

Convergence rate of $X^N - X$ for McKean equations

Luis Mario Chaparro Jaquez

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Todo list

type this result which is lemma 4.2 in the paper	2
this norm is wrong, should be $1 + \alpha$ on the lhs and everywhere else	3
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1 What is this?

An adaptation of [1, Proposition 3.1] for the case of McKean SDEs proposed in [3].
Hola The proof builds on a number of results presented in the sections below.

2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

Definition 1. For any real-valued continuous semi-martingale, the local time at zero $L_t^0(\bar{Y})$ is defined as

$$L_t^0(\bar{Y}) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|\bar{Y}| \leq \epsilon\}} d\langle \bar{Y} \rangle_s, \quad \mathbb{P}\text{-a.s.} \quad (1)$$

For all $t \geq 0$.

The first result, [1, Lemma 5.1], is not necessary to prove for this particular setting since the result holds for any semi-martingale, I include it here for self-containment reasons.

Lemma 1. For any $\epsilon \in (0, 1)$ and any real-valued, continuous semi-martingale Z we have

$$\begin{aligned} \mathbb{E}[L_t^0(Z_s)] &\leq 4\epsilon - \mathbb{E} \left[\int_0^t \left(\mathbb{1}_{\{Z_s \in (0, \epsilon)\}} + \mathbb{1}_{\{Z_s \geq \epsilon\}} e^{1-Z_s/\epsilon} \right) dZ_s \right] \\ &\quad + \frac{1}{\epsilon} \mathbb{E} \left[\int_0^t \mathbb{1}_{\{Z_s > \epsilon\}} e^{1-Z_s/\epsilon} d\langle Z \rangle_s \right]. \end{aligned}$$

Let us introduce the original and regularised Kolmogorov equations.

Definition 2. *Kolmogorov equations*

For $\beta \in (0, 1/2)$ let $b \in C_T \mathcal{C}^{-\beta}$, $u, u^N \in C_T \mathcal{C}^{(1+\beta)+}$, and $b^N \rightarrow b$ as $N \rightarrow \infty$ in $C_T \mathcal{C}^{-\beta}$. The equations

$$\begin{cases} \partial u_i + \frac{1}{2} b_i \Delta u_i = \lambda u_i - b_i \\ u_i(T) = 0, \end{cases} \quad (2)$$

$$\begin{cases} \partial u_i^N + \frac{1}{2} b_i^N \Delta u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0. \end{cases} \quad (3)$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

Lemma 2. Let

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3 Bounds for the difference of solutions to the Kolmogorov equations

We need a bound for $u - u^N$ and $\nabla u - \nabla u^N$ in L_∞ for the case in which $u \in C_T \mathcal{C}^{1+\alpha}$ for some $\alpha > \beta$ which is an adaptation of [1, Lemma 5.2].

The result builds on top of the following result:

Proposition 1 (Bound for the ρ -equivalent norm of $u - u^N$). *Let u, u^N be (mild) solutions to the Kolmogorov equations from Definition 2 then as $N \rightarrow \infty$*

$$\|u_i - u_i^N\|_{C_T \mathcal{C}^{1+\alpha}}^{(\rho)} \leq \frac{c T^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} (\|u_i\|_{C_T \mathcal{C}^{1+\alpha}} - 1)}{1 - c \rho^{\frac{\alpha+\beta-1}{2}} (\|b\|_{C_T \mathcal{C}^{-\beta}} + \lambda)} \quad (4)$$

for $\rho \geq \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_T \mathcal{C}^{1+\alpha}} + \lambda)^{\frac{2}{\alpha+\beta+1}} \quad (5)$$

and $\lambda > 0$.

Proof. See that $u^N(T) = u(T) = 0$, and in [2], set g^N, g as b^N, b respectively. See that $b^N \rightarrow b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$.

As u^N, u are mild solutions, we have

$$\begin{aligned} u_i(t) - u_i^N(t) &= P_{T-t}(u_i(T) - u_i^N(T)) \\ &\quad + \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\ &\quad - \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds \\ &= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \end{aligned}$$

$$\begin{aligned}
& - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\
& - \int_t^T P_{s-t}(b_i - b_i^N) ds \\
& = \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)] ds \\
& - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\
& - \int_t^T P_{s-t}(b_i - b_i^N) ds \\
& = \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \\
& + \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \\
& - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\
& - \int_t^T P_{s-t}(b_i - b_i^N) ds
\end{aligned}$$

Now let us compute the ρ -equivalent norm of $u - u^N$, for some $\alpha > \beta$

$$\begin{aligned}
\|u_i - u_i^N\|_{C_T C_{-\beta}}^{(\rho)} &= \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|u(t) - u^N(t)\|_{1+\alpha} \\
&\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{1+\alpha} \\
&+ \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \right\|_{1+\alpha} \\
&- \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{1+\alpha} \\
&- \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}(b_i - b_i^N) ds \right\|_{1+\alpha}.
\end{aligned}$$

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Let us take each term from the right hand side of the inequality and bound them.

For the first term, using $\gamma + 2\theta = 1 + \alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \leq c t^{-\theta} \|f\|_\gamma$ and $\|\nabla g\|_\xi \leq c \|g\|_{\xi+1}$

$$\begin{aligned}
& \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{1+\alpha} \\
& \leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_\alpha \|b_i - b_i^N\|_{-\beta} ds
\end{aligned}$$

$$\begin{aligned} &\leq c\|u_i\|_{C_T\mathcal{C}_{1+\alpha}}\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0 \leq t \leq T} e^{-\rho(T-t)}(T-t)^{\frac{1-\beta-\alpha}{2}} \\ &\leq cT^{\frac{1-\beta-\alpha}{2}}\|u_i\|_{C_T\mathcal{C}_{1+\alpha}}\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \end{aligned}$$

For the second term, see that for $N \rightarrow \infty$, we have $\|b^N\|_{C_T\mathcal{C}^{-\beta}} \leq 2\|b\|_{C_T\mathcal{C}^{-\beta}}$

$$\begin{aligned} &\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} b_i^N (\nabla u_i - \nabla u_i^N) ds \right\|_{1+\alpha} \\ &\leq c \sup_{0 \leq t \leq T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2\|b_i\|_{-\beta} \|\nabla u_i - \nabla u_i^N\|_\alpha ds \\ &\leq c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} ds \\ &\leq c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}} \end{aligned}$$

For the third term, which is the one that differs from the proof in [2] we need to use that $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta}\|f\|_\gamma$, and in this case we have $\gamma + 2\theta = 1 + \alpha$ and $\gamma = 1 + \alpha$, so that $\theta = 0$ because $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, so we will have

$$\begin{aligned} &\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t} (u_i - u_i^N) ds \right\|_{1+\alpha} \\ &\leq c\lambda \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-0} \|u_i - u_i^N\|_{1+\alpha} ds \\ &= c\lambda \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T e^{-\rho(T-s)} \sup_{0 \leq s \leq T} e^{-\rho(T-s)} \|u_i - u_i^N\|_{1+\alpha} ds \\ &= c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T\mathcal{C}_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(T-s)} e^{-\rho(T-t)} ds \\ &= c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T\mathcal{C}_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(s-t)} ds \\ &= c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T\mathcal{C}_{1+\alpha}}^{(\rho)} \sup_{0 \leq t \leq T} \rho^{-1} [1 - e^{-\rho(T-t)}] \\ &\leq c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T\mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{-1} \\ &\leq c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T\mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}} \end{aligned}$$

And for the last term

$$\begin{aligned} &\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{T-s} (b_i - b_i^N) ds \right\|_{1+\alpha} \\ &\leq c \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}} \|b_i - b_i^N\|_{-\beta} ds \end{aligned}$$

$$\begin{aligned} &\leq c \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \sup_{0 \leq t \leq T} e^{-\rho(T-t)} (s-t)^{-\frac{\alpha+\beta-1}{2}} \\ &\leq c T^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \end{aligned}$$

Putting everything together

$$\begin{aligned} \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} &\leq c T^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T \mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \\ &+ c \|b_i\|_{C_T \mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}} \\ &- c \lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}^{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}} \\ &- c T^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}}, \end{aligned}$$

and finally,

$$\begin{aligned} \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} (1 - c \rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C_T \mathcal{C}^{-\beta}} + \lambda]) \\ \leq c T^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} (\|u_i\|_{C_T \mathcal{C}^{1+\alpha}} - 1) \\ \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \leq \frac{c T^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} (\|u_i\|_{C_T \mathcal{C}^{1+\alpha}} - 1)}{(1 - c \rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C_T \mathcal{C}^{-\beta}} + \lambda])} \end{aligned}$$

As required. \square

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \leq \frac{c T^{\frac{1-\beta-\alpha}{2}} (\|u_i\|_{C_T \mathcal{C}^{1+\alpha}} - 1)}{(1 - c \rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C_T \mathcal{C}^{-\beta}} + \lambda])} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \quad (6)$$

$$\|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \leq c(\rho) \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \quad (7)$$

Check this norm

Here is the adaptation of [1, Lemma 5.2].

Proposition 2. *Bounds for $\|u - u^N\|_{L^\infty}$ and $\|\nabla u - \nabla u^N\|_{L^\infty}$*

Let $\beta \in (0, 1/2)$ and $b \in C_T \mathcal{C}^{-\beta}$. Let $u, u^N \in C_T \mathcal{C}^{(1+\beta)+}$ be (mild) solutions to the Kolmogorov equations from Definition 2.

Assume, by Proposition 1, that for some $\alpha > \beta$

$$\|u - u^N\|_{C_T \mathcal{C}^{1+\alpha}}^{(\rho)} \leq c(\rho) \|b - b^N\|_{C_T \mathcal{C}^{-\beta}}. \quad (8)$$

With $c(\rho)$ as in Proposition 1 and ρ_0 is large enough such that $c(\rho) > 0$ for all $\rho > \rho_0$. Then for all $t \in [0, T]$

$$\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C_T \mathcal{C}^{-\beta}} \quad (9)$$

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C_T \mathcal{C}^{-\beta}} \quad (10)$$

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$.

Proof. First let us prove Eq. (9).

Let $t \in [0, T]$, and see that since $u, u^N \in C_T \mathcal{C}^{(1+\beta)+}$ there exists $\alpha > \beta$ such that $u, u^N \in C_T \mathcal{C}^{1+\alpha}$, then for any $f \in \mathcal{C}^{1+\alpha}$ we have

$$\|f\|_{\mathcal{C}^{1+\alpha}} \leq c \left(\sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \neq y \in \mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \right) \quad (11)$$

so we have

$$\begin{aligned} \|u^N(t) - u(t)\|_{L^\infty} &= \sup_{x \in \mathbb{R}^d} |u^N(t, x) - u(t, x)| \\ &\leq c \|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \end{aligned} \quad (12)$$

Moreover, using the (ρ) -equivalent norm

$$\|f\|_{\mathcal{C}^{1+\alpha}} = \sup_{t \in [0, T]} e^{-\rho(T-t)} \|f(t)\|_{\mathcal{C}^{1+\alpha}}, \quad (13)$$

and Eq. (8) we see that

$$\begin{aligned} \|u^N - u\|_{C_T \mathcal{C}^{1+\alpha}} &= \sup_{t \in [0, T]} \|u^N - u\|_{\mathcal{C}^{1+\alpha}} \\ &= \sup_{t \in [0, T]} e^{\rho(T-t)} e^{-\rho(T-t)} \|u^N - u\|_{\mathcal{C}^{1+\alpha}} \\ &\leq e^{\rho T} \sup_{t \in [0, T]} e^{-\rho(T-t)} \|u^N - u\|_{\mathcal{C}^{1+\alpha}} \\ &= e^{\rho T} \|u^N - u\|_{C_T \mathcal{C}^{1+\alpha}}^{(\rho)}. \end{aligned} \quad (14)$$

Plugging Eq. (14) into Eq. (12)

$$\begin{aligned} \|u^N(t) - u(t)\|_{L^\infty} &\leq c \|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \\ &\leq \sup_{t \in [0, T]} c \|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \\ &= c \|u^N - u\|_{C_T \mathcal{C}^{\alpha+1}} \\ &\leq c e^{\rho T} \|u^N - u\|_{C_T \mathcal{C}^{\alpha+1}}^{(\rho)}. \end{aligned} \quad (15)$$

And finally by Eq. (8)

$$\|u^N(t) - u(t)\|_{L^\infty} \leq c \cdot c(\rho) \cdot e^{\rho T} \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \quad (16)$$

which proves Eq. (9).

For Eq. (10) recall that if $f \in \mathcal{C}^{1+\alpha}$ then $\nabla f \in \mathcal{C}^\alpha$. Also, by Bernstein inequality [3, Eqn. (9)]

$$\|\nabla f\|_\alpha \leq c \|f\|_{\infty+\alpha}. \quad (17)$$

Using the equivalent norm

$$\|f\|_{\mathcal{C}^{1+\alpha}} \leq c \left(\sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \in \mathbb{R}^d} |\nabla f(x)| + \sup_{x \neq y \in \mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \right) \quad (18)$$

we can see that

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq c\|u^N(t) - u(t)\|_{C^{1+\alpha}}. \quad (19)$$

And usign the same bounds that we used above for $c\|u^N(t) - u(t)\|_{C^{1+\alpha}}$ this point follows. \square

4 Bound for the difference of the auxiliary functions

This is the adaptation of result [1, Lemma 5.3].

Proposition 3. Take $\rho > \rho_0$ as in Proposition 1, $N \rightarrow \infty$, κ_ρ from Proposition 2, and $\beta \in (0, 1/2)$, then we have

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} |\psi(t,x) - \psi^N(t,x)| \leq 2\kappa_\rho \|b - b^N\|_{C_T C^{-\beta}} \quad (20)$$

Proof. Recall the definition of $\psi, \phi \in C_T \mathcal{C}^1$

$$\phi(t, x) := x + u(t, x) \quad (21)$$

$$\psi(t, \cdot) = \phi^{-1}(t, \cdot). \quad (22)$$

Note that

$$u(y) = \int_0^1 \nabla u(\alpha y) y d\alpha + u(0). \quad (23)$$

From there we have

$$u(t, y) - u(t, y') = \int_0^1 \nabla u(t, \alpha(y - y')) (y - y') d\alpha \quad (24)$$

and therefore

$$|u(t, y) - u(t, y')| \geq \left(\int_0^1 |\nabla u(t, \alpha(y - y'))|^2 d\alpha \right)^{1/2} |y - y'|, \quad (25)$$

and by 2 we finally have

$$\begin{aligned} |u(t, y) - u(t, y')| &\leq \left(\frac{1}{4} \int_0^1 d\alpha \right)^{1/2} |y - y'| \\ |u(t, y) - u(t, y')|^2 &\leq \frac{1}{4} |y - y'|^2 \end{aligned} \quad (26)$$

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5 Bound for the local time at zero of the solution to the SDEs

We need a bound for $\mathbb{E}[L_T^0(Y^N - Y)]$, for Sobolev spaces, this is result [1, Proposition 5.4] we present it here for the solutions to the SDE belonging to the appropriate Besov spaces.

Proposition 4. *Let $b \in C_T \mathcal{C}^{-\beta}$ and $b^N \rightarrow b$ in $C_T \mathcal{C}^{-\beta}$ as $N \rightarrow \infty$ for $\beta \in (0, \frac{1}{4})$ and for any $\alpha > \beta$*

Proof. Recall that Y^N, Y are solutions to the SDEs

$$Y_t = y_0 + \lambda \int_0^t u(s, \psi(s, Y_s)) ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1) dW_s \quad (27)$$

and

$$Y_t^N = y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N)) ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1) dW_s \quad (28)$$

so that the difference $Y^N - Y$ is

$$\begin{aligned} Y^N - Y_t &= (y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N)) ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1) dW_s) \\ &\quad - (y_0 + \lambda \int_0^t u(s, \psi(s, Y_s)) ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1) dW_s) \\ &= (y_0^N - y_0) + \lambda \int_0^t (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))) ds \\ &\quad - \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) - \nabla u(s, \psi(s, Y_t))) dW_s, \end{aligned} \quad (29)$$

and using Lemma 1 we have the following bound

$$\begin{aligned} \mathbb{E}[L_t^0(Y^N - Y)] &\leq 4\epsilon + 1 \\ &\quad - 2(1 + \lambda) \mathbb{E} \left[\int_0^t \left(\mathbb{1}_{\{Y_s^N - Y_s \in (0, \epsilon)\}} + \mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \right) (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))) ds \right] \\ &\quad + \frac{1}{\epsilon} \mathbb{E} \left[\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} (\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)))^2 ds \right] \end{aligned} \quad (30)$$

□

6 Proposition 3.1 (main result)

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