

Convergence rate of numerical solutions to SDEs with distributional drifts in Besov spaces

Luis Mario Chaparro Jaquez

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1 What is this?

An adaptation of ^{de angelis numerical_2020} [1, Proposition 3.1] for the case of SDEs with drift in a Besov space of negative order similar to the ones proposed in ^{issoglio and pleschke 2021} [2] and [3]. The proof builds on a number of results presented in the sections below.

El: add result about convergence of the scheme. This is done in two parts, $X^N \rightarrow X$ done in Russo Issoglio, and $X^{N,m} \rightarrow X^N$ Euler scheme convergence from De Angelis Germain Issoglio. Attention that the rate of convergence of Euler scheme depends of the smoothness of b^N .

2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

Definition 1. *Local time at zero* For any real-valued continuous semi-martingale Z , the local time at zero $L_t^0(\bar{Y})$ is defined as

$$L_t^0(Z) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|Z| \leq \epsilon\}} d\langle Z \rangle_s, \mathbb{P}\text{-a.s.} \quad (1)$$

For all $t \geq 0$.

The first result, [\[I, Lemma 5.1\]](#), is not necessary to prove for this particular setting since the result holds for any semi-martingale, I include it here for self-containment reasons. [EI: Instead of this sentence you should write something like 'The lemma below is from \[I\] and its proof can be found in \[I, Lemma 5.1\]. We include the statement here for ease of reading'](#)

Lemma 1. *Bound for local time at zero for a semi-martingale For any $\epsilon \in (0, 1)$ and any real-valued, continuous semi-martingale Z we have*

$$\begin{aligned} \mathbb{E}[L_t^0(Z_s)] &\leq 4\epsilon - 2\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Z_s \in (0, \epsilon)\}} + \mathbb{1}_{\{Z_s \geq \epsilon\}} e^{1-Z_s/\epsilon}\right) dZ_s\right] \\ &\quad + \frac{1}{\epsilon}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z_s > \epsilon\}} e^{1-Z_s/\epsilon} d\langle Z \rangle_s\right]. \end{aligned}$$

Let us introduce the original and regularised Kolmogorov equations. To shorten notation we will denote the spaces $C_T C^\gamma(\mathbb{R})$ as $C_T C^\gamma$.

Definition 2. *Kolmogorov equations For $\beta \in (0, 1/2)$ let $b \in C_T C^{-\beta}$, $u, u^N \in C_T C^{(1+\beta)+}$, and $b^N \rightarrow b$ as $N \rightarrow \infty$ in $C_T C^{-\beta}$ The equations*

$$\begin{cases} \partial_t u_i + \frac{1}{2} \Delta u_i + b_i \nabla u_i = \lambda u_i - b_i \\ u_i(T) = 0, \end{cases} \quad (2) \quad \{\text{eq:kolmogorov}\}$$

$$\begin{cases} \partial_t u_i^N + \frac{1}{2} \Delta u_i^N + b_i^N \nabla u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0. \end{cases} \quad (3) \quad \{\text{eq:kolmogorov_N}\}$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

Lemma 2. *Let u, u^N be the solutions to the Kolmogorov equations [\(2\)](#) [\(3\)](#) in $C_T C^{1+\alpha}$ respectively. We have*

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u(t,x)| \leq \frac{1}{2} \text{ and } \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u^N(t,x)| \leq \frac{1}{2} \quad (4)$$

Assumption 1. *Let $0 < \beta < 1/2$ and $b \in C_T C^{-\beta}$.*

Assumption 2. *There exists a sequence $(b^N)_N \in C_T C^{-\beta}$ such that for each N , $b^N(t, \cdot) \in C_b^\infty(\mathbb{R})$ for all $t \in [0, T]$ and such that $b^N \rightarrow b$ as $N \rightarrow \infty$.*

LM: the lemma is probably better to type it before the first time it is used since it requires of some results below

LM: TYPE THIS

3 Bounds for the difference of solutions to the Kolmogorov equations

We need a bound for $u - u^N$ and $\nabla u - \nabla u^N$ in L_∞ for the case in which $u \in C_T C^{1+\alpha}$ for some $\alpha > \beta$ which is an adaptation of [\[I, Lemma 5.2\]](#).

The result builds on top of the following result:

prop:diff_u_uN

Proposition 1. *Bound for the ρ -equivalent norm of $u - u^N$*

Let u, u^N be (mild) solutions to the Kolmogorov equations from Definition [2](#) then as [def:kolmogorov_eqns](#)
 $N \rightarrow \infty$

$$\|u_i - u_i^N\|_{C_T C^{1+\alpha}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T C^{-\beta}} (\|u_i\|_{C_T C^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}} (\|b\|_{C_T C^{-\beta}} + \lambda)} \quad (5)$$

for $\rho \geq \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_T \infty + \alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \quad (6)$$

and $\lambda > 0$.

Proof. See that $u^N(T) = u(T) = 0$, and in [\[2\]](#), set \tilde{g}^N, \tilde{g} as b^N, b respectively. See that $b^N \rightarrow b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$.

As u^N, u are mild solutions, we have

$$\begin{aligned} u_i(t) - u_i^N(t) &= P_{T-t}(u_i(T) - u_i^N(T)) \\ &\quad + \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\ &\quad - \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds \\ &= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\ &\quad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\ &\quad - \int_t^T P_{s-t}(b_i - b_i^N) ds \\ &= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)] ds \\ &\quad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\ &\quad - \int_t^T P_{s-t}(b_i - b_i^N) ds \\ &= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \\ &\quad + \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \\ &\quad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\ &\quad - \int_t^T P_{s-t}(b_i - b_i^N) ds \end{aligned}$$

Now let us compute the ρ -equivalent norm of $u - u^N$, for some $\alpha > \beta$

!!!!!!!!!!!!!!!!!!!!!!LM: this norm is wrong, should be $1 + \alpha$ on the lhs and everywhere else
!!!!!!!!!!!!!!!!!!!!!!

$$\begin{aligned}
\|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} &= \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|u(t) - u^N(t)\|_{C_T \mathcal{C}^{1+\alpha}} \\
&\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} (\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\
&+ \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} (\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\
&- \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t} (u_i - u_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\
&- \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} (b_i - b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}}.
\end{aligned}$$

Let us take each term from the right hand side of the inequality and bound them.

For the first term, using $\gamma + 2\theta = 1 + \alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta} \|f\|_{\gamma}$ and $\|\nabla g\|_{\xi} \leq c\|g\|_{\xi+1}$

$$\begin{aligned}
&\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} (\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\
&\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_{C_T \mathcal{C}^{\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} ds \\
&\leq c \|u_i\|_{C_T \mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \sup_{0 \leq t \leq T} e^{-\rho(T-t)} (T-t)^{\frac{1-\beta-\alpha}{2}} \\
&\leq c T^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T \mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}}
\end{aligned}$$

For the second term, see that for $N \rightarrow \infty$, we have $\|b^N\|_{C_T \mathcal{C}^{-\beta}} \leq 2\|b\|_{C_T \mathcal{C}^{-\beta}}$

$$\begin{aligned}
&\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} b_i^N (\nabla u_i - \nabla u_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\
&\leq c \sup_{0 \leq t \leq T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2 \|b_i\|_{C_T \mathcal{C}^{-\beta}} \|\nabla u_i - \nabla u_i^N\|_{C_T \mathcal{C}^{1+\alpha}} ds \\
&\leq c \|b_i\|_{C_T \mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} ds \\
&\leq c \|b_i\|_{C_T \mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}$$

For the third term, which is the one that differs from the proof in [\[2\]](#) ^{issoglio_pde_nodate} we need to use that $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta} \|f\|_{\gamma}$, and in this case we have $\gamma + 2\theta = 1 + \alpha$ and $\gamma = 1 + \alpha$, so that $\theta = 0$ because $u, u^N \in C_T \mathcal{C}^{1+\alpha}$, so we will have

$$\begin{aligned}
& \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{1+\alpha} \\
& \leq c\lambda \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-0} \|u_i - u_i^N\|_{1+\alpha} ds \\
& = c\lambda \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T e^{-\rho(T-s)} \sup_{0 \leq s \leq T} e^{-\rho(T-s)} \|u_i - u_i^N\|_{1+\alpha} ds \\
& = c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(T-s)} e^{-\rho(T-t)} ds \\
& = c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(s-t)} ds \\
& = c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \sup_{0 \leq t \leq T} \rho^{-1} [1 - e^{-\rho(T-t)}] \\
& \leq c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{-1} \\
& \leq c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}$$

And for the last term

$$\begin{aligned}
& \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{T-s}(b_i - b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\
& \leq c \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} ds \\
& \leq c \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \sup_{0 \leq t \leq T} e^{-\rho(T-t)} (s-t)^{-\frac{\alpha+\beta-1}{2}} \\
& \leq cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}}
\end{aligned}$$

Putting everything together

$$\begin{aligned}
\|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} & \leq cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T \mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \\
& + c \|b_i\|_{C_T \mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}} \\
& - c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}} \\
& - cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}},
\end{aligned}$$

and finally,

$$\|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} (1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C_T \mathcal{C}^{-\beta}} + \lambda])$$

$$\begin{aligned} &\leq cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T \mathcal{C}^{-\beta}} (\|u_i\|_{C^T \mathcal{C}^{1+\alpha}} - 1) \\ \|u_i - u_i^N\|_{C^T \mathcal{C}^{-\beta}}^{(\rho)} &\leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T \mathcal{C}^{-\beta}} (\|u_i\|_{C^T \mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C^T \mathcal{C}^{-\beta}} + \lambda])} \end{aligned}$$

As required. \square

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|_{C^T \mathcal{C}^{-\beta}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} (\|u_i\|_{C^T \mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C^T \mathcal{C}^{-\beta}} + \lambda])} \|b_i - b_i^N\|_{C^T \mathcal{C}^{-\beta}} \quad (7)$$

LM: Check this norm

$$\|u_i - u_i^N\|_{C^T \mathcal{C}^{-\beta}}^{(\rho)} \leq c(\rho) \|b_i - b_i^N\|_{C^T \mathcal{C}^{-\beta}} \quad (8)$$

Here is the adaptation of [de angelis numerical_2020](#) [[1](#), Lemma 5.2].

Proposition 2. *Bounds for $\|u - u^N\|_{L^\infty}$ and $\|\nabla u - \nabla u^N\|_{L^\infty}$*
Let $\beta \in (0, 1/2)$ and $b \in C^T \mathcal{C}^{-\beta}$. Let $u, u^N \in C^T \mathcal{C}^{(1+\beta)+}$ be (mild) solutions to the Kolmogorov equations from [Definition 2](#). [def:kolmogorov_eqns](#)
Assume, by [Proposition 1](#), that for some $\alpha > \beta$ [prop:diff_u_uN](#)

$$\|u - u^N\|_{C^T \mathcal{C}^{1+\alpha}}^{(\rho)} \leq c(\rho) \|b - b^N\|_{C^T \mathcal{C}^{-\beta}}. \quad (9)$$

With $c(\rho)$ as in [Proposition 1](#) and ρ_0 is large enough such that $c(\rho) > 0$ for all $\rho > \rho_0$. Then for all $t \in [0, T]$

$$\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C^T \mathcal{C}^{-\beta}} \quad (10)$$

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C^T \mathcal{C}^{-\beta}} \quad (11)$$

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$.

Proof. First let us prove [eq:uNu_bounded_by_bNb](#) ([10](#)).

Let $t \in [0, T]$, and see that since $u, u^N \in C^T \mathcal{C}^{(1+\beta)+}$ there exists $\alpha > \beta$ such that $u, u^N \in C^T \mathcal{C}^{1+\alpha}$, then for any $f \in \mathcal{C}^{1+\alpha}$ we have

$$\|f\|_{C^{1+\alpha}} \leq c \left(\sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \neq y \in \mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \right) \quad (12)$$

so we have

$$\begin{aligned} \|u^N(t) - u(t)\|_{L^\infty} &= \sup_{x \in \mathbb{R}^d} |u^N(t, x) - u(t, x)| \\ &\leq c \|u^N(t) - u(t)\|_{C^{\alpha+1}} \end{aligned} \quad (13)$$

Moreover, using the (ρ) -equivalent norm

$$\|f\|_{C^{1+\alpha}} = \sup_{t \in [0, T]} e^{-\rho(T-t)} \|f(t)\|_{C^{1+\alpha}}, \quad (14)$$

and [\(9\)](#) we see that

$$\begin{aligned}
\|u^N - u\|_{C_T C^{1+\alpha}} &= \sup_{t \in [0, T]} \|u^N - u\|_{C^{1+\alpha}} \\
&= \sup_{t \in [0, T]} e^{\rho(T-t)} e^{-\rho(T-t)} \|u^N - u\|_{C^{1+\alpha}} \\
&\leq e^{\rho T} \sup_{t \in [0, T]} e^{-\rho(T-t)} \|u^N - u\|_{C^{1+\alpha}} \\
&= e^{\rho T} \|u^N - u\|_{C_T C^{1+\alpha}}^{(\rho)}.
\end{aligned} \tag{15}$$

Plugging [\(15\)](#) into [\(13\)](#)

$$\begin{aligned}
\|u^N(t) - u(t)\|_{L^\infty} &\leq c \|u^N(t) - u(t)\|_{C^{\alpha+1}} \\
&\leq \sup_{t \in [0, T]} c \|u^N(t) - u(t)\|_{C^{\alpha+1}} \\
&= c \|u^N - u\|_{C_T C^{\alpha+1}} \\
&\leq c e^{\rho T} \|u^N - u\|_{C_T C^{\alpha+1}}^{(\rho)}.
\end{aligned} \tag{16}$$

And finally by [\(9\)](#)

$$\|u^N(t) - u(t)\|_{L^\infty} \leq c \cdot c(\rho) \cdot e^{\rho T} \|b^N - b\|_{C_T C^{-\beta}} \tag{17}$$

which proves [\(17\)](#).

For [\(11\)](#) recall that if $f \in C^{1+\alpha}$ then $\nabla f \in C^\alpha$. Also, by Bernstein inequality [\[3, Eqn. \(9\)\]](#)

$$\|\nabla f\|_\alpha \leq c \|f\|_{\infty+\alpha}. \tag{18}$$

Using the equivalent norm

$$\|f\|_{C^{1+\alpha}} \leq c \left(\sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \in \mathbb{R}^d} |\nabla f(x)| + \sup_{x \neq y \in \mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \right) \tag{19}$$

we can see that

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq c \|u^N(t) - u(t)\|_{C^{1+\alpha}}. \tag{20}$$

And usign the same bounds that we used above for $c \|u^N(t) - u(t)\|_{C^{1+\alpha}}$ this point follows. \square

4 Bound for the difference of the auxiliary functions

This is the adaptation of result [\[4, Lemma 5.3\]](#).

`prop:bound_psi-psiN`

Proposition 3. Take $\rho > \rho_0$ as in Proposition [1](#), $N \rightarrow \infty$, κ_ρ from Proposition [2](#), and $\beta \in (0, 1/2)$, then we have

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} \|\psi(t,x) - \psi^N(t,x)\| \leq 2\kappa_\rho \|b - b^N\|_{C_T C^{-\beta}} \quad (21)$$

Proof. Recall the definition of $\psi, \phi \in C_T C^1$

$$\phi(t,x) := x + u(t,x) \quad (22)$$

$$\psi(t,\cdot) = \phi^{-1}(t,\cdot). \quad (23)$$

Note that

$$u(y) = \int_0^1 \nabla u(\alpha y) y d\alpha + u(0). \quad (24)$$

From there we have

$$u(t,y) - u(t,y') = \int_0^1 \nabla u(t, \alpha(y - y'))(y - y') d\alpha \quad (25)$$

and therefore

$$\|u(t,y) - u(t,y')\| \geq \left(\int_0^1 \|\nabla u(t, \alpha(y - y'))\|^2 d\alpha \right)^{1/2} \|y - y'\|, \quad (26)$$

and by Lemma [2](#) `lemma:bounds_gradients` we finally have

$$\begin{aligned} \|u(t,y) - u(t,y')\| &\leq \left(\frac{1}{4} \int_0^1 d\alpha \right)^{1/2} \|y - y'\|' \\ \|u(t,y) - u(t,y')\|^2 &\leq \frac{1}{4} \|y - y'\|^2 \end{aligned} \quad (27)$$

LM: continue from page three in notes

□

5 Bound for the local time at zero of the solution to the SDEs

LM: Here I still need to mention how we define $Y_t = \psi(t, X_t)$, because eventually I need to use that $X_t = \psi(t, Y_t)$, probably just need to mention without defining the whole Y_t as in the paper

We need a bound for $\mathbb{E}[L_T^0(Y^N - Y)]$, for Sobolev spaces, this is result [\[1, Proposition 5.4\]](#) `de_angelis_numerical_2020` we present it here for the solutions to the SDE belonging to the appropriate Besov spaces.

First let us state the following useful result.

LM: check that the statement makes sense and has all the necessary assumptions

-n_bound_for_integral

Lemma 3. Let u, u^N be solutions to the Kolmogorov equations ~~(2)~~ ~~(3)~~ ^{eq:kolmogorov_N} then the following bound is satisfied:

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \leq 2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + \|Y_s^N - Y^N\|. \quad (28)$$

{eq:bound_u_abs}

Proof. Adding and subtracting terms, using triangle inequality and noting that for any a, b , we have $a - b \leq \|a - b\|$, then

$$\begin{aligned} \|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| &\leq \|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N))\| \\ &\quad + \|u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N))\| \\ &\quad + \|u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s))\|. \end{aligned} \quad (29)$$

The terms in the right hand side will be bounded as follows:

- For the first term, by Proposition ~~2~~ ^{prop:diff_uN_graduN}

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N))\| \leq \|u^N(s) - u(s)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C_T C^{-\beta}}, \quad (30)$$

- for the second term, observe that u, u^N are $\frac{1}{2}$ -Lipschitz and by Proposition ~~3~~ ^{prop:bound_psi_psiN} we get

$$\begin{aligned} \|u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N))\| &\leq \frac{1}{2} \|\psi^N(s, Y_s^N) - \psi(s, Y_s^N)\| \\ &\leq \kappa_\rho \|b^N - b\|_{C_T C^{-\beta}}, \end{aligned} \quad (31)$$

- and for the final term, note that ψ, ψ^N are 2-Lipschitz so that

$$\|u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \leq \frac{1}{2} \|\psi(s, Y_s^N) - \psi(s, Y_s)\| \leq \|Y_s^N - Y_s\|. \quad (32)$$

So that the following bound holds

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \leq 2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + \|Y_s^N - Y^N\|, \quad (33)$$

as required. \square

:bound_local_time_sde

Proposition 4. Let A, B be constants, $b \in C_T C^{-\beta}$ and $b^N \rightarrow b$ in $C_T C^{-\beta}$ as $N \rightarrow \infty$ for $\beta \in (0, \frac{1}{4})$ and for any $\alpha > \beta$

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq o\left(\|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1}\right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + B\|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1}. \quad (34)$$

{eq:local_time_YNY_bo}

Proof. Recall that Y^N, Y are solutions to the SDEs

$$Y_t = y_0 + \lambda \int_0^t u(s, \psi(s, Y_s)) ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1) dW_s \quad (35)$$

and

$$Y_t^N = y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N)) ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1) dW_s \quad (36)$$

so that the difference $Y^N - Y$ is

$$\begin{aligned} Y_t^N - Y_t &= \left(y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N)) ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1) dW_s \right) \\ &\quad - \left(y_0 + \lambda \int_0^t u(s, \psi(s, Y_s)) ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1) dW_s \right) \\ &= (y_0^N - y_0) + \lambda \int_0^t (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))) ds \\ &\quad + \int_0^t (\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))) dW_s, \end{aligned} \quad (37)$$

and using Lemma [lemma:local-time-at-0](#) we have the following bound

$$\begin{aligned} &\mathbb{E}[L_t^0(Y^N - Y)] \\ &\leq 4\epsilon \\ &\quad - 2\lambda \mathbb{E} \left[\int_0^t \left(\mathbb{1}_{\{Y_s^N - Y_s \in (0, \epsilon)\}} + \mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \right) \right. \\ &\quad \left. (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))) ds \right] \end{aligned} \quad (38)$$

$$+ \frac{1}{\epsilon} \mathbb{E} \left[\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} (\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)))^2 ds \right]. \quad (39)$$

LM: add the explanation of why to drop the diffusion term

First, for [eq:local_time_diff_u](#) (38), we find a bound for the factor involving the difference of u^N and u in Lemma [lemma:uN-u_bound_for_integral](#). Therefore

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \leq 2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + \|Y_s^N - Y_s\|. \quad (40)$$

Now we need to bound the result of the local time of the difference $Y_s^N - Y_s$. First notice that $Y_s^N - Y_s \geq \epsilon$, then $e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \leq 1$, also it is clear that $\mathbb{1}_{\{Y_s^N - Y_s \in (0, \epsilon)\}}$ and $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}}$ are bounded by 1, therefore $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \leq 1$. Using the previous arguments and [eq:bound_u_abs](#) (28) lead to have

$$\begin{aligned} (38) &\leq 2\lambda \mathbb{E} \left[\int_0^t 2 \left(2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + \|Y_s^N - Y_s\| \right) ds \right] \\ &\leq 4\lambda \left(2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} t + \mathbb{E} \left[\int_0^t \|Y_s^N - Y_s\| ds \right] \right). \end{aligned} \quad (41)$$

Now for [\(59\)](#), we use similar arguments as the ones in [Lemma 3](#) above, and we get the following:

$$\begin{aligned} \|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\| &\leq \|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi^N(s, Y_s^N))\| \\ &\quad + \|\nabla u(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s^N))\| \\ &\quad + \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|, \end{aligned} \quad (42)$$

where the terms on the right hand side will be bounded as follows:

- For the first term we use [Proposition 2](#) and we have

$$\begin{aligned} \|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi^N(s, Y_s^N))\| &\leq \|\nabla u^N(s) - \nabla u(s)\|_{L^\infty} \\ &\leq \kappa_\rho \|b - b^N\|_{C_T C^{-\beta}}, \end{aligned} \quad (43)$$

for the second term see that $\nabla u, \nabla u^N$ are α -Hölder continuous and using [Proposition 3](#) we have

$$\begin{aligned} \|\nabla u(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s^N))\| &\leq \|\psi^N(s, Y_s^N) - \psi(s, Y_s^N)\|^\alpha \|u\|_{C_T C^{1+\alpha}} \\ &\leq (2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}})^\alpha \|u\|_{C_T C^{1+\alpha}}. \end{aligned} \quad (44)$$

Therefore we get the bound

$$\begin{aligned} \|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\| &\leq \kappa_\rho \|b - b^N\|_{C_T C^{-\beta}} \\ &\quad + \alpha \kappa_\rho^\alpha \|b^N - b\|_{C_T C^{-\beta}}^\alpha \|u\|_{C_T C^{1+\alpha}} \\ &\quad + \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|. \end{aligned} \quad (45) \quad \boxed{\text{eq:bound_gradu_abs}}$$

Here we can also notice that $\mathbb{1}_{\{Y_s^N - Y_s < \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} < 1$, then using [\(45\)](#) and the [inequality](#)

$$(x_1 + \dots + x_k)^2 \leq k(x_1^2 + \dots + x_k^2), \quad (46)$$

for $k = 3$, we can get the bound

$$\begin{aligned} \text{eq:local_time_diff_gradu} \quad (59) &\leq \frac{1}{\epsilon} \mathbb{E} \int_0^t \left(3\kappa_\rho^2 \|b - b^N\|_{C_T C^{-\beta}}^2 + 3 \cdot 2^{2\alpha} \kappa_\rho^{2\alpha} \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \right) ds \\ &\quad + \frac{1}{\epsilon} \mathbb{E} \int_0^t 3 \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 ds \\ &\leq \frac{1}{\epsilon} 3t \|b^N - b\|_{C_T C^{-\beta}}^2 \left(\kappa_\rho^2 \|b^N - b\|_{C_T C^{-\beta}}^2 + (2\kappa_\rho)^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1} \right) \\ &\quad + \frac{1}{\epsilon} 3\mathbb{E} \left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 ds \right) \end{aligned} \quad (47) \quad \boxed{\text{eq:bound_integral_gr}}$$

Now let us denote the last term in (47) by $I_t^{N,\epsilon}$. Pick $\zeta \in (0, 1)$ such that $\alpha\zeta > \frac{1}{2}$, and since $\epsilon \in (0, 1)$ we have $\epsilon^\zeta > \epsilon$. Then split the indicator function $\mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}}$ into $\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \leq \epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}}$. Leading to the integral

$$I_t^{N,\epsilon} = \frac{1}{\epsilon} 3\mathbb{E} \left(\int_0^t (\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \leq \epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}}) e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \left| \nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)) \right|^2 ds \right) \quad (48) \quad \{\text{eq:INepsilon}\}$$

For the first term of (48) we use the fact that ∇u is α -Hölder continuous uniformly in $s \in [0, T]$ with constant $\|u\|_{C_T C^{1+\alpha}}$ and that ψ is 2-Lipschitz

$$\begin{aligned} \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 &\leq \| \psi(s, Y_s^N) - \psi(s, Y_s) \|^{\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \\ &\leq 2^\alpha \|Y_s^N - Y_s\|^\alpha \|u\|_{C_T C^{1+\alpha}}^2 \\ &= 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|Y_s^N - Y_s\|^{2\alpha} \end{aligned} \quad (49)$$

For the other term we need another way to bound it, because even though the event when $\|Y^N - Y\| > \epsilon^\zeta$ is small, we can potentially have a quantity that blows up for the bound. **EI: the explanation needs adjusting - speak to Elena** In order to solve this problem, we can use the fact that ∇u is uniformly bounded by $1/2$ thanks to Lemma 2, **lemma:bounds_gradients** and then we can bound the difference of the gradients as follows:

$$\begin{aligned} \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 &\leq \|\nabla u(s, \psi(s, Y_s^N)) + \nabla u(s, \psi(s, Y_s))\|^2 \\ &\leq \sup_{(s,x) \in [0,T] \times \mathbb{R}} \|\nabla u(s, \psi(s, Y_s^N)) + \nabla u(s, \psi(s, Y_s))\|^2 \\ &= \|2\nabla u\|_{L_\infty}^2. \end{aligned} \quad (50)$$

Therefore we have that for all $t \in [0, T]$ **LM: check where else I need to say this**

$$\begin{aligned} I_t^{N,\epsilon} &\leq \frac{1}{\epsilon} 3\mathbb{E} \left(\int_0^t (\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \leq \epsilon^\zeta\}}) e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|Y_s^N - Y_s\|^{2\alpha} ds \right) \\ &\quad + \frac{1}{\epsilon} 3\mathbb{E} \left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \|2\nabla u\|_{L_\infty}^2 ds \right) \\ &\leq \frac{1}{\epsilon} 3\mathbb{E} \left(2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|\epsilon^\zeta\|^{2\alpha} t \right) + \frac{1}{\epsilon} 3\mathbb{E} \left(4e^{1-\epsilon^{\zeta-1}} \|\nabla u\|_{L_\infty}^2 t \right) \\ &\leq \sup_{t \in [0,T]} \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \epsilon^{2\alpha\zeta} + 4e^{1-\epsilon^{\zeta-1}} \|\nabla u\|_{L_\infty}^2 \right) t \\ &= \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \epsilon^{2\alpha\zeta} + 4e^{1-\epsilon^{\zeta-1}} \|\nabla u\|_{L_\infty}^2 \right) T. \end{aligned} \quad (51) \quad \{\text{eq:INepsilon_bound}\}$$

Now by combining (41), (47) and (51), and taking the sup over $[0, T]$ we will get

$$\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] &\leq 4\epsilon \\
&+ 4\lambda 2\kappa_\rho T \|b^N - b\|_{C_T C^{-\beta}} \\
&+ 4\lambda \mathbb{E} \left[\int_0^t \|Y_s^N - Y^N\| ds \right] \\
&+ \|b^N - b\|_{C_T C^{-\beta}} \frac{1}{\epsilon} 3T \kappa_\rho^2 \|b^N - b\|_{C_T C^{-\beta}} \\
&+ \|b^N - b\|_{C_T C^{-\beta}} \frac{1}{\epsilon} 3T (2\kappa_\rho)^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1} \\
&+ \frac{3}{\epsilon} 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 T e^{2\alpha\zeta} \\
&+ \frac{3}{\epsilon} 4 \|\nabla u\|_{L_\infty}^2 T e^{1-\epsilon\zeta-1}
\end{aligned} \tag{52}$$

then we take $\epsilon = \|b^N - b\|_{C_T C^{-\beta}}$ and we get

$$\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] &\leq 4 \|b^N - b\|_{C_T C^{-\beta}} \\
&+ 4\lambda 2\kappa_\rho T \|b^N - b\|_{C_T C^{-\beta}} \\
&+ 4\lambda \mathbb{E} \left[\int_0^t \|Y_s^N - Y^N\| ds \right] \\
&+ 3T \kappa_\rho^2 \|b^N - b\|_{C_T C^{-\beta}} \\
&+ 3T (2\kappa_\rho)^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1} \\
&+ 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 T \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha\zeta-1} \\
&+ 4 \|\nabla u\|_{L_\infty}^2 T \|b^N - b\|_{C_T C^{-\beta}}^{-1} \exp \left(1 - \|b^N - b\|_{C_T C^{-\beta}}^{\zeta-1} \right)
\end{aligned} \tag{53}$$

which can be written as

$$\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] &\leq c_1 \|b^N - b\|_{C_T C^{-\beta}} + c_2 \mathbb{E} \left[\int_0^t \|Y_s^N - Y^N\| ds \right] \\
&+ c_3 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1} + c_4 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha\zeta-1} \\
&+ c_5 \exp \left(1 - \|b^N - b\|_{C_T C^{-\beta}}^{\zeta-1} \right)
\end{aligned} \tag{54} \quad \boxed{\text{eq:bound_constants}}$$

where

$$\begin{aligned}
c_1 &= 4 + 4\lambda 2\kappa_\rho T + 3\kappa_\rho^2 T \\
c_2 &= 4\lambda \\
c_3 &= 3(2\kappa_\rho)^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 T \\
c_4 &= 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 T \\
c_5 &= 4 \|\nabla u\|_{L_\infty}^2 \|b^N - b\|_{C_T C^{-\beta}}^{-1} T
\end{aligned} \tag{55} \quad \boxed{\text{eq:constants_c}}$$

Finally, observe that since $\zeta \in (0, 1)$, the term $\exp \left(1 - \|b^N - b\|_{C_T C^{-\beta}}^{\zeta-1} \right)$ decays faster than any polynomial, thus controlling c_5 , and the last term in (54) goes to zero. Also

$\alpha\zeta$ is arbitrarily close to α , and $\|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1}$ controls $\|b^N - b\|_{C_T C^{-\beta}}$ therefore we can create the bound (34)

Question: is this clear enough? Am I making sense if I am taking α fixed? EI: no if α was fixed you could not do this. But $\alpha > \beta$ in your statement, hence it works. You need to explain the details however. Maybe at this stage you could introduce $\alpha' = \alpha\zeta$ to explain, that the result works for α' but since ζ can be chosen arbitrarily close to 1 then α' is arbitrarily close to α and α was chosen such that $\alpha > \beta$ which means the result is valid for all $\alpha' > \beta$. For simplicity we write α in place of α' in the statement. Also it is better to explain the meaning of $o()$ and what terms go in there.

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq o\left(\|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1}\right) + c_2 \mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + c_4 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1} \quad (56)$$

□

6 Convergence rate of the solution to the regularised SDE and the original

In this section we present a bound for $\mathbb{E}[X^N - X]$ in terms of $\|b^N - b\|_{C_T C^{-\beta}}$.

Proposition 5. *Let assumptions 1-2 hold, then for any $\alpha > \beta$ there is a constant C_α such that*

$$\mathbb{E}[X^N - X] \leq C_\alpha \|b^N - b\|_{C_T C^{-\beta}}, \quad (57) \quad \{\text{eq:EXN-X}\}$$

as $N \rightarrow \infty$.

Proof. Note that by definition of ψ, ψ^N we have

$$\begin{aligned} |X_t^N - X_t| &= |\psi^N(t, \phi^N(t, X_t^N)) - \psi(t, \phi(t, X_t))| \\ &= |\psi^N(t, Y_t) - \psi(t, Y_t)|, \end{aligned} \quad (58)$$

then adding and subtracting, and using the triangle inequality we get

$$|X_t^N - X_t| \leq |\psi^N(t, Y_t^N) - \psi(t, Y_t^N)| + |\psi(t, Y_t^N) - \psi(t, Y_t)|. \quad (59)$$

Where the first term is bounded by $2\kappa\|b^N - b\|_{C_T C^{-\beta}}$ (Proposition 3) and since ψ is 2-Lipschitz uniformly in $t \in [0, T]$ the second term is bounded by $2|Y^N - Y|$, therefore

$$|X^N - X| \leq 2\kappa\|b^N - b\|_{C_T C^{-\beta}} + 2|Y^N - Y|. \quad (60) \quad \{\text{eq:YN-X}\}$$

By assumption the first term above goes to zero as $N \rightarrow \infty$, then we only need a bound for the second term.

By Itô-Tanaka's formula

$$|Y^N - Y| = |y_0^N - y_0| + \frac{1}{2} L_t^0(Y^N - Y) + \int_0^t \text{sgn}(Y^N - Y) d(Y^N - Y), \quad (61) \quad \{\text{eq:YNY_ito_tanaka}\}$$

by taking expectation and using the definitions of Y^N, Y we have

$$\begin{aligned} \mathbb{E}|Y^N - Y| &= \mathbb{E}|y_0^N - y_0| + \mathbb{E}\frac{1}{2}L_t^0(Y^N - Y) \\ &\quad + \lambda \mathbb{E} \int_0^t \text{sgn}(Y^N - Y)(u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)))ds, \end{aligned} \tag{62} \quad \boxed{\text{eq:EYNY}}$$

then as the first term above is a constant and for the second we have a bound in Proposition ~~4~~ `prop:bound_local_time_sde` it remains to bound the second term.

Observe that

□

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