Thursday, January 26, 2017 12:38 PM

(1) orthogonal vectors:

$u \cdot v = 0$

$\boxed{u^T v = 0}$

$u \cdot v = |u||v| \cos\theta$

(2) norm of a vector

$|v|$

$\|v\| = \sqrt{v_1^2 + \dots + v_n^2}$

$\downarrow$

$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$\boxed{v^T v = 1}$

$v \cdot v = 1$

$\boxed{\|v\| = 1} = \left(\sqrt{v_1^2 + \dots + v_n^2}\right)^2 = 1$

$\|v\|^2 = 1 \quad = v_1^2 + \dots + v_n^2 \leftarrow$

$= v^T v$

$[v_1 \dots v_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

→ orthogonal matrix

$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{bmatrix}$

$\boxed{v_i^T v_j = 0} \quad i \neq j$

$\boxed{v_i^T v_j = 1} \quad i = j$

$\boxed{V^{-1} = V^T}$

→ eigenvectors & eigenvalues are a thing

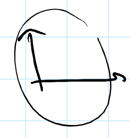
$Wv = \lambda v$

→ an $n \times n$ can have at most $\boxed{n}$ distinct eigenvalues

→ eigenvectors w/ distinct eigenvalues ⇒ linearly independent

→ n distinct eigenval ⇒ can have n distinct eigenvectors

→ span space of $\boxed{\mathbb{R}^n}$



$\mathbb{R}^2$

$[\ ], [\ ], [\ ] \dots \quad \boxed{\mathbb{R}^n}$

$\mathbb{R}^{n+1}$

HW: $Iv = \lambda v$

$\lambda = 1$

$$\lambda = 1$$

$$\underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{I}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}v_1\\ v_2\end{bmatrix} \qquad \begin{bmatrix}\ \end{bmatrix}, \begin{bmatrix}\ \end{bmatrix}, \begin{bmatrix}\ \end{bmatrix}$$

$$\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

→ $\sum_{i=1}^{N} \lambda_i = \text{Trace}(W)$

$\prod_{i=1}^{N} \lambda_i = \text{Det}(W)$

→ n linearly indep. eigenvectors form a <u>basis</u>

$$v = c_1 v_1 + c_2 v_2$$
$$u = \underbrace{W}\underbrace{v} \leftarrow$$
$$= W(c_1 v_1 + c_2 v_2)$$
$$= c_1 W v_1 + c_2 W v_2$$
$$= \underbrace{c_1 \lambda_1 v_1} + \underbrace{c_2 \lambda_2 v_2} \leftarrow$$

→ n lin. ind. eigenvec. (W)

$$W[\underbrace{v_1 \cdots v_n}] = [v_1 \cdots v_n]\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ \vdots & \ddots & & \vdots\\ 0 & \cdots & & \lambda_n\end{bmatrix}$$

$n \times n$

$$WV = V\Lambda \leftarrow$$
$$\boxed{W = V\Lambda V^{-1}} \leftarrow \qquad \underline{V^{-1}WV = \Lambda} \leftarrow$$

$V^{-1} = V^T$

→ $\boxed{\lambda = 0}$

$\prod_{i=1}^{N} \lambda_i = \text{Det}$

$= 0 \Rightarrow$ NOT INVERTIBLE $\qquad [\,|\,|\,|\,|\,]$

→ symmetric matrices

→ symmetric matrices

① eigenvectors w/ distinct eigenvalues : <u>orthogonal</u>

$u^T v = 0 \qquad v_i^T v_j = 0$

Quick proof:

$A, \quad \lambda_1 \neq \lambda_2$

$\quad\; v_1 \quad v_2 \qquad v_1^T v_2 \stackrel{?}{=} 0$

$\boxed{\lambda_1 v_1^T v_2}$

$\lambda_1 v_1^T v_2 = \boxed{(\lambda_1 v_1)^T} v_2 = (A v_1)^T v_2 \qquad (AB)^T \neq A^T B^T = B^T A^T$

$\boxed{A v_1 = \lambda v_1}$

$v_1^T A^T v_2 \qquad \boxed{A = A^T}$

$= v_1^T \underbrace{A v_2}_{\lambda_2 v_2}$

$= v_1^T \lambda_2 v_2$

$= \lambda_2 v_1^T v_2$

$\lambda_1 v_1^T v_2 = \lambda_2 v_1^T v_2$

$(\lambda_1 - \lambda_2) v_1^T v_2 = 0$

$\lambda_1 \neq \lambda_2 \qquad v_1^T v_2 = 0$

→ $V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ ←

[assume n distinct eigenval]
[symmetric matrices
symmetric matrices n lin. indep. eigenvectors
⇒ n lin. indep. eigenvectors
$V \Rightarrow V^T = V^{-1}$]
(b/c orthogonal)

$W = V \Lambda V^{-1}$

$= \boxed{V \Lambda V^T}$

$= \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{pmatrix} \qquad (-\!\!-\; v_1^T \;-\!\!-)$

$n \quad n \qquad n \quad n \qquad n$

$(\lambda_1 v_1^T)$

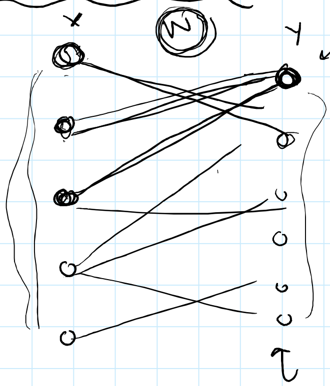
$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$

$$= (v_1\ v_2 \cdots v_n) \begin{pmatrix}\lambda_1 v_1^T \\ \vdots \\ \lambda_n v_n^T\end{pmatrix} \qquad \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix} v_1^T \\ v_2^T\end{pmatrix} = \begin{pmatrix}\lambda_1 v_1^T\\ \lambda_2 v_2^T\end{pmatrix}$$

$$= \lambda_1 v_1 v_1^T + \cdots + \lambda_n v_n v_n^T = \boxed{\sum_{i=1}^{n} \lambda_i v_i v_i^T}$$

Example problem

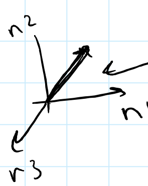
W



$$Wx = y$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} w_{11}x_1 + w_{12}x_2 + w_{13}x_3 \\ \\ \end{bmatrix}$$



$\|y\| \to \|y\|^2$

MAXIMIZE $\|y\|^2$, $\|x\|^2 = 1$

$Wx = y$ $\qquad x^T x = 1$

SOLVING PROBLEM

$\|x\|^2 = x^T x$

$\left.\begin{matrix} \|y\|^2 \\ Wx = y \end{matrix}\right\} \to \|Wx\|^2 = (Wx)^T (Wx)$

$= x^T W^T W x$

$\boxed{(AB)^T = B^T A^T}$

general def.

$\boxed{A^T = A}$

$W^T W$

$(W^T W)^T \overset{??}{=} W^T W$

$W W^T$

$W^T (W^T)^T$

$W^T W$

$W^T W = V \Lambda V^T$

max

$x^T (V \Lambda V^T) x$

$x^T \left(\sum \lambda_i v_i v_i^T \right) x$

$$x^T(\lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T + \cdots + \lambda_n v_n v_n^T)x \qquad \lambda_1 \geq \cdots \geq \lambda_n$$

$$\boxed{V^T = V^{-1}}$$

$$\leq x^T(\lambda_1 v_1 v_1^T + \lambda_1 v_2 v_2^T \cdots + \lambda_1 v_n v_n^T)x$$

$$VV^T = VV^{-1} = I$$

$$\lambda_1 x^T(v_1 v_1^T + \cdots + v_n v_n^T)x \longrightarrow \boxed{\lambda_1 x^T \underbrace{(VV^T)}_{I} x}$$

$$\underbrace{[v_1 \cdots v_n]}_{V}\begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} V^T \qquad \lambda_1 x^T x = \lambda_1$$

$$Ix = x$$

$$\lambda_1 x^T I x \qquad x^T x$$

$$\lambda_1 x^T x = 1$$

$$\lambda_1$$

λ_1

0

$$\rightarrow x^T(\lambda_1 v_1 v_1^T + \cdots + \lambda_n v_n v_n^T)x \stackrel{?}{=} \lambda_1$$

$$\rightarrow x^T \lambda_1 v_1 v_1^T x + \cdots + x^T \lambda_n v_n v_n^T x \leftarrow \quad x = v_1$$

$$\lambda_1 (v_1^T v_1 \, v_1^T v_1) \qquad \lambda_n (v_1^T v_n)(v_n^T v_1) = \lambda_1$$

$$\lambda_1$$

$$AB \neq BA$$

$$v_i^T v_j = v_j^T v_i \qquad a^T b = b^T a$$

$$v_i^T a v_j = a v_i^T v_j$$

$$AV \neq VA$$

$$\lambda \geq \cdots \geq \lambda_n$$

$\rightarrow$ to max $\|y\|^2$

$$\|Wx\|^2$$

$$x^T W^T W x$$

$$x^T(V \Lambda V^T)x$$

$$x^T(\sum_i \lambda_i v_i v_i^T)x \leq \lambda_1 x^T \overbrace{(\sum v_i v_i^T)}^{I} x$$

$$\leq \lambda_1 \underbrace{x^T x}_{1} = \lambda_1$$

$$x^T(\sum_i \lambda_i v_i v_i^T)x = \lambda_1$$

when $x = v_1$.

$\Rightarrow$ to max $\|y\|^2$, choose eigenvector of $W^T W$ w/ largest eigenval.

$$\|y\|$$

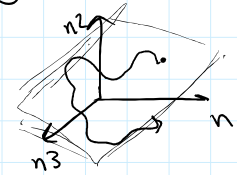
$$\|y\|^2 = y^T y$$

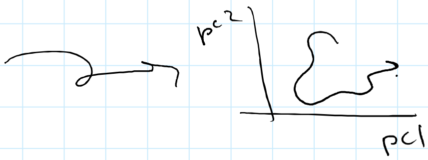
$$\|y\|$$
$$\|y\|^2 = y^T y$$

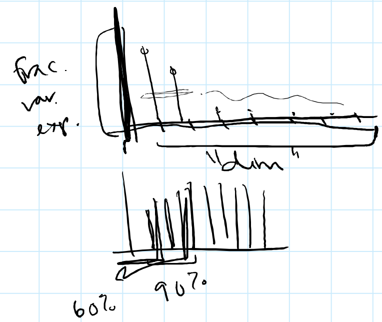
π

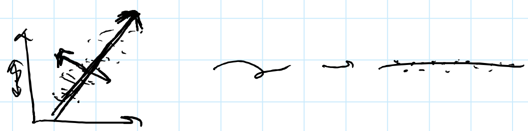
PCA

① dim red. method

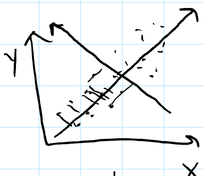






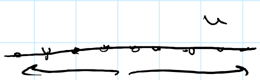


②



Data matrix: $\begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$

Goal: find line that captures greatest variance in your data



$[u_1 \cdots u_n]$ ← max variance w/ these

$$\underset{1 \times n}{[u_1 \cdots u_n]} = \underset{1 \times 2}{[p_1 \; p_2]} \underset{2 \times n}{\begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}}$$

$$u_1 = p_1 x_1 + p_2 y_1$$
$$u_2 = p_1 x_2 + p_2 y_2$$
$$\vdots$$

→ what variance again? $[u \cdots u_n]$

$$\text{var} = \frac{1}{n} \sum_{i=1}^{n} x_i^2 \leftarrow \text{zero-mean data}$$

$$\text{var}(u) = \frac{1}{n} u u^T$$

$[\quad][\quad]$

uu^T ← max this!

$$\underbrace{[s_1 \cdots s_n]}_{\text{max variance!}} = \underbrace{[u_1 \; u_2]}_{U^T} \underbrace{\begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & & y_n \end{bmatrix}}_{X}$$

2, n

s

$s = u^T X$

$\} \; s = [p_1 \, p_2] X$

s ↗ max variance! u^T $\underbrace{\quad\cdots\quad y_n}_{X}$

$\underline{s} = \underline{u^T X}$

$\mathrm{var}(u^T X) = \frac{1}{N}(u^T x)(u^T x)^T$

$= \frac{1}{N}(u^T X X^T u)$ → $\frac{1}{N} \underbrace{X X^T}_{2\times2}$ ← covariance matrix

$XX^T \to 2\times2$

$u^T W u$ ← max this!

$(XX^T)^T = XX^T$

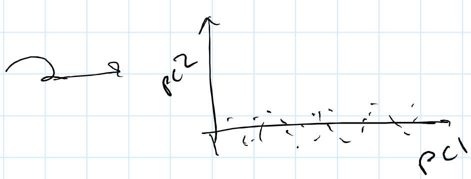
λ_1 ← largest eigenval of $W = XX^T$ covariance matrix of your data

u → eigenvector assoc. w/ eigenvalue

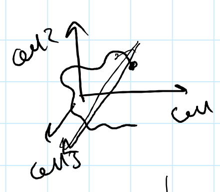
$u^T(XX^T)u$ ← variance of $u^T X$

← variance of projected data s

max at $\boxed{\lambda_1}$



$$\begin{bmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ r_1 & r_2 \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$$



time →

$$\begin{bmatrix} u_1 & \cdots & u_T \\ v_1 & \cdots & v_T \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & \cdots & p_T \\ r_1 & r_2 & \cdots & r_T \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1T} \\ \vdots & & \\ x_{N1} & \cdots & x_{NT} \end{bmatrix}$$ cells