Convergence rate of numerical solutions to SDEs with distributional drifts in Besov spaces

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Contents

1 What is this?

An adaptation of [1, Proposition 3.1] for the case of SDEs with drift in a Besov space of negative order similar to the ones proposed in [2] and [3]. The proof builds on a number of results presented in the sections below.

EI: add result about convergence of the scheme. This is done in two parts, $X^N \to X$ done in Russo Issoglio, and $X^{N,m} \to X^N$ Euler scheme convergence from De Angelis Germain Issoglio. Attention that the rate of convergence of Euler scheme depends of the smoothness of b^N.

2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

def:local_time_zero

Definition 1. *Local time at zero For any real-valued continuous semi-martingale Z, the local time at zero $L_t^0(\bar{Y})$ is defined as*

$$L_t^0(Z) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|Z| \le \epsilon\}} d\langle Z \rangle_s, \ \mathbb{P}\text{-a.s.} \tag{1}$$

For all $t \ge 0$.

The lemma below is from [de_angelis_numerical_2020 1] and its proof can be found in [de_angelis_numerical_2020 1, Lemma 5.1]. We include the statement here for ease of reading.

lemma:local-time-at-0

Lemma 1. *Bound for local time at zero for a semi-martingale For any $\epsilon \in (0,1)$ and any real-valued, continuous semi-martingale Z we have*

$$\begin{aligned} \mathbb{E}[L_t^0(Z_s)] \le & 4\epsilon - 2\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Z_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Z_s \ge \epsilon\}} e^{1 - Z_s/\epsilon}\right) dZ_s\right] \\ & + \frac{1}{\epsilon}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z > \epsilon\}} e^{1-Z_s/\epsilon} d\langle Z \rangle_s\right]. \end{aligned}$$

Let us introduce the original and regularised Kolmogorov equations. To shorten notation we will denote the spaces $C_T\mathcal{C}^\gamma(\mathbb{R})$ as $C_T\mathcal{C}^\gamma$.

def:kolmogorov_eqns

Definition 2. *Kolmogorov equations For $\beta \in (0, 1/2)$ let $b \in C_T\mathcal{C}^{-\beta}$, $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$, and $b^N \to b$ as $N \to \infty$ in $C_T\mathcal{C}^{-\beta}$ The equations*

$$\begin{cases} \partial_t u_i + \frac{1}{2}\Delta u_i + b_i \nabla u_i = \lambda u_i - b_i \\ u_i(T) = 0, \end{cases} \tag{2}$$

{eq:kolmogorov}

$$\begin{cases} \partial u_i^N + \frac{1}{2}\Delta u_i^N + b_i^N \nabla u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0. \end{cases} \tag{3}$$

{eq:kolmogorov_N}

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

lemma:bounds_gradients

Lemma 2. *Let u, u^N be the solutions to the Kolmogorov equations (eq:kolmogorov 2) (eq:kolmogorov_N 3) in $C_T\mathcal{C}^{1+\alpha}$ respectivelly. We have*

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u(t,x)| \le \frac{1}{2} and \sup_{(t,x) \in [0,T] \times \mathbb{R}} \left|\nabla u^N(t,x)\right| \le \frac{1}{2} \tag{4}$$

as:b_in_ctcb

Assumption 1. *Let $0 < \beta < 1/2$ and $b \in C_T\mathcal{C}^{-\beta}$.*

:bN_converges_in_ctcb

Assumption 2. *There exists a sequence $(b^N)_N \in C_T\mathcal{C}^{-\beta}$ such that for each N, $b^N(t,\cdot) \in C_b^\infty(\mathbb{R})$ for all $t \in [0,T]$ and such that $b^N \to b$ as $N \to \infty$.*

LM: the lemma is probably better to type it before the first time it is used since it requires of some results below

LM: TYPE THIS

LM: add Schauder estimates to reference in Prop. prop:diff_u_uN 1, add small note of them and refernce

3 Bounds for the difference of solutions to the Kolmogorov equations

We need a bound for $u - u^N$ and $\nabla u - \nabla u^N$ in L_∞ for the case in which $u \in C_T \mathcal{C}^{1+\alpha}$ for some $\alpha > \beta$ which is an adaptation of de_angelis_numerical_2020 [1, Lemma 5.2].

The result builds on top of the following result:

prop:diff_u_uN **Proposition 1.** *Let u, u^N be (mild) solutions to the Kolmogorov equations from Definition* def:kolmogorov_eqns 2 *then as $N \to \infty$*

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}} (\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda)} \tag{5}$$

for $\rho \geq \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_T + \alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \tag{6}$$

and $\lambda > 0$.

Proof. See that $u^N(T) = u(T) = 0$, and in issoglio_pde_nodate [2], set g^N, g as b^N, b respectively. See that $b^N \to b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$. As u^N, u are mild solutions, we have

$$\begin{aligned}
u_i(t) - u_i^N(t) &= P_{T-t}(u_i(T) - u_i^N(T)) + \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\
&\qquad - \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds - \int_t^T P_{s-t}(b_i - b_i^N) ds \\
&= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)] ds \\
&\qquad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds - \int_t^T P_{s-t}(b_i - b_i^N) ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds + \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \\
&\qquad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds - \int_t^T P_{s-t}(b_i - b_i^N) ds
\end{aligned}$$

Now let us compute the ρ-equivalent norm of $u - u^N$, for some $\alpha > \beta$

$$\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} &= \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|u(t) - u^N(t)\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \Bigg[\Bigg\| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \Bigg\|_{C_T\mathcal{C}^{1+\alpha}}
\end{aligned}$$

$$
\begin{aligned}
&+ \left\| \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&- \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&- \left\| \int_t^T P_{s-t}(b_i - b_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \Bigg].
\end{aligned}
$$

Let us take each term from the right hand side of the inequality and bound them.

LM: when the Schauder estimates are added, check this again For the first term, using $\gamma + 2\theta = 1 + \alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \le ct^{-\theta}\|f\|_\gamma$ and $\|\nabla g\|_\xi \le c\|g\|_{\xi+1}$

$$
\begin{aligned}
&\sup_{0\le t\le T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}\left(\nabla u_i b_i - \nabla u_i b_i^N\right) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad \le \sup_{0\le t\le T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_{C_T\mathcal{C}^\alpha} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \\
&\qquad \le c\|u_i\|_{C_T\mathcal{C}_{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\le t\le T} e^{-\rho(T-t)}(T-t)^{\frac{1-\beta-\alpha}{2}} \\
&\qquad \le cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T\mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}}
\end{aligned}
$$

For the second term, see that for $N \to \infty$, we have $\|b^N\|_{C_T\mathcal{C}^{-\beta}} \le 2\|b\|_{C_T\mathcal{C}^{-\beta}}$

$$
\begin{aligned}
&\sup_{0\le t\le T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} b_i^N (\nabla u_i - \nabla u_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad \le c \sup_{0\le t\le T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|\nabla u_i - \nabla u_i^N\|_{C_T\mathcal{C}^{1+\alpha}} ds \\
&\qquad \le c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} ds \\
&\qquad \le c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}
$$

For the third term, which is the one that difers from the proof in issoglio_pde_nodate [2] we need to use that $\|P_t f\|_{\gamma+2\theta} \le ct^{-\theta}\|f\|_\gamma$, and in this case we have $\gamma + 2\theta = 1 + \alpha$ and $\gamma = 1 + \alpha$, so that $\theta = 0$ because $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, so we will have

$$
\begin{aligned}
&\sup_{0\le t\le T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{1+\alpha} \\
&\qquad \le c\lambda \sup_{0\le t\le T} e^{-\rho(T-t)} \int_t^T (s-t)^{-0} \|u_i - u_i^N\|_{1+\alpha} ds \\
&\qquad = c\lambda \sup_{0\le t\le T} e^{-\rho(T-t)} \int_t^T e^{-\rho(T-s)} \sup_{0\le s\le T} e^{-\rho(T-s)} \|u_i - u_i^N\|_{1+\alpha} ds
\end{aligned}
$$

$$
\begin{aligned}
&= c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \int_t^T e^{-\rho(T-s)} e^{-\rho(T-t)} ds \\
&= c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \int_t^T e^{-\rho(s-t)} ds \\
&= c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \sup_{0\le t\le T} \rho^{-1}[1 - e^{-\rho(T-t)}] \\
&\le c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{-1} \\
&\le c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}
$$

And for the last term

$$
\begin{aligned}
&\sup_{0\le t\le T} e^{-\rho(T-t)} \left\| \int_t^T P_{T-s}(b_i - b_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad \le c \sup_{0\le t\le T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} ds \\
&\qquad \le c\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\le t\le T} e^{-\rho(T-t)} (s-t)^{-\frac{\alpha+\beta-1}{2}} \\
&\qquad \le cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}
\end{aligned}
$$

Putting everything together

$$
\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \le\ & cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T\mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \\
&+ c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \rho^{\frac{\alpha+\beta-1}{2}} \\
&- c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{\frac{\alpha+\beta-1}{2}} \\
&- cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}},
\end{aligned}
$$

and finally,

$$
\begin{aligned}
&\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} (1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda]) \\
&\qquad\qquad \le cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1) \\
&\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \le \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}
\end{aligned}
$$

As required. □

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{7}$$

LM: Check this norm, it was in $-\beta$ now in $1+\alpha$

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq c(\rho)\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{8}$$

Here is the adaptation of [1, Lemma 5.2].

prop:diff_uN_graduN

Proposition 2. *Bounds for $\|u - u^N\|_{L_\infty}$ and $\|\nabla u - \nabla u^N\|_{L_\infty}$*

Let $\beta \in (0,1/2)$ and $b \in C_T\mathcal{C}^{-\beta}$. Let $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ be (mild) solutions to the Kolmogorov equations from Definition 2.

Assume, by Proposition 1, that for some $\alpha > \beta$

$$\|u - u^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq c(\rho)\|b - b^N\|_{C_T\mathcal{C}^{-\beta}}. \tag{9}$$ {eq:u-uNb-bN}

With $c(\rho)$ as in Proposition 1 and ρ_0 is large enough such that $c(\rho) > 0$ for all $\rho > \rho_0$.

Then for all $t \in [0,T]$

$$\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{10}$$ {eq:uNu_bounded_by_bN

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{11}$$ {eq:graduNu_bounded_b

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$.

Proof. First let us prove (10).

Let $t \in [0,T]$, and see that since $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ there exists $\alpha > \beta$ such that $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, then for any $f \in \mathcal{C}^{1+\alpha}$ we have

$$\|f\|_{C^{1+\alpha}} \leq c\left(\sup_{x\in\mathbb{R}^d}|f(x)| + \sup_{x\neq y\in\mathbb{R}^d}\frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha}\right) \tag{12}$$

so we have

$$\begin{aligned}\|u^N(t) - u(t)\|_{L^\infty} &= \sup_{x\in\mathbb{R}^d}|u^N(t,x) - u(t,x)| \\ &\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}}\end{aligned} \tag{13}$$ {eq:u-uN_in_Linfinity

Moreover, using the (ρ)-equivalent norm

$$\|f\|_{\mathcal{C}^{1+\alpha}} = \sup_{t\in[0,T]} e^{-\rho(T-t)}\|f(t)\|_{\mathcal{C}^{1+\alpha}}, \tag{14}$$

and (9) we see that

$$\begin{aligned}
\left\|u^N - u\right\|_{C_T\mathcal{C}^{1+\alpha}} &= \sup_{t\in[0,T]} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\
&= \sup_{t\in[0,T]} e^{\rho(T-t)} e^{-\rho(T-t)} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\
&\le e^{\rho T} \sup_{t\in[0,T]} e^{-\rho(T-t)} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\
&= e^{\rho T} \left\|u^N - u\right\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} .
\end{aligned} \tag{15}$$

Plugging (15) into (13)

$$\begin{aligned}
\|u^N(t) - u(t)\|_{L^\infty} &\le c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \\
&\le \sup_{t\in[0,T]} c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{\alpha+1}} \\
&= c\left\|u^N - u\right\|_{C_T\mathcal{C}^{\alpha+1}} \\
&\le ce^{\rho T}\left\|u^N - u\right\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}} .
\end{aligned} \tag{16}$$

And finally by (9)

$$\left\|u^N(t) - u(t)\right\|_{L^\infty} \le c \cdot c(\rho) \cdot e^{\rho T} \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \tag{17}$$

which proves (10).

For (11) recall that if $f \in \mathcal{C}^{1+\alpha}$ then $\nabla f \in \mathcal{C}^\alpha$. Also, by Bernstein inequality [3, Eqn. (9)]

$$\|\nabla f\|_\alpha \le c\|f\|_{+\alpha} . \tag{18}$$

Using the equivalent norm

$$\|f\|_{\mathcal{C}^{1+\alpha}} \le c\left(\sup_{x\in\mathbb{R}^d} |f(x)| + \sup_{x\in\mathbb{R}^d} |\nabla f(x)| + \sup_{x\neq y\in\mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha}\right) \tag{19}$$

we can see that

$$\left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \le c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1+\alpha}} . \tag{20}$$

And usign the same bounds that we used above for $c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1+\alpha}}$ this point follows.

□

4 Bound for the difference of the auxiliary functions

This is the adaptation of result [1, Lemma 5.3].

prop:bound_psi-psiN **Proposition 3.** *Take $\rho > \rho_0$ as in Proposition* prop:diff_u_uN 1*, $N \to \infty$, κ_ρ from Proposition* prop:diff_uN_graduN 2*, and $\beta \in (0, 1/2)$, then we have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \|\psi(t,x) - \psi^N(t,x)\| \leq 2\kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}} \tag{21}$$

Proof. Recall the definition of $\psi, \phi \in C_T\mathcal{C}^1$

$$\phi(t,x) := x + u(t,x) \tag{22}$$

$$\psi(t,\cdot) = \phi^{-1}(t,\cdot). \tag{23}$$

Note that

$$u(y) = \int_0^1 \nabla u(\alpha y) y d\alpha + u(0). \tag{24}$$

From there we have

$$u(t,y) - u(t,y') = \int_0^1 \nabla u(t, \alpha(y-y'))(y-y')d\alpha \tag{25}$$

and therefore

$$\|u(t,y) - u(t,y')\| \geq \left(\int_0^1 \|\nabla u(t,\alpha(y-y'))\|^2 d\alpha\right)^{1/2} \|y-y'\|, \tag{26}$$

and by Lemma lemma:bounds_gradients 2 we finally have

$$\begin{aligned} \|u(t,y) - u(t,y'\|) &\leq \left(\frac{1}{4}\int_0^1 d\alpha\right)^{1/2} \|y-y\|' \\ \|u(t,y) - u(t,y')\|^2 &\leq \frac{1}{4}\|y-y'\|^2 \end{aligned} \tag{27}$$

LM: continue from page three in notes

□

5 Bound for the local time at zero of the solution to the SDEs

LM: Here I still need to mention how we define $Y_t = \psi(t, X_t)$, because eventually I need to use that $X_t = \psi(t, Y_t)$, probably just neet to mention without defining the whole Y_t as in the paper

We need a bound for $\mathbb{E}[L_T^0(Y^N - Y)]$, for Sobolev spaces, this is result [de_angelis_numerical_2020 1, Proposition 5.4] we present it here for the solutions to the SDE blonging to the appropriate Besov spaces.

First let us state the following useful result.

LM: check that the statement makes sense and has all the necessary assumptions

-n_bound_for_integral

Lemma 3. *Let u, u^N be solutions to the Kolmogorov equations (2) (3) then the following bound is satisfied:*

eq:kolmogorov_N

$$\|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s)\|) \le 2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\|. \tag{28}$$

{eq:bound_u_abs}

Proof. Adding and subtracting terms, using triangle inequality and noting that for any a, b, we have $a - b \le \|a-b\|$, then

$$\begin{aligned}\|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s)\|) \le\ & \|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi^N(s,Y_s^N)\|) \\ & + \|u(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s^N)\|) \\ & + \|u(s,\psi(s,Y_s^N)) - u(s,\psi(s,Y_s))\|.\end{aligned} \tag{29}$$

The terms in the right hand side will be bounded as follows:

- For the first term, by Proposition 2 prop:diff_uN_graduN

$$\|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi^N(s,Y_s^N)\|) \le \left\|u^N(s) - u(s)\right\|_{L^\infty} \le \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}}, \tag{30}$$

- for the second term, observe that u, u^N are $\frac{1}{2}$-Lipschitz and by Proposition 3 prop:bound_psi-psiN we get

$$\begin{aligned}\|u(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s^N)\|) &\le \frac{1}{2}\|\psi^N(s,Y_s^N) - \psi(s,Y_s^N\|) \\ &\le \kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}},\end{aligned} \tag{31}$$

- and for the final term, note that ψ, ψ^N are 2-Lipschitz

  so that

$$\|u(s,\psi(s,Y_s^N)) - u(s,\psi(s,Y_s)\|) \le \frac{1}{2}\|\psi(s,Y_s^N) - \psi(s,Y_s\|) \le \|Y_s^N - Y_s\|. \tag{32}$$

So that the following bound holds

$$\|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s)\|) \le 2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\|, \tag{33}$$

as required.

□

:bound_local_time_sde

Proposition 4. *Let A, B be constants, $b \in C_T\mathcal{C}^{-\beta}$ and $b^N \to b$ in $C_T\mathcal{C}^{-\beta}$ as $N \to \infty$ for $\beta \in (0, \frac{1}{4})$ and for any $\alpha > \beta$*

$$\mathbb{E}[L_t^0(Y^N - Y)] \le o\left(\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + B\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}. \tag{34}$$

{eq:local_time_YNY_bo

Proof. Recall that Y^N, Y are solutions to the SDEs

$$Y_t = y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1) dW_s \tag{35}$$

and

$$Y_t^N = y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1) dW_s \tag{36}$$

so that the difference $Y^N - Y$ is

$$\begin{aligned} Y_t^N - Y_t &= \left(y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1) dW_s\right) \\ &\quad - \left(y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1) dW_s\right) \\ &= (y_0^N - y_0) + \lambda \int_0^t (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)))ds \\ &\quad + \int_0^t (\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)))dW_s, \end{aligned} \tag{37}$$

and using Lemma 1 we have the following bound

$$\begin{aligned} &\mathbb{E}[L_t^0(Y^N - Y)] \\ &\leq 4\epsilon \\ &- 2\lambda \mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}}\right)\right. \\ &\qquad\qquad \left.\Big(u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\Big) ds\right] \qquad (38) \\ &+ \frac{1}{\epsilon} \mathbb{E}\left[\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \left(\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\right)^2 ds\right]. \qquad (39) \end{aligned}$$

{eq:local_time_diff_u

{eq:local_time_diff_g

LM: add the explanation of why to drop the difussion term

First, for (38), we find a bound for the factor involving the difference of u^N and u in Lemma 3. Therefore

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)\|) \leq 2\kappa_\rho \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\|. \tag{40}$$

Now we need to bound the result of the local time of the difference $Y_s^N - Y_s$. First notice that $Y_s^N - Y_s \geq \epsilon$, then $e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \leq 1$, also it is clear that $\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}}$ and $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}}$ are bounded by 1, therefore $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \leq 1$. Using the previous arguments and (28) lead to have

$$\begin{aligned} (38) &\leq 2\lambda \mathbb{E}\left[\int_0^t 2\left(2\kappa_\rho \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\|\right) ds\right] \\ &\leq 4\lambda \left(2\kappa_\rho \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}} t + \mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\| ds\right]\right). \end{aligned} \tag{41}$$

{eq:bound_integral_uN

Now for (39), we use similar arguments as the ones in Lemma 3 above, and we get the following:

$$\begin{aligned}\|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s)\|) \le \|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi^N(s,Y_s^N)\|) \\ + \|\nabla u(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s^N)\|) \\ + \|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s)\|),\end{aligned} \tag{42}$$

where the terms on the right hand side will be bounded as follows:

- For the first term we use Proposition 2 and we have

  $$\begin{aligned}\|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi^N(s,Y_s^N)\|) &\le \left\|\nabla u^N(s) - \nabla u(s)\right\|_{L^\infty} \\ &\le \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}},\end{aligned} \tag{43}$$

  for the second term see that $\nabla u, \nabla u^N$ are α-Hölder continuous and using Proposition 3 we have

  $$\begin{aligned}\|\nabla u(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s^N)\|) &\le \|\psi^N(s,Y_s^N) - \psi(s,Y_s^N)\|^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}} \\ &\le (2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}})^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}}.\end{aligned} \tag{44}$$

Therefore we get the bound

$$\begin{aligned}\|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s)\|) \le \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}} \\ + \alpha\kappa_\rho^\alpha \left\|b^N - b\right\|^\alpha_{C_T\mathcal{C}^{-\beta}} \|u\|_{C_T\mathcal{C}^{1+\alpha}} \\ + \|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|.\end{aligned} \tag{45}$$

Here we can also notice that $\mathbb{1}_{\{Y_s^N - Y_s < \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} < 1$, then using (45) and the inequality

$$(x_1 + \cdots + x_k)^2 \le k(x_1^2 + \cdots + x_k^2), \tag{46}$$

for $k = 3$, we can get the bound

$$\begin{aligned}(39) \le \frac{1}{\epsilon}\mathbb{E}\int_0^t \left(3\kappa_\rho^2 \left\|b - b^N\right\|^2_{C_T\mathcal{C}^{-\beta}} + 3\cdot 2^{2\alpha}\kappa_\rho^{2\alpha} \left\|b^N - b\right\|^{2\alpha}_{C_T\mathcal{C}^{-\beta}} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}}\right) ds \\ + \frac{1}{\epsilon}\mathbb{E}\int_0^t 3\mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|^2 ds \\ \le \frac{1}{\epsilon}3t \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \left(\kappa_\rho^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + (2\kappa_\rho)^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}\right) \\ + \frac{1}{\epsilon}3\mathbb{E}\left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \left|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right|^2 ds\right)\end{aligned} \tag{47}$$

Now let us denote the last term in (47) by $I_t^{N,\epsilon}$. Pick $\zeta \in (0,1)$ such that $\alpha\zeta > \frac{1}{2}$, and since $\epsilon \in (0,1)$ we have $\epsilon^{\zeta} > \epsilon$. Then split the indicator function $\mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}}$ into $\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^{\zeta}\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^{\zeta}\}}$. Leading to the integral

$$I_t^{N,\epsilon} = \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \left(\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^{\zeta}\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^{\zeta}\}} \right) e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \left| \nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)) \right|^2 ds \right) \tag{48}$$ {eq:INepsilon}

For the first term of (48) we use the fact that ∇u is α-Hölder continuous uniformly in $s \in [0,T]$ with constant $\|u\|_{C_T\mathcal{C}^{1+\alpha}}$ and that ψ is 2-Lipschitz

$$\begin{aligned} \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 &\le \| \|\psi(s, Y_s^N) - \psi(s, Y_s)\|^{\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}} \|^2 \\ &\le \|2^{\alpha} \|Y_s^N - Y_s\|^{\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}} \|^2 \\ &= 2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \|Y_s^N - Y_s\|^{2\alpha} \end{aligned} \tag{49}$$

For the other term we need another way to bound it, because even though the event when $\|Y^N - Y\| > \epsilon^{\zeta}$ is small, we can potentially have a quantity that blows up for the bound. EI: the explanation needs adjusting - speak to Elena In order to solve this problem, we can use the fact that ∇u is uniformly bounded by $1/2$ thanks to Lemma 2, and then we can bound the difference of the gradients as follows:

$$\begin{aligned} \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 &\le \|\nabla u(s, \psi(s, Y_s^N)) + \nabla u(s, \psi(s, Y_s))\|^2 \\ &\le \sup_{(s,x) \in [0,T] \times \mathbb{R}} \|\nabla u(s, \psi(s, Y_s^N)) + \nabla u(s, \psi(s, Y_s))\|^2 \\ &= \|2\nabla u\|^2_{L_\infty}. \end{aligned} \tag{50}$$

Therefore we have that for all $t \in [0,T]$ LM: check where else I need to say this

$$\begin{aligned} I_t^{N,\epsilon} &\le \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \left(\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^{\zeta}\}} \right) e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} 2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \|Y_s^N - Y_s\|^{2\alpha} ds \right) \\ &+ \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^{\zeta}\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \|2\nabla u\|^2_{L_\infty} ds \right) \\ &\le \frac{1}{\epsilon} 3\mathbb{E}\left(2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \|\epsilon^{\zeta}\|^{2\alpha} t \right) + \frac{1}{\epsilon} 3\mathbb{E}\left(4 e^{1 - \epsilon^{\zeta - 1}} \|\nabla u\|^2_{L_\infty} t \right) \\ &\le \sup_{t \in [0,T]} \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \epsilon^{2\alpha\zeta} + 4 e^{1 - \epsilon^{\zeta - 1}} \|\nabla u\|^2_{L_\infty} \right) t \\ &= \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \epsilon^{2\alpha\zeta} + 4 e^{1 - \epsilon^{\zeta - 1}} \|\nabla u\|^2_{L_\infty} \right) T. \end{aligned} \tag{51}$$ {eq:INepsilon_bound}

Now by combining (41), (47) and (51), and taking the sup over $[0,T]$ we will get

$$
\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] \leq 4\epsilon & \\
& + 4\lambda 2\kappa_\rho T \left\| b^N - b \right\|_{C_T\mathcal{C}^{-\beta}} \\
& + 4\lambda \mathbb{E}\left[\int_0^t \| Y_s^N - Y^N \| ds \right] \\
& + \left\| b^N - b \right\|_{C_T\mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T\kappa_\rho^2 \left\| b^N - b \right\|_{C_T\mathcal{C}^{-\beta}} \\
& + \left\| b^N - b \right\|_{C_T\mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T(2\kappa_\rho)^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \left\| b^N - b \right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}} \\
& + \frac{3}{\epsilon} 2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} T \epsilon^{2\alpha\zeta} \\
& + \frac{3}{\epsilon} 4 \|\nabla u\|^2_{L_\infty} T e^{1-\epsilon^{\zeta-1}}
\end{aligned}
\tag{52}
$$

then we take $\epsilon = \left\| b^N - b \right\|_{C_T\mathcal{C}^{-\beta}}$ and we get

$$
\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] \leq 4 \left\| b^N - b \right\|_{C_T\mathcal{C}^{-\beta}} & \\
& + 4\lambda 2\kappa_\rho T \left\| b^N - b \right\|_{C_T\mathcal{C}^{-\beta}} \\
& + 4\lambda \mathbb{E}\left[\int_0^t \| Y_s^N - Y^N \| ds \right] \\
& + 3T\kappa_\rho^2 \left\| b^N - b \right\|_{C_T\mathcal{C}^{-\beta}} \\
& + 3T(2\kappa_\rho)^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \left\| b^N - b \right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}} \\
& + 2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} T \left\| b^N - b \right\|^{2\alpha\zeta-1}_{C_T\mathcal{C}^{-\beta}} \\
& + 4 \|\nabla u\|^2_{L_\infty} T \left\| b^N - b \right\|^{-1}_{C_T\mathcal{C}^{-\beta}} \exp\left(1 - \left\| b^N - b \right\|^{\zeta-1}_{C_T\mathcal{C}^{-\beta}}\right)
\end{aligned}
\tag{53}
$$

which can be written as

$$
\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] \leq c_1 \left\| b^N - b \right\|_{C_T\mathcal{C}^{-\beta}} + c_2 \mathbb{E}\left[\int_0^t \| Y_s^N - Y^N \| ds \right] & \\
+ c_3 \left\| b^N - b \right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}} + c_4 \left\| b^N - b \right\|^{2\alpha\zeta-1}_{C_T\mathcal{C}^{-\beta}} & \\
+ c_5 \exp\left(1 - \left\| b^N - b \right\|^{\zeta-1}_{C_T\mathcal{C}^{-\beta}}\right) &
\end{aligned}
\tag{54}
$$

{eq:bound_constants}

where

$$
\begin{aligned}
c_1 &= 4 + 4\lambda 2\kappa_\rho T + 3\kappa_\rho^2 T \\
c_2 &= 4\lambda \\
c_3 &= 3(2\kappa_\rho)^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} T \\
c_4 &= 2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} T \\
c_5 &= 4 \|\nabla u\|^2_{L_\infty} \left\| b^N - b \right\|^{-1}_{C_T\mathcal{C}^{-\beta}} T
\end{aligned}
\tag{55}
$$

{eq:constants_c}

Finally, observe that since $\zeta \in (0,1)$, the term $\exp\left(1 - \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{\zeta-1}\right)$ decays faster than any polynomial, thus controlling c_5, and the last term in (54) goes to zero. Also $\alpha\zeta$ is arbitrarily close to α, and $\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}$ controls $\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}$ therefore we can create the bound (34)

Question: is this clear enough? Am I making sense if I am taking α fixed? EI: no if α was fixed you could not do this. But $\alpha > \beta$ in your statement, hence it works. You need to explain the details however. Maybe at this stage you could introduce $\alpha' = \alpha\zeta$ to explain, that the result works for α' but since ζ can be chosen arbitrarily close to 1 then α' is arbitrarily close to α and α was chosen such that $\alpha > \beta$ which means the result is valid for all $\alpha' > \beta$. For simplicity we write α in place of α' in the statement.
Also it is better to explain the meaning of $o()$ and what terms go in there.

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq o\left(\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) + c_2\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + c_4\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \tag{56}$$

□

6 Convergence rate of the solution to the regularised SDE and the original

In this section we present a bound for $\mathbb{E}[X^N - X]$ in terms of $\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}$.

prop:reg_to_original **Proposition 5.** *Let assumptions 1 2 hold, then for any $\alpha > \beta$ there is a constant C_α such that*

$$\mathbb{E}[X^N - X] \leq C_\alpha \|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}, \tag{57}$$ {eq:EXN-X}

as $N \to \infty$.

Proof. Note that by definition of ψ, ψ^N we have

$$\begin{aligned} |X_t^N - X_t| &= |\psi^N(t, \phi^N(t, X_t^N)) - \psi(t, \phi(t, X_t))| \\ &= |\psi^N(t, Y_t^N) - \psi(t, Y_t)|, \end{aligned} \tag{58}$$

then adding and subtracting, and using the triangle inequality we get

$$|X_t^N - X_t| \leq |\psi^N(t, Y_t^N) - \psi(t, Y_t^N)| + |\psi(t, Y_t^N) - \psi(t, Y_t)|. \tag{59}$$

Where the first term is bounded by $2\kappa\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}$ (Proposition 3) and since ψ is 2-Lipschitz uniformly in $t \in [0, T]$ the second term is bounded by $2|Y^N - Y|$, therefore

$$|X^N - X| \leq 2\kappa\|b^N - b\|_{C_T\mathcal{C}^{-\beta}} + 2|Y^N - Y|. \tag{60}$$ {eq:XN-X}

By assumption the first term above goes to zero as $N \to \infty$, then we only need a bound for the second term.

By Itô-Tanaka's formula

$$|Y^N - Y| = |y_0^N - y_0| + \frac{1}{2}L_t^0(Y^N - Y) + \int_0^t \operatorname{sgn}(Y^N - Y)d(Y^N - Y), \tag{61}$$ {eq:YNY_ito_tanaka}

by taking expectation and using the definitions of Y^N, Y we have

$$\mathbb{E}|Y^N - Y| = \mathbb{E}|y_0^N - y_0| + \mathbb{E}\frac{1}{2}L_t^0(Y^N - Y) \\ + \lambda\mathbb{E}\int_0^t \operatorname{sgn}(Y^N - Y)(u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s)))ds, \tag{62}$$

{eq:EYNY}

then observe that the first term above is a constant, for the second we have a bound in Proposition prop:bound_local_time_sde 4, and for the third we use LM: Add the result to bound $u^N - u$, and the fact that $\operatorname{sgn}(x) \le 1$ therefore

$$\begin{aligned} \mathbb{E}|Y^N - Y| \le & |u^N(0,x) - u(0,x)| \\ & + \frac{1}{2}\left[o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\|ds\right) + B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right] \\ & + \mathbb{E}\left[\int_0^t \left(2\kappa_\rho \|b^N - b\|_{C_T\mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\|\right)ds\right] \\ & \le |u^N(0,x) - u(0,x)| \\ & + \frac{1}{2}\left[o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\|ds\right) + B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right] \\ & + \lambda 2\kappa_\rho \|b^N - b\|_{C_T\mathcal{C}^{-\beta}} t + \lambda\mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\|ds\right), \end{aligned} \tag{63}$$

Note that the terms in orange are controlled by $o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right)$, so after merging those terms and the two involving $\mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\|ds\right)$ we get

$$\mathbb{E}|Y^N - Y| \le B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} + (A+\lambda)\mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\|ds\right) + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right). \tag{64}$$

{eq:YN-Yineq}

From there, using Gronwall's lemma we get the following bound

$$\mathbb{E}|Y^N - Y| \le B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} T e^{(A+\lambda)T} + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right). \tag{65}$$

{eq:gronwallYNY}

Now we use (eq:gronwallYNY 65) to bound (eq:XN-X 60), and as the *small-o* term controls the second term in (eq:XN-X 60) we obtain

$$\mathbb{E}\left[|X^N - X|\right] \le B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} T e^{(A+\lambda)T} + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) \tag{66}$$

□

7 Small comment about convergence rate of Euler scheme to regularized equation

It works just like in [de_angelis_numerical_2020 1]

8 Convergence rate of Euler scheme

LM: check assumptions, maybe put them into the assumptions above or smth

Proposition 6. *Let X_t^{Nm} be the Euler approximation of the solution with m time steps, and X_t the real solution. Let also $\beta_0 \in (0,1/4)$, $\beta \in (\beta_0, 1/2)$, $\alpha \in (\beta, 1-\beta)$ and $\epsilon > 0$, then it holds that*

$$\sup_{0\le t\le T} \mathbb{E}\left[\left|X_t^{Nm} - X_t\right|\right] \le c m^{-\frac{1}{2}\mu}, \tag{67}$$

{eq:euler_rate}

where

$$\mu = 1 + \frac{\beta_0 + \epsilon}{(2\alpha - 1)(\beta - \beta_0) + \beta_0 + \epsilon}. \tag{68}$$

{eq:mu}

Proof. First, by triangle inequality we have

$$\circledast := \sup_{0\le t\le T} \mathbb{E}\left[\left|X_t^{Nm} - X_t\right|\right] \le \mathbb{E}\left[\left|X_t^{Nm} - X_t^N\right|\right] + \mathbb{E}\left[\left|X_t^N - X_t\right|\right], \tag{69}$$

{eq:er01}

the first term in the right hand side is bounded by [1][Proposition 3.4] de_angelis_numerical_2020 and the second one by Proposition 5 prop:reg_to_original, so that putting those results together we get

$$\begin{aligned}\circledast &\le A_N m^{-1} + B_N m^{-1/2} + c\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \\ &\le c\Big[\left\|b^N\right\|_{\infty,L^\infty}\left(1 + \left\|\nabla b^N\right\|_{\infty,L^\infty}\right) m^{-1} + \left(\left\|\nabla b^N\right\|_{\infty,L^\infty} + \left[b^N\right]_{1/2,L^\infty}\right) m^{-1/2} + \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\Big].\end{aligned} \tag{70}$$

{eq:er_constants}

We require to find some bounds for the L^∞ (semi) norms, so that we use Schauder estimates and Bernstein inequality LM: add Bernstein inequality or reference, this is possible thanks to the definition of $b^N := p_{f_m} * b$, where $f_m \to 0$ when $m \to \infty$, and also consider the definition of the norm

$$\|g\|_{C_T\mathcal{C}^\delta} = \left\|b^N\right\|_{L^\infty} + \sup_{x\neq y} \frac{|b^N(x) - b^N(y)|}{|x-y|^\delta},$$

and the seminorm

$$[g]_{1/2,L^\infty} = \sup_{t\neq st, s\in[0,T]} \frac{\|g(t) - g(s)\|_{L^\infty}}{|t-s|^{1/2}}.$$

We have the following bounds:

$$\left\|b^N\right\|_{L^\infty} \le \left\|b^N\right\|_{C_T\mathcal{C}^\epsilon} \le c f_m^{-\frac{\epsilon+\beta}{2}} \|b\|_{C_T\mathcal{C}^{-\beta}}, \tag{71}$$

{eq:er02}

$$\left\|\nabla b^N\right\|_{L^\infty} \le \left\|\nabla b^N\right\|_{C_T\mathcal{C}^\epsilon} \le c\left\|b^N\right\|_{C_T\mathcal{C}^{\epsilon+1}} \le c f_m^{-\frac{\epsilon+\beta+1}{2}} \|b\|_{C_T\mathcal{C}^{-\beta}}, \tag{72}$$

{eq:er03}

$$\begin{aligned}\left[b^N\right]_{1/2,L^\infty} &\le \sup_{t\neq s} \frac{\|b^N(t) - b^N(s)\|_{C_T\mathcal{C}^\epsilon}}{|t-s|^{1/2}} \\ &\le \sup_{t\neq s} c f_m^{-\frac{\epsilon+\beta}{2}} \frac{\|b(t) - b(s)\|_{C_T\mathcal{C}^{-\beta}}}{|t-s|^{1/2}} \\ &= c f_m^{-\frac{\epsilon+\beta}{2}} [b]_{1/2, C_T\mathcal{C}^{-\beta}}.\end{aligned} \tag{73}$$

{eq:er04}

Plugging that into (eq:er_constants 70) we get

$$
\begin{aligned}
\circledast \leq c \Big[& \left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha - 1} \\
& + \left\| b^N \right\|_{\infty, L^\infty} \left(1 + \left\| \nabla b^N \right\|_{\infty, L^\infty} \right) m^{-1} \\
& + \left(\left\| \nabla b^N \right\|_{\infty, L^\infty} + \left[b^N \right]_{1/2, L^\infty} \right) m^{-1/2} \Big] \\
\leq j &
\end{aligned} \tag{74}
$$

{eq:er_boundlinf}

□

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