

# Convergence rate of numerical solutions to SDEs with distributional drifts in Besov spaces

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## Contents

<b>1</b>	<b>What is this?</b>	<b>1</b>
<b>2</b>	<b>Some useful definitions and results</b>	<b>1</b>
<b>3</b>	<b>Bounds for the difference of solutions to the Kolmogorov equations</b>	<b>2</b>
<b>4</b>	<b>Bound for the difference of the auxiliary functions</b>	<b>7</b>
<b>5</b>	<b>Bound for the local time at zero of the solution to the SDEs</b>	<b>8</b>
<b>6</b>	<b>Convergence rate of the solution to the regularised SDE and the original</b>	<b>14</b>

## 1 What is this?

An adaptation of [1, Proposition 3.1] for the case of SDEs with drift in a Besov space of negative order similar to the ones proposed in [2] and [3]. The proof builds on a number of results presented in the sections below.

EI: add result about convergence of the scheme. This is done in two parts,  $X^N \rightarrow X$  done in Russo Issoglio, and  $X^{N,m} \rightarrow X^N$  Euler scheme convergence from De Angelis Germain Issoglio. Attention that the rate of convergence of Euler scheme depends of the smoothness of  $b^N$ .

## 2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

**Definition 1.** *Local time at zero* For any real-valued continuous semi-martingale  $Z$ , the local time at zero  $L_t^0(\bar{Y})$  is defined as

$$L_t^0(Z) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|Z| \leq \epsilon\}} d\langle Z \rangle_s, \quad \mathbb{P}\text{-a.s.} \quad (1)$$

For all  $t \geq 0$ .

The first result, [1, Lemma 5.1], is not necessary to prove for this particular setting since the result holds for any semi-martingale, I include it here for self-containment reasons. *EI: Instead of this sentence you should write something like 'The lemma below is from [1] and its proof can be found in [1, Lemma 5.1]. We include the statement here for ease of reading'*

`lemma:local-time-at-0`

**Lemma 1.** *Bound for local time at zero for a semi-martingale* For any  $\epsilon \in (0, 1)$  and any real-valued, continuous semi-martingale  $Z$  we have

$$\begin{aligned}\mathbb{E}[L_t^0(Z_s)] &\leq 4\epsilon - 2\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Z_s \in (0, \epsilon)\}} + \mathbb{1}_{\{Z_s \geq \epsilon\}} e^{1-Z_s/\epsilon}\right) dZ_s\right] \\ &\quad + \frac{1}{\epsilon} \mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z_s > \epsilon\}} e^{1-Z_s/\epsilon} d\langle Z \rangle_s\right].\end{aligned}$$

Let us introduce the original and regularised Kolmogorov equations. To shorten notation we will denote the spaces  $C_T \mathcal{C}^\gamma(\mathbb{R})$  as  $C_T \mathcal{C}^\gamma$ .

`def:kolmogorov_eqns`

**Definition 2.** *Kolmogorov equations* For  $\beta \in (0, 1/2)$  let  $b \in C_T \mathcal{C}^{-\beta}$ ,  $u, u^N \in C_T \mathcal{C}^{(1+\beta)+}$ , and  $b^N \rightarrow b$  as  $N \rightarrow \infty$  in  $C_T \mathcal{C}^{-\beta}$ . The equations

$$\begin{cases} \partial_t u_i + \frac{1}{2} \Delta u_i + b_i \nabla u_i = \lambda u_i - b_i \\ u_i(T) = 0 \end{cases} \quad (2) \quad \{\text{eq:kolmogorov}\}$$

$$\begin{cases} \partial_t u_i^N + \frac{1}{2} \Delta u_i^N + b_i^N \nabla u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0. \end{cases} \quad (3) \quad \{\text{eq:kolmogorov\_N}\}$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

`lemma:bounds_gradients`

**Lemma 2.** Let  $u, u^N$  be the solutions to the Kolmogorov equations (2) (3) in  $C_T \mathcal{C}^{1+\alpha}$  respectively. We have

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u(t,x)| \leq \frac{1}{2} \text{ and } \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\nabla u^N(t,x)| \leq \frac{1}{2} \quad (4)$$

`as:b_in_ctcb`

**Assumption 1.** Let  $0 < \beta < 1/2$  and  $b \in C_T \mathcal{C}^{-\beta}$ .

`:bN_converges_in_ctcb`

**Assumption 2.** There exists a sequence  $(b^N)_N \in C_T \mathcal{C}^{-\beta}$  such that for each  $N$ ,  $b^N(t, \cdot) \in C_b^\infty(\mathbb{R})$  for all  $t \in [0, T]$  and such that  $b^N \rightarrow b$  as  $N \rightarrow \infty$ .

LM: the lemma is probably better to type it before the first time it is used since it requires of some results below

LM: TYPE THIS

### 3 Bounds for the difference of solutions to the Kolmogorov equations

We need a bound for  $u - u^N$  and  $\nabla u - \nabla u^N$  in  $L_\infty$  for the case in which  $u \in C_T \mathcal{C}^{1+\alpha}$  for some  $\alpha > \beta$  which is an adaptation of [1, Lemma 5.2].

The result builds on top of the following result:

`prop:diff_u_uN`

**Proposition 1.** *Bound for the  $\rho$ -equivalent norm of  $u - u^N$*

Let  $u, u^N$  be (mild) solutions to the Kolmogorov equations from Definition [2](#) then as  $N \rightarrow \infty$

$$\|u_i - u_i^N\|_{C_T C^{1+\alpha}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T C^{-\beta}} (\|u_i\|_{C_T C^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}} (\|b\|_{C_T C^{-\beta}} + \lambda)} \quad (5)$$

for  $\rho \geq \rho_0$ , where

$$\rho_0 = 2c(\|b_i\|_{C_T \infty + \alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \quad (6)$$

and  $\lambda > 0$ .

*Proof.* See that  $u^N(T) = u(T) = 0$ , and in [\[2\]](#), set  $g^N, g$  as  $b^N, b$  respectively. See that  $b^N \rightarrow b$ . Then let us reformulate the rest of the aforementioned result for  $\lambda \neq 0$ .

As  $u^N, u$  are mild solutions, we have

$$\begin{aligned} u_i(t) - u_i^N(t) &= P_{T-t}(u_i(T) - u_i^N(T)) \\ &\quad + \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\ &\quad - \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds \\ &= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\ &\quad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\ &\quad - \int_t^T P_{s-t}(b_i - b_i^N) ds \\ &= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i^N b_i^N) + (\nabla u_i^N b_i^N - \nabla u_i^N b_i^N)] ds \\ &\quad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\ &\quad - \int_t^T P_{s-t}(b_i - b_i^N) ds \\ &= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\ &\quad + \int_t^T P_{s-t}(\nabla u_i^N b_i^N - \nabla u_i^N b_i^N) ds \\ &\quad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\ &\quad - \int_t^T P_{s-t}(b_i - b_i^N) ds \end{aligned}$$

Now let us compute the  $\rho$ -equivalent norm of  $u - u^N$ , for some  $\alpha > \beta$

!!!!!!LM: this norm is wrong, should be  $1 + \alpha$  on the lhs and everywhere else  
!!!!!!

$$\begin{aligned} \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} &= \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|u(t) - u^N(t)\|_{C_T \mathcal{C}^{1+\alpha}} \\ &\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} (\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\ &\quad + \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} (\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\ &\quad - \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t} (u_i - u_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\ &\quad - \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} (b_i - b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}}. \end{aligned}$$

Let us take each term from the right hand side of the inequality and bound them.

For the first term, using  $\gamma + 2\theta = 1 + \alpha$ ,  $\gamma = -\beta$ ,  $\theta = \frac{1+\alpha+\beta}{2}$ ,  $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta} \|f\|_\gamma$  and  $\|\nabla g\|_\xi \leq c\|g\|_{\xi+1}$

$$\begin{aligned} &\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} (\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\ &\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_{C_T \mathcal{C}^\alpha} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} ds \\ &\leq c \|u_i\|_{C_T \mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \sup_{0 \leq t \leq T} e^{-\rho(T-t)} (T-t)^{\frac{1-\beta-\alpha}{2}} \\ &\leq c T^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T \mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \end{aligned}$$

For the second term, see that for  $N \rightarrow \infty$ , we have  $\|b^N\|_{C_T \mathcal{C}^{-\beta}} \leq 2\|b\|_{C_T \mathcal{C}^{-\beta}}$

$$\begin{aligned} &\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} b_i^N (\nabla u_i - \nabla u_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\ &\leq c \sup_{0 \leq t \leq T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2 \|b_i\|_{C_T \mathcal{C}^{-\beta}} \|\nabla u_i - \nabla u_i^N\|_{C_T \mathcal{C}^{1+\alpha}} ds \\ &\leq c \|b_i\|_{C_T \mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} ds \\ &\leq c \|b_i\|_{C_T \mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}} \end{aligned}$$

For the third term, which is the one that differs from the proof in [2] we need to use `issoglio_pde_nodate` that  $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta} \|f\|_\gamma$ , and in this case we have  $\gamma + 2\theta = 1 + \alpha$  and  $\gamma = 1 + \alpha$ , so that  $\theta = 0$  because  $u, u^N \in C_T \mathcal{C}^{1+\alpha}$ , so we will have

$$\begin{aligned}
& \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{1+\alpha} \\
& \leq c\lambda \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-0} \|u_i - u_i^N\|_{1+\alpha} ds \\
& = c\lambda \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T e^{-\rho(T-s)} \sup_{0 \leq s \leq T} e^{-\rho(T-s)} \|u_i - u_i^N\|_{1+\alpha} ds \\
& = c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(T-s)} e^{-\rho(T-t)} ds \\
& = c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(s-t)} ds \\
& = c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \sup_{0 \leq t \leq T} \rho^{-1} [1 - e^{-\rho(T-t)}] \\
& \leq c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{-1} \\
& \leq c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}$$

And for the last term

$$\begin{aligned}
& \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{T-s}(b_i - b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \\
& \leq c \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} ds \\
& \leq c \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \sup_{0 \leq t \leq T} e^{-\rho(T-t)} (s-t)^{-\frac{\alpha+\beta-1}{2}} \\
& \leq c T^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}}
\end{aligned}$$

Putting everything together

$$\begin{aligned}
\|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} & \leq c T^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T \mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \\
& + c \|b_i\|_{C_T \mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}} \\
& - c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}} \\
& - c T^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}},
\end{aligned}$$

and finally,

$$\|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} (1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C_T \mathcal{C}^{-\beta}} + \lambda])$$

$$\leq cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)$$

$$\|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}$$

As required.  $\square$

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \quad (7)$$

**LM:** Check this norm

$$\|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \leq c(\rho)\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \quad (8)$$

Here is the adaptation of [1, Lemma 5.2].

**Proposition 2.** Bounds for  $\|u - u^N\|_{L^\infty}$  and  $\|\nabla u - \nabla u^N\|_{L^\infty}$

Let  $\beta \in (0, 1/2)$  and  $b \in C_T\mathcal{C}^{-\beta}$ . Let  $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$  be (mild) solutions to the Kolmogorov equations from Definition 2.

Assume, by Proposition 1, that for some  $\alpha > \beta$

$$\|u - u^N\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)} \leq c(\rho)\|b - b^N\|_{C_T\mathcal{C}^{-\beta}}. \quad (9)$$

With  $c(\rho)$  as in Proposition 1 and  $\rho_0$  is large enough such that  $c(\rho) > 0$  for all  $\rho > \rho_0$ . Then for all  $t \in [0, T]$

$$\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \quad (10)$$

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \quad (11)$$

with  $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$ .

*Proof.* First let us prove (10).

Let  $t \in [0, T]$ , and see that since  $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$  there exists  $\alpha > \beta$  such that  $u, u^N \in C_T\mathcal{C}^{1+\alpha}$ , then for any  $f \in \mathcal{C}^{1+\alpha}$  we have

$$\|f\|_{C^{1+\alpha}} \leq c \left( \sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \neq y \in \mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \right) \quad (12)$$

so we have

$$\begin{aligned} \|u^N(t) - u(t)\|_{L^\infty} &= \sup_{x \in \mathbb{R}^d} |u^N(t, x) - u(t, x)| \\ &\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \end{aligned} \quad (13)$$

Moreover, using the  $(\rho)$ -equivalent norm

$$\|f\|_{\mathcal{C}^{1+\alpha}} = \sup_{t \in [0, T]} e^{-\rho(T-t)} \|f(t)\|_{\mathcal{C}^{1+\alpha}}, \quad (14)$$

and (9) we see that

$$\begin{aligned}
\|u^N - u\|_{C_T \mathcal{C}^{1+\alpha}} &= \sup_{t \in [0, T]} \|u^N - u\|_{\mathcal{C}^{1+\alpha}} \\
&= \sup_{t \in [0, T]} e^{\rho(T-t)} e^{-\rho(T-t)} \|u^N - u\|_{\mathcal{C}^{1+\alpha}} \\
&\leq e^{\rho T} \sup_{t \in [0, T]} e^{-\rho(T-t)} \|u^N - u\|_{\mathcal{C}^{1+\alpha}} \\
&= e^{\rho T} \|u^N - u\|_{C_T \mathcal{C}^{1+\alpha}}^{(\rho)}. 
\end{aligned} \tag{15} \quad \boxed{\text{eq: norm bounded by r}}$$

Plugging (15) into (13)

$$\begin{aligned}
\|u^N(t) - u(t)\|_{L^\infty} &\leq c \|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \\
&\leq \sup_{t \in [0, T]} c \|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \\
&= c \|u^N - u\|_{C_T \mathcal{C}^{\alpha+1}} \\
&\leq c e^{\rho T} \|u^N - u\|_{C_T \mathcal{C}^{\alpha+1}}^{(\rho)}. 
\end{aligned} \tag{16}$$

And finally by (9)

$$\|u^N(t) - u(t)\|_{L^\infty} \leq c \cdot c(\rho) \cdot e^{\rho T} \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \tag{17}$$

which proves (10). For (11) recall that if  $f \in \mathcal{C}^{1+\alpha}$  then  $\nabla f \in \mathcal{C}^\alpha$ . Also, by Bernstein inequality [3, Eqn. 9)]

$$\|\nabla f\|_\alpha \leq c \|f\|_{\infty+\alpha}. \tag{18}$$

Using the equivalent norm

$$\|f\|_{C^{1+\alpha}} \leq c \left( \sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \in \mathbb{R}^d} |\nabla f(x)| + \sup_{x \neq y \in \mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \right) \tag{19}$$

we can see that

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq c \|u^N(t) - u(t)\|_{\mathcal{C}^{1+\alpha}}. \tag{20}$$

And usign the same bounds that we used above for  $c \|u^N(t) - u(t)\|_{\mathcal{C}^{1+\alpha}}$  this point follows.

□

## 4 Bound for the difference of the auxiliary functions

This is the adaptation of result [de angelis numerical 2020, Lemma 5.3].

prop:bound\_psi-psiN **Proposition 3.** Take  $\rho > \rho_0$  as in Proposition T,  $N \rightarrow \infty$ ,  $\kappa_\rho$  from Proposition Q, and  $\beta \in (0, 1/2)$ , then we have prop:diff\_uN\_graduN

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} \|\psi(t,x) - \psi^N(t,x)\| \leq 2\kappa_\rho \|b - b^N\|_{C_T C^{-\beta}} \quad (21)$$

*Proof.* Recall the definition of  $\psi, \phi \in C_T \mathcal{C}^1$

$$\phi(t, x) := x + u(t, x) \quad (22)$$

$$\psi(t, \cdot) = \phi^{-1}(t, \cdot). \quad (23)$$

Note that

$$u(y) = \int_0^1 \nabla u(\alpha y) y d\alpha + u(0). \quad (24)$$

From there we have

$$u(t, y) - u(t, y') = \int_0^1 \nabla u(t, \alpha(y - y')) (y - y') d\alpha \quad (25)$$

and therefore

$$\|u(t, y) - u(t, y')\| \geq \left( \int_0^1 \|\nabla u(t, \alpha(y - y'))\|^2 d\alpha \right)^{1/2} \|y - y'\|, \quad (26)$$

and by Lemma Q we finally have lemma:bounds\_gradients

$$\begin{aligned} \|u(t, y) - u(t, y')\| &\leq \left( \frac{1}{4} \int_0^1 d\alpha \right)^{1/2} \|y - y'\| \\ \|u(t, y) - u(t, y')\|^2 &\leq \frac{1}{4} \|y - y'\|^2 \end{aligned} \quad (27)$$

LM: continue from page three in notes

□

## 5 Bound for the local time at zero of the solution to the SDEs

LM: Here I still need to mention how we define  $Y_t = \psi(t, X_t)$ , because eventually I need to use that  $X_t = \psi(t, Y_t)$ , probably just need to mention without defining the whole  $Y_t$  as in the paper

We need a bound for  $\mathbb{E}[L_T^0(Y^N - Y)]$ , for Sobolev spaces, this is result [de angelis numerical 2020] Proposition 5.4] we present it here for the solutions to the SDE belonging to the appropriate Besov spaces.

First let us state the following useful result.

LM: check that the statement makes sense and has all the necessary assumptions

-n\_bound\_for\_integral

**Lemma 3.** Let  $u, u^N$  be solutions to the Kolmogorov equations [\(2\)](#) [\(3\)](#) then the following bound is satisfied:

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \leq 2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + \|Y_s^N - Y^N\|. \quad (28)$$

{eq:bound\_u\_abs}

*Proof.* Adding and subtracting terms, using triangle inequality and noting that for any  $a, b$ , we have  $a - b \leq \|a - b\|$ , then

$$\begin{aligned} \|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| &\leq \|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N))\| \\ &\quad + \|u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N))\| \\ &\quad + \|u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s))\|. \end{aligned} \quad (29)$$

The terms in the right hand side will be bounded as follows:

- For the first term, by Proposition [prop:diff\\_uN\\_graduN](#)

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N))\| \leq \|u^N(s) - u(s)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C_T C^{-\beta}}, \quad (30)$$

- for the second term, observe that  $u, u^N$  are  $\frac{1}{2}$ -Lipschitz and by Proposition [prop:bound\\_psi-psiN](#) we get

$$\begin{aligned} \|u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N))\| &\leq \frac{1}{2} \|\psi^N(s, Y_s^N) - \psi(s, Y_s^N)\| \\ &\leq \kappa_\rho \|b^N - b\|_{C_T C^{-\beta}}, \end{aligned} \quad (31)$$

- and for the final term, note that  $\psi, \psi^N$  are 2-Lipschitz so that

$$\|u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \leq \frac{1}{2} \|\psi(s, Y_s^N) - \psi(s, Y_s)\| \leq \|Y_s^N - Y_s\|. \quad (32)$$

So that the following bound holds

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \leq 2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + \|Y_s^N - Y_s\|, \quad (33)$$

as required.  $\square$

:bound\_local\_time\_sde

**Proposition 4.** Let  $A, B$  be constants,  $b \in C_T C^{-\beta}$  and  $b^N \rightarrow b$  in  $C_T C^{-\beta}$  as  $N \rightarrow \infty$  for  $\beta \in (0, \frac{1}{4})$  and for any  $\alpha > \beta$

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq o\left(\|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1}\right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + B \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1}. \quad (34)$$

{eq:local\_time\_YNY\_bo}

*Proof.* Recall that  $Y^N, Y$  are solutions to the SDEs

$$Y_t = y_0 + \lambda \int_0^t u(s, \psi(s, Y_s)) ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1) dW_s \quad (35)$$

and

$$Y_t^N = y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N)) ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1) dW_s \quad (36)$$

so that the difference  $Y^N - Y$  is

$$\begin{aligned} Y_t^N - Y_t &= \left( y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N)) ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1) dW_s \right) \\ &\quad - \left( y_0 + \lambda \int_0^t u(s, \psi(s, Y_s)) ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1) dW_s \right) \\ &= (y_0^N - y_0) + \lambda \int_0^t (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))) ds \\ &\quad + \int_0^t (\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))) dW_s, \end{aligned} \quad (37)$$

and using Lemma [\textcolor{red}{\text{lemma:local-time-at-0}}](#) we have the following bound

$$\begin{aligned} \mathbb{E}[L_t^0(Y^N - Y)] &\leq 4\epsilon \\ &\quad - 2\lambda \mathbb{E} \left[ \int_0^t \left( \mathbb{1}_{\{Y_s^N - Y_s \in (0, \epsilon)\}} + \mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \right) \right. \\ &\quad \left. (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))) ds \right] \end{aligned} \quad (38) \quad \boxed{\text{eq:local_time_diff_u}}$$

$$\frac{1}{\epsilon} \mathbb{E} \left[ \int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} (\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)))^2 ds \right]. \quad (39) \quad \boxed{\text{eq:local_time_diff_g}}$$

**LM: add the explanation of why to drop the diffusion term**

First, for [\(38\)](#), we find a bound for the factor involving the difference of  $u^N$  and  $u$  in Lemma [\textcolor{red}{\text{lemma:un-h\\_bound\\_for\\_integral}}](#). Therefore

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \leq 2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + \|Y_s^N - Y^N\|. \quad (40)$$

Now we need to bound the result of the local time of the difference  $Y_s^N - Y_s$ . First notice that  $Y_s^N - Y_s \geq \epsilon$ , then  $e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \leq 1$ , also it is clear that  $\mathbb{1}_{\{Y_s^N - Y_s \in (0, \epsilon)\}}$  and  $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}}$  are bounded by 1, therefore  $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \leq 1$ . Using the previous arguments and [\(28\)](#) lead to have

$$\begin{aligned} &\stackrel{(38)}{\leq} 2\lambda \mathbb{E} \left[ \int_0^t 2 \left( 2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + \|Y_s^N - Y^N\| \right) ds \right] \\ &\leq 4\lambda \left( 2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} t + \mathbb{E} \left[ \int_0^t \|Y_s^N - Y^N\| ds \right] \right). \end{aligned} \quad (41) \quad \boxed{\text{eq:bound_integral_uN}}$$

Now for (39), we use similar arguments as the ones in Lemma 3 above, and we get the following:

$$\begin{aligned} \|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\| &\leq \|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi^N(s, Y_s^N))\| \\ &\quad + \|\nabla u(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s^N))\| \\ &\quad + \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|, \end{aligned} \quad (42)$$

where the terms on the right hand side will be bounded as follows:

- For the first term we use Proposition 2 and we have

$$\begin{aligned} \|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi^N(s, Y_s^N))\| &\leq \|\nabla u^N(s) - \nabla u(s)\|_{L^\infty} \\ &\leq \kappa_\rho \|b - b^N\|_{C_T C^{-\beta}}, \end{aligned} \quad (43)$$

for the second term see that  $\nabla u, \nabla u^N$  are  $\alpha$ -Hölder continuous and using Proposition 3 we have

$$\begin{aligned} \|\nabla u(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s^N))\| &\leq \|\psi^N(s, Y_s^N) - \psi(s, Y_s^N)\|^\alpha \|u\|_{C_T C^{1+\alpha}} \\ &\leq (2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}})^\alpha \|u\|_{C_T C^{1+\alpha}}. \end{aligned} \quad (44)$$

Therefore we get the bound

$$\begin{aligned} \|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\| &\leq \kappa_\rho \|b - b^N\|_{C_T C^{-\beta}} \\ &\quad + \alpha \kappa_\rho^\alpha \|b^N - b\|_{C_T C^{-\beta}}^\alpha \|u\|_{C_T C^{1+\alpha}} \\ &\quad + \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|. \end{aligned} \quad (45)$$

Here we can also notice that  $\mathbb{E} \int_0^t \mathbb{1}_{\{Y_s^N - Y_s < \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} ds < 1$ , then using (45) and the inequality

$$(x_1 + \dots + x_k)^2 \leq k(x_1^2 + \dots + x_k^2), \quad (46)$$

for  $k = 3$ , we can get the bound

$$\begin{aligned} (39) &\leq \frac{1}{\epsilon} \mathbb{E} \int_0^t \left( 3\kappa_\rho^2 \|b - b^N\|_{C_T C^{-\beta}}^2 + 3 \cdot 2^{2\alpha} \kappa_\rho^{2\alpha} \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \right) ds \\ &\quad + \frac{1}{\epsilon} \mathbb{E} \int_0^t 3 \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 ds \\ &\leq \frac{1}{\epsilon} 3t \|b^N - b\|_{C_T C^{-\beta}} \left( \kappa_\rho^2 \|b^N - b\|_{C_T C^{-\beta}} + (2\kappa_\rho)^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1} \right) \\ &\quad + \frac{1}{\epsilon} 3 \mathbb{E} \left( \int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} |\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))|^2 ds \right) \end{aligned} \quad (47)$$

Now let us denote the last term in (47) by  $I_t^{N,\epsilon}$ . Pick  $\zeta \in (0, 1)$  such that  $\alpha\zeta > \frac{1}{2}$ , and since  $\epsilon \in (0, 1)$  we have  $\epsilon^\zeta > \epsilon$ . Then split the indicator function  $\mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}}$  into  $\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \leq \epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}}$ . Leading to the integral

$$I_t^{N,\epsilon} = \frac{1}{\epsilon} 3\mathbb{E} \left( \int_0^t (\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \leq \epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}}) e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \right. \\ \left. |\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))|^2 ds \right) \quad (48) \quad \boxed{\text{eq:INepsilon}}$$

For the first term of (48) we use the fact that  $\nabla u$  is  $\alpha$ -Hölder continuous uniformly in  $s \in [0, T]$  with constant  $\|u\|_{C_T C^{1+\alpha}}$  and that  $\psi$  is 2-Lipschitz

$$\begin{aligned} \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 &\leq \|\psi(s, Y_s^N) - \psi(s, Y_s)\|^\alpha \|u\|_{C_T C^{1+\alpha}}^2 \\ &\leq 2^\alpha \|Y_s^N - Y_s\|^\alpha \|u\|_{C_T C^{1+\alpha}}^2 \\ &= 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|Y_s^N - Y_s\|^{2\alpha} \end{aligned} \quad (49)$$

For the other term we need another way to bound it, because even though the event when  $\|Y^N - Y\| > \epsilon^\zeta$  is small, we can potentially have a quantity that blows up for the bound. **EI: the explanation needs adjusting - speak to Elena** In order to solve this problem, we can use the fact that  $\nabla u$  is uniformly bounded by  $1/2$  thanks to Lemma [Lemma:bounds\\_gradients](#), and then we can bound the difference of the gradients as follows:

$$\begin{aligned} \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 &\leq \|\nabla u(s, \psi(s, Y_s^N)) + \nabla u(s, \psi(s, Y_s))\|^2 \\ &\leq \sup_{(s,x) \in [0,T] \times \mathbb{R}} \|\nabla u(s, \psi(s, Y_s^N)) + \nabla u(s, \psi(s, Y_s))\|^2 \\ &= \|2\nabla u\|_{L_\infty}^2. \end{aligned} \quad (50)$$

Therefore we have that for all  $t \in [0, T]$  **LM: check where else I need to say this**

$$\begin{aligned} I_t^{N,\epsilon} &\leq \frac{1}{\epsilon} 3\mathbb{E} \left( \int_0^t (\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \leq \epsilon^\zeta\}}) e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|Y_s^N - Y_s\|^{2\alpha} ds \right) \\ &\quad + \frac{1}{\epsilon} 3\mathbb{E} \left( \int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \|2\nabla u\|_{L_\infty}^2 ds \right) \\ &\leq \frac{1}{\epsilon} 3\mathbb{E} \left( 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|\epsilon^\zeta\|^{2\alpha} t \right) + \frac{1}{\epsilon} 3\mathbb{E} \left( 4e^{1-\epsilon^\zeta-1} \|\nabla u\|_{L_\infty}^2 t \right) \\ &\leq \sup_{t \in [0, T]} \frac{3}{\epsilon} \left( 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \epsilon^{2\alpha\zeta} + 4e^{1-\epsilon^\zeta-1} \|\nabla u\|_{L_\infty}^2 \right) t \\ &= \frac{3}{\epsilon} \left( 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \epsilon^{2\alpha\zeta} + 4e^{1-\epsilon^\zeta-1} \|\nabla u\|_{L_\infty}^2 \right) T. \end{aligned} \quad (51) \quad \boxed{\text{eq:INepsilon_bound}}$$

Now by combining (41), (47) and (51), and taking the sup over  $[0, T]$  we will get

$$\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] &\leq 4\epsilon \\
&+ 4\lambda 2\kappa_\rho T \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \\
&+ 4\lambda \mathbb{E} \left[ \int_0^t \|Y_s^N - Y^N\| ds \right] \\
&+ \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T\kappa_\rho^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \\
&+ \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T(2\kappa_\rho)^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha-1} \\
&+ \frac{3}{\epsilon} 2^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 T \epsilon^{2\alpha\zeta} \\
&+ \frac{3}{\epsilon} 4 \|\nabla u\|_{L_\infty}^2 T e^{1-\epsilon\zeta-1}
\end{aligned} \tag{52}$$

then we take  $\epsilon = \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}$  and we get

$$\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] &\leq 4 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \\
&+ 4\lambda 2\kappa_\rho T \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \\
&+ 4\lambda \mathbb{E} \left[ \int_0^t \|Y_s^N - Y^N\| ds \right] \\
&+ 3T\kappa_\rho^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \\
&+ 3T(2\kappa_\rho)^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha-1} \\
&+ 2^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 T \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha\zeta-1} \\
&+ 4 \|\nabla u\|_{L_\infty}^2 T \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{-1} \exp \left( 1 - \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{\zeta-1} \right)
\end{aligned} \tag{53}$$

which can be written as

$$\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] &\leq c_1 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} + c_2 \mathbb{E} \left[ \int_0^t \|Y_s^N - Y^N\| ds \right] \\
&+ c_3 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha-1} + c_4 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha\zeta-1} \\
&+ c_5 \exp \left( 1 - \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{\zeta-1} \right)
\end{aligned} \tag{54} \quad \boxed{\text{eq:bound_constants}}$$

where

$$\begin{aligned}
c_1 &= 4 + 4\lambda 2\kappa_\rho T + 3\kappa_\rho^2 T \\
c_2 &= 4\lambda \\
c_3 &= 3(2\kappa_\rho)^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 T \\
c_4 &= 2^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 T \\
c_5 &= 4 \|\nabla u\|_{L_\infty}^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{-1} T
\end{aligned} \tag{55} \quad \boxed{\text{eq:constants_c}}$$

Finally, observe that since  $\zeta \in (0, 1)$ , the term  $\exp \left( 1 - \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{\zeta-1} \right)$  decays faster than any polynomial, thus controlling  $c_5$ , and the last term in (54) goes to zero. Also

$\alpha\zeta$  is arbitrarily close to  $\alpha$ , and  $\|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1}$  controls  $\|b^N - b\|_{C_T C^{-\beta}}$  therefore we can create the bound (34)

Question: is this clear enough? Am I making sense if I am taking  $\alpha$  fixed? EI: no if  $\alpha$  was fixed you could not do this. But  $\alpha > \beta$  in your statement, hence it works. You need to explain the details however. Maybe at this stage you could introduce  $\alpha' = \alpha\zeta$  to explain, that the result works for  $\alpha'$  but since  $\zeta$  can be chosen arbitrarily close to 1 then  $\alpha'$  is arbitrarily close to  $\alpha$  and  $\alpha$  was chosen such that  $\alpha > \beta$  which means the result is valid for all  $\alpha' > \beta$ . For simplicity we write  $\alpha$  in place of  $\alpha'$  in the statement.

Also it is better to explain the meaning of  $o()$  and what terms go in there.

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq o\left(\|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1}\right) + c_2 \mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + c_4 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1} \quad (56)$$

□

## 6 Convergence rate of the solution to the regularised SDE and the original

In this section we present a bound for  $\mathbb{E}[X^N - X]$  in terms of  $\|b^N - b\|_{C_T C^{-\beta}}$ .

**Proposition 5.** Let assumptions ~~12~~ hold, then for any  $\alpha > \beta$  there is a constant  $C_\alpha$  such that

$$\mathbb{E}[X^N - X] \leq C_\alpha \|b^N - b\|_{C_T C^{-\beta}}, \quad (57)$$

as  $N \rightarrow \infty$ .

*Proof.* Note that by definition of  $\psi, \psi^N$  we have

$$\begin{aligned} |X_t^N - X_t| &= |\psi^N(t, \phi^N(t, X_t^N)) - \psi(t, \phi(t, X_t))| \\ &= |\psi^N(t, Y_t) - \psi(t, Y_t)|, \end{aligned} \quad (58)$$

then adding and subtracting, and using the triangle inequality we get

$$|X_t^N - X_t| \leq |\psi^N(t, Y_t^N) - \psi(t, Y_t^N)| + |\psi(t, Y_t^N) - \psi(t, Y_t)|. \quad (59)$$

Where the first term is bounded by  $2\kappa \|b^N - b\|_{C_T C^{-\beta}}$  (Proposition 3) and since  $\psi$  is 2-Lipschitz uniformly in  $t \in [0, T]$  the second term is bounded by  $2|Y^N - Y|$ , therefore

$$|X^N - X| \leq 2\kappa \|b^N - b\|_{C_T C^{-\beta}} + 2|Y^N - Y|. \quad (60)$$

By assumption the first term above goes to zero as  $N \rightarrow \infty$ , then we only need a bound for the second term.

By Itô-Tanaka's formula

$$|Y^N - Y| = |y_0^N - y_0| + \frac{1}{2} L_t^0(Y^N - Y) + \int_0^t \operatorname{sgn}(Y^N - Y) d(Y^N - Y), \quad (61)$$

by taking expectation and using the definitions of  $Y^N, Y$  we have

$$\begin{aligned} \mathbb{E}|Y^N - Y| &= \mathbb{E}|y_0^N - y_0| + \mathbb{E}\frac{1}{2}L_t^0(Y^N - Y) \\ &\quad + \lambda\mathbb{E}\int_0^t \text{sgn}(Y^N - Y)(u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)))ds, \end{aligned} \tag{62} \quad \{\text{eq:EYNY}\}$$

then as the first term above is a constant and for the second we have a bound in Proposition [prop:bound\\_local\\_time\\_sde](#) it remains to bound the second term.

Observe that  $\square$

## References

- |                                    |   |
|------------------------------------|---|
| <code>ngelis_numerical_2020</code> | [1] Tiziano De Angelis, Maximilien Germain, and Elena Issoglio. “A Numerical Scheme for Stochastic Differential Equations with Distributional Drift”. In: <i>arXiv:1906.11026 [cs, math]</i> (Oct. 22, 2020). arXiv: <a href="#">1906.11026</a> . |
| <code>issoglio_pde_nodate</code>   | [2] Elena Issoglio and Francesco Russo. “A PDE with Drift of Negative Besov Index and Related Martingale Problem”. In: (), p. 48.   |
| <code>issoglio_mckean_2021</code>  | [3] Elena Issoglio and Francesco Russo. “McKean SDEs with singular coefficients”. In: <i>arXiv:2107.14453 [math]</i> (July 30, 2021). arXiv: <a href="#">2107.14453</a> .   |