

# Convergence rate of $X^N - X$ for McKean equations

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## Todo list

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## 1 What is this?

An adaptation of [1, Proposition 3.1] for the case of McKean SDEs proposed in [3].  
The proof builds on a number of results presented in the sections below.

## 2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

**Definition 1.** *For any real-valued continuous semi-martingale, the local time at zero  $L_t^0(\bar{Y})$  is defined as*

$$L_t^0(\bar{Y}) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|\bar{Y}| \leq \epsilon\}} d\langle \bar{Y} \rangle_s, \mathbb{P}\text{-a.s.} \quad (1)$$

For all  $t \geq 0$ .

The first result, [1, Lemma 5.1], is not necessary to prove for this particular setting since the result holds for any semi-martingale, I include it here for self-containment reasons.

**Lemma 1.** [Lemma 5.1] For any  $\epsilon \in (0, 1)$  and any real-valued, continuous semi-martingale

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Let us introduce the original and regularised Kolmogorov equations.

**Definition 2** (Kolmogorov equations). *For  $\beta \in (0, 1/2)$  let  $b \in C_T \mathcal{C}^{-\beta}$ ,  $u, u^N \in C_T \mathcal{C}^{(1+\beta)+}$ , and  $b^N \rightarrow b$  as  $N \rightarrow \infty$  in  $C_T \mathcal{C}^{-\beta}$ . The equations*

$$\begin{cases} \partial u_i + \frac{1}{2} b_i \Delta u_i = \lambda u_i - b_i \\ u_i(T) = 0, \end{cases} \quad (2)$$

$$\begin{cases} \partial u_i^N + \frac{1}{2} b_i^N \Delta u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0. \end{cases} \quad (3)$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

### 3 Lemma 5.2

Adaptation of Lemma 5.2 builds on the following result:

**Proposition 1.** *Let  $u, u^N$  be (mild) solutions to the Kolmogorov equations from Definition 2 then as  $N \rightarrow \infty$*

$$\|u_i - u_i^N\|_{C_T C^{1+\alpha}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T C^{-\beta}} (\|u_i\|_{C_T C^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}} (\|b\|_{C_T C^{-\beta}} + \lambda)} \quad (4)$$

for  $\rho \geq \rho_0$ , where

$$\rho_0 = 2c(\|b_i\|_{C_T \infty + \alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \quad (5)$$

and  $\lambda \neq 0$ .

*Proof.* See that  $u^N(T) = u(T) = 0$ , and in [2], set  $g^N, g$  as  $b^N, b$  respectively. See that  $b^N \rightarrow b$ . Then let us reformulate the rest of the aforementioned result for  $\lambda \neq 0$ .

As  $u^N, u$  are mild solutions, we have

Is it just different to zero or greater?

$$\begin{aligned} u_i(t) - u_i^N(t) &= P_{T-t}(u_i(T) - u_i^N(T)) \\ &\quad + \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\ &\quad - \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds \\ &= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\ &\quad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\ &\quad - \int_t^T P_{s-t}(b_i - b_i^N) ds \\ &= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)] ds \\ &\quad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\ &\quad - \int_t^T P_{s-t}(b_i - b_i^N) ds \end{aligned}$$

$$\begin{aligned}
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \\
&+ \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N) ds
\end{aligned}$$

Now let us compute the  $\rho$ -equivalent norm of  $u - u^N$ , for some  $\alpha > \beta$

$$\begin{aligned}
\|u_i - u_i^N\|_{C_T C^{-\beta}}^{(\rho)} &= \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \|u(t) - u^N(t)\|_{1+\alpha} \\
&\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{1+\alpha} \\
&+ \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \right\|_{1+\alpha} \\
&- \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{1+\alpha} \\
&- \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}(b_i - b_i^N) ds \right\|_{1+\alpha}.
\end{aligned}$$

Let us take each term from the right hand side of the inequality and bound them.

For the first term, using  $\gamma + 2\theta = 1 + \alpha$ ,  $\gamma = -\beta$ ,  $\theta = \frac{1+\alpha+\beta}{2}$ ,  $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta} \|f\|_{\gamma}$  and  $\|\nabla g\|_{\xi} \leq c\|g\|_{\xi+1}$

$$\begin{aligned}
\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{1+\alpha} &\leq \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_{\alpha} \|b_i - b_i^N\|_{-\beta} \\
&\leq c \|u_i\|_{C_T C_{1+\alpha}} \|b_i - b_i^N\|_{C_T C^{-\beta}} \sup_{0 \leq t \leq T} e^{-\rho(T-t)} (T-t)^{\frac{1-\beta-\alpha}{2}} \\
&\leq c T^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T C_{1+\alpha}} \|b_i - b_i^N\|_{C_T C^{-\beta}}
\end{aligned}$$

For the second term, see that for  $N \rightarrow \infty$ , we have  $\|b^N\|_{C_T C^{-\beta}} \leq 2\|b\|_{C_T C^{-\beta}}$

$$\begin{aligned}
\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} b_i^N (\nabla u_i - \nabla u_i^N) ds \right\|_{1+\alpha} &\leq c \sup_{0 \leq t \leq T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2\|b_i\|_{-\beta} \|\nabla u_i - \nabla u_i^N\|_{\alpha} ds \\
&\leq c \|b_i\|_{C_T C^{-\beta}} \|u_i - u_i^N\|_{C_T C^{-\beta}}^{(\rho)} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} ds \\
&\leq c \|b_i\|_{C_T C^{-\beta}} \|u_i - u_i^N\|_{C_T C^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}$$

For the third term, which is the one that differs from the proof in [2] we need to use that  $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta} \|f\|_{\gamma}$ , and in this case we have  $\gamma + 2\theta = 1 + \alpha$  and  $\gamma = 1 + \alpha$ , so that  $\theta = 0$  because  $u, u^N \in C_T C^{1+\alpha}$ , so we will have

$$\begin{aligned}
\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{1+\alpha} &\leq c\lambda \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-0} \|u_i - u_i^N\|_{1+\alpha} ds \\
&= c\lambda \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T e^{-\rho(T-s)} \sup_{0 \leq s \leq T} e^{-\rho(T-s)} \|u_i - u_i^N\|_{1+\alpha} ds \\
&= c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T C_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(T-s)} e^{-\rho(T-t)} ds \\
&= c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T C_{1+\alpha}}^{(\rho)} \int_t^T e^{-\rho(s-t)} ds \\
&= c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T C_{1+\alpha}}^{(\rho)} \sup_{0 \leq t \leq T} \rho^{-1} [1 - e^{-\rho(T-t)}] \\
&\leq c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T C_{1+\alpha}}^{(\rho)} \rho^{-1} \\
&\leq c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T C_{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}$$

And for the last term

$$\begin{aligned}
\sup_{0 \leq t \leq T} e^{-\rho(T-t)} \left\| \int_t^T P_{T-s}(b_i - b_i^N) ds \right\|_{1+\alpha} &\leq c \sup_{0 \leq t \leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}} \|b_i - b_i^N\|_{-\beta} ds \\
&\leq c \|b_i - b_i^N\|_{C_T C^{-\beta}} \sup_{0 \leq t \leq T} e^{-\rho(T-t)} (s-t)^{-\frac{\alpha+\beta-1}{2}} \\
&\leq cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T C^{-\beta}}
\end{aligned}$$

Putting everything together

$$\begin{aligned}
\|u_i - u_i^N\|_{C^T C^{-\beta}}^{(\rho)} &\leq cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T C^{1+\alpha}} \|b_i - b_i^N\|_{C^T C^{-\beta}} \\
&\quad + c \|b_i\|_{C^T C^{-\beta}} \|u_i - u_i^N\|_{C^T C^{-\beta}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}} \\
&\quad - c\lambda \sup_{0 \leq t \leq T} \|u_i - u_i^N\|_{C_T C_{1+\alpha}}^{(\rho)} \rho^{\frac{\alpha+\beta-1}{2}} \\
&\quad - cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T C^{-\beta}} \\
\|u_i - u_i^N\|_{C^T C^{-\beta}}^{(\rho)} (1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C^T C^{-\beta}} + \lambda]) &\leq cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T C^{-\beta}} (\|u_i\|_{C_T C^{1+\alpha}} - 1) \\
\|u_i - u_i^N\|_{C^T C^{-\beta}}^{(\rho)} &\leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T C^{-\beta}} (\|u_i\|_{C_T C^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C^T C^{-\beta}} + \lambda])}
\end{aligned}$$

As required.  $\square$

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|_{C^T C^{-\beta}}^{(\rho)} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} (\|u_i\|_{C_T C^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C^T C^{-\beta}} + \lambda])} \|b_i - b_i^N\|_{C^T C^{-\beta}} \quad (6)$$

$$\|u_i - u_i^N\|_{C^T C^{-\beta}}^{(\rho)} \leq c(\rho) \|b_i - b_i^N\|_{C^T C^{-\beta}} \quad (7)$$

Check this norm

**Proposition 2** (Adaptation of Lemma 5.2). *Let  $\beta \in (0, 1/2)$  and  $b \in C_T \mathcal{C}^{-\beta}$ . Let  $u, u^N \in C_T \mathcal{C}^{(1+\beta)+}$  be (mild) solutions to the Kolmogorov equations from Definition 2. Assume that for some  $\alpha > \beta$*

$$\|u - u^N\|_{C_T \mathcal{C}^{1+\alpha}}^{(\rho)} \leq c(\rho) \|b - b^N\|_{C_T \mathcal{C}^{-\beta}}. \quad (8)$$

*With  $c(\rho)$  as in Proposition 1 and  $\rho_0$  is large enough such that  $c(\rho) > 0$  for all  $\rho > \rho_0$ . Then for all  $t \in [0, T]$*

1.  $\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C_T \mathcal{C}^{-\beta}}$
2.  $\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho \|b - b^N\|_{C_T \mathcal{C}^{-\beta}}$

*with  $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$ .*

## 4 Lemma 5.3

## 5 Proposition 5.4

## 6 Proposition 3.1 (main result)

## References

- [1] Tiziano De Angelis, Maximilien Germain, and Elena Issoglio. “A Numerical Scheme for Stochastic Differential Equations with Distributional Drift”. In: *arXiv:1906.11026 [cs, math]* (Oct. 22, 2020). arXiv: 1906.11026.
- [2] Elena Issoglio and Francesco Russo. “A PDE with Drift of Negative Besov Index and Related Martingale Problem”. In: (), p. 48.
- [3] Elena Issoglio and Francesco Russo. “McKean SDEs with singular coefficients”. In: *arXiv:2107.14453 [math]* (July 30, 2021). arXiv: 2107.14453.