Linear Algebra Part II

January 19th, 2017

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Math Tools for Neuroscience

Today's lecture

- tools for thinking about – and manipulating – matrices
- eigenvalues and eigenvectors
- matrix decompositions

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- **tools for thinking about – and manipulating – matrices**
- eigenvalues and eigenvectors
- matrix decompositions

Why linear algebra?

1.63	5.20	7.66	8.12	3.22
4.98	5.90	8.21	9.29	20.10
10.10	8.57	5.73	8.17	2.22
0.02	0.21	0.14	0.93	1.40
9.27	10.27	13.12	8.90	9.01
7.44	6.98	5.62	8.20	7.21
100.10	8.22	7.54	60.10	1.69
40.20	29.21	12.45	10.41	8.90
32.33	21.59	10.21	4.99	2.62
2.99	1.67	1.01	0.80	0.07

Datasets are matrices

time →

neuron 1	1.63	5.20	7.66	8.12	3.22
neuron 2	4.98	5.90	8.21	9.29	20.10
neuron 3	10.10	8.57	5.73	8.17	2.22
neuron 4	0.02	0.21	0.14	0.93	1.40
neuron 5	9.27	10.27	13.12	8.90	9.01
neuron 6	7.44	6.98	5.62	8.20	7.21
neuron 7	100.10	8.22	7.54	60.10	1.69
neuron 8	40.20	29.21	12.45	10.41	8.90
neuron 9	32.33	21.59	10.21	4.99	2.62
neuron 10	2.99	1.67	1.01	0.80	0.07

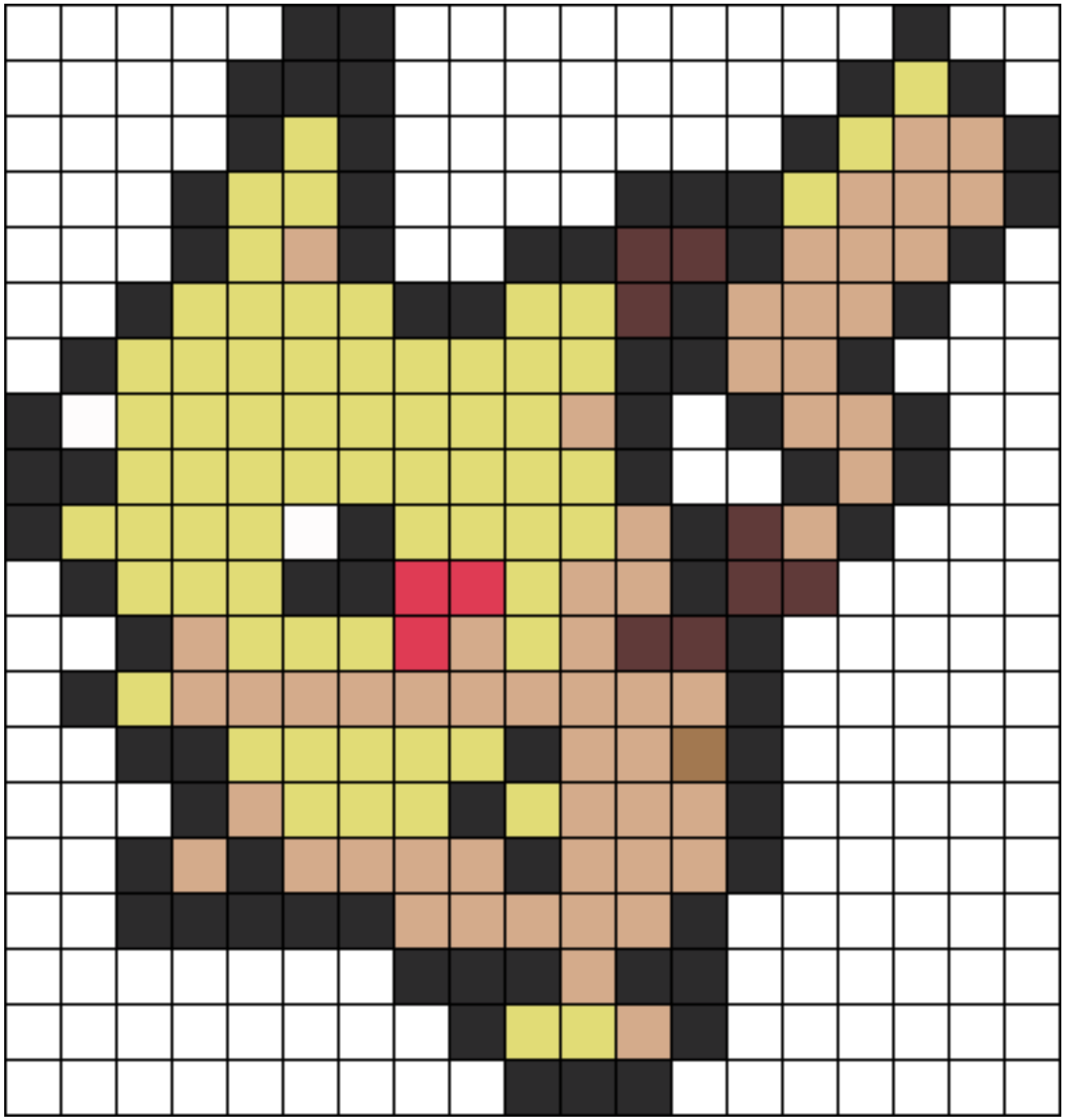
Datasets are matrices

pixels →

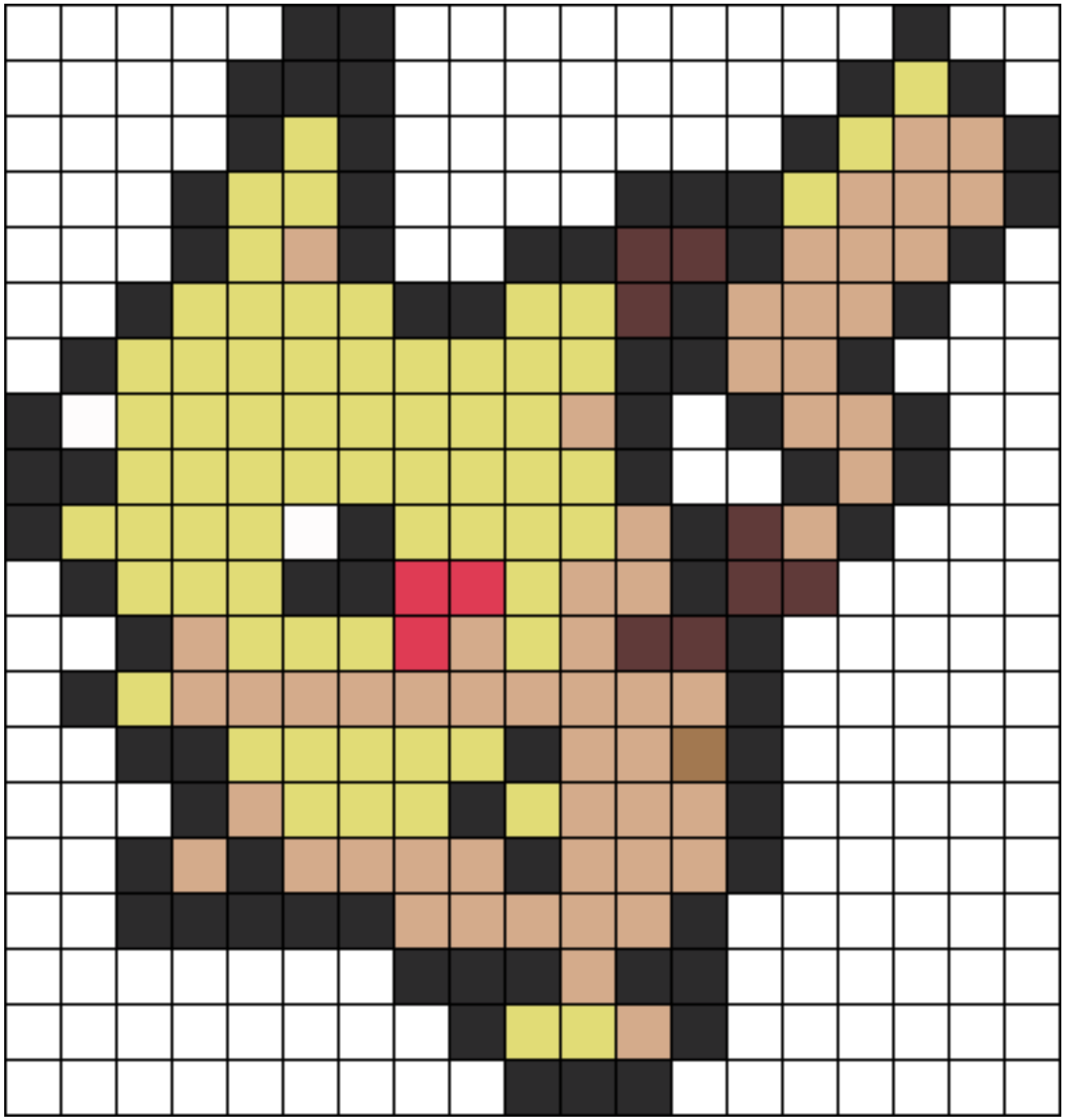
pixels ↓

50	50	49	49	0
50	51	254	189	175
50	245	239	244	200
48	213	3	50	230
46	250	49	50	70
39	247	234	65	68
120	238	243	49	63
99	87	129	56	62
108	220	221	215	210
112	143	211	219	240

Datasets are pikachus

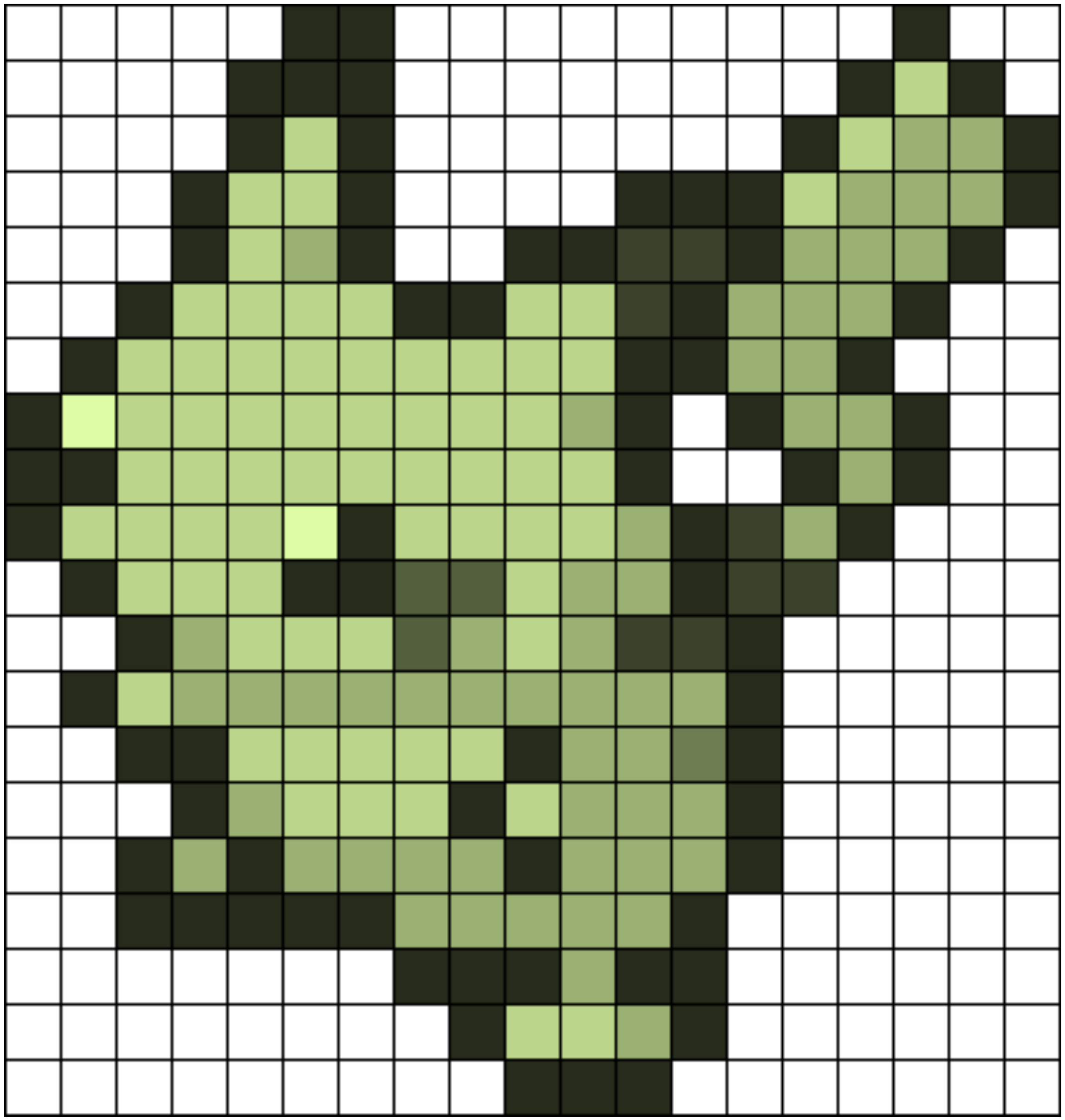


Datasets are pikachus



$= [R, G, B]$

Datasets are pikachus



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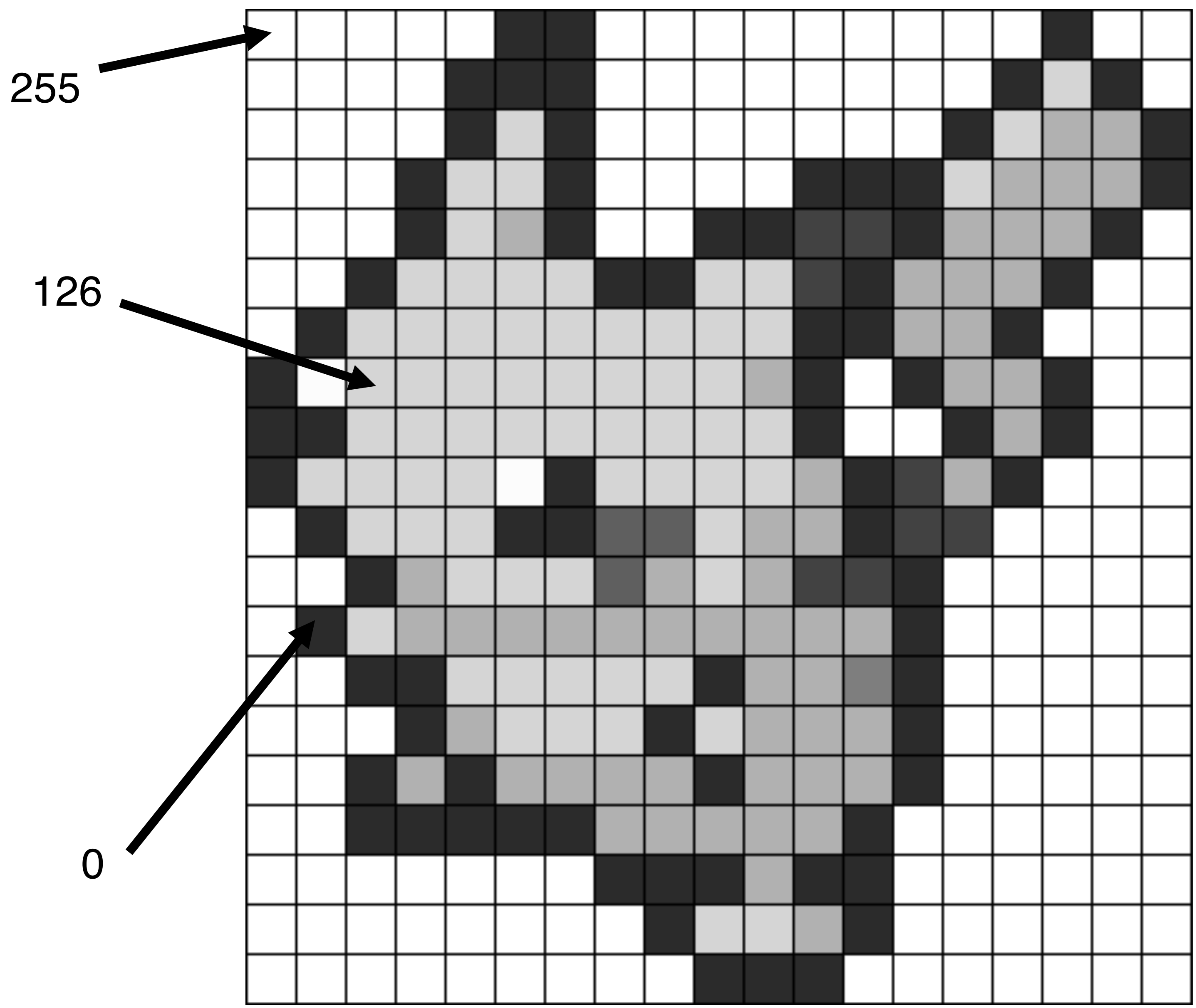
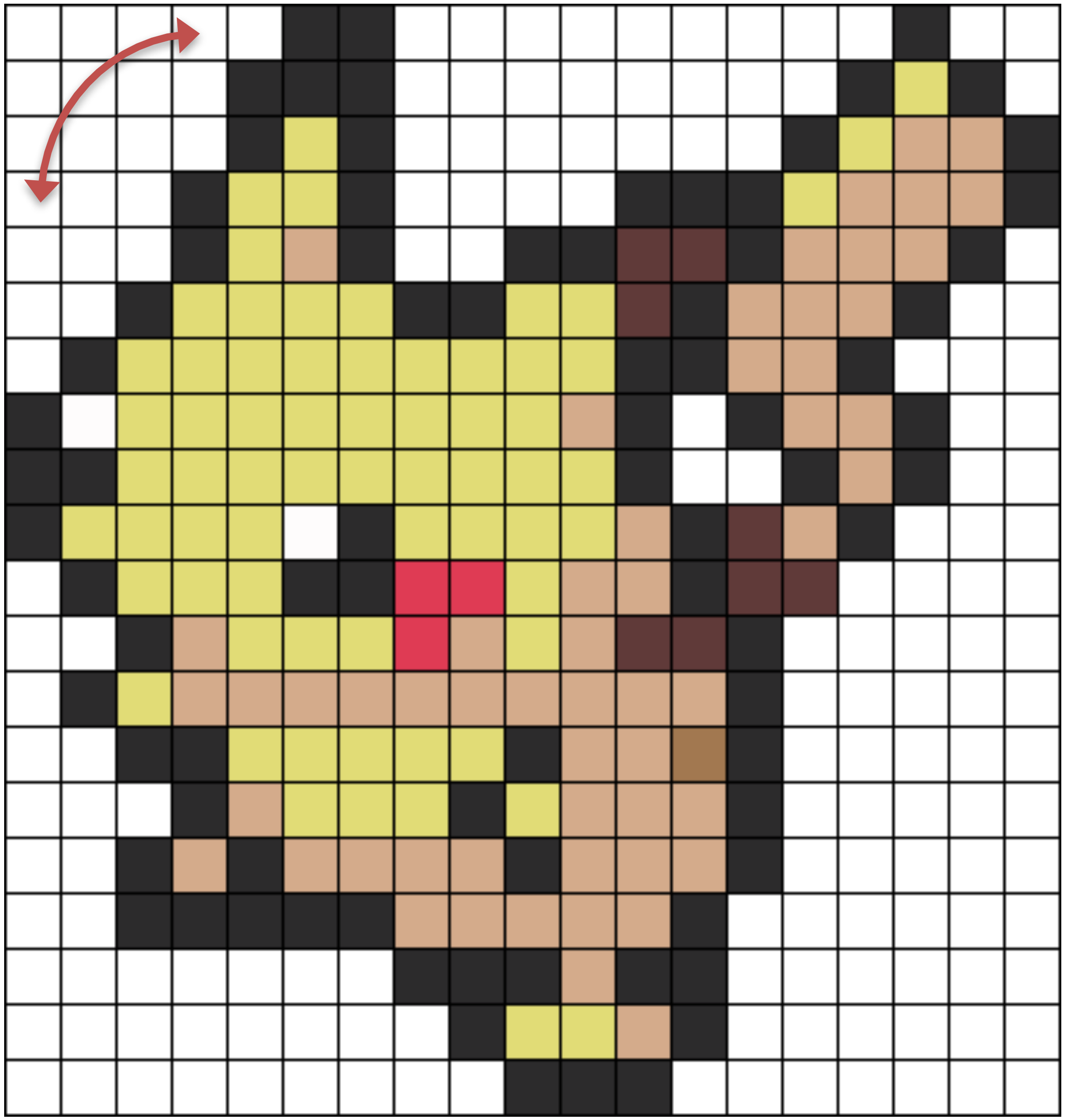


Tool # 1: `Transpose`

swap element i,j with j,i

$$X =$$



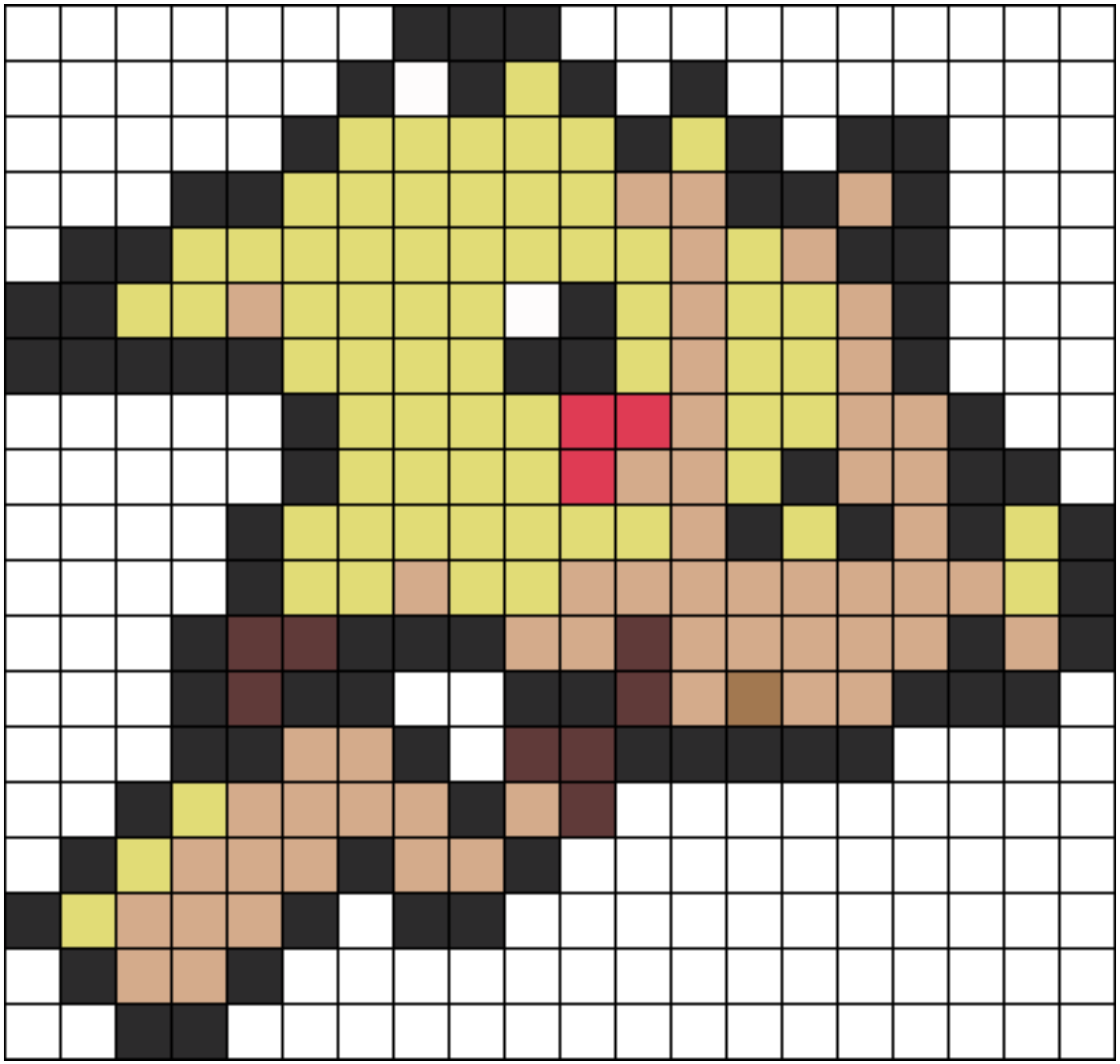
20 x 19

MATLAB: `X'`

Tool # 1: `Transpose`

swap element
i,j with j,i

$$X^T =$$



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MATLAB: `X'`

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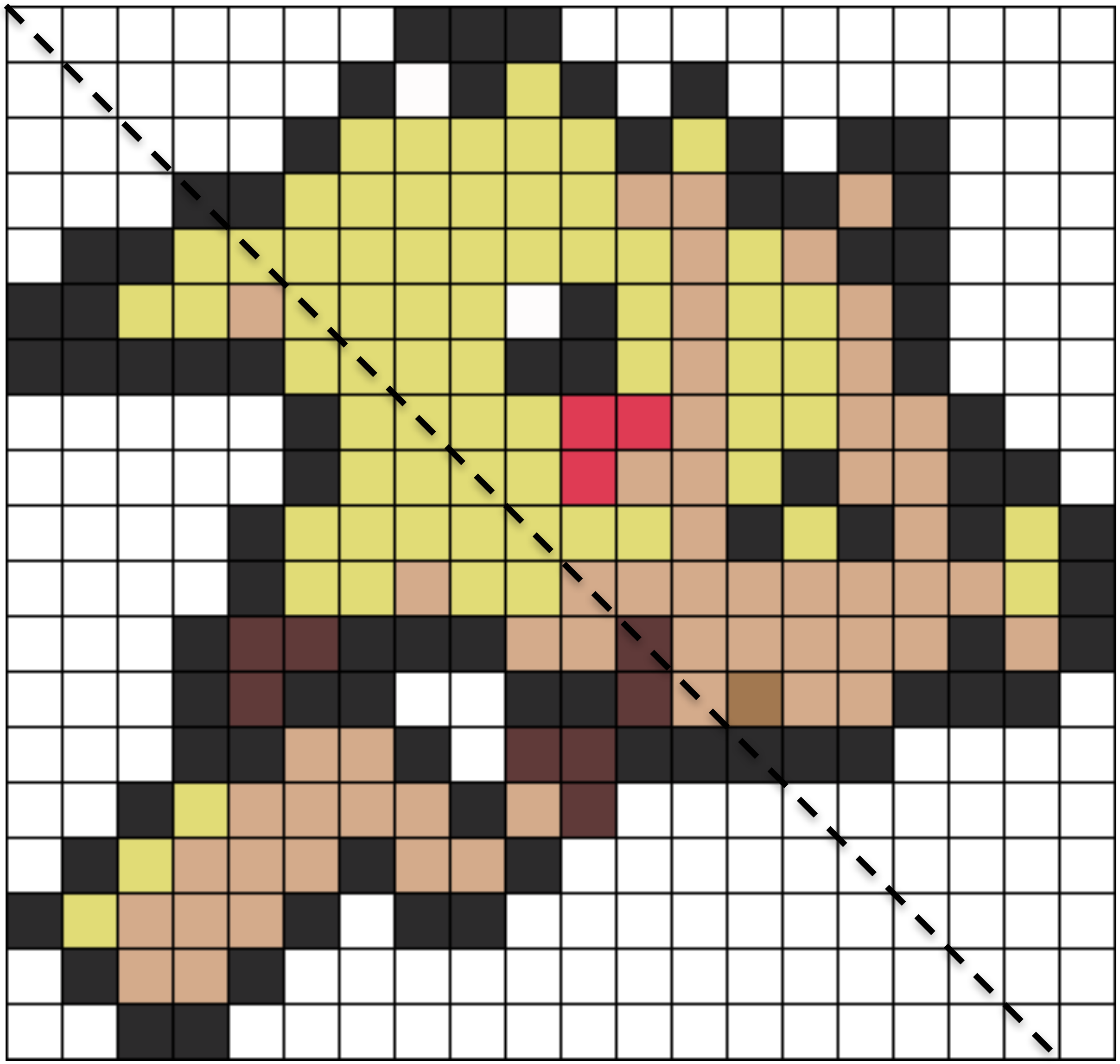

19 x 20

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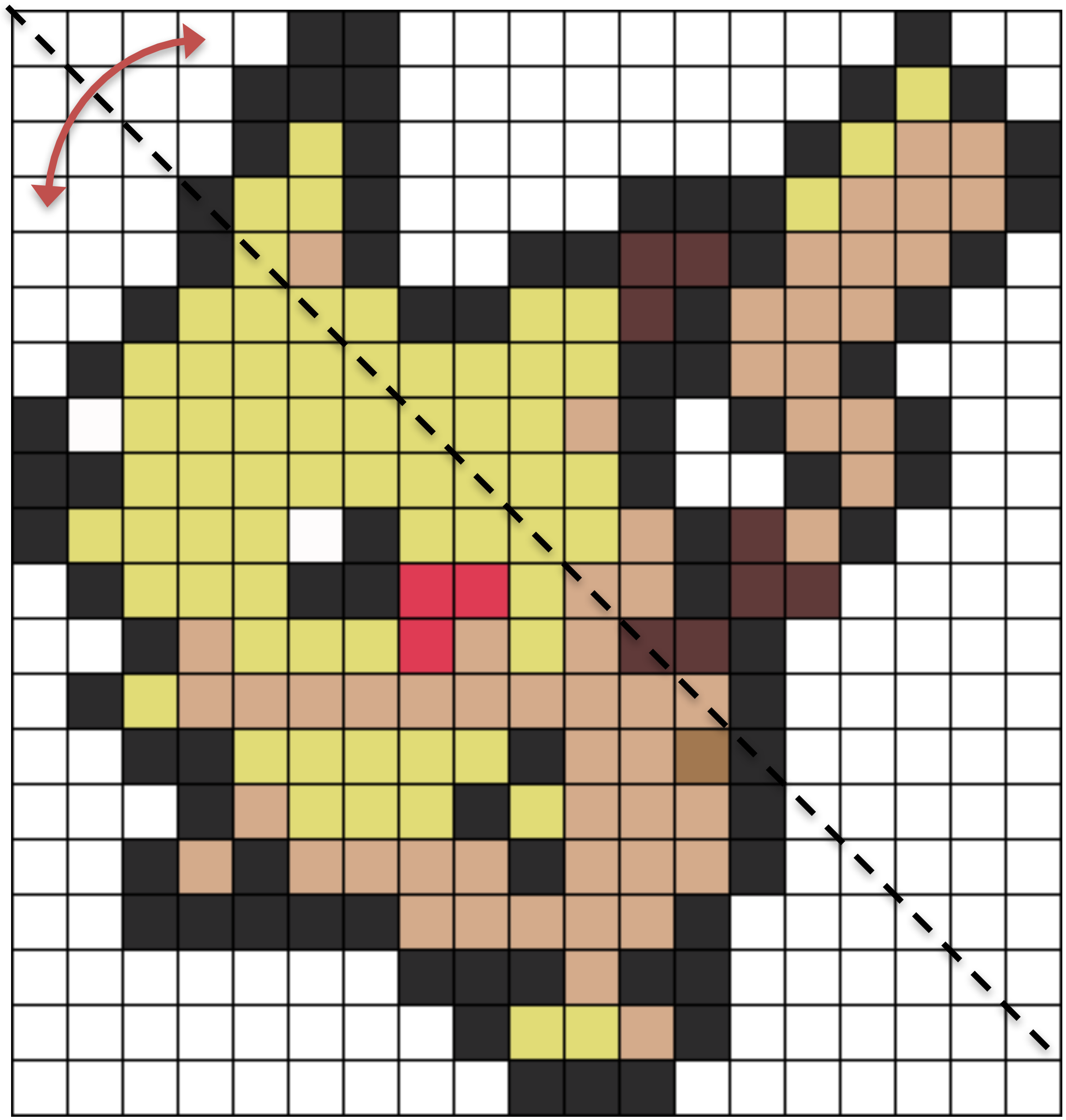
$X =$ 

20 x 19

MATLAB: `X'`

Tool # 1: `Transpose`

Why is this useful?

inner product	$x^T x$
Covariance matrix from a dataset of many measurements	XX^T
PCA	$X = U\Sigma W^T$

Tool # 1: `Transpose`

Question

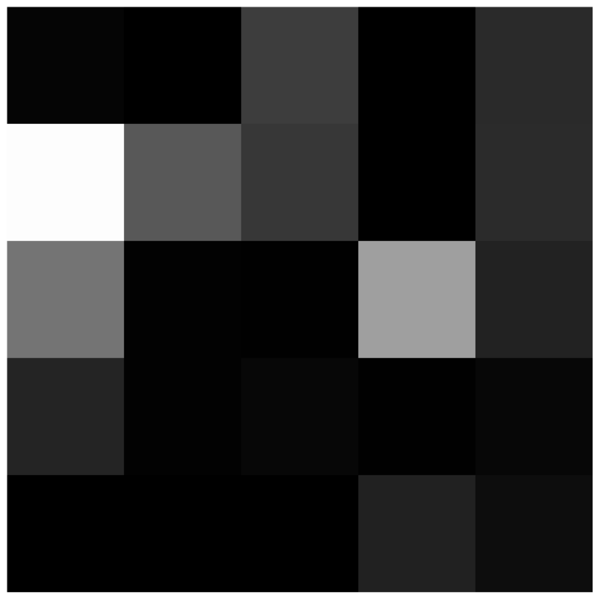
Does rank(X) = rank(X^T)?

Related property: Symmetric matrices

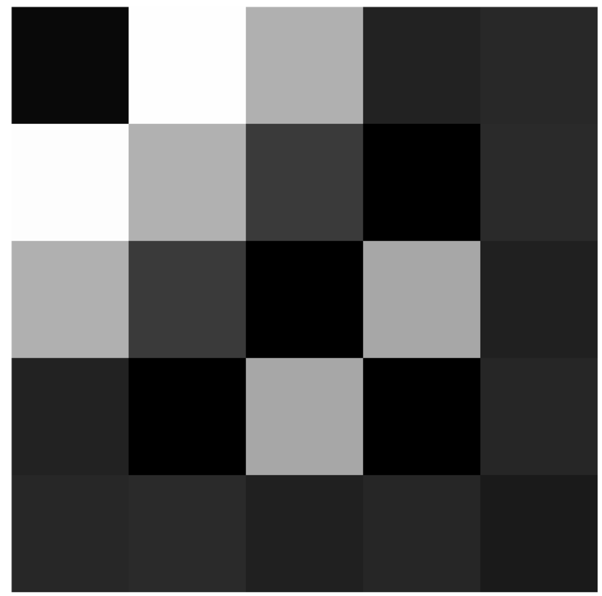
When $X = X^T$

A

B



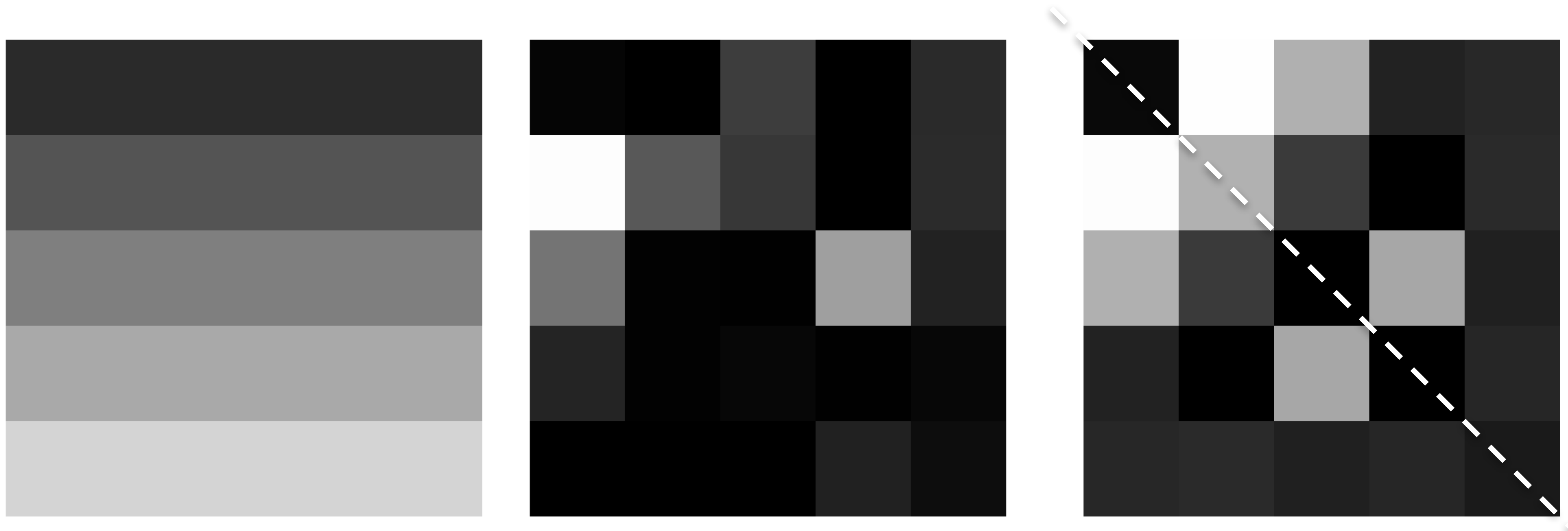
C



Related property: Symmetric matrices

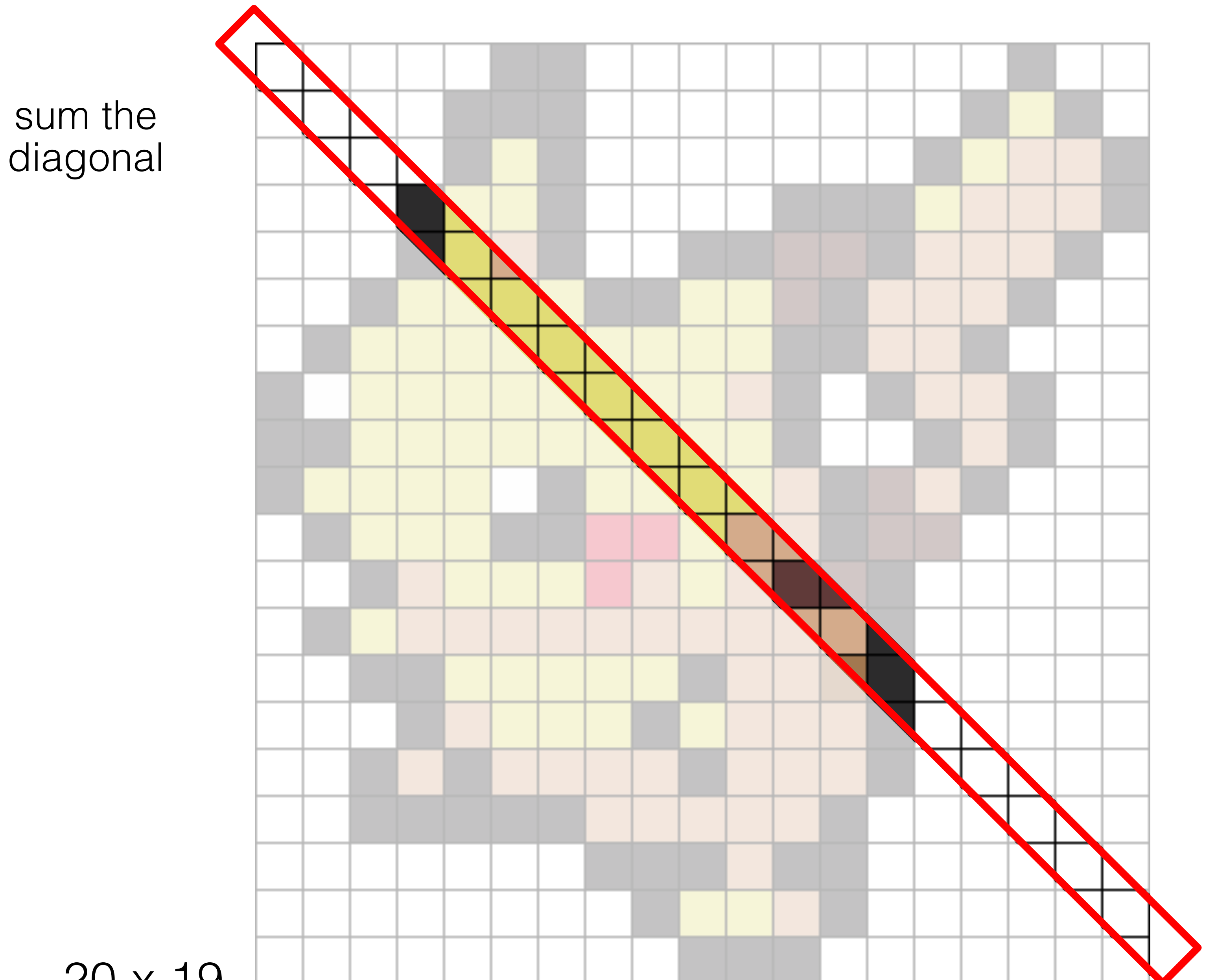
When $X = X^T$

A B **C**



can a non-square matrix be symmetric?

Tool # 2: `Trace`



MATLAB:
`trace(X)`

Tool # 2: `Trace`

True or False!

$$tr(X) = tr(X^T)$$

$$tr(A+B) = tr(A) + tr(B)$$

$$tr(X^TX) = tr(XX^T)$$

$$tr(AB) = tr(BA)$$

assume A = NxM, B = MxN

Tool # 2: `Trace`

True or False!

$tr(X) = tr(X^T)$ **True!**

$tr(A+B) = tr(A) + tr(B)$ **True!**

$tr(X^TX) = tr(XX^T)$ **True!**

$tr(AB) = tr(BA)$ **True!**

assume A = NxM, B = MxN

Tool # 3: `Norm`

length of a vector

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

[On board: Pythagorean Theorem connection]

MATLAB:
```
norm(x)
norm(x,p)
norm(X)
```

Tool # 3: `Norm`

length of a vector

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

variations on a theme

"p norm" $$||x||_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$$

"2 norm" $$||x||_2 = ||x||$$

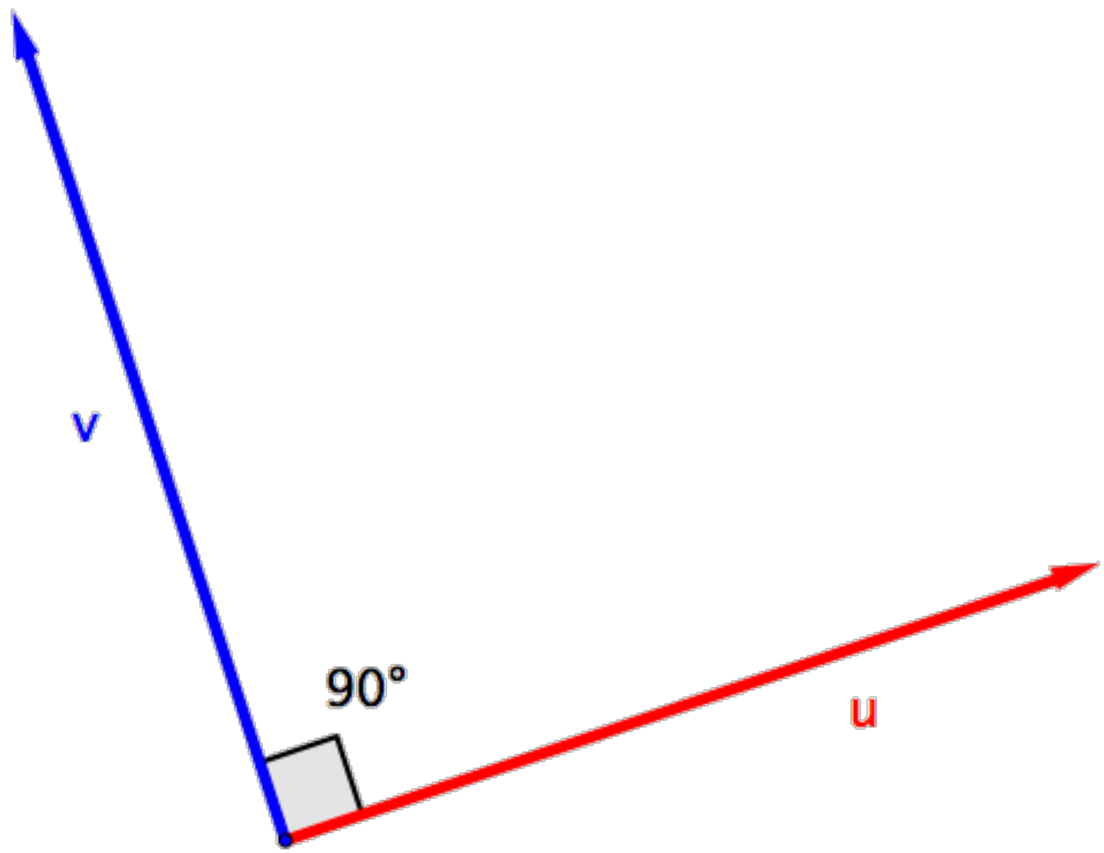
"1 norm" aka Manhattan or Taxicab norm $$||x||_1 = \sum_{i=1}^{n} |x_i|$$

MATLAB:

```
norm(x)
norm(x,p)
norm(X)
```

Big concept: `Orthogonality`

aka when vectors are perpendicular



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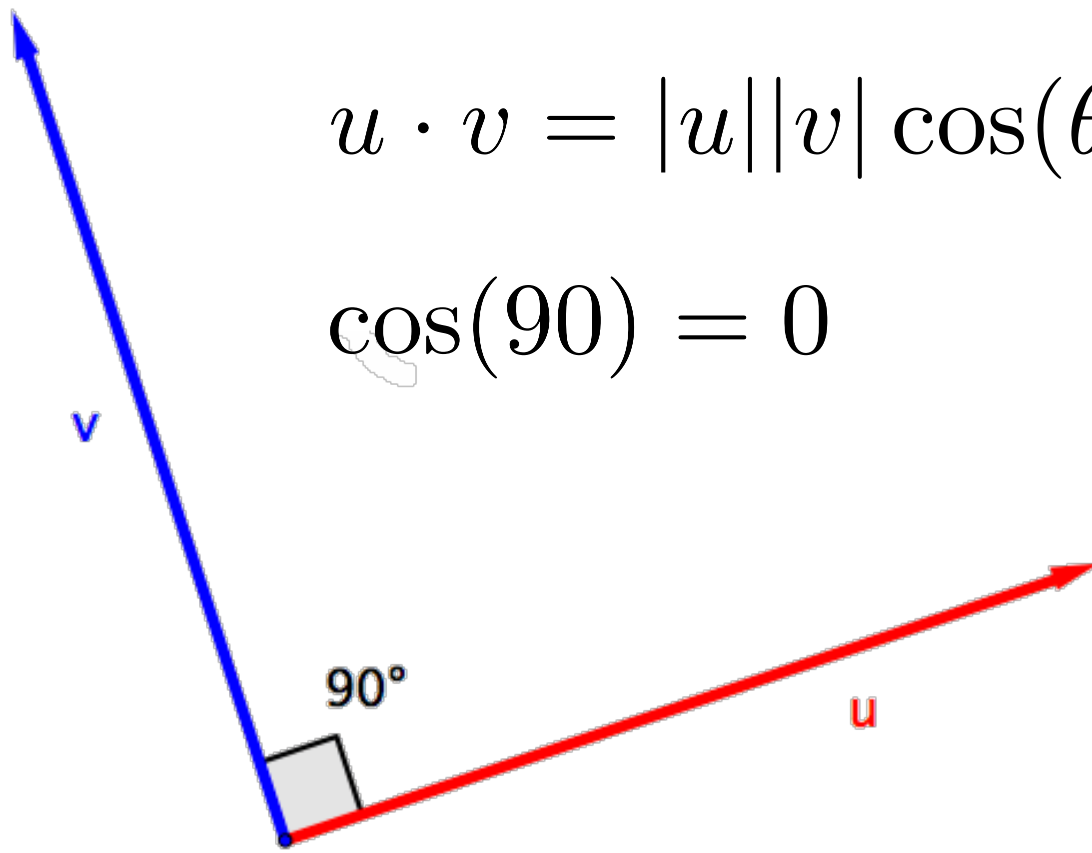


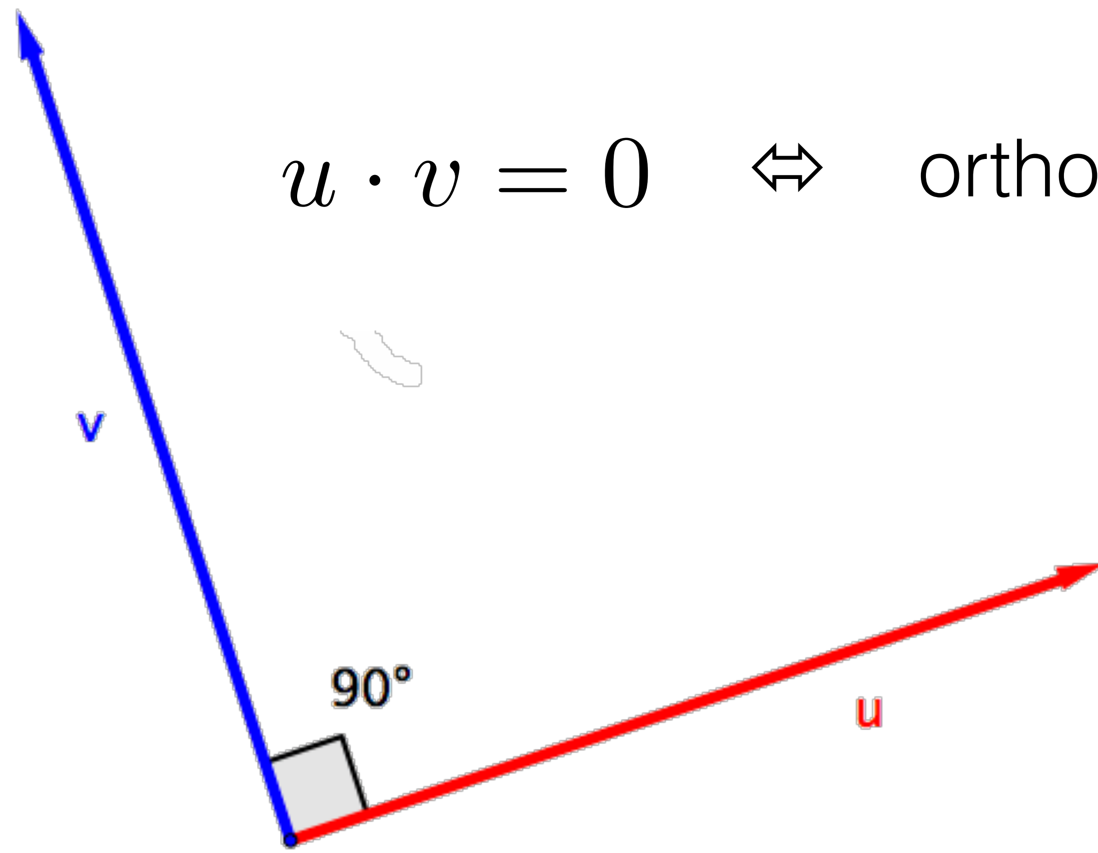
$$u \cdot v = |u||v|\cos(\theta)$$

$$\cos(90) = 0$$

Big concept: `Orthogonality`

aka when vectors are perpendicular

$$u \cdot v = 0 \quad \Leftrightarrow \quad \text{orthogonal}$$

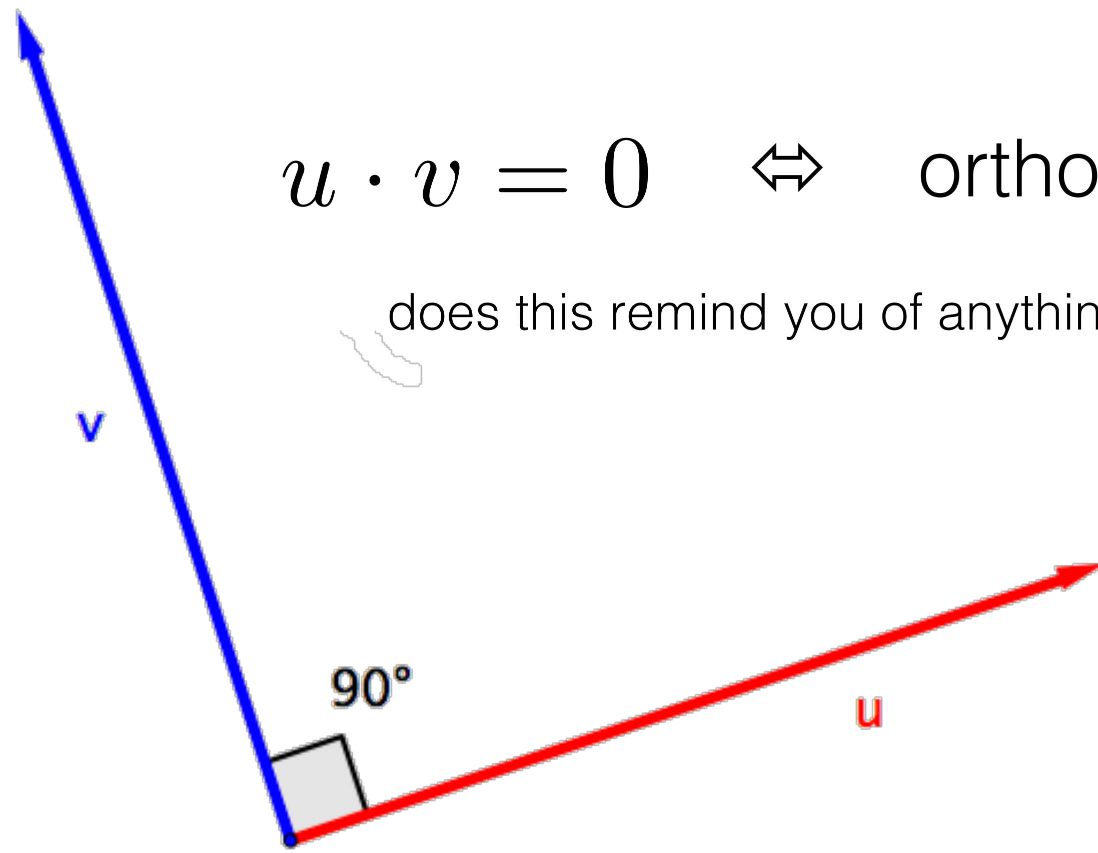


Big concept: `Orthogonality`

aka when vectors are perpendicular

$$u \cdot v = 0 \quad \Leftrightarrow \quad \text{orthogonal}$$

does this remind you of anything?



Orthogonality vs Linear Independence

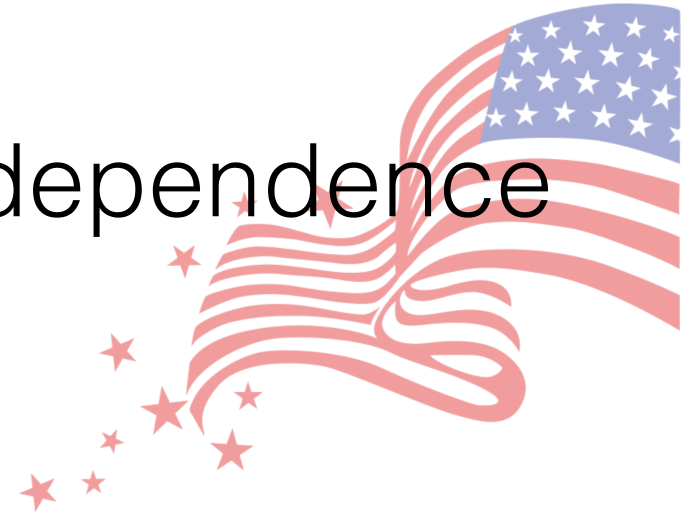
orthogonal

$$u \cdot v = 0$$

linear independence

$$au \neq v$$ for any a

Orthogonality vs Linear Independence



orthogonal	linear independence
$u \cdot v = 0$	$au \neq v$ for any a

1) find an example of two orthogonal vectors that are not linearly independent

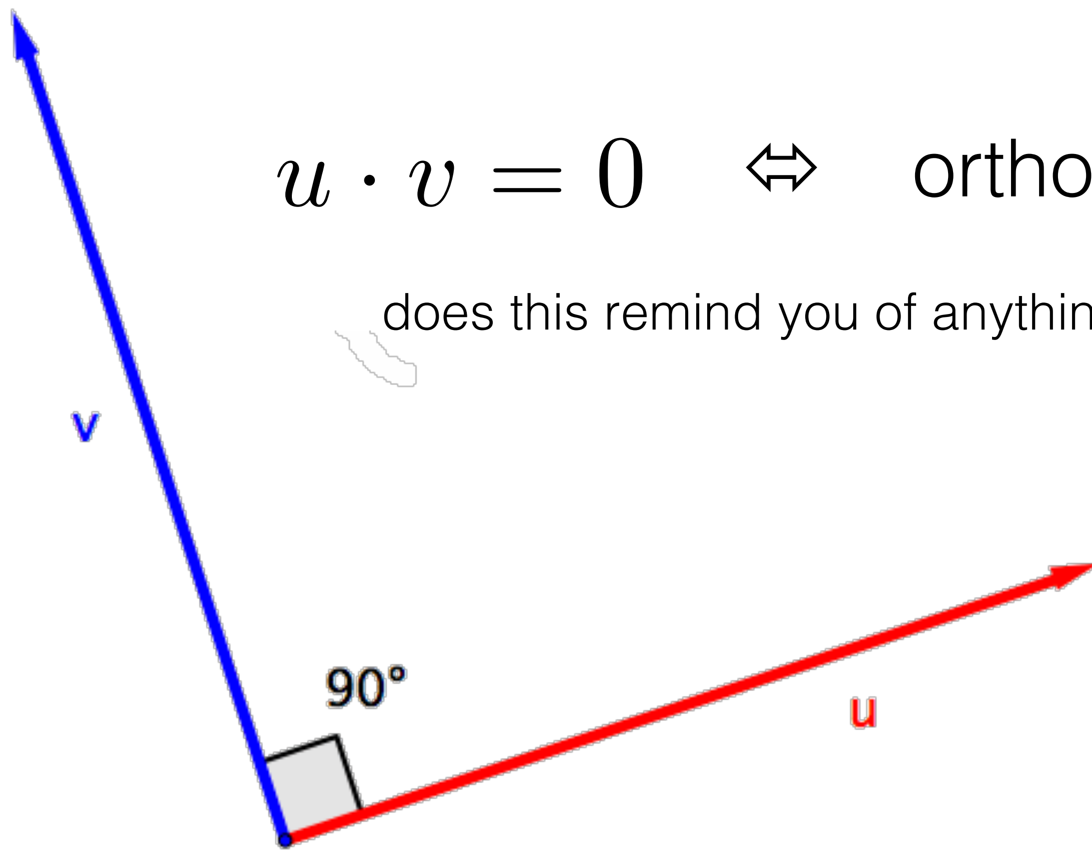
2) find an example of two linearly independent vectors that are not orthogonal

Orthogonality vs Linear Independence

aka when vectors are perpendicular

$$u \cdot v = 0 \quad \Leftrightarrow \quad \text{orthogonal}$$

does this remind you of anything?



Orthogonal matrix

when all the columns are orthogonal with norm 1

$$u \cdot v = 0 \qquad \text{and} \qquad ||u|| = ||v|| = 1$$

for all columns *u*, *v* in the matrix

Orthogonal matrix

when all the columns are orthogonal with norm 1

$$u \cdot v = 0 \qquad \text{and} \qquad \underbrace{||u|| = ||v|| = 1}$$

for all columns *u, v* in the matrix

this doesn't have anything to do with orthogonality; just historical naming weirdness

Orthogonal matrix

when all the columns are orthogonal with norm 1

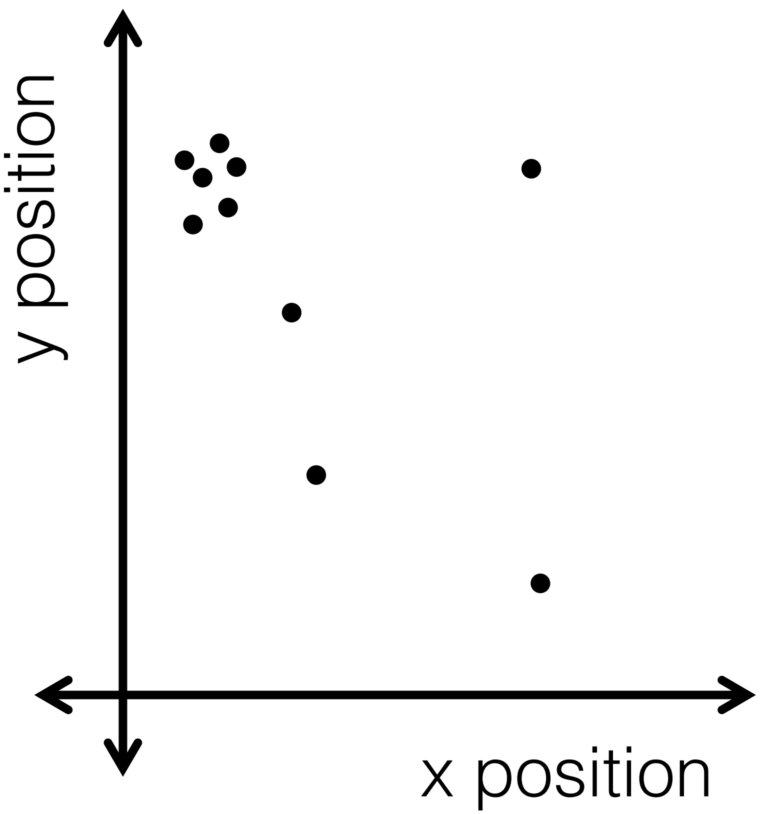
$$u \cdot v = 0 \qquad \text{and} \qquad ||u|| = ||v|| = 1$$

for all columns *u, v* in the matrix

[on board: is this matrix orthogonal?]

Big concept: `Basis`

a set of vectors that span the space



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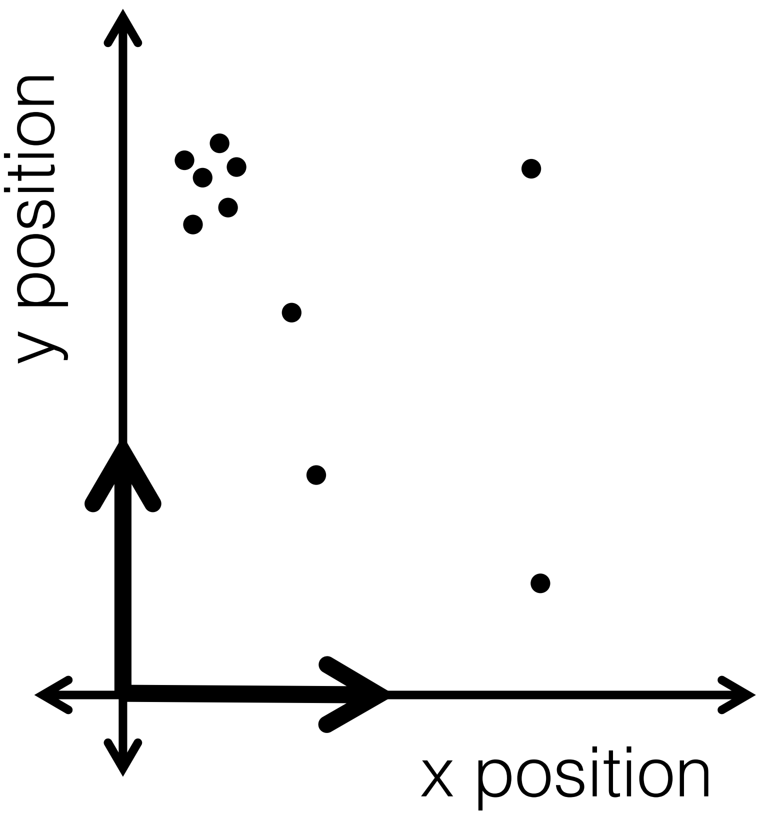

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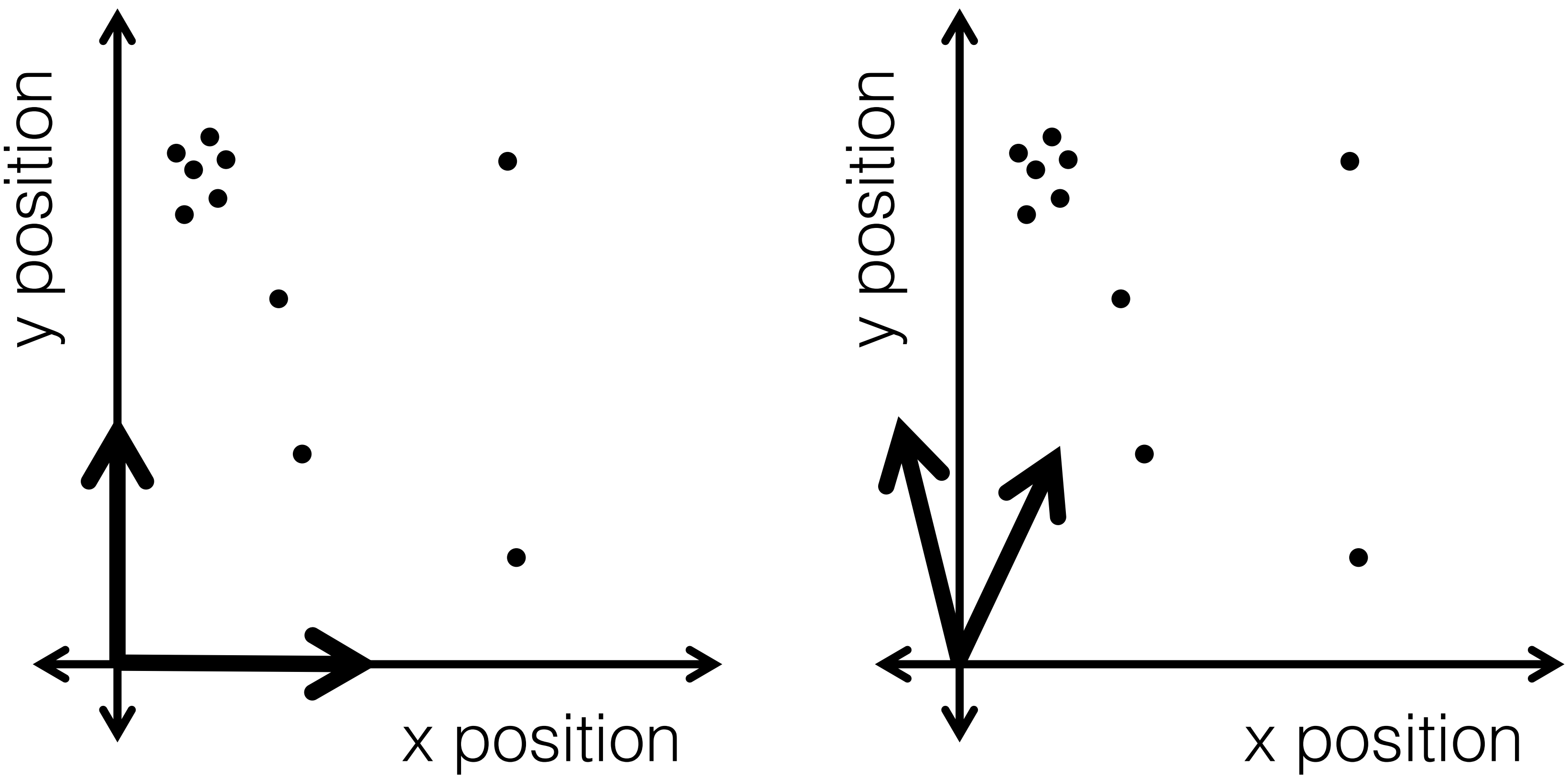
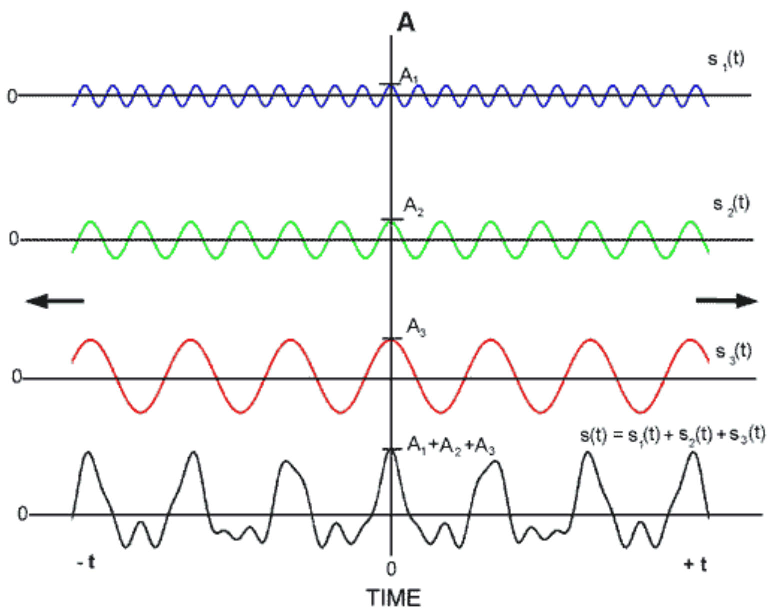


both of these are just as good of a basis set

Big concept: `Basis`

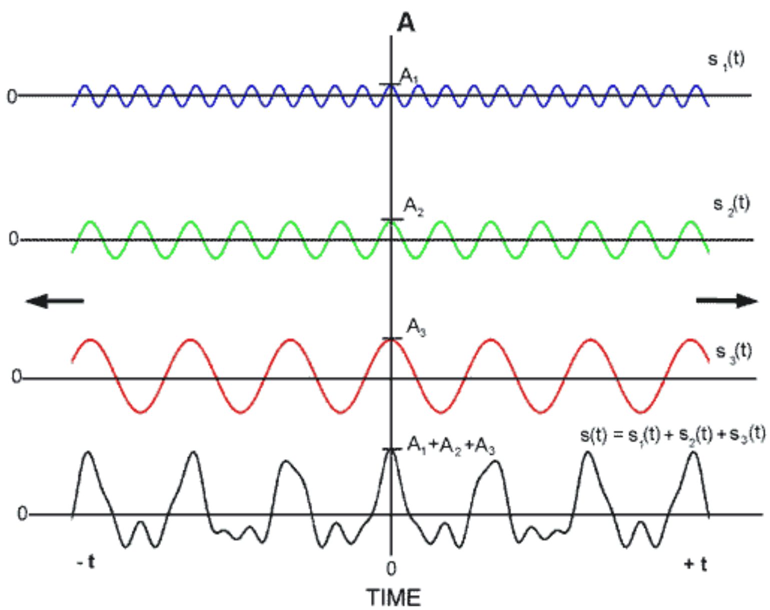
a set of vectors that span the space



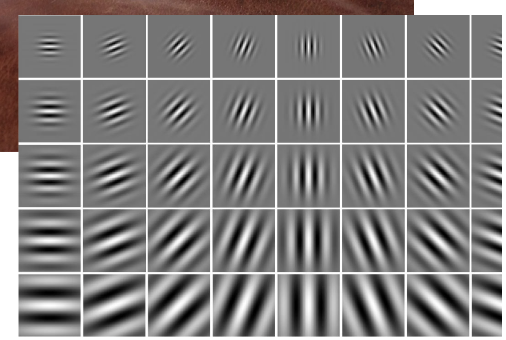
Fourier transforms

Big concept: `Basis`

a set of vectors that span the space



Frequencies are a basis for any function (Fourier transforms)



Gabor filters (V1-like simple cell receptive fields) are a basis for natural images