Convergence rate of $X^N - X$ for McKean equations

Luis Mario Chaparro Jaquez

June 6, 2022

Contents

1 What is this?

An adaptation of [1, Proposition 3.1] for the case of SDEs with drift in a Besov space of negative order proposed in [2] and [3]. The proof builds on a number of results presented in the sections below.

EI: add result about convergence of the scheme. This is done in two parts, $X^N \to X$ done in Russo Issoglio, and $X^{N,m} \to X^N$ Euler scheme convergence from De Angelis Germain Issoglio. Attention that the rate of convergence of Euler scheme depends of the smoothness of b^N.

2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

Definition 1. *Local time at zero For any real-valued continuous semi-martingale Z, the local time at zero $L_t^0(\bar{Y})$ is defined as*

$$L_t^0(Z) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|Z| \le \epsilon\}} d\langle Z \rangle_s, \ \mathbb{P}\text{-}a.s. \tag{1}$$

For all $t \ge 0$.

1

The first result, [1, Lemma 5.1], is not necessary to prove for this particular setting since the result holds for any semi-martingale, I include it here for self-containment reasons. EI: Instead of this sentence you should write something like 'The lemma below is from [1] and its proof can be found in [1, Lemma 5.1]. We include the statement here for ease of reading'

Lemma 1. *Bound for local time at zero for a semi-martingale For any $\epsilon \in (0,1)$ and any real-valued, continuous semi-martingale Z we have*

$$\mathbb{E}[L_t^0(Z_s)] \leq 4\epsilon - 2\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Z_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Z_s \geq \epsilon\}}e^{1-Z_s/\epsilon}\right) dZ_s\right]$$
$$+ \frac{1}{\epsilon}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z > \epsilon\}}e^{1-Z_s/\epsilon}d\langle Z\rangle_s\right].$$

Let us introduce the original and regularised Kolmogorov equations. To shorten notation we will denote the spaces $C_T\mathcal{C}^\gamma(\mathbb{R})$ as $C_T\mathcal{C}^\gamma$.

Definition 2. *Kolmogorov equations For $\beta \in (0,1/2)$ let $b \in C_T\mathcal{C}^{-\beta}$, $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$, and $b^N \to b$ as $N \to \infty$ in $C_T\mathcal{C}^{-\beta}$ The equations*

$$\begin{cases} \partial_t u_i + \frac{1}{2}\Delta u_i + b_i\nabla u_i = \lambda u_i - b_i \\ u_i(T) = 0, \end{cases} \qquad (2)$$

$$\begin{cases} \partial_t u_i^N + \frac{1}{2}\Delta u_i^N + b_i^N\nabla u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0. \end{cases} \qquad (3)$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

Lemma 2. *Let u, u^N be the solutions to the Kolmogorov equations (2) (3) in $C_T\mathcal{C}^{1+\alpha}$ respectivelly. We have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u(t,x)| \leq \frac{1}{2} \text{ and } \sup_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u^N(t,x)| \leq \frac{1}{2} \qquad (4)$$

Assumption 1. *Let $0 < \beta < 1/2$ and $b \in C_T\mathcal{C}^{-\beta}$.*

Assumption 2. *There exists a sequence $(b^N)_N \in C_T\mathcal{C}^{-\beta}$ such that for each N, $b^N(t,\cdot) \in C_b^\infty(\mathbb{R})$ for all $t \in [0,T]$ and such that $b^N \to b$ as $N \to \infty$.*

LM: the lemma is probably better to type it before the first time it is used since it requires of some results below
LM: TYPE THIS

3 Bounds for the difference of solutions to the Kolmogorov equations

We need a bound for $u - u^N$ and $\nabla u - \nabla u^N$ in L_∞ for the case in which $u \in C_T\mathcal{C}^{1+\alpha}$ for some $\alpha > \beta$ which is an adaptation of [1, Lemma 5.2].

The result builds on top of the following result:

Proposition 1. *Bound for the ρ-equivalent norm of $u - u^N$*
Let u, u^N be (mild) solutions to the Kolmogorov equations from Definition *2 then as*
$N \to \infty$

$$\|u_i - u_i^N\|_{C_T \mathcal{C}^{1+\alpha}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}}(\|u_i\|_{C_T \mathcal{C}^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}}(\|b\|_{C_T \mathcal{C}^{-\beta}} + \lambda)} \tag{5}$$

for $\rho \geq \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_T \infty + \alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \tag{6}$$

and $\lambda > 0$.

Proof. See that $u^N(T) = u(T) = 0$, and in [2], set g^N, g as b^N, b respectively. See that
$b^N \to b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$.
As u^N, u are mild solutions, we have

$$
\begin{aligned}
u_i(t) - u_i^N(t) &= P_{T-t}(u_i(T) - u_i^N(T)) \\
&+ \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\
&- \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N) ds \\
&= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)] ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N) ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \\
&+ \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N) ds
\end{aligned}
$$

Now let us compute the ρ-equivalent norm of $u - u^N$, for some $\alpha > \beta$

!!!!!!!!!!!!!!!!!!!!!!!LM: this norm is wrong, should be $1 + \alpha$ on the lhs and everywhere else !!!!!!!!!!!!!!!!!!!!

$$\|u_i - u_i^N\|_{C_T C^{-\beta}}^{(\rho)} = \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|u(t) - u^N(t)\|_{C_T C^{1+\alpha}}$$

$$\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \|_{C_T C^{1+\alpha}}$$

$$+ \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \|_{C_T C^{1+\alpha}}$$

$$- \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|\lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \|_{C_T C^{1+\alpha}}$$

$$- \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(b_i - b_i^N) ds \|_{C_T C^{1+\alpha}}.$$

Let us take each term from the right hand side of the inequality and bound them.

For the first term, using $\gamma + 2\theta = 1 + \alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta}\|f\|_\gamma$ and $\|\nabla g\|_\xi \leq c\|g\|_{\xi+1}$

$$\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{C_T C^{1+\alpha}}$$

$$\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_{C_T C^\alpha} \|b_i - b_i^N\|_{C_T C^{-\beta}}$$

$$\leq c\|u_i\|_{C_T C_{1+\alpha}} \|b_i - b_i^N\|_{C_T C^{-\beta}} \sup_{0 \leq t \leq T} e^{-\rho(T-t)} (T-t)^{\frac{1-\beta-\alpha}{2}}$$

$$\leq cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T C^{1+\alpha}} \|b_i - b_i^N\|_{C_T C^{-\beta}}$$

For the second term, see that for $N \to \infty$, we have $\|b^N\|_{C_T C^{-\beta}} \leq 2\|b\|_{C_T C^{-\beta}}$

$$\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} b_i^N (\nabla u_i - \nabla u_i^N) ds \right\|_{C_T C^{1+\alpha}}$$

$$\leq c \sup_{0 \leq t \leq T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2\|b_i\|_{C_T C^{-\beta}} \|\nabla u_i - \nabla u_i^N\|_{C_T C^{1+\alpha}} ds$$

$$\leq c\|b_i\|_{C_T C^{-\beta}} \|u_i - u_i^N\|_{C_T C^{-\beta}}^{(\rho)} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} ds$$

$$\leq c\|b_i\|_{C_T C^{-\beta}} \|u_i - u_i^N\|_{C_T C^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}$$

For the third term, which is the one that difers from the proof in [2] we need to use that $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta}\|f\|_\gamma$, and in this case we have $\gamma + 2\theta = 1 + \alpha$ and $\gamma = 1 + \alpha$, so that $\theta = 0$ because $u, u^N \in C_T C^{1+\alpha}$, so we will have

4

$$\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{1+\alpha}$$

$$\leq c\lambda \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-0} \|u_i - u_i^N\|_{1+\alpha} ds$$

$$= c\lambda \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T e^{-\rho(T-s)} \sup_{0 \leq s \leq T} e^{-\rho(T-s)} \|u_i - u_i^N\|_{1+\alpha} ds$$

$$= c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(T-s)} e^{-\rho(T-t)} ds$$

$$= c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(s-t)} ds$$

$$= c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \sup_{0 \leq t \leq T} \rho^{-1} [1 - e^{-\rho(T-t)}]$$

$$\leq c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{-1}$$

$$\leq c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}$$

And for the last term

$$\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{T-s}(b_i - b_i^N) ds \right\|_{C_T \mathcal{C}^{1+\alpha}}$$

$$\leq c \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} ds$$

$$\leq c \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \sup_{0 \leq t \leq T} e^{-\rho(T-t)} (s-t)^{-\frac{\alpha+\beta-1}{2}}$$

$$\leq cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T \mathcal{C}^{-\beta}}$$

Putting everything together

$$\|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \leq cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T \mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}}$$

$$+ c\|b_i\|_{C_T \mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}$$

$$- c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T \mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}$$

$$- cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T \mathcal{C}^{-\beta}},$$

and finally,

$$\|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} (1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C_T \mathcal{C}^{-\beta}} + \lambda])$$

$$\leq cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T \mathcal{C}^{-\beta}}(\|u_i\|_{C_T \mathcal{C}^{1+\alpha}} - 1)$$

$$\|u_i - u_i^N\|^{(\rho)}_{C_T \mathcal{C}^{-\beta}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T \mathcal{C}^{-\beta}}(\|u_i\|_{C_T \mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T \mathcal{C}^{-\beta}} + \lambda])}$$

As required. □

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|^{(\rho)}_{C_T \mathcal{C}^{-\beta}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}(\|u_i\|_{C_T \mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T \mathcal{C}^{-\beta}} + \lambda])}\|b_i - b_i^N\|_{C^T \mathcal{C}^{-\beta}} \tag{7}$$

LM: Check this norm

$$\|u_i - u_i^N\|^{(\rho)}_{C_T \mathcal{C}^{-\beta}} \leq c(\rho)\|b_i - b_i^N\|_{C^T \mathcal{C}^{-\beta}} \tag{8}$$

Here is the adaptation of [1, Lemma 5.2]. _{de_angelis_numerical_2020}

Proposition 2. *Bounds for* $\|u - u^N\|_{L_\infty}$ *and* $\|\nabla u - \nabla u^N\|_{L_\infty}$
Let $\beta \in (0, 1/2)$ *and* $b \in C_T \mathcal{C}^{-\beta}$*. Let* $u, u^N \in C_T \mathcal{C}^{(1+\beta)+}$ *be (mild) solutions to the Kolmogorov equations from Definition 2.*
Assume, by Proposition 1, that for some $\alpha > \beta$

$$\|u - u^N\|^{(\rho)}_{C_T \mathcal{C}^{1+\alpha}} \leq c(\rho)\|b - b^N\|_{C_T \mathcal{C}^{-\beta}}. \tag{9}$$

With $c(\rho)$ *as in Proposition 1 and* ρ_0 *is large enough such that* $c(\rho) > 0$ *for all* $\rho > \rho_0$*. Then for all* $t \in [0, T]$

$$\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C_T \mathcal{C}^{-\beta}} \tag{10}$$

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C_T \mathcal{C}^{-\beta}} \tag{11}$$

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$*.*

Proof. First let us prove (10).
Let $t \in [0, T]$, and see that since $u, u^N \in C_T \mathcal{C}^{(1+\beta)+}$ there exists $\alpha > \beta$ such that $u, u^N \in C_T \mathcal{C}^{1+\alpha}$, then for any $f \in \mathcal{C}^{1+\alpha}$ we have

$$\|f\|_{C^{1+\alpha}} \leq c \left(\sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \neq y \in \mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \right) \tag{12}$$

so we have

$$\begin{aligned}\|u^N(t) - u(t)\|_{L^\infty} &= \sup_{x \in \mathbb{R}^d} |u^N(t, x) - u(t, x)| \\ &\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}}\end{aligned} \tag{13}$$

Moreover, using the (ρ)-equivalent norm

$$\|f\|_{\mathcal{C}^{1+\alpha}} = \sup_{t \in [0,T]} e^{-\rho(T-t)}\|f(t)\|_{\mathcal{C}^{1+\alpha}}, \tag{14}$$

6

and (9) we see that

$$
\begin{aligned}
\left\|u^N - u\right\|_{C_T \mathcal{C}^{1+\alpha}} &= \sup_{t \in [0,T]} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\
&= \sup_{t \in [0,T]} e^{\rho(T-t)} e^{-\rho(T-t)} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\
&\leq e^{\rho T} \sup_{t \in [0,T]} e^{-\rho(T-t)} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\
&= e^{\rho T} \left\|u^N - u\right\|_{C_T \mathcal{C}^{1+\alpha}}^{(\rho)}.
\end{aligned}
\tag{15}
$$

Plugging (15) into (13)

$$
\begin{aligned}
\|u^N(t) - u(t)\|_{L^\infty} &\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \\
&\leq \sup_{t \in [0,T]} c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{\alpha+1}} \\
&= c\left\|u^N - u\right\|_{C_T \mathcal{C}^{\alpha+1}} \\
&\leq c e^{\rho T} \left\|u^N - u\right\|_{C_T \mathcal{C}^{\alpha+1}}^{(\rho)}.
\end{aligned}
\tag{16}
$$

And finally by (9)

$$
\left\|u^N(t) - u(t)\right\|_{L^\infty} \leq c \cdot c(\rho) \cdot e^{\rho T} \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}}
\tag{17}
$$

which proves (10).

For (11) recall that if $f \in \mathcal{C}^{1+\alpha}$ then $\nabla f \in \mathcal{C}^\alpha$. Also, by Bernstein inequality [3, Eqn. (9)]

$$
\|\nabla f\|_\alpha \leq c\|f\|_{\infty + \alpha}.
\tag{18}
$$

Using the equivalent norm

$$
\|f\|_{C^{1+\alpha}} \leq c \left(\sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \in \mathbb{R}^d} |\nabla f(x)| + \sup_{x \neq y \in \mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \right)
\tag{19}
$$

we can see that

$$
\left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \leq c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1+\alpha}}.
\tag{20}
$$

And usign the same bounds that we used above for $c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1+\alpha}}$ this point follows.

$\square$

4   Bound for the difference of the auxiliary functions

This is the adaptation of result [1, Lemma 5.3].

Proposition 3. *Take $\rho > \rho_0$ as in Proposition 1, $N \to \infty$, κ_ρ from Proposition 2, and $\beta \in (0, 1/2)$, then we have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \left|\psi(t,x) - \psi^N(t,x)\right| \leq 2\kappa_\rho \left\|b - b^N\right\|_{C_T \mathcal{C}^{-\beta}} \tag{21}$$

Proof. Recall the definition of $\psi, \phi \in C_T \mathcal{C}^1$

$$\phi(t,x) := x + u(t,x) \tag{22}$$

$$\psi(t,\cdot) = \phi^{-1}(t,\cdot). \tag{23}$$

Note that

$$u(y) = \int_0^1 \nabla u(\alpha y) y \, d\alpha + u(0). \tag{24}$$

From there we have

$$u(t,y) - u(t,y') = \int_0^1 \nabla u(t, \alpha(y - y'))(y - y') \, d\alpha \tag{25}$$

and therefore

$$|u(t,y) - u(t,y')| \geq \left(\int_0^1 |\nabla u(t, \alpha(y - y'))|^2 \, d\alpha\right)^{1/2} |y - y'|, \tag{26}$$

and by Lemma 2 we finally have

$$|u(t,y) - u(t,y')| \leq \left(\frac{1}{4} \int_0^1 d\alpha\right)^{1/2} |y - y'|$$
$$|u(t,y) - u(t,y')|^2 \leq \frac{1}{4} |y - y'|^2 \tag{27}$$

LM: continue from page three in notes

$\square$

5 Bound for the local time at zero of the solution to the SDEs

LM: Here I still need to mention how we define $Y_t = \psi(t, X_t)$, because eventually I need to use that $X_t = \psi(t, Y_t)$, probably just neet to mention without defining the whole Y_t as in the paper

We need a bound for $\mathbb{E}[L_T^0(Y^N - Y)]$, for Sobolev spaces, this is result [1, Proposition 5.4] we present it here for the solutions to the SDE blonging to the appropriate Besov spaces.

First let us state the following useful result.

LM: check that the statement makes sense and has all the necessary assumptions

Lemma 3. *Let u, u^N be solutions to the Kolmogorov equations* (2) (3) *then the following bound is satisfied:*

$$\left| u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)) \right| \leq 2\kappa_\rho \left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta}} + \left| Y_s^N - Y^N \right|. \qquad (28) \quad \boxed{\texttt{\{eq:bound\_u\_abs\}}}$$

Proof. Adding and subtracting terms, using triangle inequality and noting that for any a, b, we have $a - b \leq |a - b|$, then

$$
\begin{aligned}
\left| u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)) \right| &\leq \left| u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N)) \right| \\
&\quad + \left| u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N)) \right| \qquad (29) \\
&\quad + \left| u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s)) \right|.
\end{aligned}
$$

The terms in the right hand side will be bounded as follows:

- For the first term, by Proposition 2

$$\left| u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N)) \right| \leq \left\| u^N(s) - u(s) \right\|_{L^\infty} \leq \kappa_\rho \left\| b - b^N \right\|_{C_T \mathcal{C}^{-\beta}}, \qquad (30)$$

- for the second term, observe that u, u^N are $\frac{1}{2}$-Lipschitz and by Proposition 3 we get

$$
\begin{aligned}
\left| u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N)) \right| &\leq \frac{1}{2} \left| \psi^N(s, Y_s^N) - \psi(s, Y_s^N) \right| \qquad (31) \\
&\leq \kappa_\rho \left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta}},
\end{aligned}
$$

- and for the final term, note that ψ, ψ^N are 2-Lipschitz so that

$$\left| u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s)) \right| \leq \frac{1}{2} \left| \psi(s, Y_s^N) - \psi(s, Y_s) \right| \leq \left| Y_s^N - Y_s \right|. \qquad (32)$$

So that the following bound holds

$$\left| u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)) \right| \leq 2\kappa_\rho \left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta}} + \left| Y_s^N - Y^N \right|, \qquad (33)$$

as required.

$\square$

Proposition 4. *Let A, B be constants, $b \in C_T \mathcal{C}^{-\beta}$ and $b^N \to b$ in $C_T \mathcal{C}^{-\beta}$ as $N \to \infty$ for $\beta \in (0, \frac{1}{4})$ and for any $\alpha > \beta$*

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq o\left(\left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha-1} \right) + A\mathbb{E}\left(\int_0^t \left| Y^N - Y \right| ds \right) + B \left\| b^N - b \right\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha-1}.$$

$$(34) \quad \boxed{\texttt{\{eq:local\_time\_YNY\_bo}}$$

9

Proof. Recall that Y^N, Y are solutions to the SDEs

$$Y_t = y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1)dW_s \qquad (35)$$

and

$$Y_t^N = y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1)dW_s \qquad (36)$$

so that the difference $Y^N - Y$ is

$$
\begin{aligned}
Y_t^N - Y_t &= \left(y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1)dW_s \right) \\
&\quad - \left(y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1)dW_s \right) \\
&= (y_0^N - y_0) + \lambda \int_0^t (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)))ds \\
&\quad + \int_0^t (\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)))dW_s,
\end{aligned} \qquad (37)
$$

and using Lemma $\overset{\texttt{lemma:local-time-at-0}}{1}$ we have the following bound

$$
\begin{aligned}
&\mathbb{E}[L_t^0(Y^N - Y)] \\
&\leq 4\epsilon \\
&\quad - 2\lambda \mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \right) \right.
\end{aligned}
$$
$$\left. \left(u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)) \right) ds \right] \quad (38) \qquad \boxed{\texttt{\{eq:local\_time\_diff\_u}}}$$
$$+ \frac{1}{\epsilon} \mathbb{E}\left[\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \left(\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)) \right)^2 ds \right]. \quad (39) \qquad \boxed{\texttt{\{eq:local\_time\_diff\_g}}}$$

LM: add the explanation of why to drop the difussion term

First, for ($\overset{\texttt{eq:local\_time\_diff\_u}}{38}$), we find a bound for the factor involving the difference of u^N and u in Lemma $\overset{\texttt{lemma:uN-n\_bound\_for\_integral}}{3}$. Therefore

$$\left| u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)) \right| \leq 2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + |Y_s^N - Y^N|. \qquad (40)$$

Now we need to bound the result of the local time of the difference $Y_s^N - Y_s$. First notice that $Y_s^N - Y_s \geq \epsilon$, then $e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \leq 1$, also it is clear that $\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}}$ and $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}}$ are bounded by 1, therefore $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \leq 1$. Using the previous arguments and ($\overset{\texttt{eq:bound\_u\_abs}}{28}$) lead to have

$$
\begin{aligned}
(\overset{\texttt{eq:local\_time\_diff\_u}}{38}) &\leq 2\lambda \mathbb{E}\left[\int_0^t 2 \left(2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} + |Y_s^N - Y^N| \right) ds \right] \\
&\leq 4\lambda \left(2\kappa_\rho \|b^N - b\|_{C_T C^{-\beta}} t + \mathbb{E}\left[\int_0^t |Y_s^N - Y^N| ds \right] \right).
\end{aligned} \qquad (41) \qquad \boxed{\texttt{\{eq:bound\_integral\_uN}}}
$$

10

Now for (39), we use similar arguments as the ones in Lemma 3 above, and we get the following:

$$
\begin{aligned}
\left|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\right| \leq & \left|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi^N(s, Y_s^N))\right| \\
& + \left|\nabla u(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s^N))\right| \\
& + \left|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\right|,
\end{aligned}
\tag{42}
$$

where the terms on the right hand side will be bounded as follows:

- For the first term we use Proposition 2 and we have

$$
\begin{aligned}
\left|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi^N(s, Y_s^N))\right| &\leq \left\|\nabla u^N(s) - \nabla u(s)\right\|_{L^\infty} \\
&\leq \kappa_\rho \left\|b - b^N\right\|_{C_T \mathcal{C}^{-\beta}},
\end{aligned}
\tag{43}
$$

for the second term see that $\nabla u, \nabla u^N$ are α-Hölder continuous and using Proposition 3 we have

$$
\begin{aligned}
\left|\nabla u(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s^N))\right| &\leq \left|\psi^N(s, Y_s^N) - \psi(s, Y_s^N)\right|^\alpha \|u\|_{C_T \mathcal{C}^{1+\alpha}} \\
&\leq (2\kappa_\rho \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}})^\alpha \|u\|_{C_T \mathcal{C}^{1+\alpha}}.
\end{aligned}
\tag{44}
$$

Therefore we get the bound

$$
\begin{aligned}
\left|\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\right| \leq & \kappa_\rho \left\|b - b^N\right\|_{C_T \mathcal{C}^{-\beta}} \\
& + \alpha \kappa_\rho^\alpha \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}}^\alpha \|u\|_{C_T \mathcal{C}^{1+\alpha}} \\
& + \left|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\right|.
\end{aligned}
\tag{45}
$$

{eq:bound_gradu_abs}

Here we can also notice that $\mathbb{1}_{\{Y_s^N - Y_s < \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} < 1$, then using (45) and the inequality

$$
(x_1 + \cdots + x_k)^2 \leq k(x_1^2 + \cdots + x_k^2),
\tag{46}
$$

for $k = 3$, we can get the bound

$$
\begin{aligned}
(39) \leq & \frac{1}{\epsilon} \mathbb{E} \int_0^t \left(3\kappa_\rho^2 \left\|b - b^N\right\|_{C_T \mathcal{C}^{-\beta}}^2 + 3 \cdot 2^{2\alpha} \kappa_\rho^{2\alpha} \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2\right) ds \\
& + \frac{1}{\epsilon} \mathbb{E} \int_0^t 3 \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \left|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\right|^2 ds \\
\leq & \frac{1}{\epsilon} 3t \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}} \left(\kappa_\rho^2 \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}} + (2\kappa_\rho)^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 \left\|b^N - b\right\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha - 1}\right) \\
& + \frac{1}{\epsilon} 3 \mathbb{E} \left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \left|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\right|^2 ds\right)
\end{aligned}
\tag{47}
$$

{eq:bound_integral_gr

11

Now let us denote the last term in ($\overset{\texttt{eq:bound\_integral\_graduNu}}{47}$) by $I_t^{N,\epsilon}$. Pick $\zeta \in (0,1)$ such that $\alpha\zeta > \frac{1}{2}$, and since $\epsilon \in (0,1)$ we have $\epsilon^\zeta > \epsilon$. Then split the indicator function $\mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}}$ into $\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}}$. Leading to the integral

$$
I_t^{N,\epsilon} = \frac{1}{\epsilon} 3\mathbb{E} \left(\int_0^t \left(\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}} \right) e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \right.
$$
$$
\left. \left| \nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)) \right|^2 ds \right)
$$
(48) {eq:INepsilon}

For the first term of ($\overset{\texttt{eq:INepsilon}}{48}$) we use the fact that ∇u is α-Hölder continuous uniformly in $s \in [0,T]$ with constant $\|u\|_{C_T C^{1+\alpha}}$ and that ψ is 2-Lipschitz

$$
\left| \nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)) \right|^2 \le \left| \left| \psi(s, Y_s^N) - \psi(s, Y_s) \right|^\alpha \|u\|_{C_T C^{1+\alpha}} \right|^2
$$
$$
\le \left| 2^\alpha \left| Y_s^N - Y_s \right|^\alpha \|u\|_{C_T C^{1+\alpha}} \right|^2
$$
(49)
$$
= 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \left| Y_s^N - Y_s \right|^{2\alpha}
$$

For the other term we need another way to bound it, because even though the event when $\left| Y^N - Y \right| > \epsilon^\zeta$ is small, we can potentially have a quantity that blows up for the bound. EI: the explanation needs adjusting - speak to Elena In order to solve this problem, we can use the fact that ∇u is uniformly bounded by $1/2$ thanks to Lemma $\overset{\texttt{lemma:bounds\_gradients}}{2}$, and then we can bound the difference of the gradients as follows:

$$
\left| \nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)) \right|^2 \le \left| \nabla u(s, \psi(s, Y_s^N)) + \nabla u(s, \psi(s, Y_s)) \right|^2
$$
$$
\le \sup_{(s,x) \in [0,T] \times \mathbb{R}} \left| \nabla u(s, \psi(s, Y_s^N)) + \nabla u(s, \psi(s, Y_s)) \right|^2
$$
$$
= \|2\nabla u\|_{L_\infty}^2.
$$
(50)

Therefore we have that for all $t \in [0,T]$ LM: check where else I need to say this

$$
I_t^{N,\epsilon} \le \frac{1}{\epsilon} 3\mathbb{E} \left(\int_0^t \left(\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^\zeta\}} \right) e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} 2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \left| Y_s^N - Y_s \right|^{2\alpha} ds \right)
$$
$$
+ \frac{1}{\epsilon} 3\mathbb{E} \left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \|2\nabla u\|_{L_\infty}^2 ds \right)
$$
$$
\le \frac{1}{\epsilon} 3\mathbb{E} \left(2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \left| \epsilon^\zeta \right|^{2\alpha} t \right) + \frac{1}{\epsilon} 3\mathbb{E} \left(4e^{1 - \epsilon^{\zeta-1}} \|\nabla u\|_{L_\infty}^2 t \right)
$$
(51) {eq:INepsilon_bound}
$$
\le \sup_{t \in [0,T]} \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \epsilon^{2\alpha\zeta} + 4e^{1 - \epsilon^{\zeta-1}} \|\nabla u\|_{L_\infty}^2 \right) t
$$
$$
= \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|_{C_T C^{1+\alpha}}^2 \epsilon^{2\alpha\zeta} + 4e^{1 - \epsilon^{\zeta-1}} \|\nabla u\|_{L_\infty}^2 \right) T.
$$

Now by combining ($\overset{\texttt{eq:bound\_integral...}}{41}$), ($\overset{\texttt{eq:...graduNu}}{47}$) and ($\overset{\texttt{eq:...bound}}{51}$), and taking the sup over $[0,T]$ we will get

12

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq 4\epsilon$$
$$+ 4\lambda 2\kappa_\rho T \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}$$
$$+ 4\lambda \mathbb{E}\left[\int_0^t |Y_s^N - Y^N| ds\right]$$
$$+ \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T\kappa_\rho^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \qquad (52)$$
$$+ \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T(2\kappa_\rho)^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha-1}$$
$$+ \frac{3}{\epsilon} 2^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 T\epsilon^{2\alpha\zeta}$$
$$+ \frac{3}{\epsilon} 4\|\nabla u\|_{L_\infty}^2 Te^{1-\epsilon^{\zeta-1}}$$

then we take $\epsilon = \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}$ and we get

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq 4\|b^N - b\|_{C_T \mathcal{C}^{-\beta}}$$
$$+ 4\lambda 2\kappa_\rho T \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}$$
$$+ 4\lambda \mathbb{E}\left[\int_0^t |Y_s^N - Y^N| ds\right]$$
$$+ 3T\kappa_\rho^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} \qquad (53)$$
$$+ 3T(2\kappa_\rho)^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha-1}$$
$$+ 2^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 T\|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha\zeta-1}$$
$$+ 4\|\nabla u\|_{L_\infty}^2 T\|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{-1} \exp\left(1 - \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{\zeta-1}\right)$$

which can be written as

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq c_1 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}} + c_2 \mathbb{E}\left[\int_0^t |Y_s^N - Y^N| ds\right]$$
$$+ c_3 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha-1} + c_4 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{2\alpha\zeta-1} \qquad (54) \quad \boxed{\texttt{\{eq:bound\_constants\}}}$$
$$+ c_5 \exp\left(1 - \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{\zeta-1}\right)$$

where

$$c_1 = 4 + 4\lambda 2\kappa_\rho T + 3\kappa_\rho^2 T$$
$$c_2 = 4\lambda$$
$$c_3 = 3(2\kappa_\rho)^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 T \qquad (55) \quad \boxed{\texttt{\{eq:constants\_c\}}}$$
$$c_4 = 2^{2\alpha} \|u\|_{C_T \mathcal{C}^{1+\alpha}}^2 T$$
$$c_5 = 4\|\nabla u\|_{L_\infty}^2 \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{-1} T$$

Finally, observe that since $\zeta \in (0,1)$, the term $\exp\left(1 - \|b^N - b\|_{C_T \mathcal{C}^{-\beta}}^{\zeta-1}\right)$ decays faster than any polynomial, thus controlling c_5, and the last term in (54) goes to zero. Also

13

$\alpha\zeta$ is arbitrarily close to α, and $\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}$ controls $\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}$ therefore we can create the bound (34)

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq o\left(\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) + c_2\mathbb{E}\left(\int_0^t |Y^N - Y| ds\right) + c_4\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}$$
(56)

$\square$

6 Convergence rate of the solution to the regularised SDE and the original

In this section we present a bound for $\mathbb{E}[X^N - X]$ in terms of $\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}$.

Proposition 5. *Let assumptions 1 2 hold, then for any $\alpha > \beta$ there is a constant C_α such that*

$$\mathbb{E}[X^N - X] \leq C_\alpha\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}},$$
(57) {eq:EXN-X}

as $N \to \infty$.

Proof. Note that by definition of ψ, ψ^N we have

$$\begin{aligned}|X_t^N - X_t| &= |\psi^N(t, \phi^N(t, X_t^N)) - \psi(t, \phi(t, X_t))| \\ &= |\psi^N(t, Y_t) - \psi(t, Y_t)|,\end{aligned}$$
(58)

then adding and subtracting, and using the triangle inequality we get

$$|X_t^N - X_t| \leq |\psi^N(t, Y_t^N) - \psi(t, Y_t^N)| + |\psi(t, Y_t^N) - \psi(t, Y_t)|.$$
(59)

Where the first term is bounded by $2\kappa\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}$ (Proposition 3) and since ψ is 2-Lipschitz uniformly in $t \in [0, T]$ the second term is bounded by $2|Y^N - Y|$, therefore

$$|X^N - X| \leq 2\kappa\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + 2|Y^N - Y|.$$
(60) {eq:XN-X}

By assumption the first term above goes to zero as $N \to \infty$, then we only need a bound for the second term.

By Itô-Tanaka's formula

$$|Y^N - Y| = |y_0^N - y_0| + \frac{1}{2}L_t^0(Y^N - Y) + \int_0^t \text{sgn}(Y^N - Y)d(Y^N - Y),$$
(61) {eq:YNY_ito_tanaka}

14

by taking expectation and using the definitions of Y^N, Y we have

$$\mathbb{E}|Y^N - Y| = \mathbb{E}|y_0^N - y_0| + \mathbb{E}\frac{1}{2}L_t^0(Y^N - Y)$$
$$+ \lambda\mathbb{E}\int_0^t \text{sgn}(Y^N - Y)(u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)))ds, \tag{62}$$

`{eq:EYNY}`

then as the first term above is a constant and for the second we have a bound in Proposition 4 `prop:bound_local_time_sde` it remains to bound the second term.

Observe that □

References

`ngelis_numerical_2020`

[1] Tiziano De Angelis, Maximilien Germain, and Elena Issoglio. "A Numerical Scheme for Stochastic Differential Equations with Distributional Drift". In: *arXiv:1906.11026 [cs, math]* (Oct. 22, 2020). arXiv: 1906.11026.

`issoglio_pde_nodate`

[2] Elena Issoglio and Francesco Russo. "A PDE with Drift of Negative Besov Index and Related Martingale Problem". In: (), p. 48.

`issoglio_mckean_2021`

[3] Elena Issoglio and Francesco Russo. "McKean SDEs with singular coefficients". In: *arXiv:2107.14453 [math]* (July 30, 2021). arXiv: 2107.14453.