Rank of a Matrix

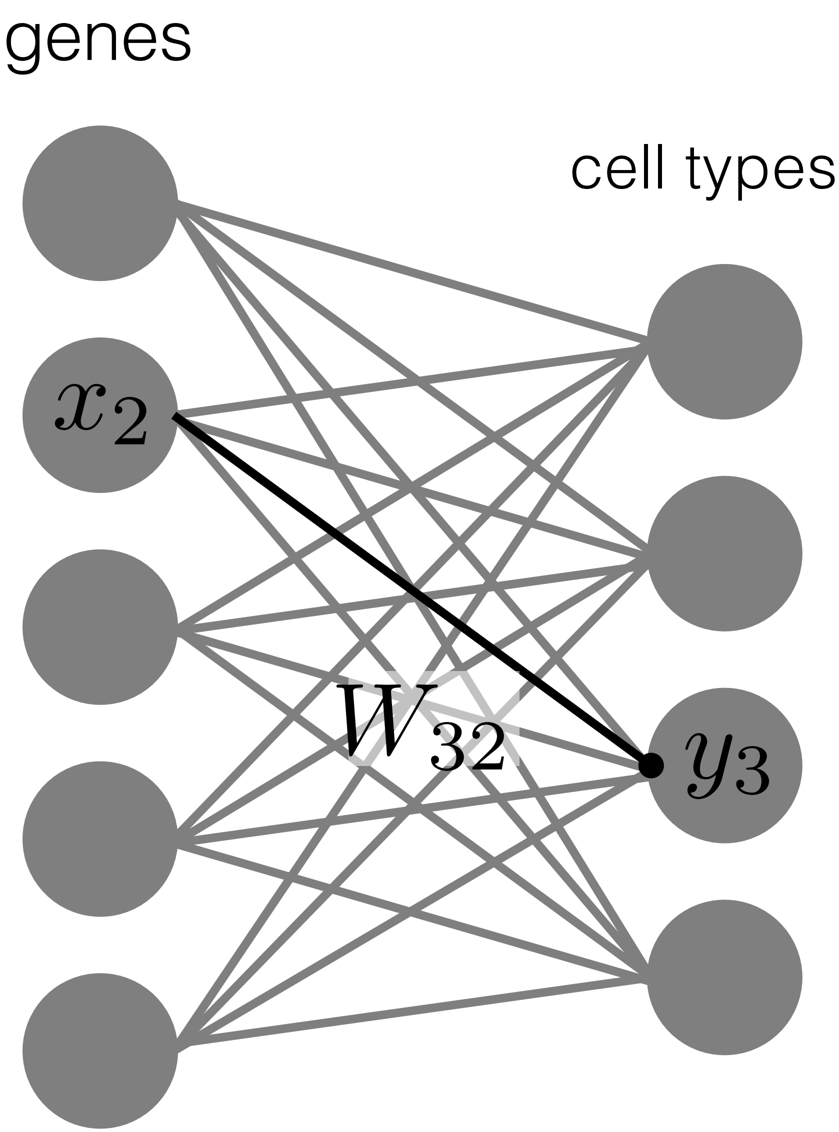
- Are there special matrices whose columns don't span the full plane?

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$$



- You can only get vectors along the (1,2) direction (i.e. outputs live in 1 dimension, so we call the matrix *rank 1*)

Example: Development of cell types

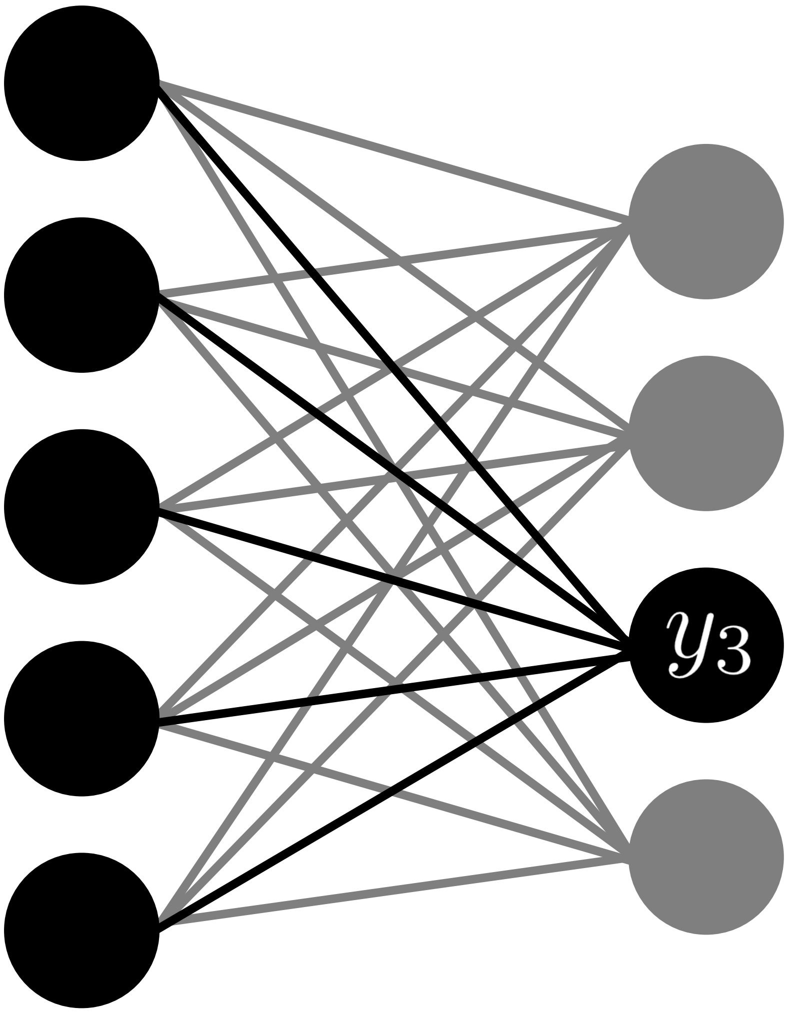


$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} & \cdots & W_{1N} \\ W_{21} & W_{22} & \cdots & W_{2N} \\ \vdots & \vdots & & \vdots \\ W_{M1} & W_{M2} & \cdots & W_{MN} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

- W_{32} is the influence of gene 2 on developing cell type 3

Example: Development of cell types
inner product point of view

- *How many cells of type 3 will be created?*

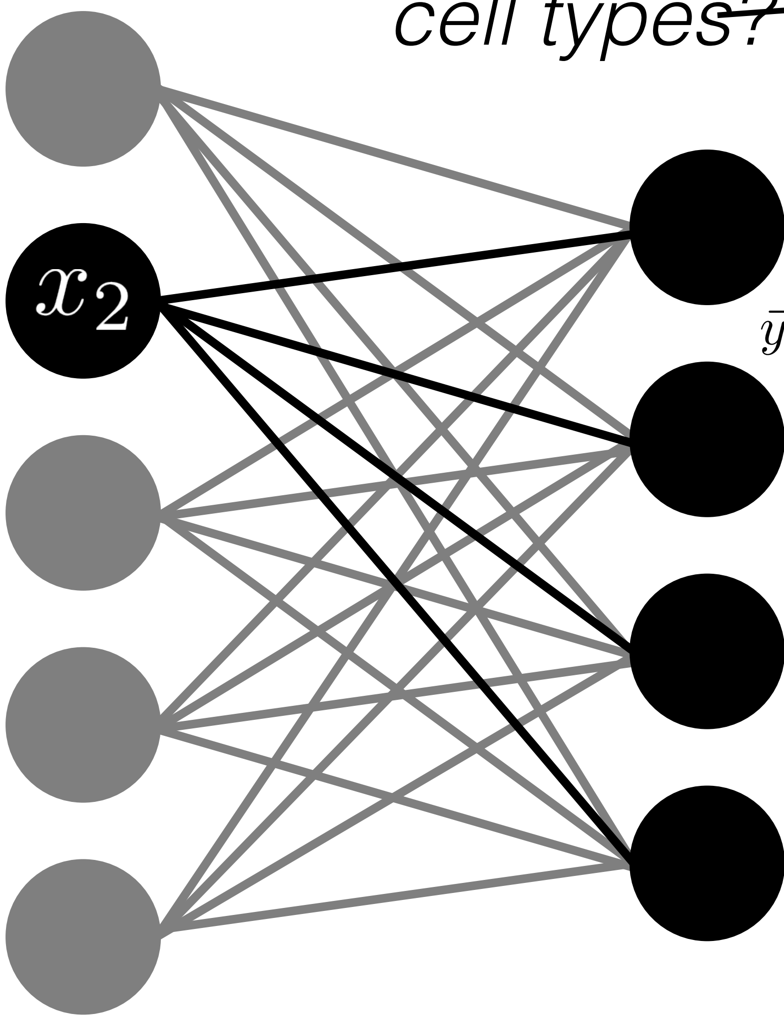


$$y_3 = \sum_{j=1}^{5} W_{3j} x_j$$

- The response is the dot product of the 3rd row of W with the vector x (gene expressions)

Example: Development of cell types: **outer product point of view**

- *How does gene 2 contribute to the distribution of cell types?*



$$\vec{y} = x_1 \overbrace{\overrightarrow{W}}^{\text{1st column of W}}{}^{(1)} + x_2 \overrightarrow{W}^{(2)} + \cdots + \underbrace{x_j \overrightarrow{W}^{(j)}}_{\text{Contribution of } x_j \textit{ to cell types}} + \cdots + x_N \overrightarrow{W}^{(N)}$$

Product of 2 Matrices

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

N X P   P X M   N X M

- MATLAB: ‘inner matrix dimensions must agree’
- **Note:** Matrix multiplication doesn’t (generally) commute, **AB** ≠ **BA**

Matrix times Matrix:
by inner products

$$
\begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1P} \\
A_{21} & A_{22} & \cdots & A_{2P} \\
\vdots & \vdots & & \vdots \\
A_{i1} & A_{i2} & \cdots & A_{iP} \\
\vdots & \vdots & & \vdots \\
A_{N1} & A_{N2} & \cdots & A_{NP}
\end{pmatrix}
\begin{pmatrix}
B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1M} \\
B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2M} \\
\vdots & \vdots & & \vdots & & \vdots \\
B_{P1} & B_{P2} & \cdots & B_{Pj} & \cdots & B_{PM}
\end{pmatrix}
=
\begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{1M} \\
C_{21} & C_{22} & \cdots & C_{2M} \\
\vdots & \vdots & C_{ij} & \vdots \\
C_{N1} & C_{N2} & \cdots & C_{NM}
\end{pmatrix}
$$

- C_{ij} is the inner product of the i^{th} row of A with the j^{th} column of B

Matrix times Matrix:
by inner products

$$
\begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1P} \\
A_{21} & A_{22} & \cdots & A_{2P} \\
\vdots & \vdots & & \vdots \\
A_{i1} & A_{i2} & \cdots & A_{iP} \\
\vdots & \vdots & & \vdots \\
A_{N1} & A_{N2} & \cdots & A_{NP}
\end{pmatrix}
\begin{pmatrix}
B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1M} \\
B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2M} \\
\vdots & \vdots & & \vdots & & \vdots \\
B_{P1} & B_{P2} & \cdots & B_{Pj} & \cdots & B_{PM}
\end{pmatrix}
=
\begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{1M} \\
C_{21} & C_{22} & \cdots & C_{2M} \\
\vdots & \vdots & C_{ij} & \vdots \\
C_{N1} & C_{N2} & \cdots & C_{NM}
\end{pmatrix}
$$

- C_{ij} is the inner product of the i^{th} row of A with the j^{th} column of B

Matrix times Matrix:
by inner products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{iP} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{Pj} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & C_{ij} & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

- C_{ij} is the inner product of the i^{th} row of A with the j^{th} column of B

Matrix times Matrix:
by outer products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

$$\overleftrightarrow{C} =$$

Matrix times Matrix:
by outer products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

$$\overleftrightarrow{C} = \begin{pmatrix} \\ A^{\mathrm{c}1} \\ \\ \end{pmatrix} \begin{pmatrix} \quad B^{\mathrm{r}1} \quad \end{pmatrix} +$$

Matrix times Matrix:
by outer products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

$$\overleftrightarrow{C} = \begin{pmatrix} \\ A^{\mathrm{c}1} \\ \\ \end{pmatrix} \begin{pmatrix} \quad B^{\mathrm{r}1} \quad \end{pmatrix} + \begin{pmatrix} \\ A^{\mathrm{c}2} \\ \\ \end{pmatrix} \begin{pmatrix} \quad B^{\mathrm{r}2} \quad \end{pmatrix} +$$

Matrix times Matrix:
by outer products

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1P} \\ A_{21} & A_{22} & \cdots & A_{2P} \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NP} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1M} \\ B_{21} & B_{22} & \cdots & B_{2M} \\ \vdots & \vdots & & \vdots \\ B_{P1} & B_{P2} & \cdots & B_{PM} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1M} \\ C_{21} & C_{22} & \cdots & C_{2M} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NM} \end{pmatrix}$$

$$\overleftrightarrow{C} = \begin{pmatrix} \\ A^{\mathrm{c}1} \\ \\ \end{pmatrix} \begin{pmatrix} \quad B^{\mathrm{r}1} \quad \end{pmatrix} + \begin{pmatrix} \\ A^{\mathrm{c}2} \\ \\ \end{pmatrix} \begin{pmatrix} \quad B^{\mathrm{r}2} \quad \end{pmatrix} + \cdots + \begin{pmatrix} \\ A^{\mathrm{cP}} \\ \\ \end{pmatrix} \begin{pmatrix} \quad B^{\mathrm{rP}} \quad \end{pmatrix}$$

- **C** is a sum of outer products of the columns of **A** with the rows of **B**

BREAK

ME: Let’s do some problems later
ME TO ME: It is later

Part 2: Matrix Properties

- (A few) special matrices
- The determinant

Special matrices: **diagonal matrix**

$$\overleftrightarrow{D} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$$

$$\overleftrightarrow{D}\,\overrightarrow{x} = \begin{pmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{pmatrix}$$

- This acts like scalar multiplication

Special matrices: **identity matrix**

$$\overleftrightarrow{1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

for all $\overleftrightarrow{A}, \quad \overleftrightarrow{1}\,\overleftrightarrow{A} = \overleftrightarrow{A}\,\overleftrightarrow{1} = \overleftrightarrow{A}$

Special matrices: **inverse matrix**

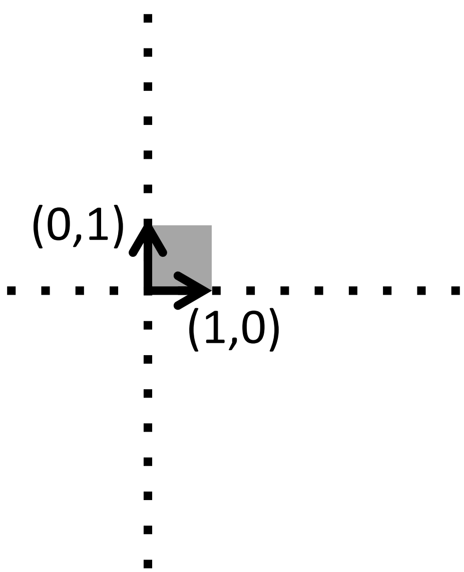
$$\overleftrightarrow{A}\,\overleftrightarrow{A}^{-1} = \overleftrightarrow{A}^{-1}\overleftrightarrow{A} = \overleftrightarrow{1}$$

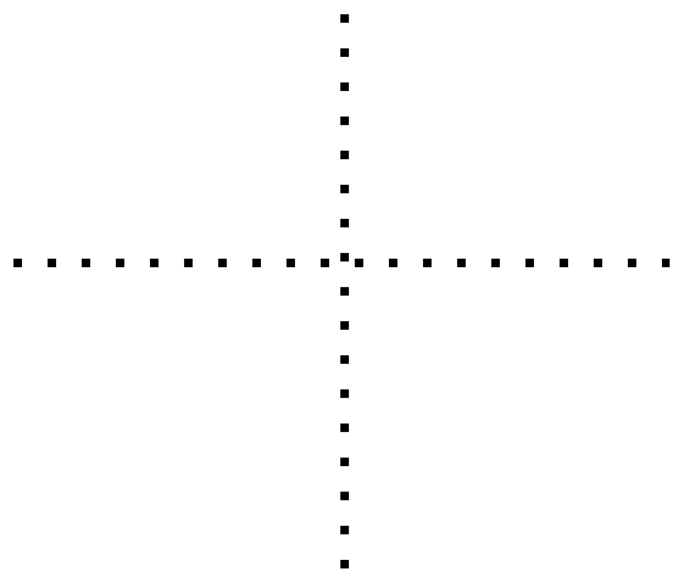
- Does the inverse always exist?

Part 2: Matrix Properties

- (A few) special matrices
- The determinant

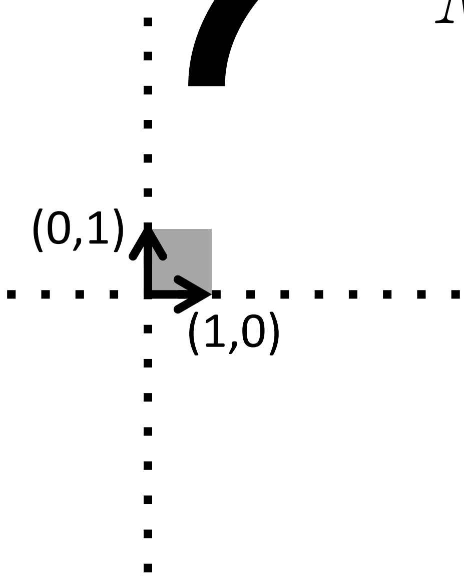
How does a matrix transform a square?

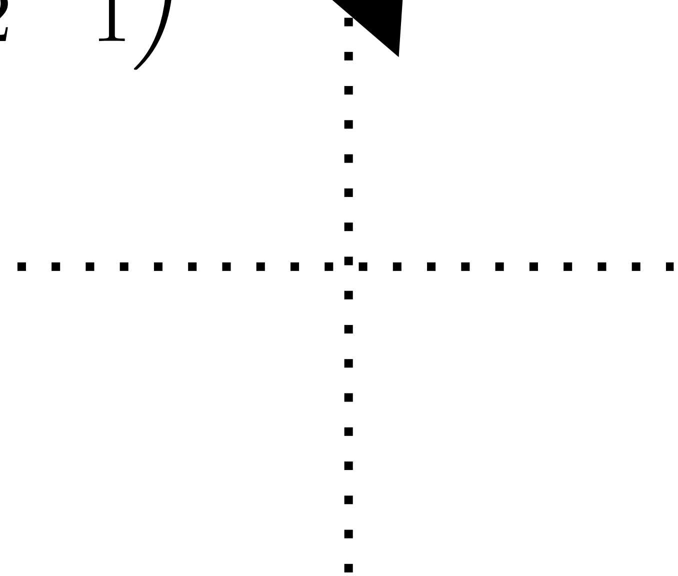




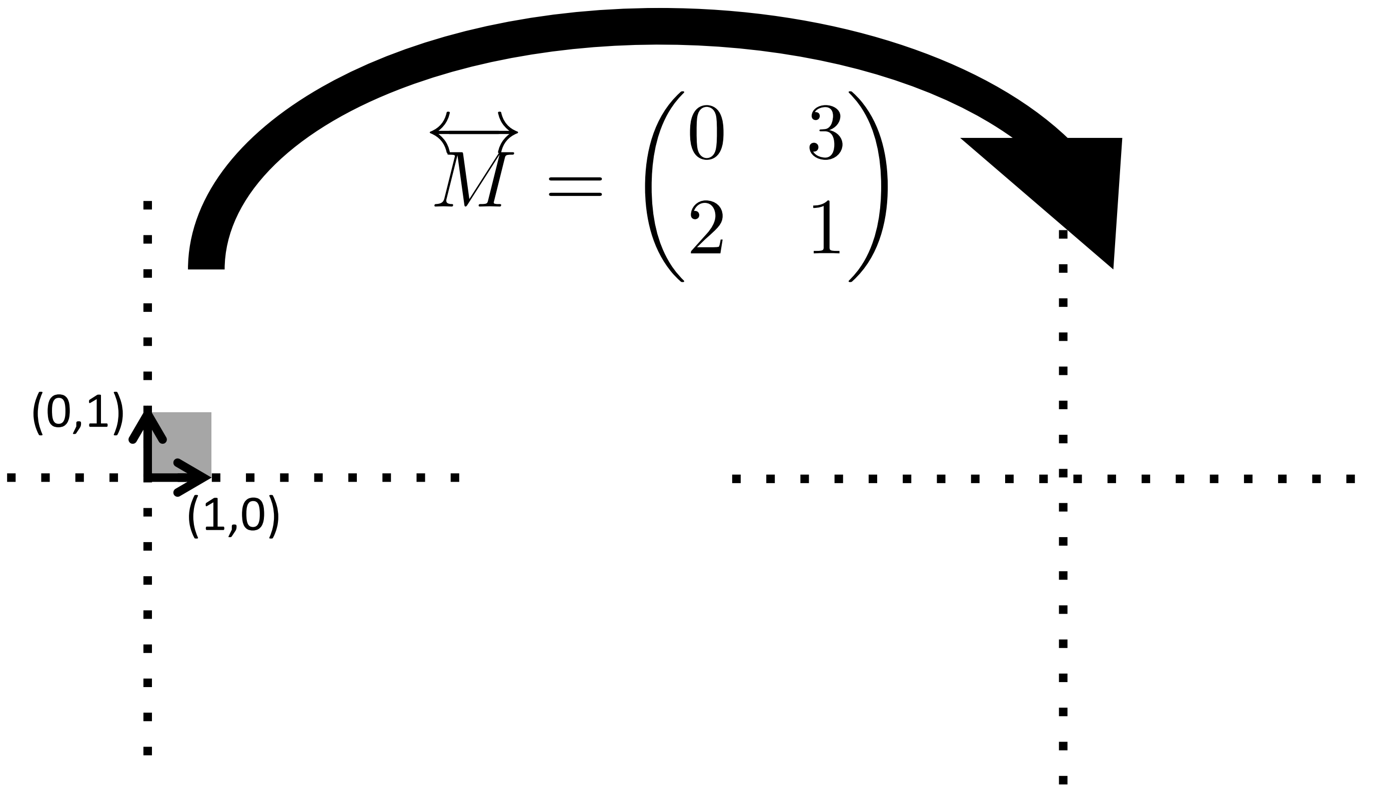
How does a matrix transform a square?

$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$





How does a matrix transform a square?

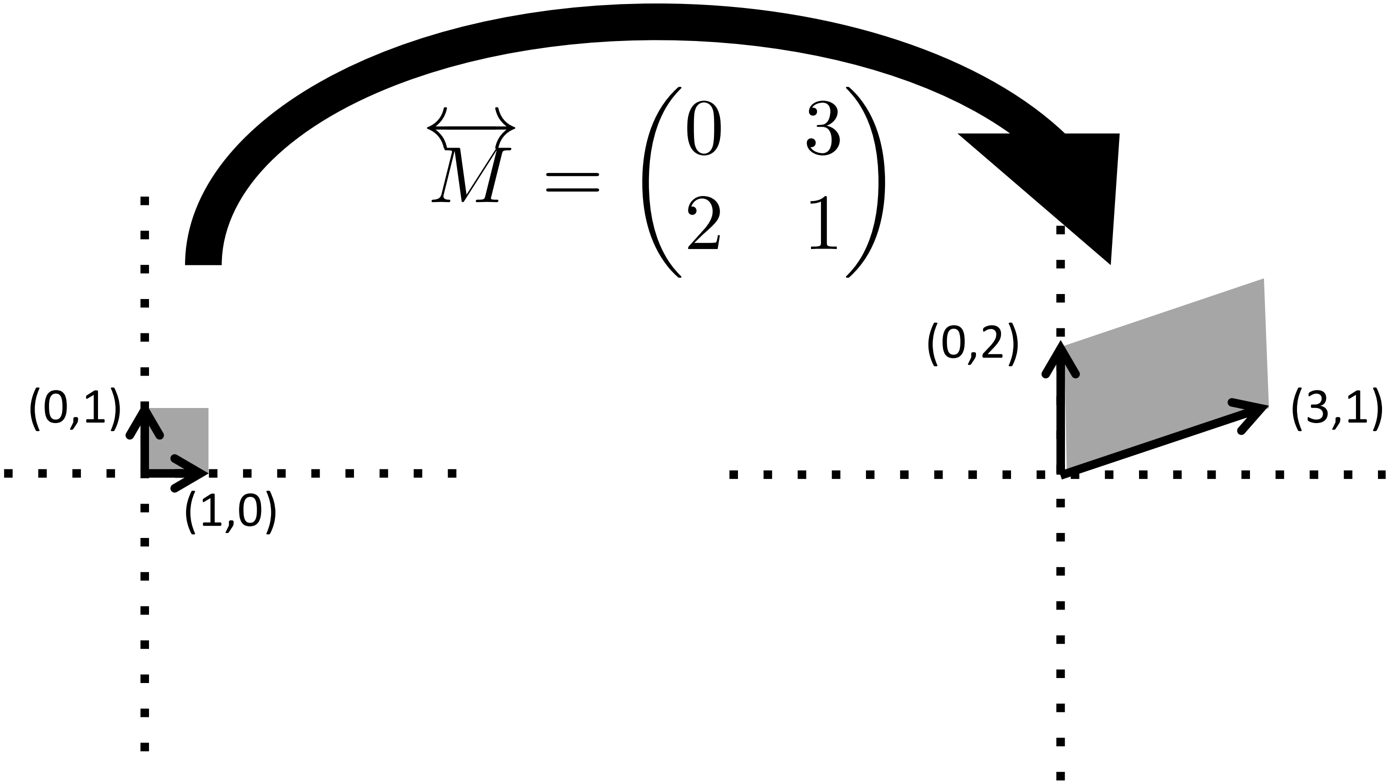


$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$

$$Mx?$$

How does a matrix transform a square?

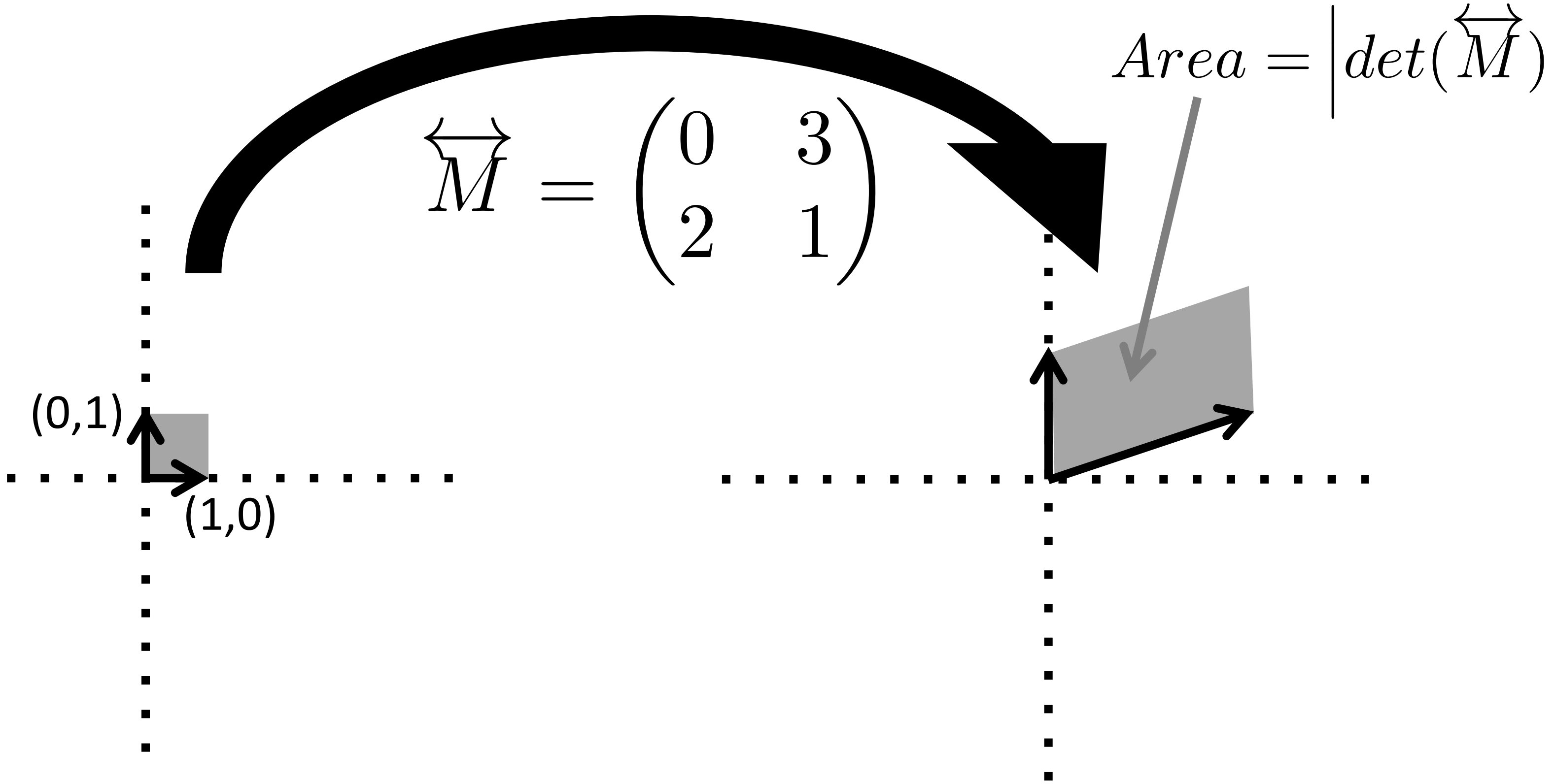
$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$



Geometric definition of the determinant: How does a matrix transform a square?

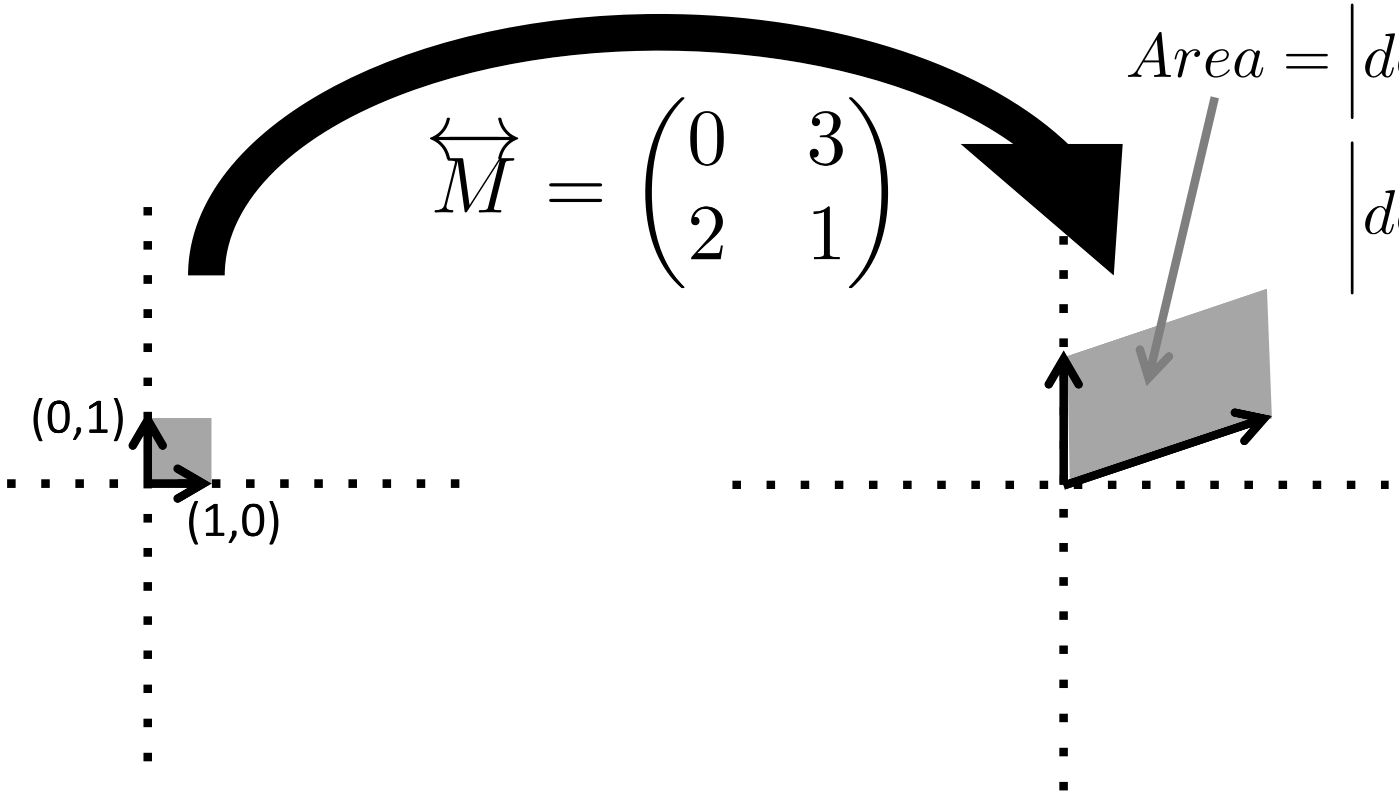
$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$

$$Area = \left| det(\overleftrightarrow{M}) \right|$$



MATLAB: `det(M)`

Geometric definition of the determinant: How does a matrix transform a square?



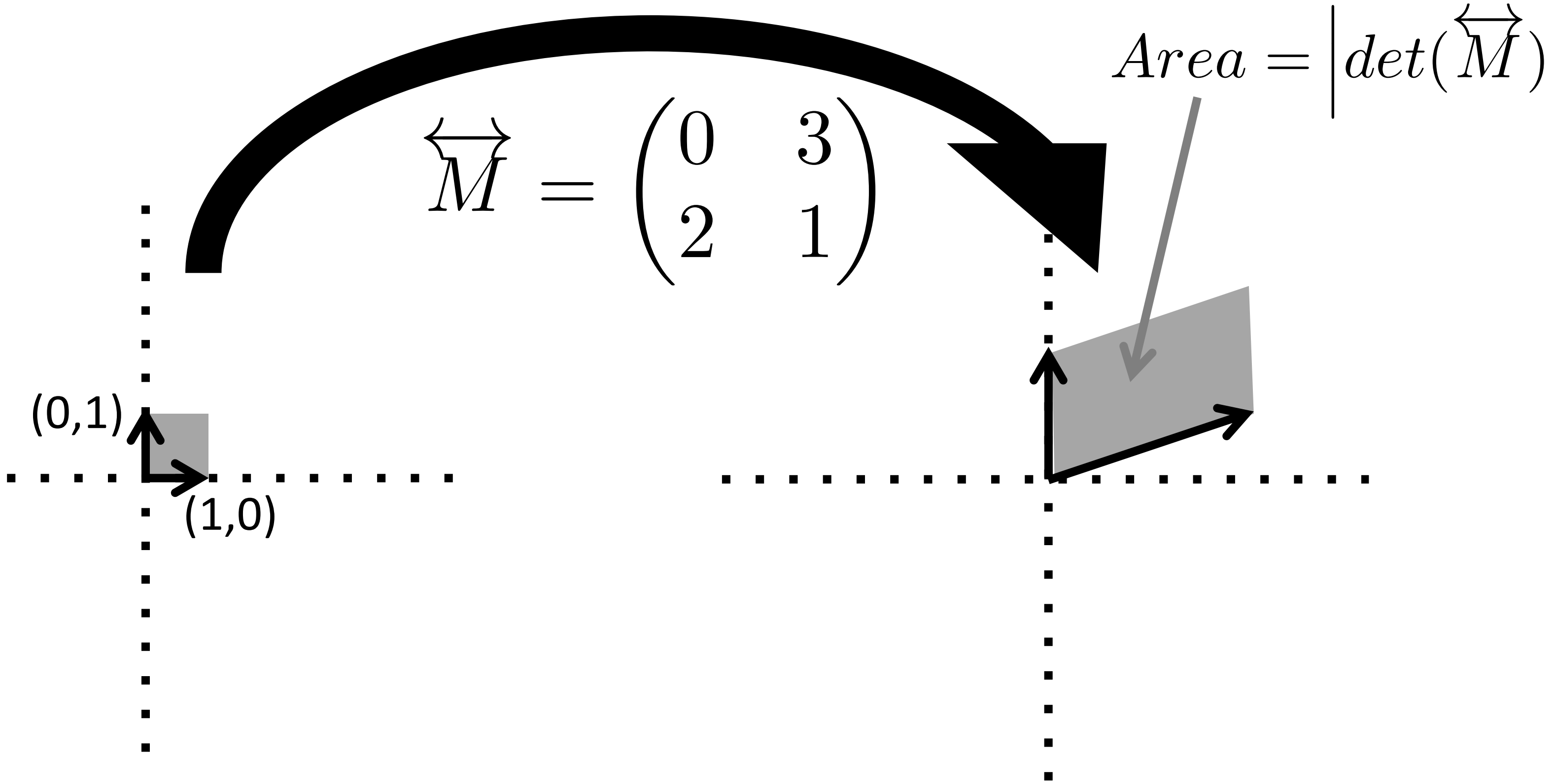
$$\overleftrightarrow{M} = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}$$

$$Area = \left| det(\overleftrightarrow{M}) \right|$$

$$\left| det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = |ad - bc|$$

MATLAB: `det(M)`

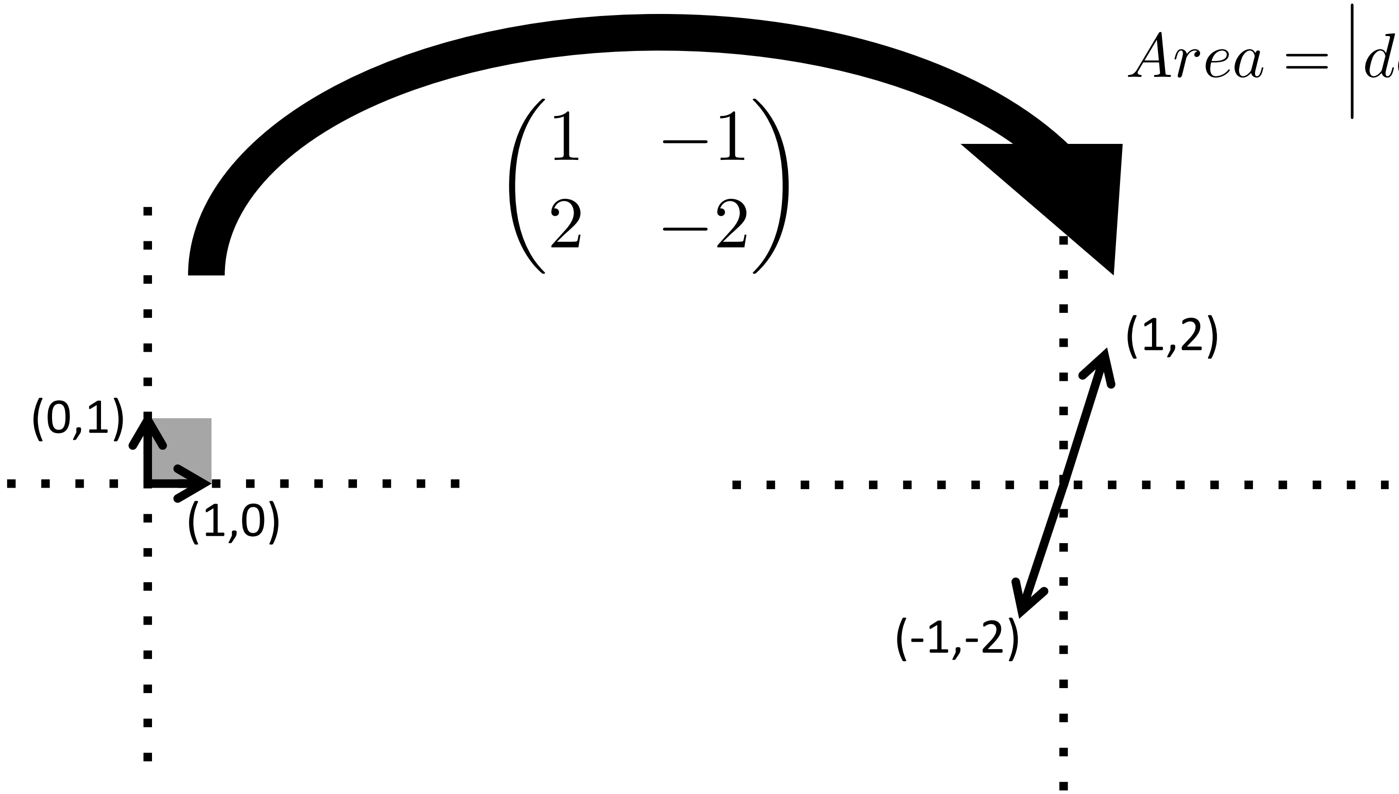
Geometric definition of the determinant: How does a matrix transform a square?



what about $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$?

Geometric definition of the determinant: How does a matrix transform a square?

$$Area = \left| det(\overleftrightarrow{M}) \right|$$



no area!

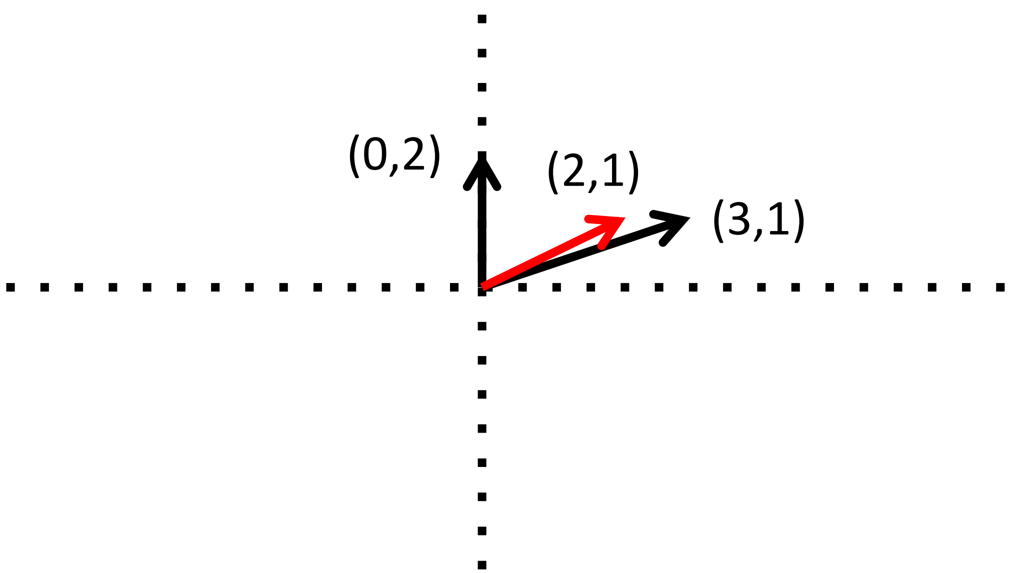
this means the determinant is zero

what about $\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$?

What do matrices do to vectors?

$$\overleftrightarrow{M}$$

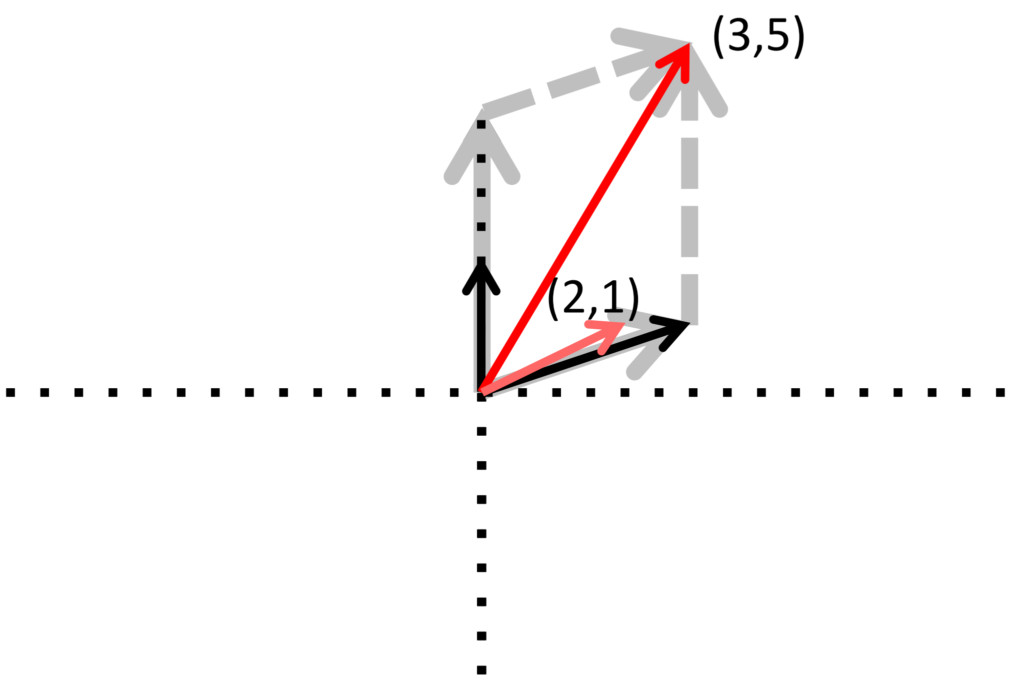
$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$



Recall

$$\overleftrightarrow{M}$$

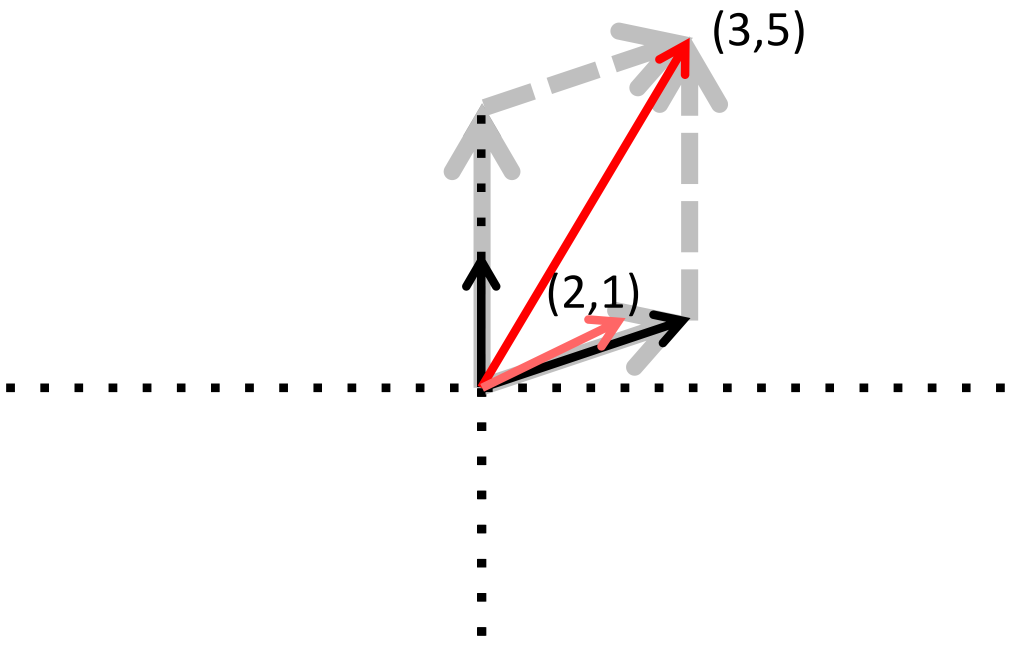
$$\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$



What do matrices do to vectors?

$$\overleftrightarrow{M}$$

$$\begin{pmatrix}0 & 3\\ 2 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}0\\2\end{pmatrix} + 1\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix}$$



- The new vector is:
  1) **rotated**
  2) **scaled**

The End

Next lecture: Linear Algebra II

- eigenvalues and eigenvectors
- PCA and matrix decompositions

Acknowledgements

Some of these slides were adapted from a Linear Algebra lecture presented at the 2014 Woods Hole *Methods in Computational Neuroscience* course by Mark Goldman and Emily Mackevicius.