Convergence rate of $X^N - X$ for McKean equations

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Todo list

1 What is this?

An adaptation of [1, Proposition 3.1] for the case of McKean SDEs proposed in [3].

The proof builds on a number of results presented in the sections below.

2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

Definition 1. *For any real-valued continuous semi-martingale, the local time at zero $L_t^0(\bar{Y})$ is defined as*

$$L_t^0(\bar{Y}) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|\bar{Y}| \leq \epsilon\}} d\langle \bar{Y} \rangle_s, \ \mathbb{P}\text{-}a.s. \tag{1}$$

For all $t \geq 0$.

The first result, [1, Lemma 5.1], is not necessary to prove for this particular setting since the result holds for any semi-martingale, I include it here for self-containment reasons.

Lemma 1. *[Lemma 5.1] For any $\epsilon \in (0, 1)$ and any real-valued, continuous semi-martingale*

Type down this lemma

Let us introduce the original and regularised Kolmogorov equations.

Definition 2 (Kolmogorov equations)**.** *For $\beta \in (0, 1/2)$ let $b \in C_T\mathcal{C}^{-\beta}$, $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$, and $b^N \to b$ as $N \to \infty$ in $C_T\mathcal{C}^{-\beta}$ The equations*

$$\begin{cases} \partial u_i + \frac{1}{2} b_i \Delta u_i = \lambda u_i - b_i \\ u_i(T) = 0, \end{cases} \tag{2}$$

$$\begin{cases} \partial u_i^N + \frac{1}{2} b_i^N \Delta u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0. \end{cases} \tag{3}$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

3 Lemma 5.2

Adaptation of Lemma 5.2 builds on the following result:

Proposition 1. *Let u, u^N be (mild) solutions to the Kolmogorov equations from Definition 2 then as $N \to \infty$*

$$\|u_i - u_i^N\|_{C_T\mathcal{C}^{1+\alpha}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}} (\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda)} \tag{4}$$

for $\rho \geq \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_T\infty+\alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \tag{5}$$

and $\lambda \neq 0$.

Is it just different to zero or greater?

Proof. See that $u^N(T) = u(T) = 0$, and in [2], set g^N, g as b^N, b respectively. See that $b^N \to b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$.

As u^N, u are mild solutions, we have

$$\begin{aligned}
u_i(t) - u_i^N(t) &= P_{T-t}(u_i(T) - u_i^N(T)) \\
&+ \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\
&- \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N) ds \\
&= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)] ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N) ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \\
&+ \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N) ds
\end{aligned}
$$

Now let us compute the ρ-equivalent norm of $u - u^N$, for some $\alpha > \beta$

$$
\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} &= \sup_{0\le t\le T} e^{-\rho(T-t)} \|u(t) - u^N(t)\|_{1+\alpha} \\
&\le \sup_{0\le t\le T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \|_{1+\alpha} \\
&+ \sup_{0\le t\le T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \|_{1+\alpha} \\
&- \sup_{0\le t\le T} e^{-\rho(T-t)} \| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \|_{1+\alpha} \\
&- \sup_{0\le t\le T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(b_i - b_i^N) ds \|_{1+\alpha}.
\end{aligned}
$$

Let us take each term from the right hand side of the inequality and bound them.

For the first term, using $\gamma + 2\theta = 1 + \alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \le ct^{-\theta}\|f\|_\gamma\|$ and $\|\nabla g\|_\xi \le c\|g\|_{\xi+1}$

$$
\begin{aligned}
\sup_{0\le t\le T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \|_{1+\alpha} &\le \sup_{0\le t\le T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_\alpha \|b_i - b_i^N\|_{-\beta} \\
&\le c\|u_i\|_{C_T\mathcal{C}_{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\le t\le T} e^{-\rho(T-t)} (T-t)^{\frac{1-\beta-\alpha}{2}} \\
&\le cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T\mathcal{C}_{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}}
\end{aligned}
$$

For the second term, see that for $N \to \infty$, we have $\|b^N\|_{C_T\mathcal{C}^{-\beta}} \le 2\|b\|_{C_T\mathcal{C}^{-\beta}}$

$$
\begin{aligned}
\sup_{0\le t\le T} e^{-\rho(T-t)} \| \int_t^T P_{s-t} b_i^N (\nabla u_i - \nabla u_i^N) ds \|_{1+\alpha} &\le c \sup_{0\le t\le T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2\|b_i\|_{-\beta} \|\nabla u_i - \nabla u_i^N\|_\alpha ds \\
&\le c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} ds \\
&\le c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}
$$

For the third term, which is the one that difers from the proof in [2] we need to use that $\|P_t f\|_{\gamma+2\theta} \le ct^{-\theta}\|f\|_\gamma$, and in this case we have $\gamma + 2\theta = 1 + \alpha$ and $\gamma = 1 + \alpha$, so that $\theta = 0$ because $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, so we will have

$$i = 9$$

$\square$

Proposition 2 (Adaptation of Lemma 5.2)**.** *Let $\beta \in (0, 1/2)$ and $b \in C_T\mathcal{C}^{-\beta}$. Let $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ be (mild) solutions to the Kolmogorov equations from Definition 2*

Assume that for some $\alpha > \beta$

$$\|u - u^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq c(\rho)\|b - b^N\|_{C_T\mathcal{C}^{-\beta}}. \tag{6}$$

With $c(\rho)$ as in Proposition 1 and ρ_0 is large enough such that $c(\rho) > 0$ for all $\rho > \rho_0$. Then for all $t \in [0, T]$

1. $\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}}$
2. $\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}}$

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$.

4 Lemma 5.3

5 Proposition 5.4

6 Proposition 3.1 (main result)

References

[1] Tiziano De Angelis, Maximilien Germain, and Elena Issoglio. "A Numerical Scheme for Stochastic Differential Equations with Distributional Drift". In: *arXiv:1906.11026 [cs, math]* (Oct. 22, 2020). arXiv: 1906.11026.

[2] Elena Issoglio and Francesco Russo. "A PDE with Drift of Negative Besov Index and Related Martingale Problem". In: (), p. 48.

[3] Elena Issoglio and Francesco Russo. "McKean SDEs with singular coefficients". In: *arXiv:2107.14453 [math]* (July 30, 2021). arXiv: 2107.14453.