Convergence rate of numerical solutions to SDEs with distributional drifts in Besov spaces

Luis Mario Chaparro Jaquez

July 20, 2022

Contents

1 What is this?

An adaptation of [1, Proposition 3.1] for the case of SDEs with drift in a Besov space of negative order similar to the ones proposed in [2] and [3]. The proof builds on a number of results presented in the sections below.

EI: add result about convergence of the scheme. This is done in two parts, $X^N \to X$ done in Russo Issoglio, and $X^{N,m} \to X^N$ Euler scheme convergence from De Angelis Germain Issoglio. Attention that the rate of convergence of Euler scheme depends of the smoothness of b^N.

2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

Definition 1. *For any real-valued continuous semi-martingale Z, the local time at zero $L_t^0(\bar{Y})$ is defined as*

$$L_t^0(Z) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|Z| \le \epsilon\}} d\langle Z \rangle_s, \ \mathbb{P}\text{-}a.s. \tag{1}$$

For all $t \ge 0$.

The lemma below is from [1] and its proof can be found in [1, Lemma 5.1]. We include the statement here for ease of reading.

Lemma 1. *For any $\epsilon \in (0,1)$ and any real-valued, continuous semi-martingale Z we have*

$$\mathbb{E}[L_t^0(Z_s)] \le 4\epsilon - 2\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Z_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Z_s \ge \epsilon\}} e^{1 - Z_s/\epsilon} \right) dZ_s \right]$$
$$+ \frac{1}{\epsilon} \mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z > \epsilon\}} e^{1 - Z_s/\epsilon} d\langle Z \rangle_s \right].$$

Let us introduce the original and regularised Kolmogorov equations. To shorten notation we will denote the spaces $C_T \mathcal{C}^\gamma(\mathbb{R})$ as $C_T \mathcal{C}^\gamma$.

LM: add Feynamn-Kac formula

Definition 2. *For $\beta \in (0, 1/2)$ let $b \in C_T \mathcal{C}^{-\beta}$, $u, u^N \in C_T \mathcal{C}^{(1+\beta)+}$, and $b^N \to b$ as $N \to \infty$ in $C_T \mathcal{C}^{-\beta}$ The equations*

$$\begin{cases} \partial_t u_i + \frac{1}{2}\Delta u_i + b_i \nabla u_i = \lambda u_i - b_i \\ u_i(T) = 0, \end{cases} \tag{2}$$

$$\begin{cases} \partial u_i^N + \frac{1}{2}\Delta u_i^N + b_i^N \nabla u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0. \end{cases} \tag{3}$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

Lemma 2. *Let u, u^N be the solutions to the Kolmogorov equations (2) (3) in $C_T \mathcal{C}^{1+\alpha}$ respectivelly. We have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u(t,x)| \le \frac{1}{2} \quad and \quad \sup_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u^N(t,x)| \le \frac{1}{2} \tag{4}$$

Assumption 1. *Let $0 < \beta < 1/2$ and $b \in C_T \mathcal{C}^{-\beta}$.*

Assumption 2. *There exists a sequence $(b^N)_N \in C_T \mathcal{C}^{-\beta}$ such that for each N, $b^N(t,\cdot) \in C_b^\infty(\mathbb{R})$ for all $t \in [0,T]$ and such that $b^N \to b$ as $N \to \infty$.*

Definition 3. *The function is called heat kernel*

$$p_f = \frac{1}{\sqrt{4\pi f}} e^{-\frac{|x-y|^2}{4f}}. \tag{5}$$

This is the fundamental solution to the heat equation.
The convolution of the heat kernel with a (generalised) function, is called heat semigroup:

$$P_f g = p_f * g = \int_\Omega p_f(x) g(x-y) dx \tag{6}$$

Lemma 3. *Given a function $f \in \mathcal{C}^\gamma$ for some $\gamma \in \mathbb{R}$, then for any $\theta \ge 0$ there exists a constant c such that*

$$\|P_t f\|_{\gamma+2\theta} \le ct^{-\theta} \|f\|_\gamma. \tag{7}$$

Moreover, for $f \in \mathcal{C}^\gamma$ and any $\theta \in (0,1)$ we have

$$\|P_t f - f\|_\gamma \le ct^\theta \|f\|_{\gamma+2\theta}. \tag{8}$$

Lemma 4. *Given a function $f \in \mathcal{C}^\gamma$ for some $\gamma \in \mathbb{R}$, there exists a constant $c > 0$ such that*

$$\|\nabla g\|_\gamma \le c \|g\|_{\gamma+1}. \tag{9}$$

3 Bounds for the difference of solutions to the Kolmogorov equations

We need a bound for $u - u^N$ and $\nabla u - \nabla u^N$ in L_∞ for the case in which $u \in C_T \mathcal{C}^{1+\alpha}$ for some $\alpha > \beta$ which is an adaptation of [1, Lemma 5.2].

The result builds on top of the following result:

Proposition 1. *Let u, u^N be (mild) solutions to the Kolmogorov equations from Definition 2 then as $N \to \infty$*

$$\|u_i - u_i^N\|_{C_T \mathcal{C}^{1+\alpha}}^{(\rho)} \le \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} (\|u_i\|_{C_T \mathcal{C}^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}} (\|b\|_{C_T \mathcal{C}^{-\beta}} + \lambda)} \tag{10}$$

for $\rho \ge \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_T+\alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \tag{11}$$

and $\lambda > 0$.

Proof. See that $u^N(T) = u(T) = 0$, and in [2], set g^N, g as b^N, b respectively. See that $b^N \to b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$. As u^N, u are mild solutions, we have

$$u_i(t) - u_i^N(t) = P_{T-t}(u_i(T) - u_i^N(T)) + \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds$$

$$- \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds$$

$$= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds - \int_t^T P_{s-t}(b_i - b_i^N) ds$$

$$= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)] ds$$

$$- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds - \int_t^T P_{s-t}(b_i - b_i^N) ds$$

$$= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds + \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds$$

$$- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds - \int_t^T P_{s-t}(b_i - b_i^N) ds$$

Now let us compute the ρ-equivalent norm of $u - u^N$, for some $\alpha > \beta$

$$\left\| u_i - u_i^N \right\|_{C_T \mathcal{C}^{1+\alpha}}^{(\rho)} = \sup_{0 \le t \le T} e^{-\rho(T-t)} \left\| u(t) - u^N(t) \right\|_{C_T \mathcal{C}^{1+\alpha}}$$

$$\le \sup_{0 \le t \le T} e^{-\rho(T-t)} \left[\left\| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \right.$$

$$+ \left\| \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}}$$

$$- \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}}$$

$$\left. - \left\| \int_t^T P_{s-t}(b_i - b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}} \right].$$

Let us take each term from the right hand side of the inequality and bound them.

LM: change this sentence accoring to the addition of Schauder estimates

Using 3 For the first term, using $\gamma + 2\theta = 1 + \alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \le c t^{-\theta} \|f\|_\gamma$ and $\|\nabla g\|_\xi \le c \|g\|_{\xi+1}$

$$\sup_{0 \le t \le T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}\left(\nabla u_i b_i - \nabla u_i b_i^N \right) ds \right\|_{C_T \mathcal{C}^{1+\alpha}}$$

$$\le \sup_{0 \le t \le T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_{C_T \mathcal{C}^\alpha} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}}$$

$$\le c \|u_i\|_{C_T \mathcal{C}_{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \sup_{0 \le t \le T} e^{-\rho(T-t)} (T-t)^{\frac{1-\beta-\alpha}{2}}$$

$$\le c T^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T \mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}}$$

For the second term, see that for $N \to \infty$, we have $\|b^N\|_{C_T \mathcal{C}^{-\beta}} \le 2\|b\|_{C_T \mathcal{C}^{-\beta}}$

$$\sup_{0 \le t \le T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} b_i^N \left(\nabla u_i - \nabla u_i^N \right) ds \right\|_{C_T \mathcal{C}^{1+\alpha}}$$

$$\le c \sup_{0 \le t \le T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2 \|b_i\|_{C_T \mathcal{C}^{-\beta}} \|\nabla u_i - \nabla u_i^N\|_{C_T \mathcal{C}^{1+\alpha}} ds$$

$$\leq c\|b_i\|_{C_T\mathcal{C}^{-\beta}}\|u_i-u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)}\int_t^T (s-t)^{-\theta}e^{-\rho(T-t)}ds$$

$$\leq c\|b_i\|_{C_T\mathcal{C}^{-\beta}}\|u_i-u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)}\rho^{\frac{\alpha+\beta-1}{2}}$$

For the third term, which is the one that difers from the proof in [2] we need to use that $\|P_t f\|_{\gamma+2\theta}\leq ct^{-\theta}\|f\|_\gamma$, and in this case we have $\gamma+2\theta=1+\alpha$ and $\gamma=1+\alpha$, so that $\theta=0$ because $u,u^N\in C_T\mathcal{C}^{1+\alpha}$, so we will have

$$\sup_{0\leq t\leq T}e^{-\rho(T-t)}\left\|\lambda\int_t^T P_{s-t}(u_i-u_i^N)ds\right\|_{1+\alpha}$$

$$\leq c\lambda \sup_{0\leq t\leq T}e^{-\rho(T-t)}\int_t^T (s-t)^{-0}\|u_i-u_i^N\|_{1+\alpha}ds$$

$$= c\lambda \sup_{0\leq t\leq T}e^{-\rho(T-t)}\int_t^T e^{-\rho(T-s)}\sup_{0\leq s\leq T}e^{-\rho(T-s)}\|u_i-u_i^N\|_{1+\alpha}ds$$

$$= c\lambda \sup_{0\leq t\leq T}\|u_i-u_i^N\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)}\int_t^T e^{-\rho(T-s)}e^{-\rho(T-t)}ds$$

$$= c\lambda \sup_{0\leq t\leq T}\|u_i-u_i^N\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)}\int_t^T e^{-\rho(s-t)}ds$$

$$= c\lambda \sup_{0\leq t\leq T}\|u_i-u_i^N\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)}\sup_{0\leq t\leq T}\rho^{-1}[1-e^{-\rho(T-t)}]$$

$$\leq c\lambda \sup_{0\leq t\leq T}\|u_i-u_i^N\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)}\rho^{-1}$$

$$\leq c\lambda \sup_{0\leq t\leq T}\|u_i-u_i^N\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)}\rho^{\frac{\alpha+\beta-1}{2}}$$

And for the last term

$$\sup_{0\leq t\leq T}e^{-\rho(T-t)}\left\|\int_t^T P_{T-s}(b_i-b_i^N)ds\right\|_{C_T\mathcal{C}^{1+\alpha}}$$

$$\leq c\sup_{0\leq t\leq T}e^{-\rho(T-t)}\int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}}\|b_i-b_i^N\|_{C_T\mathcal{C}^{-\beta}}ds$$

$$\leq c\|b_i-b_i^N\|_{C_T\mathcal{C}^{-\beta}}\sup_{0\leq t\leq T}e^{-\rho(T-t)}(s-t)^{-\frac{\alpha+\beta-1}{2}}$$

$$\leq cT^{\frac{1-\beta-\alpha}{2}}\|b_i-b_i^N\|_{C^T\mathcal{C}^{-\beta}}$$

Putting everything together

$$\|u_i-u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)}\leq cT^{\frac{1-\beta-\alpha}{2}}\|u_i\|_{C_T\mathcal{C}^{1+\alpha}}\|b_i-b_i^N\|_{C_T\mathcal{C}^{-\beta}}$$

$$+ c\|b_i\|_{C_T\mathcal{C}^{-\beta}}\|u_i-u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)}\rho^{\frac{\alpha+\beta-1}{2}}$$

$$- c\lambda \sup_{0\leq t\leq T}\|u_i-u_i^N\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)}\rho^{\frac{\alpha+\beta-1}{2}}$$

$$- cT^{\frac{1-\beta-\alpha}{2}}\|b_i-b_i^N\|_{C^T\mathcal{C}^{-\beta}},$$

and finally,

$$\|u_i-u_i^N\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)}\left(1-c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}}+\lambda]\right)$$

$$\leq cT^{\frac{1-\beta-\alpha}{2}}\|b_i-b_i^N\|_{C^T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}}-1)$$

$$\|u_i - u_i^N\|_{C_T C^{1+\alpha}}^{(\rho)} \le \frac{cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T C^{-\beta}}(\|u_i\|_{C_T C^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T C^{-\beta}} + \lambda])}$$

As required. □

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|_{C_T C^{1+\alpha}}^{(\rho)} \le \frac{cT^{\frac{1-\beta-\alpha}{2}}(\|u_i\|_{C_T C^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T C^{-\beta}} + \lambda])}\|b_i - b_i^N\|_{C^T C^{-\beta}} \tag{12}$$

LM: Check this norm, it was in $-\beta$ now in $1 + \alpha$

$$\|u_i - u_i^N\|_{C_T C^{1+\alpha}}^{(\rho)} \le c(\rho)\|b_i - b_i^N\|_{C^T C^{-\beta}} \tag{13}$$

Here is the adaptation of [de_angelis_numerical_2020 1, Lemma 5.2].

uN_graduN **Proposition 2.** *Let $\beta \in (0,1/2)$ and $b \in C_T C^{-\beta}$. Let $u, u^N \in C_T C^{(1+\beta)+}$ be (mild) solutions to the Kolmogorov equations from Definition* def:kolmogorov_eqns 2.
Assume, by Proposition prop:diff_u_uN *1, that for some $\alpha > \beta$*

$$\|u - u^N\|_{C_T C^{1+\alpha}}^{(\rho)} \le c(\rho)\|b - b^N\|_{C_T C^{-\beta}}. \tag{14}$$ {eq:u-uNb

With $c(\rho)$ as in Proposition prop:diff_u_uN *1 and ρ_0 is large enough such that $c(\rho) > 0$ for all $\rho > \rho_0$.*
Then for all $t \in [0,T]$

$$\|u^N(t) - u(t)\|_{L^\infty} \le \kappa_\rho \|b - b^N\|_{C_T C^{-\beta}} \tag{15}$$ {eq:uNu_b

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \le \kappa_\rho \|b - b^N\|_{C_T C^{-\beta}} \tag{16}$$ {eq:gradu

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$.

Proof. First let us prove (eq:uNu_bounded_by_bNb 15).

Let $t \in [0,T]$, and see that since $u, u^N \in C_T C^{(1+\beta)+}$ there exists $\alpha > \beta$ such that $u, u^N \in C_T C^{1+\alpha}$, then for any $f \in C^{1+\alpha}$ we have

$$\|f\|_{C^{1+\alpha}} \le c\left(\sup_{x \in \mathbb{R}^d}|f(x)| + \sup_{x \ne y \in \mathbb{R}^d}\frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha}\right) \tag{17}$$

so we have

$$\|u^N(t) - u(t)\|_{L^\infty} = \sup_{x \in \mathbb{R}^d}|u^N(t,x) - u(t,x)| \\ \le c\|u^N(t) - u(t)\|_{C^{\alpha+1}} \tag{18}$$ {eq:u-uN_

Moreover, using the (ρ)-equivalent norm

$$\|f\|_{C^{1+\alpha}} = \sup_{t \in [0,T]} e^{-\rho(T-t)}\|f(t)\|_{C^{1+\alpha}}, \tag{19}$$

and (eq:u-uNb-bN 14) we see that

$$\begin{aligned}\|u^N - u\|_{C_T C^{1+\alpha}} &= \sup_{t \in [0,T]}\|u^N - u\|_{C^{1+\alpha}} \\ &= \sup_{t \in [0,T]} e^{\rho(T-t)}e^{-\rho(T-t)}\|u^N - u\|_{C^{1+\alpha}} \\ &\le e^{\rho T}\sup_{t \in [0,T]} e^{-\rho(T-t)}\|u^N - u\|_{C^{1+\alpha}} \\ &= e^{\rho T}\|u^N - u\|_{C_T C^{1+\alpha}}^{(\rho)}.\end{aligned} \tag{20}$$ {eq:norm_

5

Plugging (20) into (18)

$$
\begin{aligned}
\left\| u^N(t) - u(t) \right\|_{L^\infty} &\le c \left\| u^N(t) - u(t) \right\|_{\mathcal{C}^{\alpha+1}} \\
&\le \sup_{t \in [0,T]} c \left\| u^N(t) - u(t) \right\|_{\mathcal{C}^{\alpha+1}} \\
&= c \left\| u^N - u \right\|_{C_T \mathcal{C}^{\alpha+1}} \\
&\le c e^{\rho T} \left\| u^N - u \right\|_{C_T \mathcal{C}^{\alpha+1}}^{(\rho)} .
\end{aligned}
\tag{21}
$$

And finally by (14)

$$
\left\| u^N(t) - u(t) \right\|_{L^\infty} \le c \cdot c(\rho) \cdot e^{\rho T} \left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta}}
\tag{22}
$$

which proves (15).

For (16) recall that if $f \in \mathcal{C}^{1+\alpha}$ then $\nabla f \in \mathcal{C}^\alpha$. Also, by Bernstein inequality (9)

$$
\left\| \nabla f \right\|_\alpha \le c \left\| f \right\|_{1+\alpha} .
\tag{23}
$$

Using the equivalent norm

$$
\left\| f \right\|_{C^{1+\alpha}} \le c \left(\sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \in \mathbb{R}^d} |\nabla f(x)| + \sup_{x \neq y \in \mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \right)
\tag{24}
$$

we can see that

$$
\left\| \nabla u^N(t) - \nabla u(t) \right\|_{L^\infty} \le c \left\| u^N(t) - u(t) \right\|_{\mathcal{C}^{1+\alpha}} .
\tag{25}
$$

And usign the same bounds that we used above for $c \left\| u^N(t) - u(t) \right\|_{\mathcal{C}^{1+\alpha}}$ this point follows.

$\square$

4   Bound for the difference of the auxiliary functions

This is the adaptation of result [1, Lemma 5.3].

Proposition 3. *Take $\rho > \rho_0$ as in Proposition 1, $N \to \infty$, κ_ρ from Proposition 2, and $\beta \in (0, 1/2)$, then we have*

$$
\sup_{(t,x) \in [0,T] \times \mathbb{R}} \left\| \psi(t,x) - \psi^N(t,x) \right\| \le 2\kappa_\rho \left\| b - b^N \right\|_{C_T \mathcal{C}^{-\beta}}
\tag{26}
$$

Proof. Recall the definition of $\psi, \phi \in C_T \mathcal{C}^1$

$$
\phi(t,x) := x + u(t,x)
\tag{27}
$$

$$
\psi(t, \cdot) = \phi^{-1}(t, \cdot).
\tag{28}
$$

Note that

$$
u(y) = \int_0^1 \nabla u(\alpha y) y \, d\alpha + u(0).
\tag{29}
$$

From there we have

$$
u(t,y) - u(t,y') = \int_0^1 \nabla u(t, \alpha(y - y'))(y - y') \, d\alpha
\tag{30}
$$

and therefore

6

$$\|u(t,y) - u(t,y')\| \geq \left(\int_0^1 \|\nabla u(t,\alpha(y-y'))\|^2 d\alpha \right)^{1/2} \|y - y'\|, \tag{31}$$

and by Lemma `lemma:bounds_gradients` 2 we finally have

$$\|u(t,y) - u(t,y'\|) \leq \left(\frac{1}{4} \int_0^1 d\alpha \right)^{1/2} \|y - y\|'$$

$$\|u(t,y) - u(t,y')\|^2 \leq \frac{1}{4} \|y - y'\|^2 \tag{32}$$

LM: continue from page three in notes

□

5 Bound for the local time at zero of the solution to the SDEs

LM: Here I still need to mention how we define $Y_t = \psi(t, X_t)$, because eventually I need to use that $X_t = \psi(t, Y_t)$, probably just neet to mention without defining the whole Y_t as in the paper `de_angelis_numerical_2020`

We need a bound for $\mathbb{E}[L_T^0(Y^N - Y)]$, for Sobolev spaces, this is result [1, Proposition 5.4] we present it here for the solutions to the SDE blonging to the appropriate Besov spaces.

First let us state the following useful result.

LM: check that the statement makes sense and has all the necessary assumptions

`_integral` **Lemma 5.** *Let u, u^N be solutions to the Kolmogorov equations* `eq:kolmogorov` (2) `eq:kolmogorov_N` (3) *then the following bound is satisfied:*

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \leq 2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + \|Y_s^N - Y^N\|. \tag{33} \quad \boxed{\text{\{eq:bound}}$$

Proof. Adding and subtracting terms, using triangle inequality and noting that for any a, b, we have $a - b \leq \|a - b\|$, then

$$\begin{aligned}
\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| &\leq \|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N))\| \\
&\quad + \|u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N))\| \\
&\quad + \|u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s))\|.
\end{aligned} \tag{34}$$

The terms in the right hand side will be bounded as follows:

- For the first term, by Proposition `prop:diff_uN_graduN` 2

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N))\| \leq \|u^N(s) - u(s)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C_T C^{-\beta}}, \tag{35}$$

- for the second term, observe that u, u^N are $\frac{1}{2}$-Lipschitz and by Proposition `prop:bound_psi-psiN` 3 we get

$$\begin{aligned}
\|u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N))\| &\leq \frac{1}{2} \|\psi^N(s, Y_s^N) - \psi(s, Y_s^N)\| \\
&\leq \kappa_\rho \|b^N - b\|_{C_T C^{-\beta}},
\end{aligned} \tag{36}$$

- and for the final term, note that ψ, ψ^N are 2-Lipschitz
  so that

$$\|u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \leq \frac{1}{2} \|\psi(s, Y_s^N) - \psi(s, Y_s)\| \leq \|Y_s^N - Y_s\|. \tag{37}$$

So that the following bound holds

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \leq 2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + \|Y_s^N - Y^N\|, \tag{38}$$

as required.

□

Proposition 4. *Let A, B be constants, $b \in C_T C^{-\beta}$ and $b^N \to b$ in $C_T C^{-\beta}$ as $N \to \infty$ for $\beta \in (0, \frac{1}{4})$ and for any $\alpha > \beta$*

$$\mathbb{E}[L_t^0(Y^N - Y)] \le o\left(\left\|b^N - b\right\|_{C_T C^{-\beta}}^{2\alpha-1}\right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\|ds\right) + B\left\|b^N - b\right\|_{C_T C^{-\beta}}^{2\alpha-1}. \tag{39} \quad \text{\{eq:local}$$

Proof. Recall that Y^N, Y are solutions to the SDEs

$$Y_t = y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1)dW_s \tag{40}$$

and

$$Y_t^N = y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1)dW_s \tag{41}$$

so that the difference $Y^N - Y$ is

$$\begin{aligned}
Y_t^N - Y_t &= \left(y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1)dW_s\right) \\
&\quad - \left(y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1)dW_s\right) \\
&= (y_0^N - y_0) + \lambda \int_0^t (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)))ds \\
&\quad + \int_0^t (\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)))dW_s,
\end{aligned} \tag{42}$$

and using Lemma $\overset{\text{\tiny lemma:local-time-at-0}}{1}$ we have the following bound

$$\begin{aligned}
&\mathbb{E}[L_t^0(Y^N - Y)] \\
&\le 4\epsilon \\
&\quad - 2\lambda\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Y_s^N - Y_s \ge \epsilon\}}e^{1 - \frac{Y_s^N - Y_s}{\epsilon}}\right)\right. \\
&\hspace{6cm} \left.\left(u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\right)ds\right] \\
&\quad + \frac{1}{\epsilon}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}}e^{1 - \frac{Y_s^N - Y_s}{\epsilon}}\left(\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\right)^2 ds\right].
\end{aligned}$$

$$\hspace{11cm} (43) \quad \text{\{eq:local}$$
$$\hspace{11cm} (44) \quad \text{\{eq:local}$$

LM: add the explanation of why to drop the difussion term

First, for $\overset{\text{\tiny eq:local\_time\_diff\_u}}{(43)}$, we find a bound for the factor involving the difference of u^N and u in Lemma $\overset{\text{\tiny lemma:uN-n\_bound\_for\_integral}}{5}$. Therefore

$$\|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\| \le 2\kappa_\rho \left\|b^N - b\right\|_{C_T C^{-\beta}} + \|Y_s - Y^N\|. \tag{45}$$

Now we need to bound the result of the local time of the difference $Y_s^N - Y_s$. First notice that $Y_s^N - Y_s \ge \epsilon$, then $e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \le 1$, also it is clear that $\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}}$ and $\mathbb{1}_{\{Y_s^N - Y_s \ge \epsilon\}}$ are bounded by 1, therefore $\mathbb{1}_{\{Y_s^N - Y_s \ge \epsilon\}}e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \le 1$. Using the previous arguments and $\overset{\text{\tiny eq:bound\_u\_abs}}{(33)}$ lead to have

$$\begin{aligned}
\overset{\text{\tiny eq:local\_time\_diff\_u}}{(43)} &\le 2\lambda\mathbb{E}\left[\int_0^t 2\left(2\kappa_\rho \left\|b^N - b\right\|_{C_T C^{-\beta}} + \|Y_s - Y^N\|\right)ds\right] \\
&\le 4\lambda\left(2\kappa_\rho \left\|b^N - b\right\|_{C_T C^{-\beta}} t + \mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\|ds\right]\right).
\end{aligned} \tag{46} \quad \text{\{eq:bound}$$

Now for $\overset{\text{\tiny eq:local\_time\_diff\_gradu}}{(44)}$, we use similar arguments as the ones in Lemma $\overset{\text{\tiny lemma:uN-n\_bound\_for\_integral}}{5}$ above, and we get the following:

$$\begin{aligned}
\|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\| &\le \|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi^N(s, Y_s^N))\| \\
&\quad + \|\nabla u(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s^N))\|) \\
&\quad + \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|,
\end{aligned} \tag{47}$$

where the terms on the right hand side will be bounded as follows:

8

- For the first term we use Proposition 2 [prop:diff_uN_graduN] and we have

$$\|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi^N(s, Y_s^N)\|) \le \|\nabla u^N(s) - \nabla u(s)\|_{L^\infty}$$
$$\le \kappa_\rho \|b - b^N\|_{C_T \mathcal{C}^{-\beta}}, \tag{48}$$

for the second term see that $\nabla u, \nabla u^N$ are α-Hölder continuous and using Proposition 3 [prop:bound_psi-psiN] we have

$$\|\nabla u(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s^N)\|) \le \|\psi^N(s, Y_s^N) - \psi(s, Y_s^N)\|^\alpha \|u\|_{C_T \mathcal{C}^{1+\alpha}}$$
$$\le (2\kappa_\rho \|b^N - b\|_{C_T \mathcal{C}^{-\beta}})^\alpha \|u\|_{C_T \mathcal{C}^{1+\alpha}}. \tag{49}$$

Therefore we get the bound

$$\|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|) \le \kappa_\rho \|b - b^N\|_{C_T \mathcal{C}^{-\beta}}$$
$$+ \alpha \kappa_\rho^\alpha \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^\alpha \|u\|_{C_T \mathcal{C}^{1+\alpha}} \tag{50} \quad \{eq:bound}$$
$$+ \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|.$$

Here we can also notice that $\mathbb{1}_{\{Y_s^N - Y_s < \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} < 1$, then using (50) [eq:bound_gradu_abs] and the inequality

$$(x_1 + \cdots + x_k)^2 \le k(x_1^2 + \cdots + x_k^2), \tag{51}$$

for $k = 3$, we can get the bound

$$(44) \le \frac{1}{\epsilon} \mathbb{E} \int_0^t \left(3\kappa_\rho^2 \|b - b^N\|_{C_T \mathcal{C}^{-\beta}}^2 + 3 \cdot 2^{2\alpha} \kappa_\rho^{2\alpha} \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2\right) ds$$
$$+ \frac{1}{\epsilon} \mathbb{E} \int_0^t 3 \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 ds$$
$$\le \frac{1}{\epsilon} 3t \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \left(\kappa_\rho^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} + (2\kappa_\rho)^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha - 1}\right)$$
$$+ \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \left|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\right|^2 ds\right) \tag{52} \quad \{eq:bound}$$

Now let us denote the last term in (52) [eq:bound_integral_graduNu] by $I_t^{N,\epsilon}$. Pick $\zeta \in (0,1)$ such that $\alpha\zeta > \frac{1}{2}$, and since $\epsilon \in (0,1)$ we have $\epsilon^\zeta > \epsilon$. Then split the indicator function $\mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}}$ into $\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}}$. Leading to the integral

$$I_t^{N,\epsilon} = \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \left(\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}}\right) e^{1 - \frac{Y_s^N - Y_s}{\epsilon}}\right.$$
$$\left. \left|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\right|^2 ds\right) \tag{53} \quad \{eq:INeps}$$

For the first term of (53) [eq:INepsilon] we use the fact that ∇u is α-Hölder continuous uniformly in $s \in [0, T]$ with constant $\|u\|_{C_T \mathcal{C}^{1+\alpha}}$ and that ψ is 2-Lipschitz

$$\|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 \le \|\|\psi(s, Y_s^N) - \psi(s, Y_s)\|^\alpha \|u\|_{C_T \mathcal{C}^{1+\alpha}}\|^2$$
$$\le \|2^\alpha \|Y_s^N - Y_s\|^\alpha \|u\|_{C_T \mathcal{C}^{1+\alpha}}\|^2 \tag{54}$$
$$= 2^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 \|Y_s^N - Y_s\|^{2\alpha}$$

For the other term we need another way to bound it, because even though the event when $\|Y^N - Y\| > \epsilon^\zeta$ is small, we can potentially have a quantity that blows up for the bound. EI: the explanation needs adjusting - speak to Elena In order to solve this problem, we can use the fact that ∇u is uniformly bounded by $1/2$ thanks to Lemma 2 [Lemma:bounds_gradients], and then we can bound the difference of the gradients as follows:

$$\|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\|^2 \le \|\nabla u(s, \psi(s, Y_s^N)) + \nabla u(s, \psi(s, Y_s))\|^2$$
$$\le \sup_{(s,x) \in [0,T] \times \mathbb{R}} \|\nabla u(s, \psi(s, Y_s^N)) + \nabla u(s, \psi(s, Y_s))\|^2 \tag{55}$$
$$= \|2\nabla u\|_{L_\infty}^2.$$

Therefore we have that for all $t \in [0,T]$ LM: check where else I need to say this

$$I_t^{N,\epsilon} \leq \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \left(\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \leq \epsilon^\zeta\}} \right) e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|Y_s^N - Y_s\|^{2\alpha} ds \right)$$

$$+ \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \|2\nabla u\|_{L_\infty}^2 ds \right)$$

$$\leq \frac{1}{\epsilon} 3\mathbb{E}\left(2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|\epsilon^\zeta\|^{2\alpha} t \right) + \frac{1}{\epsilon} 3\mathbb{E}\left(4 e^{1 - \epsilon^{\zeta - 1}} \|\nabla u\|_{L_\infty}^2 t \right) \qquad (56)$$

$$\leq \sup_{t \in [0,T]} \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \epsilon^{2\alpha\zeta} + 4 e^{1 - \epsilon^{\zeta - 1}} \|\nabla u\|_{L_\infty}^2 \right) t$$

$$= \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \epsilon^{2\alpha\zeta} + 4 e^{1 - \epsilon^{\zeta - 1}} \|\nabla u\|_{L_\infty}^2 \right) T.$$

Now by combining (46), (52) and (56), and taking the sup over $[0,T]$ we will get

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq 4\epsilon$$

$$+ 4\lambda 2\kappa_\rho T \|b^N - b\|_{C_T C^{-\beta}}$$

$$+ 4\lambda \mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\| ds \right]$$

$$+ \|b^N - b\|_{C_T C^{-\beta}} \frac{1}{\epsilon} 3T \kappa_\rho^2 \|b^N - b\|_{C_T C^{-\beta}} \qquad (57)$$

$$+ \|b^N - b\|_{C_T C^{-\beta}} \frac{1}{\epsilon} 3T (2\kappa_\rho)^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha - 1}$$

$$+ \frac{3}{\epsilon} 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 T \epsilon^{2\alpha\zeta}$$

$$+ \frac{3}{\epsilon} 4 \|\nabla u\|_{L_\infty}^2 T e^{1 - \epsilon^{\zeta - 1}}$$

then we take $\epsilon = \|b^N - b\|_{C_T C^{-\beta}}$ and we get

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq 4\|b^N - b\|_{C_T C^{-\beta}}$$

$$+ 4\lambda 2\kappa_\rho T \|b^N - b\|_{C_T C^{-\beta}}$$

$$+ 4\lambda \mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\| ds \right]$$

$$+ 3T \kappa_\rho^2 \|b^N - b\|_{C_T C^{-\beta}} \qquad (58)$$

$$+ 3T (2\kappa_\rho)^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha - 1}$$

$$+ 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 T \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha\zeta - 1}$$

$$+ 4 \|\nabla u\|_{L_\infty}^2 T \|b^N - b\|_{C_T C^{-\beta}}^{-1} \exp\left(1 - \|b^N - b\|_{C_T C^{-\beta}}^{\zeta - 1} \right)$$

which can be written as

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq c_1 \|b^N - b\|_{C_T C^{-\beta}} + c_2 \mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\| ds \right]$$

$$+ c_3 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha - 1} + c_4 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha\zeta - 1} \qquad (59)$$

$$+ c_5 \exp\left(1 - \|b^N - b\|_{C_T C^{-\beta}}^{\zeta - 1} \right)$$

where

$$c_1 = 4 + 4\lambda 2\kappa_\rho T + 3\kappa_\rho^2 T$$
$$c_2 = 4\lambda$$
$$c_3 = 3(2\kappa_\rho)^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 T \qquad (60)$$
$$c_4 = 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 T$$
$$c_5 = 4 \|\nabla u\|_{L_\infty}^2 \|b^N - b\|_{C_T C^{-\beta}}^{-1} T$$

{eq:const

Finally, observe that since $\zeta \in (0,1)$, the term $\exp\left(1 - \|b^N - b\|_{C_T C^{-\beta}}^{\zeta - 1}\right)$ decays faster than any polynomial, thus controlling c_5, and the last term in (59) goes to zero. Also $\alpha\zeta$ is arbitrarily close to α, and $\|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1}$ controls $\|b^N - b\|_{C_T C^{-\beta}}$ therefore we can create the bound (39)

Question: is this clear enough? Am I making sense if I am taking α fixed? EI: no if α was fixed you could not do this. But $\alpha > \beta$ in your statement, hence it works. You need to explain the details however. Maybe at this stage you could introduce $\alpha' = \alpha\zeta$ to explain, that the result works for α' but since ζ can be chosen arbitrarily close to 1 then α' is arbitrarily close to α and α was chosen such that $\alpha > \beta$ which means the result is valid for all $\alpha' > \beta$. For simplicity we write α in place of α' in the statement.
Also it is better to explain the meaning of $o()$ and what terms go in there.

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq o\left(\|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1}\right) + c_2 \mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + c_4 \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1} \qquad (61)$$

$\square$

6 Convergence rate of the solution to the regularised SDE and the original

In this section we present a bound for $\mathbb{E}[X^N - X]$ in terms of $\|b^N - b\|_{C_T C^{-\beta}}$.

Proposition 5. *Let assumptions I 2 hold, then for any $\alpha > \beta$ there is a constant C_α such that*

$$\mathbb{E}[X^N - X] \leq C_\alpha \|b^N - b\|_{C_T C^{-\beta}}^{2\alpha-1}, \qquad (62)$$

{eq:EXN-X

as $N \to \infty$.

Proof. Note that by definition of ψ, ψ^N we have

$$|X_t^N - X_t| = |\psi^N(t, \phi^N(t, X_t^N)) - \psi(t, \phi(t, X_t))|$$
$$= |\psi^N(t, Y_t^N) - \psi(t, Y_t)|, \qquad (63)$$

then adding and subtracting, and using the triangle inequality we get

$$|X_t^N - X_t| \leq |\psi^N(t, Y_t^N) - \psi(t, Y_t^N)| + |\psi(t, Y_t^N) - \psi(t, Y_t)|. \qquad (64)$$

Where the first term is bounded by $2\kappa \|b^N - b\|_{C_T C^{-\beta}}$ (Proposition 3) and since ψ is 2-Lipschitz uniformly in $t \in [0, T]$ the second term is bounded by $2|Y^N - Y|$, therefore

$$|X^N - X| \leq 2\kappa \|b^N - b\|_{C_T C^{-\beta}} + 2|Y^N - Y|. \qquad (65)$$

{eq:XN-X}

By assumption the first term above goes to zero as $N \to \infty$, then we only need a bound for the second term.

By Itô-Tanaka's formula

$$|Y^N - Y| = |y_0^N - y_0| + \frac{1}{2} L_t^0(Y^N - Y) + \int_0^t \text{sgn}(Y^N - Y) d(Y^N - Y), \qquad (66)$$

{eq:YNY_i

by taking expectation and using the definitions of Y^N, Y we have

$$\mathbb{E}|Y^N - Y| = \mathbb{E}|y_0^N - y_0| + \mathbb{E}\frac{1}{2} L_t^0(Y^N - Y)$$
$$+ \lambda \mathbb{E} \int_0^t \text{sgn}(Y^N - Y)(u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))) ds, \qquad (67)$$

{eq:EYNY}

then observe that the first term above is a constant, for the second we have a bound in Lemma 5 and for the third we use Lemma 5, and the fact that $\text{sgn}(x) \leq 1$ therefore

$$
\begin{aligned}
\mathbb{E}|Y^N - Y| &\leq |u^N(0,x) - u(0,x)| \\
&+ \frac{1}{2}\left[o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right] \\
&+ \mathbb{E}\left[\int_0^t \left(2\kappa_\rho \|b^N - b\|_{C_T\mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\|\right) ds\right] \\
&\leq |u^N(0,x) - u(0,x)| \\
&+ \frac{1}{2}\left[o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right] \\
&+ \lambda 2\kappa_\rho \|b^N - b\|_{C_T\mathcal{C}^{-\beta}} t + \lambda\mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\| ds\right),
\end{aligned}
\tag{68}
$$

Note that the terms in orange are controlled by $o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right)$, so after merging those terms and the two involving $\mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\| ds\right)$ we get

$$
\mathbb{E}|Y^N - Y| \leq B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} + (A+\lambda)\mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\| ds\right) + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right).
\tag{69}
$$

From there, using Gronwall's lemma we get the following bound

$$
\mathbb{E}|Y^N - Y| \leq B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} T e^{(A+\lambda)T} + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right).
\tag{70}
$$

Now we use (70) to bound (65), and as the *small-o* term controls the second term in (65) we obtain

$$
\mathbb{E}\left[|X^N - X|\right] \leq B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} T e^{(A+\lambda)T} + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right)
\tag{71}
$$

$\square$

7 Small comment about convergence rate of Euler scheme to regularized equation

It works just like in [1] LM: make all the comment

8 Convergence rate of Euler scheme

LM: check assumptions, maybe put them into the assumptions above or smth

Proposition 6. *Let X_t^{Nm} be the Euler approximation of the solution with m time steps, and X_t the real solution. Let also $\beta_0 \in (0, 1/4)$, $\beta \in (\beta_0, 1/2)$, $\alpha \in (1/2, 1-\beta_0)$ and $\epsilon > 0$, then it holds that*

$$
\sup_{0 \leq t \leq T} \mathbb{E}\left[|X_t^{Nm} - X_t|\right] \leq cm^{-\frac{1}{2}-\mu},
\tag{72}
$$

where

$$
\mu = \frac{1}{2} \cdot \frac{\beta_0 + \epsilon}{(2\alpha-1)(\beta-\beta_0) + \beta_0 + \epsilon}.
\tag{73}
$$

Proof. First, by triangle inequality we have

$$
\circledast := \sup_{0 \leq t \leq T} \mathbb{E}\left[|X_t^{Nm} - X_t|\right] \leq \mathbb{E}\left[|X_t^{Nm} - X_t^N|\right] + \mathbb{E}\left[|X_t^N - X_t|\right],
\tag{74}
$$

the first term in the right hand side is bounded by [1][Proposition 3.4] and the second one by Proposition 5, so that putting those results together we get

$$
\begin{aligned}
\circledast &\le A_N m^{-1} + B_N m^{-1/2} + c \left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha-1} \\
&\le c \left[\left\| b^N \right\|_{\infty,L^\infty} \left(1 + \left\| \nabla b^N \right\|_{\infty,L^\infty} \right) m^{-1} + \left(\left\| \nabla b^N \right\|_{\infty,L^\infty} + \left[b^N \right]_{1/2,L^\infty} \right) m^{-1/2} + \left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta_0}}^{2\alpha-1} \right].
\end{aligned}
\tag{75}
$$

We require to find some bounds for the L^∞ (semi) norms, so that we use Schauder estimates and Bernstein inequality LM: add Bernstein inequality or reference, this is possible thanks to the definition of $b^N := p_{f_m} * b$, where $f_m \to 0$ when $m \to \infty$, and also consider the definition of the norm

$$
\|g\|_{C_T \mathcal{C}^\delta} = \left\| b^N \right\|_{L^\infty} + \sup_{x \ne y} \frac{|b^N(x) - b^N(y)|}{|x-y|^\delta},
$$

and the seminorm

$$
[g]_{1/2,L^\infty} = \sup_{t \ne s t, s \in [0,T]} \frac{\|g(t) - g(s)\|_{L^\infty}}{|t-s|^{1/2}}.
$$

We have the following bounds:

$$
\left\| b^N \right\|_{L^\infty} \le \left\| b^N \right\|_{C_T \mathcal{C}^\epsilon} \le c f_m^{-\frac{\epsilon+\beta}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}},
\tag{76}
$$

$$
\left\| \nabla b^N \right\|_{L^\infty} \le \left\| \nabla b^N \right\|_{C_T \mathcal{C}^\epsilon} \le c \left\| b^N \right\|_{C_T \mathcal{C}^{\epsilon+1}} \le c f_m^{-\frac{\epsilon+\beta_0+1}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}},
\tag{77}
$$

$$
\begin{aligned}
\left[b^N \right]_{1/2,L^\infty} &\le \sup_{t \ne s} \frac{\|b^N(t) - b^N(s)\|_{C_T \mathcal{C}^\epsilon}}{|t-s|^{1/2}} \\
&\le \sup_{t \ne s} c f_m^{-\frac{\epsilon+\beta_0}{2}} \frac{\|b(t) - b(s)\|_{C_T \mathcal{C}^{-\beta_0}}}{|t-s|^{1/2}} \\
&= c f_m^{-\frac{\epsilon+\beta_0}{2}} [b]_{1/2, C_T \mathcal{C}^{-\beta_0}}.
\end{aligned}
\tag{78}
$$

Plugging that into (75) we get

$$
\begin{aligned}
\circledast &\le c \Bigg[\left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta_0}}^{2\alpha-1} \\
&\quad + \left\| b^N \right\|_{\infty,L^\infty} \left(1 + \left\| \nabla b^N \right\|_{\infty,L^\infty} \right) m^{-1} \\
&\quad + \left(\left\| \nabla b^N \right\|_{\infty,L^\infty} + \left[b^N \right]_{1/2,L^\infty} \right) m^{-1/2} \Bigg] \\
&\le c \Bigg[\left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta_0}}^{2\alpha-1} \\
&\quad + f_m^{-\frac{\epsilon+\beta_0}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}} \left(1 + f_m^{-\frac{\epsilon+\beta_0+1}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}} \right) m^{-1} \\
&\quad + \left(f_m^{-\frac{\epsilon+\beta_0+1}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}} + f_m^{-\frac{\epsilon+\beta_0}{2}} [b]_{1/2, C_T \mathcal{C}^{-\beta_0}} \right) m^{-1/2} \Bigg] \\
&\le c \Bigg[\left(f_m^{\nu/2} \|b\|_{C_T \mathcal{C}^{-\beta+\nu}} \right)^{2\alpha-1} \\
&\quad + f_m^{-\frac{\epsilon+\beta_0}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}} \left(1 + f_m^{-\frac{\epsilon+\beta_0+1}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}} \right) m^{-1} \\
&\quad + \left(f_m^{-\frac{\epsilon+\beta_0+1}{2}} \|b\|_{C_T \mathcal{C}^{-\beta_0}} + f_m^{-\frac{\epsilon+\beta_0}{2}} [b]_{1/2, C_T \mathcal{C}^{-\beta_0}} \right) m^{-1/2} \Bigg]
\end{aligned}
\tag{79}
$$

Where the first term in the last inequality comes from Schauder estimates LM: add this earlier and refer to it. Also, since the norms are finite they can be absorbed by a constant and we have:

$$
\circledast \le c \left[f_m^{\nu/2(2\alpha-1)} + \left(f_m^{-\frac{\epsilon+\beta_0}{2}} + f_m^{-\frac{2\epsilon+2\beta_0+1}{2}} \right) m^{-1} + \left(f_m^{-\frac{\epsilon+\beta_0+1}{2}} + f_m^{-\frac{\epsilon+\beta_0}{2}} \right) m^{-1/2} \right]
\tag{80}
$$

for some $\nu \in [0,1)$.

Now, since we would like to have $f_m = m^{-\theta}$, and that converges to zero as $m \to \infty$, we know that $f_m^{\nu/2}$ converges even faster. Because of that, we want $\nu/2$ to be as large as possible. Observe that we can go up to $b \in C_T \mathcal{C}^{-\beta_0}$, i.e. $-\beta + \nu = -\beta_0$ or $\nu = \beta - \beta_0$. Considering only the slowest terms in (80) and substituting ν we get

$$\circledast \le f_m^{\frac{\beta-\beta_0}{2}(2\alpha-1)} + m^{-1/2} f_m^{-\frac{\epsilon+\beta_0}{2}} \tag{81}$$

The optimal of this quantity is when the two terms on the left hand side are equal, this is

$$m^{-1/2} = f_m^{\frac{(2\alpha-1)(\beta-\beta_0)+\epsilon+\beta_0}{2}}$$
$$f_m = m^{-\frac{1}{(2\alpha-1)(\beta-\beta_0)+\epsilon+\beta_0}} \tag{82}$$

So, if we plug it into the rate (any of the two terms in (81)), we get

$$\circledast \le cm^{-\frac{1}{2}+\frac{1}{2}\frac{\beta_0+\epsilon}{(2\alpha-1)(\beta-\beta_0)+\beta_0+\epsilon}} \tag{83}$$

□

Note that for any fixed $\beta_0 \in (0,1/4)$ we can vary the following parameters inside their constraints $\alpha \in (1/2.1-\beta_0), \beta \in (\beta_0,1/2), \epsilon > 0$, so take the exponent of the right hand side on (81),

$$-\frac{1}{2} + \frac{1}{2}\frac{\beta_0+\epsilon}{(2\alpha-1)(\beta-\beta_0)+\beta_0+\epsilon}.$$

The best case scenario is when the second term is close to zero, having a convergence rate of $1/2 + \epsilon'$, and the worst case is for the second term is close to $1/2$ leaving us with a convergence rate of ϵ', for ϵ' small.

For the best case let us compute the infimum of the term, i. e:

$$\inf_{\substack{\alpha \in (1/2,1-\beta_0) \\ \beta \in (\beta_0,1/2) \\ \epsilon>0}} \left\{ \frac{\beta_0+\epsilon}{\beta_0+\epsilon+(2\alpha-1)(\beta-\beta_0)} \right\},$$

for which, naturally, we want the denomanator to be as large as it can be, so we select $\beta \approx 1/2$, and $\alpha \approx 1-\beta_0$, therefore we get

$$\inf_{\substack{\alpha \in (1/2,1-\beta_0) \\ \beta \in (\beta_0,1/2) \\ \epsilon>0}} \left\{ \frac{\beta_0+\epsilon}{\beta_0+\epsilon+(2\alpha-1)(\beta-\beta_0)} \right\} = \inf_{\epsilon>0} \left\{ \frac{\beta_0+\epsilon}{\beta_0+\epsilon+(1-2\beta_0)(1/2-\beta_0)} + \epsilon' \right\}.$$

And the above is minimised for $\epsilon \approx 0$.

So, the best rate of the scheme is

$$m^{-1/2\left(1-\frac{\beta_0}{(1/2-\beta_0)(1-2\beta_0)+\beta_0}\right)}.$$

In particular consider the two important cases of the limits of β_0, i. e:

- For $\beta_0 \approx 0$ we have $m^{-1/2\cdot\left(1-\frac{0}{1/2\cdot1}+\epsilon'\right)} = m^{-1/2-\epsilon'}$,

- and for $\beta_0 \approx \frac{1}{4}$ we have $m^{-1/2\cdot\left(1-\frac{1/4}{1/4\cdot(1-1/2)+1/4}+\epsilon'\right)} = m^{-1/2\cdot(1-2/3+\epsilon')} = m^{-1/6-\epsilon'}$.

For the case in which we get the worst convergence rate, note that we with to have

$$\sup_{\substack{\alpha \in (1/2,1-\beta_0) \\ \beta \in (\beta_0,1/2) \\ \epsilon>0}} \left\{ \frac{\beta_0+\epsilon}{\beta_0+\epsilon+(2\alpha-1)(\beta-\beta_0)} \right\} = 1$$

which is possible by selecting either $\beta \approx \beta_0$ or $\alpha \approx 1/2$, since we make the third term in the denominator equal zero, i. e:

$$\sup_{\substack{\alpha \in (1/2, 1-\beta_0) \\ \beta \in (\beta_0, 1/2) \\ \epsilon > 0}} \left\{ \frac{\beta_0 + \epsilon}{\beta_0 + \epsilon + (2\alpha - 1)(\beta - \beta_0)} \right\} = \sup_{\epsilon > 0} \left\{ \frac{\beta_0 + \epsilon}{\beta_0 + \epsilon + (1 - 1)(\beta_0 - \beta_0)} + \epsilon' \right\}$$

$$= \sup_{\epsilon > 0} \left\{ \frac{\beta_0 + \epsilon}{\beta_0 + \epsilon} + \epsilon' \right\}$$

$$= 1 + \epsilon',$$

for any ϵ and any β_0. And therefore we have for this case a rate $m^{-1/2 \cdot (1 - 1 + \epsilon')} = m^{-\epsilon'}$.

References

[1] Tiziano De Angelis, Maximilien Germain, and Elena Issoglio. "A Numerical Scheme for Stochastic Differential Equations with Distributional Drift". In: *arXiv:1906.11026 [cs, math]* (Oct. 22, 2020). arXiv: 1906.11026.

[2] Elena Issoglio and Francesco Russo. "A PDE with Drift of Negative Besov Index and Related Martingale Problem". In: (), p. 48.

[3] Elena Issoglio and Francesco Russo. "McKean SDEs with singular coefficients". In: *arXiv:2107.14453 [math]* (July 30, 2021). arXiv: 2107.14453.