Convergence rate of $X^N - X$ for McKean equations

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Todo list

Contents

1 What is this?

An adaptation of [1, Proposition 3.1] for the case of McKean SDEs EI:not yet for McKean SDEs, here you are only looking at SDEs with distributional coefficients in Besov spaces. SDEs framed as martingale problems SDEs with drift in a Besov space of negative order proposed in [2] and [3].

The proof builds on a number of results presented in the sections below.

2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

Definition 1. *Local time at zero For any real-valued continuous semi-martingale $\bar{Y}$, the local time at zero $L_t^0(\bar{Y})$ is defined as*

$$L_t^0(\bar{Y}) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|\bar{Y}| \leq \epsilon\}} d\langle \bar{Y} \rangle_s, \ \mathbb{P}\text{-}a.s. \tag{1}$$

For all $t \geq 0$.

The first result, [1, Lemma 5.1], is not necessary to prove for this particular setting since the result holds for any semi-martingale, I include it here for self-containment reasons. EI:Instead of this sentence you should write something like 'The lemma below is from [1] and its proof can be found in [1, Lemma 5.1]. We include the statement here for ease of reading'

Lemma 1. *Bound for local time at zero for a semi-martingale For any $\epsilon \in (0,1)$ and any real-valued, continuous semi-martingale Z we have*

$$\mathbb{E}[L_t^0(Z_s)] \leq 4\epsilon - 2\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Z_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Z_s \geq \epsilon\}} e^{1-Z_s/\epsilon}\right) dZ_s\right]$$
$$+ \frac{1}{\epsilon}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z > \epsilon\}} e^{1-Z_s/\epsilon} d\langle Z\rangle_s\right].$$

EI:It is confusing to call the semimartingale $\bar{Y}$ above and Z in the lemma. It is not wrong, but not the best use of notation.

Let us introduce the original and regularised Kolmogorov equations.

Definition 2. *Kolmogorov equations*

For $\beta \in (0, 1/2)$ let $b \in C_T\mathcal{C}^{-\beta}$, $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$, and $b^N \to b$ as $N \to \infty$ in $C_T\mathcal{C}^{-\beta}$ The equations EI:EI: mistake in PDEs, correct please!!

$$\begin{cases} \partial u_i + \frac{1}{2} b_i \Delta u_i = \lambda u_i - b_i \\ u_i(T) = 0, \end{cases} \tag{2}$$

$$\begin{cases} \partial u_i^N + \frac{1}{2} b_i^N \Delta u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0. \end{cases} \tag{3}$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

Lemma 2. *Bounds for $\nabla u, \nabla u^N$ Let*

type this result whi is lemma in the pap

3 Bounds for the difference of solutions to the Kolmogorov equations

We need a bound for $u - u^N$ and $\nabla u - \nabla u^N$ in L_∞ for the case in which $u \in C_T\mathcal{C}^{1+\alpha}$ for some $\alpha > \beta$ which is an adaptation of [de_angelis_numerical_2020][1, Lemma 5.2].

The result builds on top of the following result:

Proposition 1. *Bound for the ρ-equivalent norm of $u - u^N$*

Let u, u^N be (mild) solutions to the Kolmogorov equations from [def:kolmogorov_eqns][2] then as $N \to \infty$

$$\|u_i - u_i^N\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}}(\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda)} \tag{4}$$

for $\rho \geq \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_T\infty+\alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \tag{5}$$

and $\lambda > 0$.

Proof. See that $u^N(T) = u(T) = 0$, and in [issoglio_pde_nodate][2], set g^N, g as b^N, b respectively. See that $b^N \to b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$.

As u^N, u are mild solutions, we have

$$u_i(t) - u_i^N(t) = P_{T-t}(u_i(T) - u_i^N(T))$$
$$+ \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds$$
$$- \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds$$
$$= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds$$

$$-\lambda \int_t^T P_{s-t}(u_i - u_i^N) ds$$

$$-\int_t^T P_{s-t}(b_i - b_i^N) ds$$

$$= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)] ds$$

$$-\lambda \int_t^T P_{s-t}(u_i - u_i^N) ds$$

$$-\int_t^T P_{s-t}(b_i - b_i^N) ds$$

$$= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds$$

$$+ \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds$$

$$-\lambda \int_t^T P_{s-t}(u_i - u_i^N) ds$$

$$-\int_t^T P_{s-t}(b_i - b_i^N) ds$$

Now let us compute the ρ-equivalent norm of $u - u^N$, for some $\alpha > \beta$

$$\|u_i - u_i^N\|_{C_T \mathcal{C}^{-\beta}}^{(\rho)} = \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|u(t) - u^N(t)\|_{1+\alpha}$$

$$\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|\int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds\|_{1+\alpha}$$

$$+ \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|\int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds\|_{1+\alpha}$$

$$- \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|\lambda \int_t^T P_{s-t}(u_i - u_i^N) ds\|_{1+\alpha}$$

$$- \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|\int_t^T P_{s-t}(b_i - b_i^N) ds\|_{1+\alpha}.$$

Let us take each term from the right hand side of the inequality and bound them.
For the first term, using $\gamma + 2\theta = 1 + \alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta}\|f\|_\gamma\|$ and $\|\nabla g\|_\xi \leq c\|g\|_{\xi+1}$

$$\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\|\int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds\right\|_{1+\alpha}$$

$$\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_\alpha \|b_i - b_i^N\|_{-\beta}$$

$$\leq c\|u_i\|_{C_T \mathcal{C}_{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}} \sup_{0 \leq t \leq T} e^{-\rho(T-t)}(T-t)^{\frac{1-\beta-\alpha}{2}}$$

$$\leq cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T \mathcal{C}_{1+\alpha}} \|b_i - b_i^N\|_{C_T \mathcal{C}^{-\beta}}$$

For the second term, see that for $N \to \infty$, we have $\|b^N\|_{C_T \mathcal{C}^{-\beta}} \leq 2\|b\|_{C_T \mathcal{C}^{-\beta}}$

$$\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\|\int_t^T P_{s-t} b_i^N (\nabla u_i - \nabla u_i^N) ds\right\|_{1+\alpha}$$

$$\leq c \sup_{0\leq t\leq T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2\|b_i\|_{-\beta} \|\nabla u_i - \nabla u_i^N\|_\alpha ds$$

$$\leq c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} ds$$

$$\leq c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}$$

For the third term, which is the one that difers from the proof in [2] we need to use that $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta}\|f\|_\gamma$, and in this case we have $\gamma + 2\theta = 1 + \alpha$ and $\gamma = 1 + \alpha$, so that $\theta = 0$ because $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, so we will have

$$\sup_{0\leq t\leq T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{1+\alpha}$$

$$\leq c\lambda \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-0} \|u_i - u_i^N\|_{1+\alpha} ds$$

$$= c\lambda \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T e^{-\rho(T-s)} \sup_{0\leq s\leq T} e^{-\rho(T-s)} \|u_i - u_i^N\|_{1+\alpha} ds$$

$$= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|_{C_T\mathcal{C}_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(T-s)} e^{-\rho(T-t)} ds$$

$$= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|_{C_T\mathcal{C}_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(s-t)} ds$$

$$= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|_{C_T\mathcal{C}_{1+\alpha}}^{(\rho)} \sup_{0\leq t\leq T} \rho^{-1}[1 - e^{-\rho(T-t)}]$$

$$\leq c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|_{C_T\mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{-1}$$

$$\leq c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|_{C_T\mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}$$

And for the last term

$$\sup_{0\leq t\leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{T-s}(b_i - b_i^N) ds \right\|_{1+\alpha}$$

$$\leq c \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}} \|b_i - b_i^N\|_{-\beta} ds$$

$$\leq c\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\leq t\leq T} e^{-\rho(T-t)} (s-t)^{-\frac{\alpha+\beta-1}{2}}$$

$$\leq cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}$$

Putting everything together

$$\|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \leq cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T\mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}}$$

$$+ c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}$$

$$- c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|_{C_T\mathcal{C}_{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}$$

$$- cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}},$$

and finally,

$$\|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} (1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])$$

$$\leq cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)$$

$$\|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}$$

As required. □

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{6}$$

The orange box: "Check thi norm"

$$\|u_i - u_i^N\|_{C_T\mathcal{C}^{-\beta}}^{(\rho)} \leq c(\rho)\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{7}$$

Here is the adaptation of [1, Lemma 5.2].

Proposition 2. *Bounds for* $\|u - u^N\|_{L_\infty}$ *and* $\|\nabla u - \nabla u^N\|_{L_\infty}$
Let $\beta \in (0, 1/2)$ *and* $b \in C_T\mathcal{C}^{-\beta}$. *Let* $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ *be (mild) solutions to the Kolmogorov equations from [2].*
Assume, by 1, that for some $\alpha > \beta$

$$\|u - u^N\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)} \leq c(\rho)\|b - b^N\|_{C_T\mathcal{C}^{-\beta}}. \tag{8}$$

With $c(\rho)$ *as in 1 and* ρ_0 *is large enough such that* $c(\rho) > 0$ *for all* $\rho > \rho_0$.
Then for all $t \in [0, T]$

$$\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{9}$$

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{10}$$

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$.

Proof. First let us prove 9.

Let $t \in [0, T]$, and see that since $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ there exists $\alpha > \beta$ such that $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, then for any $f \in \mathcal{C}^{1+\alpha}$ we have

$$\|f\|_{C^{1+\alpha}} \leq c\left(\sup_{x\in\mathbb{R}^d}|f(x)| + \sup_{x\neq y\in\mathbb{R}^d}\frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha}\right) \tag{11}$$

so we have

$$\|u^N(t) - u(t)\|_{L^\infty} = \sup_{x\in\mathbb{R}^d}|u^N(t,x) - u(t,x)|$$
$$\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \tag{12}$$

Moreover, using the (ρ)-equivalent norm

$$\|f\|_{\mathcal{C}^{1+\alpha}} = \sup_{t\in[0,T]}e^{-\rho(T-t)}\|f(t)\|_{\mathcal{C}^{1+\alpha}}, \tag{13}$$

and 8 we see that

$$\begin{aligned}
\|u^N - u\|_{C_T\mathcal{C}^{1+\alpha}} &= \sup_{t\in[0,T]}\|u^N - u\|_{\mathcal{C}^{1+\alpha}} \\
&= \sup_{t\in[0,T]}e^{\rho(T-t)}e^{-\rho(T-t)}\|u^N - u\|_{\mathcal{C}^{1+\alpha}} \\
&\leq e^{\rho T}\sup_{t\in[0,T]}e^{-\rho(T-t)}\|u^N - u\|_{\mathcal{C}^{1+\alpha}} \\
&= e^{\rho T}\|u^N - u\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)}.
\end{aligned} \tag{14}$$

Plugging 14 into 12

5

$$\begin{aligned}
\left\|u^N(t) - u(t)\right\|_{L^\infty} &\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \\
&\leq \sup_{t\in[0,T]} c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{\alpha+1}} \\
&= c\left\|u^N - u\right\|_{C_T\mathcal{C}^{\alpha+1}} \\
&\leq ce^{\rho T}\left\|u^N - u\right\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}.
\end{aligned} \tag{15}$$

And finally by $\overset{\text{eq:u-uNb-bN}}{8}$

$$\left\|u^N(t) - u(t)\right\|_{L^\infty} \leq c \cdot c(\rho) \cdot e^{\rho T}\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \tag{16}$$

which proves $\overset{\text{eq:uNu\_bounded\_by\_bNb}}{9}$.

For $\overset{\text{eq:graduNu\_bounded\_by\_bNb}}{10}$ recall that if $f \in \mathcal{C}^{1+\alpha}$ then $\nabla f \in \mathcal{C}^\alpha$. Also, by Bernstein inequality $[\overset{\text{issoglio\_mckean\_2021}}{3}, \text{Eqn. (9)}]$

$$\|\nabla f\|_\alpha \leq c\|f\|_{\infty+\alpha}. \tag{17}$$

Using the equivalent norm

$$\|f\|_{C^{1+\alpha}} \leq c\left(\sup_{x\in\mathbb{R}^d}|f(x)| + \sup_{x\in\mathbb{R}^d}|\nabla f(x)| + \sup_{x\neq y\in\mathbb{R}^d}\frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha}\right) \tag{18}$$

we can see that

$$\left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \leq c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1+\alpha}}. \tag{19}$$

And usign the same bounds that we used above for $c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1+\alpha}}$ this point follows.

$\square$

4 Bound for the difference of the auxiliary functions

This is the adaptation of result $[\overset{\text{de\_angelis\_numerical\_2020}}{1}, \text{Lemma 5.3}]$.

Proposition 3. *Bound for* $|\psi(t,x) - \psi^N(t,x)|$
Take $\rho > \rho_0$ *as in* $[\overset{\text{prop:diff\_u\_uN}}{1}$, $N \to \infty$, κ_ρ *from* $\overset{\text{prop:diff\_uN\_graduN}}{2}$, *and* $\beta \in (0,1/2)$, *then we have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}\left|\psi(t,x) - \psi^N(t,x)\right| \leq 2\kappa_\rho\left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}} \tag{20}$$

Proof. Recall the definition of $\psi, \phi \in C_T\mathcal{C}^1$

$$\phi(t,x) := x + u(t,x) \tag{21}$$

$$\psi(t,\cdot) = \phi^{-1}(t,\cdot). \tag{22}$$

Note that

$$u(y) = \int_0^1 \nabla u(\alpha y)y\,d\alpha + u(0). \tag{23}$$

From there we have

$$u(t,y) - u(t,y') = \int_0^1 \nabla u(t,\alpha(y-y'))(y-y')\,d\alpha \tag{24}$$

and therefore

$$|u(t,y) - u(t,y')| \geq \left(\int_0^1 |\nabla u(t,\alpha(y-y'))|^2\,d\alpha\right)^{1/2}|y-y'|, \tag{25}$$

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and by lemma:bounds_gradients 2 we finally have

$$|u(t,y) - u(t,y')| \leq \left(\frac{1}{4}\int_0^1 d\alpha\right)^{1/2}|y - y'|$$

$$|u(t,y) - u(t,y')|^2 \leq \frac{1}{4}|y - y'|^2$$

(26)

$\square$

5   Bound for the local time at zero of the solution to the SDEs

We need a bound for $\mathbb{E}[L_T^0(Y^N - Y)]$, for Sobolev spaces, this is result [1, Proposition 5.4] de_angelis_numerical_2020 we present it here for the solutions to the SDE blonging to the appropriate Besov spaces.

Proposition 4. *Bound for local time of solutions to the SDE Let $b \in C_T\mathcal{C}^{-\beta}$ and $b^N \to b$ in $C_T\mathcal{C}^{-\beta}$ as $N \to \infty$ for $\beta \in (0, \frac{1}{4})$ and for any $\alpha > \beta$*

Proof. EI:Make sure that you write λ and not $1 + \lambda$ Recall that Y^N, Y are solutions to the SDEs

$$Y_t = y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1)dW_s \tag{27}$$

and

$$Y_t^N = y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1)dW_s \tag{28}$$

so that the difference $Y^N - Y$ is

$$\begin{aligned}
Y^N - Y_t &= (y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1)dW_s) \\
&\quad - (y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1)dW_s) \\
&= (y_0^N - y_0) + \lambda \int_0^t (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)))ds \\
&\quad - \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) - \nabla u(s, \psi(s, Y_t)))dW_s,
\end{aligned} \tag{29}$$

and using lemma:local-time-at-0 1 we have the following bound

$$\begin{aligned}
&\mathbb{E}[L_t^0(Y^N - Y)] \\
&\leq 4\epsilon + 1 \\
&\quad - 2(1 + \lambda)\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}}\right) (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)))ds\right] \\
&\quad + \frac{1}{\epsilon}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} (\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s)))^2 ds\right].
\end{aligned} \tag{30}$$

{eq:local

For the second and third terms of eq:local_time_diff 30 let us bound the factors involving the differences of u, u^N and $\nabla u, \nabla u^N$. First, for u, u^N adding and subtracting terms and using triangle inequality we have

$$\begin{aligned}
\left|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\right| &\leq \left|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N))\right| \\
&\quad + \left|u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N))\right| \\
&\quad + \left|u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s))\right|.
\end{aligned} \tag{31}$$

$\square$

The terms in the right hand side will be bounded as follows:

- For the first term, by prop:diff_uN_graduN 2

$$\left|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi^N(s,Y_s^N))\right| \le \left\|u^N(s) - u(s)\right\|_{L^\infty} \le \kappa_\rho\left\|b - b^N\right\|_{C_T C^{-\beta}}, \tag{32}$$

- for the second term, observe that u, u^N are $\frac{1}{2}$-Lipschitz and by prop:bound_psi-psiN 3 we get

$$\begin{aligned}\left|u(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s^N))\right| &\le \frac{1}{2}\left|\psi^N(s,Y_s^N) - \psi(s,Y_s^N)\right| \\ &\le \kappa_\rho\left\|b^N - b\right\|_{C_T C^{-\beta}}, \end{aligned} \tag{33}$$

- and for the final term, note that ψ, ψ^N are 2-Lipschitz
  so that

$$\left|u(s,\psi(s,Y_s^N)) - u(s,\psi(s,Y_s))\right| \le \frac{1}{2}\left|\psi(s,Y_s^N) - \psi(s,Y_s)\right| \le \left|Y_s^N - Y_s\right|. \tag{34}$$

So that the following bound holds

$$\left|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s))\right| \le 2\kappa_\rho\left\|b^N - b\right\|_{C_T C^{-\beta}}, + \left|Y_s^N - Y^N\right|. \tag{35} \quad \text{\{eq:bound}}$$

Now for the third term in eq:local_time_diff 30 by adding and subtracting terms and using the triangle inequality

$$\begin{aligned}\left|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right| &\le \left|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi^N(s,Y_s^N))\right| \\ &+ \left|\nabla u(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s^N))\right| \\ &+ \left|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right|\end{aligned}. \tag{36}$$

The terms on the right hand side will be bounded as follows:

- For the first term we use prop:diff_uN_graduN 2 and we have

$$\begin{aligned}\left|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi^N(s,Y_s^N))\right| &\le \left\|\nabla u^N(s) - \nabla u(s)\right\|_{L^\infty} \\ &\le \kappa_\rho\left\|b - b^N\right\|_{C_T C^{-\beta}}, \end{aligned} \tag{37}$$

for the second term see that $\nabla u, \nabla u^N$ are α-Hölder continuous and using prop:bound_psi-psiN 3 we have

$$\begin{aligned}\left|\nabla u(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s^N))\right| &\le \left|\psi^N(s,Y_s^N) - \psi(s,Y_s^N)\right|^\alpha \|u\|_{C_T C^{1+\alpha}} \\ &\le (2\kappa_\rho\left\|b^N - b\right\|_{C_T C^{-\beta}})^\alpha \|u\|_{C_T C^{1+\alpha}}. \end{aligned} \tag{38}$$

Therefore we get the bound

$$\begin{aligned}\left|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right| &\le \kappa_\rho\left\|b - b^N\right\|_{C_T C^{-\beta}} \\ &+ 2^\alpha \kappa_\rho^\alpha\left\|b^N - b\right\|_{C_T C^{-\beta}}^\alpha \|u\|_{C_T C^{1+\alpha}} \\ &+ \left|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right|\end{aligned} . \tag{39} \quad \text{\{eq:bound}}$$

Using the bounds for eq:bound_time eq:bound_gradu_abs 35 and 39 and the inequality

$$(x_1 + \cdots + x_k)^2 \le k(x_1 + \cdots + x_k), \tag{40}$$

for some k, we get

$$\mathbb{E}[L_t^0(Y^N - Y)]$$

$$\leq 4\epsilon + 4(1+\lambda)\left(2\kappa_\rho\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}t + \mathbb{E}\left[\int_0^t |Y_s^N - Y^N|ds\right]\right)$$

$$+ \frac{1}{\epsilon}3t\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}\left(\kappa_\rho^2\|b^N - b\|_{C_T\mathcal{C}^{-\beta}} + 4(2\kappa_\rho)^{2\alpha}\|u\|_{C_T\mathcal{C}^{1+\alpha}}^2\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) \qquad (41)$$

$$+ \frac{1}{\epsilon}3\mathbb{E}\left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}}e^{1-\frac{Y_s^N - Y_s}{\epsilon}}\big|\nabla u(s, \psi(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\big|^2 ds\right)$$

EI:add explanation of why you use absolute value of difference of $u^N - u$, because in Eq. (30) there is no absolute value

EI:explain hoy you get 4 in fron of $(1+\lambda)$ by bounding the indicator functions under the integral in Eq. (30)

6 Convergence rate of the solution to the regularised SDE and the original

References

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