Convergence rate of numerical solutions to SDEs with distributional drifts in Besov spaces

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Contents

1 What is this?

An adaptation of [1, Proposition 3.1] for the case of SDEs with drift in a Besov space of negative order similar to the ones proposed in [2] and [3]. The proof builds on a number of results presented in the sections below.

EI: add result about convergence of the scheme. This is done in two parts, $X^N \to X$ done in Russo Issoglio, and $X^{N,m} \to X^N$ Euler scheme convergence from De Angelis Germain Issoglio. Attention that the rate of convergence of Euler scheme depends of the smoothness of b^N.

2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

time_zero **Definition 1.** *For any real-valued continuous semi-martingale Z, the local time at zero $L^0_t(\bar{Y})$ is defined as*

$$L^0_t(Z) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t \mathbb{1}_{\{|Z|\le\epsilon\}} d\langle Z\rangle_s,\ \mathbb{P}\text{-}a.s. \tag{1}$$

For all $t \ge 0$.

The lemma below is from [1] and its proof can be found in [1, Lemma 5.1]. We include the statement here for ease of reading.

time-at-0 **Lemma 1.** *For any $\epsilon \in (0,1)$ and any real-valued, continuous semi-martingale Z we have*

$$\begin{aligned}\mathbb{E}[L^0_t(Z_s)] \le& 4\epsilon - 2\mathbb{E}\Big[\int_0^t \Big(\mathbb{1}_{\{Z_s\in(0,\epsilon)\}} + \mathbb{1}_{\{Z_s\ge\epsilon\}} e^{1-Z_s/\epsilon}\Big) dZ_s\Big] \\ &+ \frac{1}{\epsilon}\mathbb{E}\Big[\int_0^t \mathbb{1}_{\{Z>\epsilon\}} e^{1-Z_s/\epsilon} d\langle Z\rangle_s\Big].\end{aligned}$$

Let us introduce the original and regularised Kolmogorov equations. To shorten notation we will denote the spaces $C_T\mathcal{C}^\gamma(\mathbb{R})$ as $C_T\mathcal{C}^\gamma$.

LM: add Feynamn-Kac formula

Definition 2. *For $\beta \in (0, 1/2)$ let $b \in C_T\mathcal{C}^{-\beta}$, $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$, and $b^N \to b$ as $N \to \infty$ in $C_T\mathcal{C}^{-\beta}$ The equations*

$$\begin{cases} \partial_t u_i + \frac{1}{2}\Delta u_i + b_i \nabla u_i = \lambda u_i - b_i \\ u_i(T) = 0, \end{cases} \tag{2}$$

$$\begin{cases} \partial u_i^N + \frac{1}{2}\Delta u_i^N + b_i^N \nabla u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0. \end{cases} \tag{3}$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

Lemma 2. *Let u, u^N be the solutions to the Kolmogorov equations (2) (3) in $C_T\mathcal{C}^{1+\alpha}$ respectivelly. We have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u(t,x)| \le \frac{1}{2} \text{and} \sup_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u^N(t,x)| \le \frac{1}{2} \tag{4}$$

Assumption 1. *Let $0 < \beta < 1/2$ and $b \in C_T\mathcal{C}^{-\beta}$.*

Assumption 2. *There exists a sequence $(b^N)_N \in C_T\mathcal{C}^{-\beta}$ such that for each N, $b^N(t,\cdot) \in C_b^\infty(\mathbb{R})$ for all $t \in [0,T]$ and such that $b^N \to b$ as $N \to \infty$.*

Definition 3. *The function is called heat kernel*

$$p_f = \frac{1}{\sqrt{4\pi f}} e^{-\frac{|x-y|^2}{4f}}. \tag{5}$$

This is the fundamental solution to the heat equation.
The convolution of the heat kernel with a (generalised) function, is called heat semigroup:

$$P_f g = p_f * g = \int_\Omega p_f(x) g(x-y) dx \tag{6}$$

Lemma 3. *Given a function $f \in \mathcal{C}^\gamma$ for some $\gamma \in \mathbb{R}$, then for any $\theta \ge 0$ there exists a constant c such that*

$$\|P_t f\|_{\gamma+2\theta} \le c t^{-\theta} \|f\|_\gamma. \tag{7}$$

Moreover, for $f \in \mathcal{C}^\gamma$ and any $\theta \in (0,1)$ we have

$$\|P_t f - f\|_\gamma \le c t^\theta \|f\|_{\gamma+2\theta}. \tag{8}$$

Lemma 4. *Given a function $f \in \mathcal{C}^\gamma$ for some $\gamma \in \mathbb{R}$, there exists a constant $c > 0$ such that*

$$\|\nabla g\|_\gamma \le c\|g\|_{\gamma+1}. \tag{9}$$

LM: add the definition of mild solution for the Kolmogorov eqns

LM: theorem: exists a unique mild solution $u \in C_T\mathcal{C}^{1+\alpha}$ for all $\alpha \in (\beta, 1-\beta)$ cite Issoglio & Russo PDE Martingale problem

LM: from that theorem it follows that $\nabla u \in C_T\mathcal{C}^\alpha$, thus is α-Holder continuous

3 Bounds for the difference of solutions to the Kolmogorov equations

We need a bound for $u - u^N$ and $\nabla u - \nabla u^N$ in L_∞ for the case in which $u \in C_T\mathcal{C}^{1+\alpha}$ for some $\alpha \in (\beta, 1-\beta)$ which is an adaptation of [1, Lemma 5.2].

The result builds on top of the following result:

diff_u_uN **Proposition 1.** *Let u, u^N be (mild) solutions to the Kolmogorov equations from Definition* def:kolmogorov_eqns 2 *then as $N \to \infty$*

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \le \frac{cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}}(\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda)} \tag{10}$$

for $\rho \ge \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_T+\alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \tag{11}$$

and $\lambda > 0$.

Proof. See that $u^N(T) = u(T) = 0$, and in [issoglio_pde_nodate 2], set g^N, g as b^N, b respectively. See that $b^N \to b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$. As u^N, u are mild solutions, we have

$$\begin{aligned}
u_i(t) - u_i^N(t) &= P_{T-t}(u_i(T) - u_i^N(T)) + \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N)ds \\
&\qquad\qquad - \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N)ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N)ds - \lambda\int_t^T P_{s-t}(u_i - u_i^N)ds - \int_t^T P_{s-t}(b_i - b_i^N)ds \\
&= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)]ds \\
&\qquad - \lambda\int_t^T P_{s-t}(u_i - u_i^N)ds - \int_t^T P_{s-t}(b_i - b_i^N)ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N)ds + \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N)ds \\
&\qquad - \lambda\int_t^T P_{s-t}(u_i - u_i^N)ds - \int_t^T P_{s-t}(b_i - b_i^N)ds
\end{aligned}$$

Now let us compute the ρ-equivalent norm of $u - u^N$, for some $\alpha > \beta$

$$\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} &= \sup_{0\le t\le T} e^{-\rho(T-t)}\|u(t) - u^N(t)\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\le \sup_{0\le t\le T} e^{-\rho(T-t)}\Bigg[\left\|\int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N)ds\right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad + \left\|\int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N)ds\right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad - \left\|\lambda\int_t^T P_{s-t}(u_i - u_i^N)ds\right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad - \left\|\int_t^T P_{s-t}(b_i - b_i^N)ds\right\|_{C_T\mathcal{C}^{1+\alpha}}\Bigg].
\end{aligned}$$

Let us take each term from the right hand side of the inequality and bound them.
LM: change this sentence accoring to the addition of Schauder estimates
Using lemma:schauder_estimates 3 For the first term, using $\gamma + 2\theta = 1 + \alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \le ct^{-\theta}\|f\|_\gamma$ and $\|\nabla g\|_\xi \le c\|g\|_{\xi+1}$

$$\begin{aligned}
&\sup_{0\le t\le T} e^{-\rho(T-t)}\left\|\int_t^T P_{s-t}\left(\nabla u_i b_i - \nabla u_i b_i^N\right)ds\right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad \le \sup_{0\le t\le T} e^{-\rho(T-t)}\int_t^T (s-t)^{-\theta}\|\nabla u_i\|_{C_T\mathcal{C}^\alpha}\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \\
&\qquad \le c\|u_i\|_{C_T\mathcal{C}_{1+\alpha}}\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\le t\le T} e^{-\rho(T-t)}(T-t)^{\frac{1-\beta-\alpha}{2}}
\end{aligned}$$

$$\leq cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T\mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}}$$

For the second term, see that for $N \to \infty$, we have $\|b^N\|_{C_T\mathcal{C}^{-\beta}} \leq 2\|b\|_{C_T\mathcal{C}^{-\beta}}$

$$\begin{aligned}
\sup_{0\leq t\leq T} e^{-\rho(T-t)} &\left\| \int_t^T P_{s-t} b_i^N (\nabla u_i - \nabla u_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\leq c \sup_{0\leq t\leq T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|\nabla u_i - \nabla u_i^N\|_{C_T\mathcal{C}^{1+\alpha}} ds \\
&\leq c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} ds \\
&\leq c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}$$

For the third term, which is the one that difers from the proof in [2] we need to use that $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta}\|f\|_{\gamma}$, and in this case we have $\gamma + 2\theta = 1+\alpha$ and $\gamma = 1+\alpha$, so that $\theta = 0$ because $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, so we will have

$$\begin{aligned}
\sup_{0\leq t\leq T} e^{-\rho(T-t)} &\left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{1+\alpha} \\
&\leq c\lambda \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-0} \|u_i - u_i^N\|_{1+\alpha} ds \\
&= c\lambda \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T e^{-\rho(T-s)} \sup_{0\leq s\leq T} e^{-\rho(T-s)} \|u_i - u_i^N\|_{1+\alpha} ds \\
&= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \int_t^T e^{-\rho(T-s)} e^{-\rho(T-t)} ds \\
&= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \int_t^T e^{-\rho(s-t)} ds \\
&= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \sup_{0\leq t\leq T} \rho^{-1}[1 - e^{-\rho(T-t)}] \\
&\leq c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{-1} \\
&\leq c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}$$

And for the last term

$$\begin{aligned}
\sup_{0\leq t\leq T} e^{-\rho(T-t)} &\left\| \int_t^T P_{T-s}(b_i - b_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\leq c \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} ds \\
&\leq c\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\leq t\leq T} e^{-\rho(T-t)} (s-t)^{-\frac{\alpha+\beta-1}{2}} \\
&\leq cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}
\end{aligned}$$

Putting everything together

$$\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq\ & cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T\mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \\
&+ c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \rho^{\frac{\alpha+\beta-1}{2}} \\
&- c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}$$

$$- cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}},$$

and finally,

$$\begin{aligned}\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} (1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda]) \\ \leq cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1) \\ \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}\end{aligned}$$

As required. □

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{12}$$

LM: Check this norm, it was in $-\beta$ now in $1 + \alpha$

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq c(\rho) \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{13}$$

Here is the adaptation of [1, Lemma 5.2]. de_angelis_numerical_2020

uN_graduN **Proposition 2.** *Let $\beta \in (0, 1/2)$ and $b \in C_T\mathcal{C}^{-\beta}$. Let $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ be (mild) solutions to the Kolmogorov equations from Definition 2.* def:kolmogorov_eqns

Assume, by Proposition 1, that for some $\alpha > \beta$ prop:diff_u_uN

$$\|u - u^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq c(\rho) \|b - b^N\|_{C_T\mathcal{C}^{-\beta}}. \tag{14}$$

{eq:u-uNb

With $c(\rho)$ as in Proposition 1 and ρ_0 is large enough such that $c(\rho) > 0$ for all $\rho > \rho_0$. prop:diff_u_uN

Then for all $t \in [0, T]$

$$\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{15}$$

{eq:uNu_b

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{16}$$

{eq:gradu

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$.

Proof. First let us prove (15). eq:uNu_bounded_by_bNb

Let $t \in [0, T]$, and see that since $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ there exists $\alpha > \beta$ such that $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, then for any $f \in \mathcal{C}^{1+\alpha}$ we have

$$\|f\|_{C^{1+\alpha}} \leq c \left(\sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \neq y \in \mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \right) \tag{17}$$

so we have

$$\begin{aligned}\|u^N(t) - u(t)\|_{L^\infty} &= \sup_{x \in \mathbb{R}^d} |u^N(t, x) - u(t, x)| \\ &\leq c \|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}}\end{aligned} \tag{18}$$

{eq:u-uN_

Moreover, using the (ρ)-equivalent norm

$$\|f\|_{\mathcal{C}^{1+\alpha}} = \sup_{t \in [0, T]} e^{-\rho(T-t)} \|f(t)\|_{\mathcal{C}^{1+\alpha}}, \tag{19}$$

and (14) we see that

$$\begin{aligned}\left\|u^N-u\right\|_{C_T\mathcal{C}^{1+\alpha}} &= \sup_{t\in[0,T]}\left\|u^N-u\right\|_{\mathcal{C}^{1+\alpha}}\\ &= \sup_{t\in[0,T]} e^{\rho(T-t)}e^{-\rho(T-t)}\left\|u^N-u\right\|_{\mathcal{C}^{1+\alpha}}\\ &\le e^{\rho T}\sup_{t\in[0,T]} e^{-\rho(T-t)}\left\|u^N-u\right\|_{\mathcal{C}^{1+\alpha}}\\ &= e^{\rho T}\left\|u^N-u\right\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}}.\end{aligned} \tag{20}$$

Plugging (20) into (18)

$$\begin{aligned}\|u^N(t)-u(t)\|_{L^\infty} &\le c\|u^N(t)-u(t)\|_{\mathcal{C}^{\alpha+1}}\\ &\le \sup_{t\in[0,T]} c\left\|u^N(t)-u(t)\right\|_{\mathcal{C}^{\alpha+1}}\\ &= c\left\|u^N-u\right\|_{C_T\mathcal{C}^{\alpha+1}}\\ &\le ce^{\rho T}\left\|u^N-u\right\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}.\end{aligned} \tag{21}$$

And finally by (14)

$$\left\|u^N(t)-u(t)\right\|_{L^\infty} \le c\cdot c(\rho)\cdot e^{\rho T}\left\|b^N-b\right\|_{C_T\mathcal{C}^{-\beta}} \tag{22}$$

which proves (15).
For (16) recall that if $f\in\mathcal{C}^{1+\alpha}$ then $\nabla f\in\mathcal{C}^{\alpha}$. Also, by Bernstein inequality (9)

$$\|\nabla f\|_{\alpha} \le c\|f\|_{1+\alpha}. \tag{23}$$

Using the equivalent norm

$$\|f\|_{\mathcal{C}^{1+\alpha}} \le c\left(\sup_{x\in\mathbb{R}^d}|f(x)| + \sup_{x\in\mathbb{R}^d}|\nabla f(x)| + \sup_{x\neq y\in\mathbb{R}^d}\frac{|\nabla f(x)-\nabla f(y)|}{|x-y|^{\alpha}}\right) \tag{24}$$

we can see that

$$\left\|\nabla u^N(t)-\nabla u(t)\right\|_{L^\infty} \le c\left\|u^N(t)-u(t)\right\|_{\mathcal{C}^{1+\alpha}}. \tag{25}$$

And usign the same bounds that we used above for $c\left\|u^N(t)-u(t)\right\|_{\mathcal{C}^{1+\alpha}}$ this point follows.

□

4 Bound for the difference of the auxiliary functions

This is the adaptation of result [1, Lemma 5.3].

Proposition 3. *Take $\rho>\rho_0$ as in Proposition 1, $N\to\infty$, κ_ρ from Proposition 2, and $\beta\in(0,1/2)$, then we have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}\|\psi(t,x)-\psi^N(t,x)\| \le 2\kappa_\rho\left\|b-b^N\right\|_{C_T\mathcal{C}^{-\beta}} \tag{26}$$

Proof. Recall the definition of $\psi,\phi\in C_T\mathcal{C}^1$

$$\phi(t,x) := x+u(t,x) \tag{27}$$

$$\psi(t,\cdot) = \phi^{-1}(t,\cdot). \tag{28}$$

Note that

$$u(y) = \int_0^1 \nabla u(\alpha y) y d\alpha + u(0). \tag{29}$$

From there we have

$$u(t,y) - u(t,y') = \int_0^1 \nabla u(t,\alpha(y-y'))(y-y') d\alpha \tag{30}$$

and therefore

$$\|u(t,y) - u(t,y')\| \geq \left(\int_0^1 \|\nabla u(t,\alpha(y-y'))\|^2 d\alpha \right)^{1/2} \|y-y'\|, \tag{31}$$

and by Lemma lemma:bounds_gradients 2 we finally have

$$\begin{aligned} \|u(t,y) - u(t,y'\|) &\leq \left(\frac{1}{4} \int_0^1 d\alpha \right)^{1/2} \|y-y\|' \\ \|u(t,y) - u(t,y')\|^2 &\leq \frac{1}{4} \|y-y'\|^2 \end{aligned} \tag{32}$$

LM: continue from page three in notes

□

5 Bound for the local time at zero of the solution to the SDEs

LM: Here I still need to mention how we define $Y_t = \psi(t, X_t)$, because eventually I need to use that $X_t = \psi(t, Y_t)$, probably just neet to mention without defining the whole Y_t as in the paper

We need a bound for $\mathbb{E}[L_T^0(Y^N - Y)]$, for Sobolev spaces, this is result de_angelis_numerical_2020 [1, Proposition 5.4] we present it here for the solutions to the SDE blonging to the appropriate Besov spaces.

First let us state the following useful result.

LM: check that the statement makes sense and has all the necessary assumptions

_integral **Lemma 5.** *Let u, u^N be solutions to the Kolmogorov equations eq:kolmogorov eq:kolmogorov_N (2) (3) then the following bound is satisfied:*

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)\|) \leq 2\kappa_\rho \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\|. \tag{33}$$ {eq:bound

Proof. Adding and subtracting terms, using triangle inequality and noting that for any a, b, we have $a - b \leq \|a - b\|$, then

$$\begin{aligned} \|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)\|) \leq& \|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N))\|) \\ &+ \|u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N))\|) \\ &+ \|u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s))\|. \end{aligned} \tag{34}$$

The terms in the right hand side will be bounded as follows:

- For the first term, by Proposition prop:diff_uN_graduN 2

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N))\|) \leq \left\|u^N(s) - u(s)\right\|_{L^\infty} \leq \kappa_\rho \left\|b - b^N\right\|_{C_T \mathcal{C}^{-\beta}}, \tag{35}$$

- for the second term, observe that u, u^N are $\frac{1}{2}$-Lipschitz and by Proposition prop:bound_psi-psiN 3 we get

$$\begin{aligned} \|u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N))\|) &\leq \frac{1}{2} \|\psi^N(s, Y_s^N) - \psi(s, Y_s^N)\|) \\ &\leq \kappa_\rho \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}}, \end{aligned} \tag{36}$$

- and for the final term, note that ψ, ψ^N are 2-Lipschitz so that

$$\|u(s,\psi(s,Y_s^N)) - u(s,\psi(s,Y_s)\|) \le \frac{1}{2}\|\psi(s,Y_s^N) - \psi(s,Y_s\|) \le \|Y_s^N - Y_s\|. \tag{37}$$

So that the following bound holds

$$\|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s)\|) \le 2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\|, \tag{38}$$

as required.

□

_time_sde **Proposition 4.** *Let A,B be constants, $b \in C_T\mathcal{C}^{-\beta}$ and $b^N \to b$ in $C_T\mathcal{C}^{-\beta}$ as $N \to \infty$ for $\beta \in (0,\frac{1}{4})$ and for any $\alpha \in (\beta, 1-\beta)$*

$$\mathbb{E}[L_t^0(Y^N - Y)] \le o\left(\left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}\right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\|ds\right) + B\left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}. \tag{39}$$ {eq:local

Proof. Recall that Y^N, Y are solutions to the SDEs

$$Y_t = y_0 + \lambda \int_0^t u(s,\psi(s,Y_s))ds + \int_0^t (\nabla u(s,\psi(s,Y_t)) + 1)dW_s \tag{40}$$

and

$$Y_t^N = y_0^N + \lambda \int_0^t u^N(s,\psi^N(s,Y_s^N))ds + \int_0^t (\nabla u^N(s,\psi^N(s,Y_t^N)) + 1)dW_s \tag{41}$$

so that the difference $Y^N - Y$ is

$$\begin{aligned} Y_t^N - Y_t &= \left(y_0^N + \lambda \int_0^t u^N(s,\psi^N(s,Y_s^N))ds + \int_0^t (\nabla u^N(s,\psi^N(s,Y_t^N)) + 1)dW_s\right) \\ &\quad - \left(y_0 + \lambda \int_0^t u(s,\psi(s,Y_s))ds + \int_0^t (\nabla u(s,\psi(s,Y_t)) + 1)dW_s\right) \\ &= (y_0^N - y_0) + \lambda \int_0^t (u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s)))ds \\ &\quad + \int_0^t (\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s)))dW_s, \end{aligned} \tag{42}$$

and using Lemma 1 (lemma:local-time-at-0) we have the following bound

$$\begin{aligned} &\mathbb{E}[L_t^0(Y^N - Y)] \\ &\le 4\epsilon \\ &- 2\lambda\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Y_s^N - Y_s \ge \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}}\right)\right. \\ &\qquad\qquad \left.\left(u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s))\right)ds\right] \end{aligned} \tag{43}$$ {eq:local

$$+ \frac{1}{\epsilon}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \left(\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right)^2 ds\right]. \tag{44}$$ {eq:local

LM: add the explanation of why to drop the difusion term

First, for (43) (eq:local_time_diff_u), we find a bound for the factor involving the difference of u^N and u in Lemma 5 (lemma:uN-n_bound_for_integral). Therefore

$$\|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s)\|) \le 2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\|. \tag{45}$$

Now we need to bound the result of the local time of the difference $Y_s^N - Y_s$. First notice that $Y_s^N - Y_s \ge \epsilon$, then $e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \le 1$, also it is clear that $\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}}$ and $\mathbb{1}_{\{Y_s^N - Y_s \ge \epsilon\}}$ are bounded by 1, therefore $\mathbb{1}_{\{Y_s^N - Y_s \ge \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \le 1$. Using the previous arguments and (33) (eq:bound_u_abs) lead to have

$$
\begin{aligned}
(43) &\le 2\lambda \mathbb{E}\left[\int_0^t 2\left(2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + \left\|Y_s^N - Y^N\right\|\right) ds\right] \\
&\le 4\lambda\left(2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} t + \mathbb{E}\left[\int_0^t \left\|Y_s^N - Y^N\right\| ds\right]\right).
\end{aligned} \tag{46}
$$

eq:local_time_diff_u {eq:bound

Now for (eq:local_time_diff_gradu 44), we use similar arguments as the ones in Lemma lemma:uN-n_bound_for_integral 5 above, and we get the following:

$$
\begin{aligned}
\|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\| \le{}& \|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi^N(s,Y_s^N))\| \\
&+ \|\nabla u(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s^N))\| \\
&+ \|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|,
\end{aligned} \tag{47}
$$

where the terms on the right hand side will be bounded as follows:

- For the first term we use Proposition prop:diff_uN_graduN 2 and we have

$$
\begin{aligned}
\|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi^N(s,Y_s^N))\|) &\le \left\|\nabla u^N(s) - \nabla u(s)\right\|_{L^\infty} \\
&\le \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}},
\end{aligned} \tag{48}
$$

  for the second term see that $\nabla u, \nabla u^N$ are α-Hölder continuous and using Proposition prop:bound_psi-psiN 3 we have

  LM: mention for which alpha this is possible and refer to the remark we have to add above

$$
\begin{aligned}
\|\nabla u(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s^N))\|) &\le \|\psi^N(s,Y_s^N) - \psi(s,Y_s^N)\|^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\le (2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}})^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}}.
\end{aligned} \tag{49}
$$

Therefore we get the bound

$$
\begin{aligned}
\|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|) \le{}& \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ \alpha \kappa_\rho^\alpha \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}} \\
&+ \|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|.
\end{aligned} \tag{50}
$$

{eq:bound

Here we can also notice that $\mathbb{1}_{\{Y_s^N - Y_s < \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} < 1$, then using (eq:bound_gradu_abs 50) and the inequality

$$
(x_1 + \cdots + x_k)^2 \le k(x_1^2 + \cdots + x_k^2), \tag{51}
$$

for $k = 3$, we can get the bound

$$
\begin{aligned}
(44) \le{}& \frac{1}{\epsilon}\mathbb{E}\int_0^t \left(3\kappa_\rho^2 \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}}^2 + 3\cdot 2^{2\alpha} \kappa_\rho^{2\alpha} \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}}^2\right) ds \\
&+ \frac{1}{\epsilon}\mathbb{E}\int_0^t 3\mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|^2 ds \\
\le{}& \frac{1}{\epsilon} 3t \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \left(\kappa_\rho^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + (2\kappa_\rho)^{2\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}}^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) \\
&+ \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \left|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right|^2 ds\right)
\end{aligned} \tag{52}
$$

eq:local_time_diff_gradu {eq:bound

Now let us denote the last term in (eq:bound_integral_graduNu 52) by $I_t^{N,\epsilon}$. Pick $\zeta \in (0,1)$ such that $\alpha\zeta > \frac{1}{2}$, and since $\epsilon \in (0,1)$ we have $\epsilon^\zeta > \epsilon$. Then split the indicator function $\mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}}$ into $\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}}$. Leading to the integral

$$
\begin{aligned}
I_t^{N,\epsilon} = \frac{1}{\epsilon} 3\mathbb{E}\Bigg(\int_0^t &\left(\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}}\right) e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \\
&\left|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right|^2 ds\Bigg)
\end{aligned} \tag{53}
$$

{eq:INeps

For the first term of (53) we use the fact that ∇u is α-Hölder continuous uniformly in $s \in [0,T]$ with constant $\|u\|_{C_T\mathcal{C}^{1+\alpha}}$ and that ψ is 2-Lipschitz

$$
\begin{aligned}
\|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|^2 &\le \| \|\psi(s,Y_s^N) - \psi(s,Y_s)\|^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}} \|^2 \\
&\le \|2^\alpha \|Y_s^N - Y_s\|^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}} \|^2 \\
&= 2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \|Y_s^N - Y_s\|^{2\alpha}
\end{aligned}
\tag{54}
$$

For the other term we need another way to bound it, because even though the event when $\|Y^N - Y\| > \epsilon^\zeta$ is small, we can potentially have a quantity that blows up for the bound. EI: the explanation needs adjusting - speak to Elena In order to solve this problem, we can use the fact that ∇u is uniformly bounded by $1/2$ thanks to Lemma 2, and then we can bound the difference of the gradients as follows:

$$
\begin{aligned}
\|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|^2 &\le \|\nabla u(s,\psi(s,Y_s^N)) + \nabla u(s,\psi(s,Y_s))\|^2 \\
&\le \sup_{(s,x)\in[0,T]\times\mathbb{R}} \|\nabla u(s,\psi(s,Y_s^N)) + \nabla u(s,\psi(s,Y_s))\|^2 \\
&= \|2\nabla u\|^2_{L_\infty}.
\end{aligned}
\tag{55}
$$

Therefore we have that for all $t \in [0,T]$ LM: check where else I need to say this

$$
\begin{aligned}
I_t^{N,\epsilon} &\le \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \left(\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^\zeta\}} \right) e^{1-\frac{Y_s^N - Y_s}{\epsilon}} 2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \|Y_s^N - Y_s\|^{2\alpha} ds \right) \\
&+ \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \|2\nabla u\|^2_{L_\infty} ds \right) \\
&\le \frac{1}{\epsilon} 3\mathbb{E}\left(2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \|\epsilon^\zeta\|^{2\alpha} t \right) + \frac{1}{\epsilon} 3\mathbb{E}\left(4e^{1-\epsilon^{\zeta-1}} \|\nabla u\|^2_{L_\infty} t \right) \\
&\le \sup_{t\in[0,T]} \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \epsilon^{2\alpha\zeta} + 4e^{1-\epsilon^{\zeta-1}} \|\nabla u\|^2_{L_\infty} \right) t \\
&= \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \epsilon^{2\alpha\zeta} + 4e^{1-\epsilon^{\zeta-1}} \|\nabla u\|^2_{L_\infty} \right) T.
\end{aligned}
\tag{56}
$$

{eq:INeps

Now by combining (46), (52) and (56), and taking the sup over $[0,T]$ we will get

$$
\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] \le 4\epsilon \\
&+ 4\lambda 2\kappa_\rho T \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ 4\lambda \mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\| ds \right] \\
&+ \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T\kappa_\rho^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T(2\kappa_\rho)^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}} \\
&+ \frac{3}{\epsilon} 2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} T\epsilon^{2\alpha\zeta} \\
&+ \frac{3}{\epsilon} 4 \|\nabla u\|^2_{L_\infty} T e^{1-\epsilon^{\zeta-1}}
\end{aligned}
\tag{57}
$$

then we take $\epsilon = \|b^N - b\|_{C_T\mathcal{C}^{-\beta}}$ and we get

$$\begin{aligned}\mathbb{E}[L_t^0(Y^N - Y)] \le{}& 4\|b^N - b\|_{C_T\mathcal{C}^{-\beta}} \\ &+ 4\lambda 2\kappa_\rho T\|b^N - b\|_{C_T\mathcal{C}^{-\beta}} \\ &+ 4\lambda\mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\| ds\right] \\ &+ 3T\kappa_\rho^2\|b^N - b\|_{C_T\mathcal{C}^{-\beta}} \\ &+ 3T(2\kappa_\rho)^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}}\|b^N - b\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}} \\ &+ 2^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}} T\|b^N - b\|^{2\alpha\zeta-1}_{C_T\mathcal{C}^{-\beta}} \\ &+ 4\|\nabla u\|^2_{L_\infty} T\|b^N - b\|^{-1}_{C_T\mathcal{C}^{-\beta}}\exp\left(1 - \|b^N - b\|^{\zeta-1}_{C_T\mathcal{C}^{-\beta}}\right)\end{aligned} \tag{58}$$

which can be written as

$$\begin{aligned}\mathbb{E}[L_t^0(Y^N - Y)] \le{}& c_1\|b^N - b\|_{C_T\mathcal{C}^{-\beta}} + c_2\mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\| ds\right] \\ &+ c_3\|b^N - b\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}} + c_4\|b^N - b\|^{2\alpha\zeta-1}_{C_T\mathcal{C}^{-\beta}} \\ &+ c_5\exp\left(1 - \|b^N - b\|^{\zeta-1}_{C_T\mathcal{C}^{-\beta}}\right)\end{aligned} \tag{59}$$ {eq:bound

where

$$\begin{aligned}c_1 &= 4 + 4\lambda 2\kappa_\rho T + 3\kappa_\rho^2 T \\ c_2 &= 4\lambda \\ c_3 &= 3(2\kappa_\rho)^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}} T \\ c_4 &= 2^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}} T \\ c_5 &= 4\|\nabla u\|^2_{L_\infty}\|b^N - b\|^{-1}_{C_T\mathcal{C}^{-\beta}} T\end{aligned} \tag{60}$$ {eq:const

Finally, observe that since $\zeta \in (0,1)$, the term $\exp\left(1 - \|b^N - b\|^{\zeta-1}_{C_T\mathcal{C}^{-\beta}}\right)$ decays faster than any polynomial, thus controlling c_5, and the last term in (59) goes to zero. Also $\alpha\zeta$ is arbitrarily close to α, and $\|b^N - b\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}$ controls $\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}$ therefore we can create the bound (39)

Question: is this clear enough? Am I making sense if I am taking α fixed? EI: no if α was fixed you could not do this. But $\alpha > \beta$ in your statement, hence it works. You need to explain the details however. Maybe at this stage you could introduce $\alpha' = \alpha\zeta$ to explain, that the result works for α' but since ζ can be chosen arbitrarily close to 1 then α' is arbitrarily close to α and α was chosen such that $\alpha > \beta$ which means the result is valid for all $\alpha' > \beta$. For simplicity we write α in place of α' in the statement.
Also it is better to explain the meaning of $o()$ and what terms go in there.

$$\mathbb{E}[L_t^0(Y^N - Y)] \le o\left(\|b^N - b\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}\right) + c_2\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + c_4\|b^N - b\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}} \tag{61}$$

□

6 Convergence rate of the solution to the regularised SDE and the original

In this section we present a bound for $\mathbb{E}[X^N - X]$ in terms of $\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}$.

Proposition 5. *Let assumptions 1 2 hold, then for any $\alpha \in (1/2, 1-\beta)$ there is a constant C_α such that*

$$\mathbb{E}[X^N - X] \le C_\alpha\|b^N - b\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}, \tag{62}$$ {eq:EXN-X

as $N \to \infty$.

Proof. Note that by definition of ψ, ψ^N we have

$$
\begin{aligned}
|X_t^N - X_t| &= |\psi^N(t, \phi^N(t, X_t^N)) - \psi(t, \phi(t, X_t))| \\
&= |\psi^N(t, Y_t^N) - \psi(t, Y_t)|,
\end{aligned} \tag{63}
$$

then adding and subtracting, and using the triangle inequality we get

$$
|X_t^N - X_t| \le |\psi^N(t, Y_t^N) - \psi(t, Y_t^N)| + |\psi(t, Y_t^N) - \psi(t, Y_t)|. \tag{64}
$$

Where the first term is bounded by $2\kappa \|b^N - b\|_{C_T\mathcal{C}^{-\beta}}$ (Proposition 3) and since ψ is 2-Lipschitz uniformly in $t \in [0, T]$ the second term is bounded by $2|Y^N - Y|$, therefore

$$
|X^N - X| \le 2\kappa \|b^N - b\|_{C_T\mathcal{C}^{-\beta}} + 2|Y^N - Y|. \tag{65}
$$

By assumption the first term above goes to zero as $N \to \infty$, then we only need a bound for the second term.

By Itô-Tanaka's formula

$$
|Y^N - Y| = |y_0^N - y_0| + \frac{1}{2} L_t^0(Y^N - Y) + \int_0^t \operatorname{sgn}(Y^N - Y) d(Y^N - Y), \tag{66}
$$

by taking expectation and using the definitions of Y^N, Y we have

$$
\begin{aligned}
\mathbb{E}|Y^N - Y| = \mathbb{E}|y_0^N - y_0| &+ \mathbb{E}\frac{1}{2} L_t^0(Y^N - Y) \\
&+ \lambda \mathbb{E} \int_0^t \operatorname{sgn}(Y^N - Y)(u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)))ds,
\end{aligned} \tag{67}
$$

then observe that the first term above is a constant, for the second we have a bound in Proposition 4 and for the third we use Lemma 5, and the fact that $\operatorname{sgn}(x) \le 1$ therefore

$$
\begin{aligned}
\mathbb{E}|Y^N - Y| \le |u^N(0, x) - u(0, x)| & \\
&+ \frac{1}{2}\left[o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha - 1} \right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds \right) + B \|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right] \\
&+ \mathbb{E}\left[\int_0^t \left(2\kappa_\rho \|b^N - b\|_{C_T\mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\| \right) ds \right] \\
&\le |u^N(0, x) - u(0, x)| \\
&+ \frac{1}{2}\left[o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha - 1} \right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds \right) + B \|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right] \\
&+ \lambda 2\kappa_\rho \|b^N - b\|_{C_T\mathcal{C}^{-\beta}} t + \lambda \mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\| ds \right),
\end{aligned} \tag{68}
$$

Note that the terms in orange are controlled by $o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right)$, so after merging those terms and the two involving $\mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\| ds \right)$ we get

$$
\mathbb{E}|Y^N - Y| \le B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} + (A + \lambda)\mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\| ds \right) + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right). \tag{69}
$$

From there, using Gronwall's lemma we get the following bound

$$
\mathbb{E}|Y^N - Y| \le B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} T e^{(A+\lambda)T} + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right). \tag{70}
$$

Now we use (70) to bound (65), and as the *small-o* term controls the second term in (65) we obtain

$$
\mathbb{E}\left[|X^N - X| \right] \le B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} T e^{(A+\lambda)T} + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right) \tag{71}
$$

□

7 Small comment about convergence rate of Euler scheme to regularized equation

It works just like in [1] LM: make all the comment

8 Convergence rate of Euler scheme

LM: check assumptions, maybe put them into the assumptions above or smth
LM: mention the SDE somewhere before

Theorem 1. *Let X_t be the solution to the SDE.......with drift coefficient $b \in C_T\mathcal{C}^{-\beta}$, and X_t^{Nm} be the Euler approximation of the solution with m time steps. Let also $\beta_0 \in (0,1/4)$, then it holds that*

$$\sup_{0\le t\le T} \mathbb{E}\left[\left|X_t^{Nm} - X_t\right|\right] \le c m^{-\frac{1}{2}+\mu+\epsilon}, \tag{72}$$

{eq:euler

where

$$\mu = \frac{1}{2} \cdot \frac{\beta_0}{(1/2-\beta_0)(1-2\beta_0)+\beta_0}. \tag{73}$$

{eq:mu}

Proof. First, by triangle inequality we have

$$\circledast := \sup_{0\le t\le T} \mathbb{E}\left[\left|X_t^{Nm} - X_t\right|\right] \le \mathbb{E}\left[\left|X_t^{Nm} - X_t^N\right|\right] + \mathbb{E}\left[\left|X_t^N - X_t\right|\right], \tag{74}$$

{eq:er01}

the first term in the right hand side is bounded by [1][Proposition 3.4] and the second one by Proposition 5, so that putting those results together we get

$$\begin{aligned}\circledast &\le A_N m^{-1} + B_N m^{-1/2} + c\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \\ &\le c\Big[\left\|b^N\right\|_{\infty,L^\infty}\left(1+\left\|\nabla b^N\right\|_{\infty,L^\infty}\right) m^{-1} + \left(\left\|\nabla b^N\right\|_{\infty,L^\infty} + \left[b^N\right]_{1/2,L^\infty}\right) m^{-1/2} + \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta_0}}^{2\alpha-1}\Big].\end{aligned} \tag{75}$$

{eq:er_co

We require to find some bounds for the L^∞ (semi) norms, so that we use Schauder estimates and Bernstein inequality LM: add Bernstein inequality or reference, this is possible thanks to the definition of $b^N := p_{f_m} * b$, where $f_m \to 0$ when $m \to \infty$, and also consider the definition of the norm

$$\|g\|_{C_T\mathcal{C}^\delta} = \left\|b^N\right\|_{L^\infty} + \sup_{x\ne y}\frac{|b^N(x) - b^N(y)|}{|x-y|^\delta},$$

and the seminorm

$$[g]_{1/2,L^\infty} = \sup_{t\ne st, s\in[0,T]} \frac{\|g(t)-g(s)\|_{L^\infty}}{|t-s|^{1/2}}.$$

We have the following bounds:

$$\left\|b^N\right\|_{L^\infty} \le \left\|b^N\right\|_{C_T\mathcal{C}^\epsilon} \le c f_m^{-\frac{\epsilon+\beta}{2}} \|b\|_{C_T\mathcal{C}^{-\beta_0}}, \tag{76}$$

{eq:er02}

$$\left\|\nabla b^N\right\|_{L^\infty} \le \left\|\nabla b^N\right\|_{C_T\mathcal{C}^\epsilon} \le c\left\|b^N\right\|_{C_T\mathcal{C}^{\epsilon+1}} \le c f_m^{-\frac{\epsilon+\beta_0+1}{2}} \|b\|_{C_T\mathcal{C}^{-\beta_0}}, \tag{77}$$

{eq:er03}

$$\begin{aligned}\left[b^N\right]_{1/2,L^\infty} &\le \sup_{t\ne s}\frac{\|b^N(t)-b^N(s)\|_{C_T\mathcal{C}^\epsilon}}{|t-s|^{1/2}} \\ &\le \sup_{t\ne s} c f_m^{-\frac{\epsilon+\beta_0}{2}} \frac{\|b(t)-b(s)\|_{C_T\mathcal{C}^{-\beta_0}}}{|t-s|^{1/2}} \\ &= c f_m^{-\frac{\epsilon+\beta_0}{2}} [b]_{1/2,C_T\mathcal{C}^{-\beta_0}}.\end{aligned} \tag{78}$$

{eq:er04}

Plugging that into (75) we get

$$
\begin{aligned}
\circledast \leq c \Big[& \left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta_0}}^{2\alpha-1} \\
& + \left\| b^N \right\|_{\infty, L^\infty} \left(1 + \left\| \nabla b^N \right\|_{\infty, L^\infty}\right) m^{-1} \\
& + \left(\left\| \nabla b^N \right\|_{\infty, L^\infty} + \left[b^N \right]_{1/2, L^\infty} \right) m^{-1/2} \Big] \\
\leq c \Big[& \left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta_0}}^{2\alpha-1} \\
+ & f_m^{-\frac{\epsilon+\beta_0}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}} \left(1 + f_m^{-\frac{\epsilon+\beta_0+1}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}}\right) m^{-1} \\
+ & \left(f_m^{-\frac{\epsilon+\beta_0+1}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}} + f_m^{-\frac{\epsilon+\beta_0}{2}} [b]_{1/2, C_T \mathcal{C}^{-\beta_0}} \right) m^{-1/2} \Big] \\
\leq c \Big[& \left(f_m^{\nu/2} \|b\|_{C_T \mathcal{C}^{-\beta+\nu}} \right)^{2\alpha-1} \\
+ & f_m^{-\frac{\epsilon+\beta_0}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}} \left(1 + f_m^{-\frac{\epsilon+\beta_0+1}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}}\right) m^{-1} \\
+ & \left(f_m^{-\frac{\epsilon+\beta_0+1}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}} + f_m^{-\frac{\epsilon+\beta_0}{2}} [b]_{1/2, C_T \mathcal{C}^{-\beta_0}} \right) m^{-1/2} \Big]
\end{aligned} \tag{79}
$$

Where the first term in the last inequality comes from Schauder estimates LM: add this earlier and refer to it. Also, since the norms are finite they can be absorbed by a constant and we have:

$$
\circledast \leq c \left[f_m^{\nu/2(2\alpha-1)} + \left(f_m^{-\frac{\epsilon+\beta_0}{2}} + f_m^{-\frac{2\epsilon+2\beta_0+1}{2}} \right) m^{-1} + \left(f_m^{-\frac{\epsilon+\beta_0+1}{2}} + f_m^{-\frac{\epsilon+\beta_0}{2}} \right) m^{-1/2} \right] \tag{80}
$$

for some $\nu \in [0, 1)$.

Now, since we would like to have $f_m = m^{-\theta}$, and that converges to zero as $m \to \infty$, we know that $f_m^{\nu/2}$ converges even faster. Because of that, we want $\nu/2$ to be as large as possible. Observe that we can go up to $b \in C_T \mathcal{C}^{-\beta_0}$, i.e. $-\beta + \nu = -\beta_0$ or $\nu = \beta - \beta_0$. Considering only the slowest terms in (80) and substituting ν we get

$$
\circledast \leq f_m^{\frac{\beta-\beta_0}{2}(2\alpha-1)} + m^{-1/2} f_m^{-\frac{\epsilon+\beta_0}{2}} \tag{81}
$$

The optimal of this quantity is when the two terms on the left hand side are equal, this is

$$
\begin{aligned}
m^{-1/2} &= f_m^{\frac{(2\alpha-1)(\beta-\beta_0)+\epsilon+\beta_0}{2}} \\
f_m &= m^{-\frac{1}{(2\alpha-1)(\beta-\beta_0)+\epsilon+\beta_0}}
\end{aligned} \tag{82}
$$

So, if we plug it into the rate (any of the two terms in (81)), we get

$$
\circledast \leq c m^{-\frac{1}{2} + \frac{1}{2} \frac{\beta_0+\epsilon}{(2\alpha-1)(\beta-\beta_0)+\beta_0+\epsilon}} \tag{83}
$$

Note that in order to attain the best rate we requier to minimize the second term of the exponent of (83), i. e:

$$
\inf_{\substack{\alpha \in (1/2, 1-\beta_0) \\ \beta \in (\beta_0, 1/2) \\ \epsilon > 0}} \left\{ \frac{\beta_0 + \epsilon}{\beta_0 + \epsilon + (2\alpha - 1)(\beta - \beta_0)} \right\} = \frac{\beta_0}{(1/2 - \beta_0)(1 - 2\beta_0) + \beta_0 + \epsilon}.
$$

Therefore, we have

$$
\mathbb{E}\left[\left| X_t^{Nm} - X_t \right| \right] \leq m^{-\frac{1}{2} + \frac{1}{2} \cdot \frac{\beta_0}{(1/2-\beta_0)(1-2\beta_0)+\beta_0} + \epsilon}, \tag{84}
$$

as required. □

LM: this is not needed or needs adjustment

Note that for any fixed $\beta_0 \in (0,1/4)$ we can vary the following parameters inside their constraints $\alpha \in (1/2.1-\beta_0), \beta \in (\beta_0, 1/2), \epsilon > 0$, so take the exponent of the right hand side on (81), eq:er_slowest

$$-\frac{1}{2}+\frac{1}{2}\frac{\beta_0+\epsilon}{(2\alpha-1)(\beta-\beta_0)+\beta_0+\epsilon}.$$

The best case scenario is when the second term is close to zero, having a convergence rate of $1/2+\epsilon'$, and the worst case is for the second term is close to $1/2$ leaving us with a convergence rate of ϵ', for ϵ' small.

For the best case let us compute the infimum of the term, i. e:

$$\inf_{\substack{\alpha\in(1/2,1-\beta_0)\\ \beta\in(\beta_0,1/2)\\ \epsilon>0}}\left\{\frac{\beta_0+\epsilon}{\beta_0+\epsilon+(2\alpha-1)(\beta-\beta_0)}\right\},$$

for which, naturally, we want the denomanator to be as large as it can be, so we select $\beta \approx 1/2$, and $\alpha \approx 1-\beta_0$, therefore we get

$$\inf_{\substack{\alpha\in(1/2,1-\beta_0)\\ \beta\in(\beta_0,1/2)\\ \epsilon>0}}\left\{\frac{\beta_0+\epsilon}{\beta_0+\epsilon+(2\alpha-1)(\beta-\beta_0)}\right\}=\inf_{\epsilon>0}\left\{\frac{\beta_0+\epsilon}{\beta_0+\epsilon+(1-2\beta_0)(1/2-\beta_0)}+\epsilon'\right\}.$$

And the above is minimised for $\epsilon \approx 0$.

So, the best rate of the scheme is

$$m^{-1/2\left(1-\frac{\beta_0}{(1/2-\beta_0)(1-2\beta_0)+\beta_0}\right)}.$$

In particular consider the two important cases of the limits of β_0, i. e:

- For $\beta_0 \approx 0$ we have $m^{-1/2\cdot\left(1-\frac{0}{1/2\cdot 1}+\epsilon'\right)} = m^{-1/2-\epsilon'}$,
- and for $\beta_0 \approx \frac{1}{4}$ we have $m^{-1/2\cdot\left(1-\frac{1/4}{1/4\cdot(1-1/2)+1/4}+\epsilon'\right)} = m^{-1/2\cdot(1-2/3+\epsilon')} = m^{-1/6-\epsilon'}$.

For the case in which we get the worst convergence rate, note that we with to have

$$\sup_{\substack{\alpha\in(1/2,1-\beta_0)\\ \beta\in(\beta_0,1/2)\\ \epsilon>0}}\left\{\frac{\beta_0+\epsilon}{\beta_0+\epsilon+(2\alpha-1)(\beta-\beta_0)}\right\}=1$$

which is possible by selecting either $\beta \approx \beta_0$ or $\alpha \approx 1/2$, since we make the third term in the denominator equal zero, i. e:

$$\begin{aligned}\sup_{\substack{\alpha\in(1/2,1-\beta_0)\\ \beta\in(\beta_0,1/2)\\ \epsilon>0}}\left\{\frac{\beta_0+\epsilon}{\beta_0+\epsilon+(2\alpha-1)(\beta-\beta_0)}\right\}&=\sup_{\epsilon>0}\left\{\frac{\beta_0+\epsilon}{\beta_0+\epsilon+(1-1)(\beta_0-\beta_0)}+\epsilon'\right\}\\ &=\sup_{\epsilon>0}\left\{\frac{\beta_0+\epsilon}{\beta_0+\epsilon}+\epsilon'\right\}\\ &=1+\epsilon',\end{aligned}$$

for any ϵ and any β_0. And therefore we have for this case a rate $m^{-1/2\cdot(1-1+\epsilon')} = m^{-\epsilon'}$.

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