Convergence rate of $X^N - X$ for McKean equations

Luis Mario Chaparro Jaquez

April 8, 2022

Todo list

1 What is this?

An adaptation of [1, Proposition 3.1] for the case of McKean SDEs proposed in [3].

The proof builds on a number of results presented in the sections below.

2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

Definition 1. *For any real-valued continuous semi-martingale, the local time at zero $L^0_t(\bar{Y})$ is defined as*

$$L^0_t(\bar{Y}) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t \mathbb{1}_{\{|\bar{Y}|\le\epsilon\}} d\langle \bar{Y}\rangle_s,\ \mathbb{P}\text{-}a.s. \tag{1}$$

For all $t \geq 0$.

The first result, [1, Lemma 5.1], is not necessary to prove for this particular setting since the result holds for any semi-martingale, I include it here for self-containment reasons.

Lemma 1. *[Lemma 5.1] For any $\epsilon \in (0,1)$ and any real-valued, continuous semi-martingale*

Type down this lemma

Let us introduce the original and regularised Kolmogorov equations.

Definition 2 (Kolmogorov equations)**.** *For $\beta \in (0, 1/2)$ let $b \in C_T\mathcal{C}^{-\beta}$, $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$, and $b^N \to b$ as $N \to \infty$ in $C_T\mathcal{C}^{-\beta}$ The equations*

$$\begin{cases} \partial u_i + \frac{1}{2} b_i \Delta u_i = \lambda u_i - b_i \\ u_i(T) = 0, \end{cases} \tag{2}$$

$$\begin{cases}\partial u_i^N + \frac{1}{2} b_i^N \Delta u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0.\end{cases} \tag{3}$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

3 Bounds for $\|u - u^N\|_{L_\infty}$ and $\|\nabla u - \nabla u^N\|_{L_\infty}$

We need a bound for $u - u^N$ and $\nabla u - \nabla u^N$ in L_∞ for the case in which $u \in C_T\mathcal{C}^{1+\alpha}$ for some $\alpha > \beta$ which is an adaptation of [1, Lemma 5.2].

The result builds on top of the following result:

Proposition 1 (Bound for the ρ-equivalent norm of $u - u^N$)**.** *Let u, u^N be (mild) solutions to the Kolmogorov equations from Definition 2 then as $N \to \infty$*

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}} (\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda)} \tag{4}$$

for $\rho \geq \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_T\infty+\alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \tag{5}$$

and $\lambda > 0$.

Proof. See that $u^N(T) = u(T) = 0$, and in [2], set g^N, g as b^N, b respectively. See that $b^N \to b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$.

As u^N, u are mild solutions, we have

$$\begin{aligned}
u_i(t) - u_i^N(t) &= P_{T-t}(u_i(T) - u_i^N(T)) \\
&+ \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\
&- \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N) ds \\
&= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)] ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds
\end{aligned}$$

$$
\begin{aligned}
&- \int_t^T P_{s-t}(b_i - b_i^N)ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N)ds \\
&+ \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N)ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N)ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N)ds
\end{aligned}
$$

Now let us compute the ρ-equivalent norm of $u - u^N$, for some $\alpha > \beta$

this norm is wrong, should be $1+\alpha$ on the lhs and everywhere else

$$
\begin{aligned}
\|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} &= \sup_{0\le t\le T} e^{-\rho(T-t)}\|u(t) - u^N(t)\|_{1+\alpha} \\
&\le \sup_{0\le t\le T} e^{-\rho(T-t)}\|\int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N)ds\|_{1+\alpha} \\
&+ \sup_{0\le t\le T} e^{-\rho(T-t)}\|\int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N)ds\|_{1+\alpha} \\
&- \sup_{0\le t\le T} e^{-\rho(T-t)}\|\lambda\int_t^T P_{s-t}(u_i - u_i^N)ds\|_{1+\alpha} \\
&- \sup_{0\le t\le T} e^{-\rho(T-t)}\|\int_t^T P_{s-t}(b_i - b_i^N)ds\|_{1+\alpha}.
\end{aligned}
$$

Let us take each term from the right hand side of the inequality and bound them.

For the first term, using $\gamma + 2\theta = 1+\alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \le ct^{-\theta}\|f\|_{\gamma}\|$ and $\|\nabla g\|_{\xi} \le c\|g\|_{\xi+1}$

$$
\begin{aligned}
\sup_{0\le t\le T} e^{-\rho(T-t)}\|\int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N)ds\|_{1+\alpha} &\le \sup_{0\le t\le T} e^{-\rho(T-t)}\int_t^T (s-t)^{-\theta}\|\nabla u_i\|_{\alpha}\|b_i - b_i^N\|_{-\beta} \\
&\le c\|u_i\|_{C_T\mathcal{C}_{1+\alpha}}\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\le t\le T} e^{-\rho(T-t)}(T-t)^{\frac{1-\beta-\alpha}{2}} \\
&\le cT^{\frac{1-\beta-\alpha}{2}}\|u_i\|_{C_T\mathcal{C}_{1+\alpha}}\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}}
\end{aligned}
$$

For the second term, see that for $N \to \infty$, we have $\|b^N\|_{C_T\mathcal{C}^{-\beta}} \le 2\|b\|_{C_T\mathcal{C}^{-\beta}}$

$$
\begin{aligned}
\sup_{0\le t\le T} e^{-\rho(T-t)}\|\int_t^T P_{s-t}b_i^N(\nabla u_i - \nabla u_i^N)ds\|_{1+\alpha} &\le c\sup_{0\le t\le T}\int_t^T (s-t)^{-\theta}e^{-\rho(T-t)}2\|b_i\|_{-\beta}\|\nabla u_i - \nabla u_i^N\|_{\alpha}ds \\
&\le c\|b_i\|_{C_T\mathcal{C}^{-\beta}}\|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)}\int_t^T (s-t)^{-\theta}e^{-\rho(T-t)}ds \\
&\le c\|b_i\|_{C_T\mathcal{C}^{-\beta}}\|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)}\rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}
$$

For the third term, which is the one that difers from the proof in [2] we need to use that $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta}\|f\|_\gamma$, and in this case we have $\gamma + 2\theta = 1+\alpha$ and $\gamma = 1+\alpha$, so that $\theta = 0$ because $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, so we will have

$$\begin{aligned}
\sup_{0\leq t\leq T} e^{-\rho(T-t)}\|\lambda \int_t^T P_{s-t}(u_i - u_i^N)ds\|_{1+\alpha} &\leq c\lambda \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-0}\|u_i - u_i^N\|_{1+\alpha}ds \\
&= c\lambda \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T e^{-\rho(T-s)} \sup_{0\leq s\leq T} e^{-\rho(T-s)}\|u_i - u_i^N\|_{1+\alpha}ds \\
&= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \int_t^T e^{-\rho(T-s)}e^{-\rho(T-t)}ds \\
&= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \int_t^T e^{-\rho(s-t)}ds \\
&= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \sup_{0\leq t\leq T} \rho^{-1}[1 - e^{-\rho(T-t)}] \\
&\leq c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}}\rho^{-1} \\
&\leq c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}}\rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}$$

And for the last term

$$\begin{aligned}
\sup_{0\leq t\leq T} e^{-\rho(T-t)}\| \int_t^T P_{T-s}(b_i - b_i^N)ds\|_{1+\alpha} &\leq c \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}}\|b_i - b_i^N\|_{-\beta}ds \\
&\leq c\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\leq t\leq T} e^{-\rho(T-t)}(s-t)^{-\frac{\alpha+\beta-1}{2}} \\
&\leq cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}
\end{aligned}$$

Putting everything together

$$\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} &\leq cT^{\frac{1-\beta-\alpha}{2}}\|u_i\|_{C_T\mathcal{C}^{1+\alpha}}\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \\
&\quad + c\|b_i\|_{C_T\mathcal{C}^{-\beta}}\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{\frac{\alpha+\beta-1}{2}} \\
&\quad - c\lambda \sup_{0\leq t\leq T}\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}}\rho^{\frac{\alpha+\beta-1}{2}} \\
&\quad - cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \\
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda]) &\leq cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1) \\
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} &\leq \frac{cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}
\end{aligned}$$

As required. $\square$

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{6}$$

Check this norm

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq c(\rho)\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{7}$$

Proposition 2 (Bounds for $\|u - u^N\|_{L_\infty}$ and $\|\nabla u - \nabla u^N\|_{L_\infty}$)**.** *Let $\beta \in (0, 1/2)$ and $b \in C_T\mathcal{C}^{-\beta}$. Let $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ be (mild) solutions to the Kolmogorov equations from Definition 2*

Assume that for some $\alpha > \beta$

$$\|u - u^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq c(\rho)\|b - b^N\|_{C_T\mathcal{C}^{-\beta}}. \tag{8}$$

With $c(\rho)$ as in Proposition 1 and ρ_0 is large enough such that $c(\rho) > 0$ for all $\rho > \rho_0$. Then for all $t \in [0, T]$

$$\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{9}$$

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{10}$$

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$.

4 Lemma 5.3

5 Bound for the local time at zero of $Y^N - Y$

We need a bound for $\mathbb{E}[L_T^0(Y^N - Y)]$, for Sobolev spaces, this is result [1, Proposition 5.4] we present it here for the solutions to the SDE blonging to the appropriate Besov spaces.

Proposition 3. *Let $b \in C_T\mathcal{C}^{-\beta}$ and $b^N \to b$ in $C_T\mathcal{C}^{-\beta}$ as $N \to \infty$ for $\beta \in (0, \frac{1}{4})$ and for any $\alpha > \beta$*

Proof. Recall that Y^N, Y are solutions to the SDEs

$$Y_t = y_0 + \lambda\int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1)dW_s \tag{11}$$

and

$$Y_t^N = y_0^N + \lambda\int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1)dW_s \tag{12}$$

so that the difference $Y^N - Y$ is

$$Y^N - Y_t = (y_0^N + \lambda\int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1)dW_s) - (y_0 + \lambda\int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \tag{13}$$

□

6 Proposition 3.1 (main result)

References

[1] Tiziano De Angelis, Maximilien Germain, and Elena Issoglio. “A Numerical Scheme for Stochastic Differential Equations with Distributional Drift”. In: *arXiv:1906.11026 [cs, math]* (Oct. 22, 2020). arXiv: 1906.11026.

[2] Elena Issoglio and Francesco Russo. “A PDE with Drift of Negative Besov Index and Related Martingale Problem”. In: (), p. 48.

[3] Elena Issoglio and Francesco Russo. “McKean SDEs with singular coefficients”. In: *arXiv:2107.14453 [math]* (July 30, 2021). arXiv: 2107.14453.