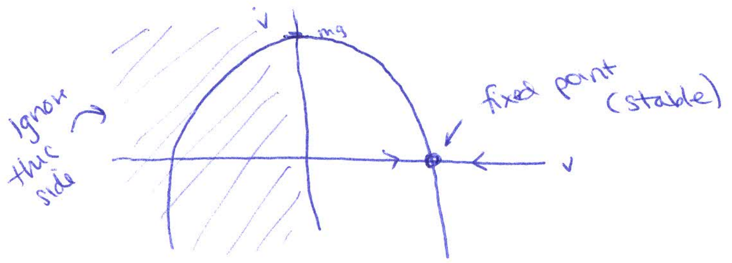
Problem 1 Solution

$$m\frac{dv}{dt} = mg - kv^2$$

Plot of $\dot{v}$ vs. v:



fixed point $\Rightarrow$

$$0 = mg - kv^2$$

$$kv^2 = mg$$

$$\boxed{v^* = \sqrt{\frac{mg}{k}}}$$ [AKA: terminal velocity]

stability:

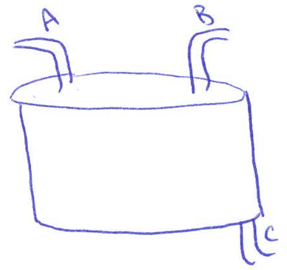
$$f(v) = mg - kv^2$$

$$f'(v) = -2kv$$

$$f'(v^*) = -2kv^* < 0 \Rightarrow \text{STABLE}$$

(just like the picture above says!)

Problem 2 Solution



A: 30 g/L at 2 L/min (60 g/min)
B: 45 g/L at 1 L/min (45 g/min)
C: leaves at 3 L/min $\left(\frac{S}{100} \cdot 3 \text{ g/min}\right)$

S is grams in tank
100 L in tank

A) $\frac{dS}{dt} = \underbrace{60}_{\text{enter through pipe A}} + \underbrace{45}_{\text{enter through pipe B}} - \underbrace{\frac{3S}{100}}_{\text{leaving through pipe C}}$

$$\frac{dS}{dt} = 105 - \frac{3S}{100}$$

B) Use separation of variables:

$$\int \frac{dS}{105 - \frac{3S}{100}} = \int dt$$

$$-\frac{100}{3} \ln\left(105 - \frac{3S}{100}\right) = t + C$$

$$\ln\left(105 - \frac{3S}{100}\right) = -\frac{3t}{100} + C'$$

$$105 - \frac{3S}{100} = A e^{-3t/100}$$

$$S(t) = \frac{100}{3}\left[105 - Ae^{-3t/100}\right]$$

$$S(0) = \frac{100}{3}\left[105 - A\right] = 0$$

$$A = 105$$

$$S(t) = \frac{100 \cdot 105}{3}\left[1 - e^{-3t/100}\right]$$

⇒ Sugar steadily increases to the steady-state concentration

Problem 3 Solution

$$\tau \frac{dV}{dt} = \alpha (V - V_{rest})(V - V_{crit}) + RI$$

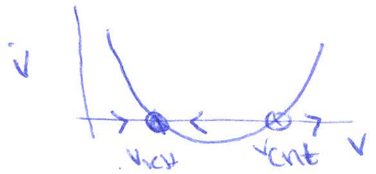
A) Assume $I = 0$.

If $V = V_{rest}$, then $\frac{dV}{dt} = 0 \Rightarrow$ steady-state (fixed point)

If $V = V_{crit}$, then $\frac{dV}{dt} = 0$

If $V_{rest} < V < V_{crit}$, $\frac{dV}{dt} < 0 \rightarrow$ decays to V_{rest} (V_{rest} is stable)

Qualitative picture:

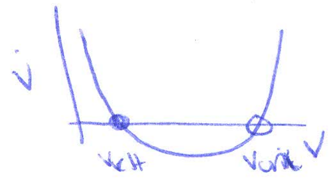


V_{rest} - stable fixed point

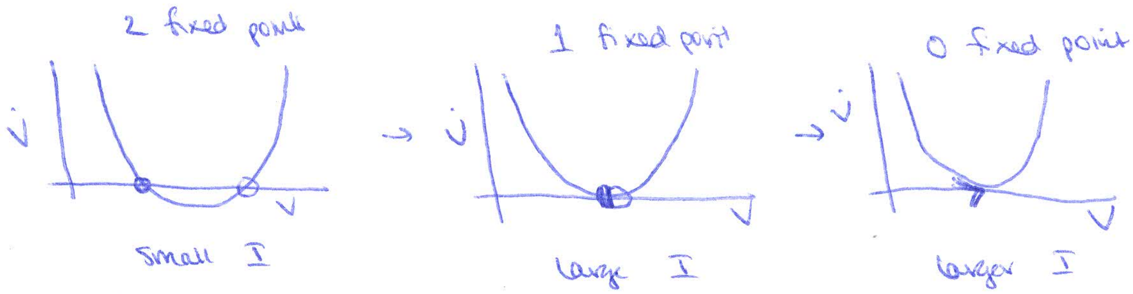
V_{crit} - unstable fixed point

$\Rightarrow$ 2 fixed points, unlike LIF model

B) Consider original portrait:



Increasing I will "lift" the parabola up:



$\rightarrow$ with I large, $\dot{V}$ is always positive $\Rightarrow$ voltage is always increasing $\Rightarrow$ consistent "spiking"

$\rightarrow$ the above change in # of f.p. is called a "blue-sky bifurcation" or "saddle-node bifurcation"

Problem 4 Solution

$$\dot{x} = x + y$$
$$\dot{y} = 4x - 2y$$

A) $z = \begin{bmatrix} x \\ y \end{bmatrix}$, $\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$

$\Rightarrow \quad \dot{z} = Az$

B) Finding the fixed point:

$$\left.\begin{aligned} 0 = x + y &\rightarrow x = -y \\ 0 = 4x - 2y &\Rightarrow y = 2x \end{aligned}\right\} \rightarrow x = 2x \Rightarrow \boxed{x = 0} \text{ and } \boxed{y = 0}$$

To find eigenvectors & eigenvalues, must solve this eq: $Av = \lambda v$

$\Rightarrow$ solve $\det(A - \lambda I) = 0$

$\Rightarrow$ $\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix}\right) = 0$

$$(1-\lambda)(-2-\lambda) - 4 = 0$$
$$-2 - \lambda + 2\lambda + \lambda^2 - 4 = 0$$
$$\lambda^2 + \lambda - 6 = 0$$
$$(\lambda + 3)(\lambda - 2) = 0$$
$$\boxed{\lambda_1 = -3, \quad \lambda_2 = 2}$$

To find eigenvector for each eigenvalue:

$$A\vec{v} = \lambda \vec{v}$$

For $\lambda_1 = -3$:

$$\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -3 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

By carrying out matrix algebra

$v_1 + v_2 = -3v_1$ $\leftarrow$ "top row" $4v_1 = -v_2$

$4v_1 - 2v_2 = -3v_2 \rightarrow 4v_1 = -v_2$

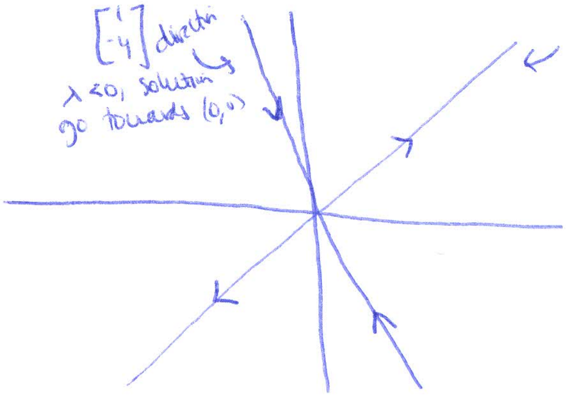
$$\Big\} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \begin{matrix} \leftarrow v_1 \\ \leftarrow v_2 \end{matrix}$$

For $\lambda = 2$:

$$\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

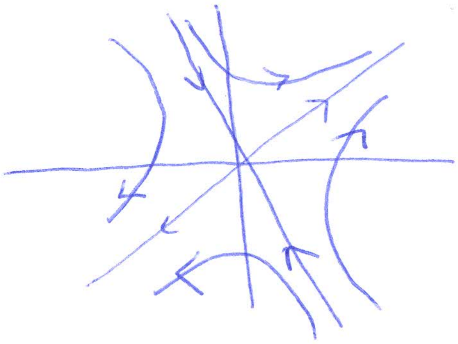
$$\left.\begin{array}{l} v_1 + v_2 = 2v_1 \;\rightarrow\; v_1 = v_2 \\ 4v_1 - 2v_2 = 2v_2 \;\rightarrow\; 4v_1 = 4v_2 \end{array}\right\} = \begin{array}{c} \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}\begin{array}{l} \leftarrow v_1 \\ v_2 \end{array}$$

c)



← $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ direction: positive eigenvalue
⇒ solutions go AWAY from $(0,0)$

D



← "saddle"

(towards origin in one direction, away from origin in another)