In class, I said that symmetric matrices (with $n$ linearly independent eigenvectors, if the matrix is $n \times n$) can be decomposed in the following way:

symmetric matrix $\rightarrow$ $$A = EDE^T$$ $\leftarrow$ diagonal matrix of eigenvalues

$\uparrow$ matrix of eigenvectors

To show this clearly:

First, remember that $Av_i = \lambda_i v_i$ for $i = 1, \ldots, n$ (all eigenvectors)

This actually gives us $n$ equations that we can re-write in matrix-vector notation:

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{bmatrix}$$

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To convince yourself of this, think about how this is equal to:

$$\begin{bmatrix} Av_1 & Av_2 & \cdots & Av_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix}$$

and how this is just a compact form of our original statement ($Av_i = \lambda_i v_i$ for $i = 1, \ldots, n$).

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Now, if $E = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, then

$$AE = ED$$

$$\Rightarrow AEE^T = EDE^T \quad \rightarrow EE^T = I \text{ (see hmwk problem)}$$

$$\Rightarrow A = EDE^T.$$