Convergence rate of numerical solutions to SDEs with distributional drifts in Besov spaces

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Contents

1 What is this?

An adaptation of [de_angelis_numerical_2020 1, Proposition 3.1] for the case of SDEs with drift in a Besov space of negative order similar to the ones proposed in [issoglio_pde_2021 2] and [issoglio_mckean 3]. The proof builds on a number of results presented in the sections below.

EI: add result about convergence of the scheme. This is done in two parts, $X^N \to X$ done in Russo Issoglio, and $X^{N,m} \to X^N$ Euler scheme convergence from De Angelis Germain Issoglio. Attention that the rate of convergence of Euler scheme depends of the smoothness of b^N.

2 Some useful definitions, results and setting of the problem

2.1 Function spaces we use

2.2 Some assumptions

b_in_ctcb **Assumption 1.** *Let $0 < \beta < 1/2$ and $b \in C_T\mathcal{C}^{-\beta}$.*

s_in_ctcb **Assumption 2.** *There exists a sequence $(b^N)_N \in C_T\mathcal{C}^{-\beta}$ such that for each N, $b^N(t,\cdot) \in C_b^\infty(\mathbb{R})$ for all $t \in [0,T]$ and such that $b^N \to b$ as $N \to \infty$.*

:vT_lives **Assumption 3.** *Let*

2.3 Some results and definitions

Here we present some results and definitions to refer on the text.

Definition 1. *Let us consider the SDE*

$$X_t = X_0 + \int_0^t b(t, X_t)dt + W_t \tag{1}$$

{eq:sde}

where $b \in C_T\mathcal{S}'$, and W_t is a Brownian motion.

In particular we care about $b \in C_T\mathcal{C}^{-\beta}$, but it is useful to consider the above equation in the most general case, since we have some definitions in $\mathcal{S}'$.

time_zero **Definition 2.** *For any real-valued continuous semi-martingale Z, the local time at zero $L_t^0(\bar{Y})$ is defined as*

$$L_t^0(Z) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t \mathbb{1}_{\{|Z|\le\epsilon\}} d\langle Z\rangle_s, \ \mathbb{P}\text{-}a.s. \tag{2}$$

For all $t \ge 0$.

The lemma below is from [1] and its proof can be found in [1, Lemma 5.1]. We include the statement here for ease of reading.

time-at-0 **Lemma 1.** *For any $\epsilon \in (0,1)$ and any real-valued, continuous semi-martingale Z we have*

$$\begin{aligned}\mathbb{E}[L_t^0(Z_s)] \le & 4\epsilon - 2\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Z_s\in(0,\epsilon)\}} + \mathbb{1}_{\{Z_s\ge\epsilon\}} e^{1-Z_s/\epsilon}\right) dZ_s\right] \\ & + \frac{1}{\epsilon}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z>\epsilon\}} e^{1-Z_s/\epsilon} d\langle Z\rangle_s\right].\end{aligned}$$

Let us introduce the original and regularised Kolmogorov equations. To shorten notation we will denote the spaces $C_T\mathcal{C}^\gamma(\mathbb{R})$ as $C_T\mathcal{C}^\gamma$.

LM: add Feynamn-Kac formula

orov_eqns **Definition 3.** *For $\beta \in (0, 1/2)$ let $b \in C_T\mathcal{C}^{-\beta}$, $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$, and $b^N \to b$ as $N \to \infty$ in $C_T\mathcal{C}^{-\beta}$ The equations*

$$\begin{cases}\partial_t u_i + \frac{1}{2}\Delta u_i + b_i \nabla u_i = \lambda u_i - b_i \\ u_i(T) = 0,\end{cases} \tag{3}$$

{eq:kolmo

$$\begin{cases}\partial u_i^N + \frac{1}{2}\Delta u_i^N + b_i^N \nabla u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0.\end{cases} \tag{4}$$

{eq:kolmo

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

gradients **Lemma 2.** *Let u, u^N be the solutions to the Kolmogorov equations (3) (4) in $C_T\mathcal{C}^{1+\alpha}$ respectivelly. We have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |\nabla u(t,x)| \le \frac{1}{2}, \tag{5}$$

and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \left|\nabla u^N(t,x)\right| \le \frac{1}{2}. \tag{6}$$

at_kernel **Definition 4.** *The following function is called heat kernel*

$$p_f = \frac{1}{\sqrt{4\pi f}} e^{-\frac{|x-y|^2}{4f}}. \tag{7}$$

{eq:heat_

This is the fundamental solution to the heat equation.
The convolution of the heat kernel with a (generalised) function, is called heat semigroup:

$$\left(P_f g\right)(y) = \left(p_f * g\right)(y) = \int_{\mathbb{R}} p_f(x) g(y-x) dx \tag{8}$$

{eq:heat_

The following definition is from [2, Definition 3.2].

_solution **Definition 5.** *Let $b, u : [0,T] \to \mathcal{C}^{-\beta}$, we say that u is a mild solution of (3) if it satisfies*

$$v(t) = P_{T-t} v_T + \int_t^T P_{s-t}\left(\nabla v(s) b(s)\right) ds - \int_t^T P_{s-t}\left(G(v)(s)\right) ds, \tag{9}$$ {eq:mild_

for all $t \in [0,T]$

The uniqueness is given too in [2, Theorem 3.7]

Theorem 1. *Let b satisfy Assumption 1, and u_T satisfy Assumption 3.* LM: *add the proper assumption on u* *Then there exists a mild solution to (3) in $C_T\mathcal{C}^{(1-\beta)-}$ which is unique in $C_T\mathcal{C}^{\beta+}$.*

LM: it might seem the comments below have been addressed, but not really, we are in a space that is different

LM: add the definition of mild solution for the Kolmogorov eqns

LM: theorem: exists a unique mild solution $u \in C_T\mathcal{C}^{1+\alpha}$ for all $\alpha \in (\beta, 1-\beta)$ cite Issoglio & Russo PDE Martingale problem

LM: from that theorem it follows that $\nabla u \in C_T\mathcal{C}^{\alpha}$, thus is α-Holder continuous

Remark 1. LM: *write this remark well* *Thanks to the above theorem, as $u \in C_T\mathcal{C}^{1+\alpha}$, we have that $\nabla u \in C_T\mathcal{C}^{\alpha}$. Thus, ∇u is indeed α-Hölder continuous.*

estimates **Lemma 3.** *Given a function $f \in \mathcal{C}^{\gamma}$ for some $\gamma \in \mathbb{R}$, then for any $\theta \geq 0$ there exists a constant c such that*

$$\|P_t f\|_{\gamma+2\theta} \leq c t^{-\theta} \|f\|_{\gamma}. \tag{10}$$ {eq:se_Pt

Moreover, for $f \in \mathcal{C}^{\gamma}$ and any $\theta \in (0,1)$ we have

$$\|P_t f - f\|_{\gamma} \leq c t^{\theta} \|f\|_{\gamma+2\theta}. \tag{11}$$ {eq:se_Pt

nequality **Lemma 4.** *Given a function $f \in \mathcal{C}^{\gamma}$ for some $\gamma \in \mathbb{R}$, there exists a constant $c > 0$ such that*

$$\|\nabla g\|_{\gamma} \leq c \|g\|_{\gamma+1}. \tag{12}$$ {eq:berns

3 Bounds for the difference of solutions to the Kolmogorov equations

We need a bound for $u - u^N$ and $\nabla u - \nabla u^N$ in L_∞ for the case in which $u \in C_T\mathcal{C}^{1+\alpha}$ for some $\alpha \in (\beta, 1-\beta)$ which is an adaptation of [1, Lemma 5.2].

The result builds on top of the following result:

diff_u_uN **Proposition 1.** *Let u, u^N be (mild) solutions to the Kolmogorov equations from Definition 3 then as $N \to \infty$*

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}} (\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda)} \tag{13}$$

for $\rho \geq \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_{T+\alpha}} + \lambda)^{\frac{2}{\alpha+\beta+1}} \tag{14}$$

and $\lambda > 0$.

Proof. See that $u^N(T) = u(T) = 0$, and in [2], set g^N, g as b^N, b respectively. See that $b^N \to b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$. As u^N, u are mild solutions, we have

$$u_i(t) - u_i^N(t) = P_{T-t}(u_i(T) - u_i^N(T)) + \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\ - \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds$$

$$
\begin{aligned}
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds - \int_t^T P_{s-t}(b_i - b_i^N) ds \\
&= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)] ds \\
&\qquad\qquad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds - \int_t^T P_{s-t}(b_i - b_i^N) ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds + \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \\
&\qquad\qquad - \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds - \int_t^T P_{s-t}(b_i - b_i^N) ds
\end{aligned}
$$

Now let us compute the ρ-equivalent norm of $u - u^N$, for some $\alpha > \beta$

$$
\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} &= \sup_{0 \le t \le T} e^{-\rho(T-t)} \|u(t) - u^N(t)\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\le \sup_{0 \le t \le T} e^{-\rho(T-t)} \Bigg[\left\| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad + \left\| \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad - \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad - \left\| \int_t^T P_{s-t}(b_i - b_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \Bigg].
\end{aligned}
$$

Let us take each term from the right hand side of the inequality and bound them.
LM: change this sentence accoring to the addition of Schauder estimates
Using lemma:schauder_estimates 3 For the first term, using $\gamma + 2\theta = 1 + \alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \le ct^{-\theta}\|f\|_\gamma$ and $\|\nabla g\|_\xi \le c\|g\|_{\xi+1}$

$$
\begin{aligned}
&\sup_{0 \le t \le T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}\left(\nabla u_i b_i - \nabla u_i b_i^N\right) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad \le \sup_{0 \le t \le T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_{C_T\mathcal{C}^{\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \\
&\qquad \le c\|u_i\|_{C_T\mathcal{C}_{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0 \le t \le T} e^{-\rho(T-t)} (T-t)^{\frac{1-\beta-\alpha}{2}} \\
&\qquad \le cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T\mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}}
\end{aligned}
$$

For the second term, see that for $N \to \infty$, we have $\|b^N\|_{C_T\mathcal{C}^{-\beta}} \le 2\|b\|_{C_T\mathcal{C}^{-\beta}}$

$$
\begin{aligned}
&\sup_{0 \le t \le T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} b_i^N (\nabla u_i - \nabla u_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\qquad \le c \sup_{0 \le t \le T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|\nabla u_i - \nabla u_i^N\|_{C_T\mathcal{C}^{1+\alpha}} ds \\
&\qquad \le c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} ds \\
&\qquad \le c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}
$$

For the third term, which is the one that difers from the proof in issoglio_pde_nodate [2] we need to use that $\|P_t f\|_{\gamma+2\theta} \le ct^{-\theta}\|f\|_\gamma$, and in this case we have $\gamma + 2\theta = 1 + \alpha$ and $\gamma = 1 + \alpha$, so that $\theta = 0$ because $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, so we will have

$$
\begin{aligned}
\sup_{0\le t\le T} e^{-\rho(T-t)} \left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{1+\alpha} \\
&\le c\lambda \sup_{0\le t\le T} e^{-\rho(T-t)} \int_t^T (s-t)^{-0} \|u_i - u_i^N\|_{1+\alpha} ds \\
&= c\lambda \sup_{0\le t\le T} e^{-\rho(T-t)} \int_t^T e^{-\rho(T-s)} \sup_{0\le s\le T} e^{-\rho(T-s)} \|u_i - u_i^N\|_{1+\alpha} ds \\
&= c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \int_t^T e^{-\rho(T-s)} e^{-\rho(T-t)} ds \\
&= c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \int_t^T e^{-\rho(s-t)} ds \\
&= c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \sup_{0\le t\le T} \rho^{-1}[1 - e^{-\rho(T-t)}] \\
&\le c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{-1} \\
&\le c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}
$$

And for the last term

$$
\begin{aligned}
\sup_{0\le t\le T} e^{-\rho(T-t)} \left\| \int_t^T P_{T-s}(b_i - b_i^N) ds \right\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\le c \sup_{0\le t\le T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} ds \\
&\le c\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\le t\le T} e^{-\rho(T-t)} (s-t)^{-\frac{\alpha+\beta-1}{2}} \\
&\le cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}
\end{aligned}
$$

Putting everything together

$$
\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \le{} & cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T\mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \\
& + c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \rho^{\frac{\alpha+\beta-1}{2}} \\
& - c\lambda \sup_{0\le t\le T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{\frac{\alpha+\beta-1}{2}} \\
& - cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}},
\end{aligned}
$$

and finally,

$$
\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} (1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda]) \\
&\le cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1) \\
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} &\le \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}} [\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}
\end{aligned}
$$

As required. □

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{15}$$

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq c(\rho)\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{16}$$

Here is the adaptation of [1, Lemma 5.2].

Proposition 2. *Let $\beta \in (0,1/2)$ and $b \in C_T\mathcal{C}^{-\beta}$. Let $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ be (mild) solutions to the Kolmogorov equations from Definition 3.*

Assume, by Proposition 1, that for some $\alpha > \beta$

$$\|u - u^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq c(\rho)\|b - b^N\|_{C_T\mathcal{C}^{-\beta}}. \tag{17}$$

With $c(\rho)$ as in Proposition 1 and ρ_0 is large enough such that $c(\rho) > 0$ for all $\rho > \rho_0$.

Then for all $t \in [0,T]$

$$\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{18}$$

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{19}$$

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$.

Proof. First let us prove (18).

Let $t \in [0,T]$, and see that since $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ there exists $\alpha > \beta$ such that $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, then for any $f \in \mathcal{C}^{1+\alpha}$ we have

$$\|f\|_{C^{1+\alpha}} \leq c\left(\sup_{x\in\mathbb{R}^d}|f(x)| + \sup_{x\neq y\in\mathbb{R}^d}\frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha}\right) \tag{20}$$

so we have

$$\begin{aligned}\|u^N(t) - u(t)\|_{L^\infty} &= \sup_{x\in\mathbb{R}^d}|u^N(t,x) - u(t,x)| \\ &\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}}\end{aligned} \tag{21}$$

Moreover, using the (ρ)-equivalent norm

$$\|f\|_{\mathcal{C}^{1+\alpha}} = \sup_{t\in[0,T]} e^{-\rho(T-t)}\|f(t)\|_{\mathcal{C}^{1+\alpha}}, \tag{22}$$

and (17) we see that

$$\begin{aligned}\left\|u^N - u\right\|_{C_T\mathcal{C}^{1+\alpha}} &= \sup_{t\in[0,T]}\left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\ &= \sup_{t\in[0,T]} e^{\rho(T-t)}e^{-\rho(T-t)}\left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\ &\leq e^{\rho T}\sup_{t\in[0,T]} e^{-\rho(T-t)}\left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\ &= e^{\rho T}\left\|u^N - u\right\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}}.\end{aligned} \tag{23}$$

Plugging (23) into (21)

$$\begin{aligned}\|u^N(t) - u(t)\|_{L^\infty} &\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \\ &\leq \sup_{t\in[0,T]} c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{\alpha+1}} \\ &= c\left\|u^N - u\right\|_{C_T\mathcal{C}^{\alpha+1}} \\ &\leq ce^{\rho T}\left\|u^N - u\right\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}.\end{aligned} \tag{24}$$

And finally by (17)

$$\left\|u^N(t)-u(t)\right\|_{L^\infty} \le c\cdot c(\rho)\cdot e^{\rho T}\left\|b^N-b\right\|_{C_T\mathcal{C}^{-\beta}} \tag{25}$$

which proves (18).
For (19) recall that if $f\in\mathcal{C}^{1+\alpha}$ then $\nabla f\in\mathcal{C}^{\alpha}$. Also, by Bernstein inequality (12)

$$\|\nabla f\|_{\alpha}\le c\|f\|_{1+\alpha}. \tag{26}$$

Using the equivalent norm

$$\|f\|_{C^{1+\alpha}}\le c\left(\sup_{x\in\mathbb{R}^d}|f(x)|+\sup_{x\in\mathbb{R}^d}|\nabla f(x)|+\sup_{x\neq y\in\mathbb{R}^d}\frac{|\nabla f(x)-\nabla f(y)|}{|x-y|^{\alpha}}\right) \tag{27}$$

we can see that

$$\left\|\nabla u^N(t)-\nabla u(t)\right\|_{L^\infty}\le c\left\|u^N(t)-u(t)\right\|_{\mathcal{C}^{1+\alpha}}. \tag{28}$$

And usign the same bounds that we used above for $c\left\|u^N(t)-u(t)\right\|_{\mathcal{C}^{1+\alpha}}$ this point follows.

□

4 Bound for the difference of the auxiliary functions

This is the adaptation of result [1, Lemma 5.3].

Proposition 3. *Take $\rho>\rho_0$ as in Proposition 1, $N\to\infty$, κ_ρ from Proposition 2, and $\beta\in(0,1/2)$, then we have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}}\|\psi(t,x)-\psi^N(t,x)\|\le 2\kappa_\rho\left\|b-b^N\right\|_{C_T\mathcal{C}^{-\beta}} \tag{29}$$

Proof. Recall the definition of $\psi,\phi\in C_T\mathcal{C}^1$

$$\phi(t,x):=x+u(t,x) \tag{30}$$

$$\psi(t,\cdot)=\phi^{-1}(t,\cdot). \tag{31}$$

Note that

$$u(y)=\int_0^1\nabla u(\alpha y)y\,d\alpha+u(0). \tag{32}$$

From there we have

$$u(t,y)-u(t,y')=\int_0^1\nabla u(t,\alpha(y-y'))(y-y')\,d\alpha \tag{33}$$

and therefore

$$\|u(t,y)-u(t,y')\|\ge\left(\int_0^1\|\nabla u(t,\alpha(y-y'))\|^2 d\alpha\right)^{1/2}\|y-y'\|, \tag{34}$$

and by Lemma 2 we finally have

$$\begin{aligned}\|u(t,y)-u(t,y'\|)&\le\left(\frac{1}{4}\int_0^1 d\alpha\right)^{1/2}\|y-y\|'\\ \|u(t,y)-u(t,y')\|^2&\le\frac{1}{4}\|y-y'\|^2\end{aligned} \tag{35}$$

LM: continue from page three in notes

□

5 Bound for the local time at zero of the solution to the SDEs

LM: Here I still need to mention how we define $Y_t = \psi(t, X_t)$, because eventually I need to use that $X_t = \psi(t, Y_t)$, probably just neet to mention without defining the whole Y_t as in the paper

We need a bound for $\mathbb{E}[L^0_T(Y^N - Y)]$, for Sobolev spaces, this is result [1, Proposition 5.4] we present it here for the solutions to the SDE blonging to the appropriate Besov spaces.

First let us state the following useful result.

LM: check that the statement makes sense and has all the necessary assumptions

Lemma 5. *Let u, u^N be solutions to the Kolmogorov equations (3) (4) then the following bound is satisfied:*

$$\|u^N(s,\psi^N(s,Y^N_s)) - u(s,\psi(s,Y_s)\|) \le 2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + \|Y^N_s - Y^N\|. \tag{36}$$

Proof. Adding and subtracting terms, using triangle inequality and noting that for any a, b, we have $a - b \le \|a - b\|$, then

$$\begin{aligned}\|u^N(s,\psi^N(s,Y^N_s)) - u(s,\psi(s,Y_s)\|) \le{}& \|u^N(s,\psi^N(s,Y^N_s)) - u(s,\psi^N(s,Y^N_s)\|) \\ &+ \|u(s,\psi^N(s,Y^N_s)) - u(s,\psi(s,Y^N_s)\|) \\ &+ \|u(s,\psi(s,Y^N_s)) - u(s,\psi(s,Y_s))\|.\end{aligned} \tag{37}$$

The terms in the right hand side will be bounded as follows:

- For the first term, by Proposition 2

$$\|u^N(s,\psi^N(s,Y^N_s)) - u(s,\psi^N(s,Y^N_s)\|) \le \left\|u^N(s) - u(s)\right\|_{L^\infty} \le \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}}, \tag{38}$$

- for the second term, observe that u, u^N are $\frac{1}{2}$-Lipschitz and by Proposition 3 we get

$$\begin{aligned}\|u(s,\psi^N(s,Y^N_s)) - u(s,\psi(s,Y^N_s)\|) &\le \frac{1}{2}\|\psi^N(s,Y^N_s) - \psi(s,Y^N_s\|) \\ &\le \kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}},\end{aligned} \tag{39}$$

- and for the final term, note that ψ, ψ^N are 2-Lipschitz

  so that

$$\|u(s,\psi(s,Y^N_s)) - u(s,\psi(s,Y_s)\|) \le \frac{1}{2}\|\psi(s,Y^N_s) - \psi(s,Y_s\|) \le \|Y^N_s - Y_s\|. \tag{40}$$

So that the following bound holds

$$\|u^N(s,\psi^N(s,Y^N_s)) - u(s,\psi(s,Y_s)\|) \le 2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + \|Y^N_s - Y^N\|, \tag{41}$$

as required.

□

Proposition 4. *Let A, B be constants, $b \in C_T\mathcal{C}^{-\beta}$ and $b^N \to b$ in $C_T\mathcal{C}^{-\beta}$ as $N \to \infty$ for $\beta \in (0, \frac{1}{4})$ and for any $\alpha \in (\beta, 1-\beta)$*

$$\mathbb{E}[L^0_t(Y^N - Y)] \le o\left(\left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}\right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + B\left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}. \tag{42}$$

Proof. Recall that Y^N, Y are solutions to the SDEs

$$Y_t = y_0 + \lambda \int_0^t u(s,\psi(s,Y_s))ds + \int_0^t (\nabla u(s,\psi(s,Y_t)) + 1)dW_s \tag{43}$$

and

$$Y^N_t = y^N_0 + \lambda \int_0^t u^N(s,\psi^N(s,Y^N_s))ds + \int_0^t (\nabla u^N(s,\psi^N(s,Y^N_t)) + 1)dW_s \tag{44}$$

so that the difference $Y^N - Y$ is

$$\begin{aligned}
Y_t^N - Y_t &= \left(y_0^N + \lambda \int_0^t u^N(s,\psi^N(s,Y_s^N))ds + \int_0^t (\nabla u^N(s,\psi^N(s,Y_t^N))+1)dW_s\right) \\
&\quad - \left(y_0 + \lambda \int_0^t u(s,\psi(s,Y_s))ds + \int_0^t (\nabla u(s,\psi(s,Y_t))+1)dW_s\right) \\
&= (y_0^N - y_0) + \lambda \int_0^t (u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s)))ds \\
&\quad + \int_0^t (\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s)))dW_s,
\end{aligned} \tag{45}$$

and using Lemma 1 we have the following bound

$$\begin{aligned}
&\mathbb{E}[L_t^0(Y^N - Y)] \\
&\leq 4\epsilon \\
&- 2\lambda \mathbb{E}\Bigg[\int_0^t \left(\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}}\right) \\
&\qquad\qquad \left(u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s))\right) ds\Bigg] \qquad (46) \quad \boxed{\texttt{\{eq:local}} \\
&+ \frac{1}{\epsilon}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \left(\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right)^2 ds\right]. \qquad (47) \quad \boxed{\texttt{\{eq:local}}
\end{aligned}$$

LM: add the explanation of why to drop the difusion term

First, for (46), we find a bound for the factor involving the difference of u^N and u in Lemma 5. Therefore

$$\|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s)\|) \leq 2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\|. \tag{48}$$

Now we need to bound the result of the local time of the difference $Y_s^N - Y_s$. First notice that $Y_s^N - Y_s \geq \epsilon$, then $e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \leq 1$, also it is clear that $\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}}$ and $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}}$ are bounded by 1, therefore $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \leq 1$. Using the previous arguments and (36) lead to have

$$\begin{aligned}
(46) &\leq 2\lambda \mathbb{E}\left[\int_0^t 2\left(2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\|\right) ds\right] \\
&\leq 4\lambda \left(2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} t + \mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\| ds\right]\right).
\end{aligned} \tag{49}$$ $\boxed{\texttt{\{eq:bound}}$

Now for (47), we use similar arguments as the ones in Lemma 5 above, and we get the following:

$$\begin{aligned}
\|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\| &\leq \|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi^N(s,Y_s^N))\| \\
&\quad + \|\nabla u(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s^N))\| \\
&\quad + \|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|,
\end{aligned} \tag{50}$$

where the terms on the right hand side will be bounded as follows:

- For the first term we use Proposition 2 and we have

$$\begin{aligned}
\|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi^N(s,Y_s^N)\|) &\leq \left\|\nabla u^N(s) - \nabla u(s)\right\|_{L^\infty} \\
&\leq \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}},
\end{aligned} \tag{51}$$

  for the second term see that $\nabla u, \nabla u^N$ are α-Hölder continuous and using Proposition 3 we have

  LM: mention for which alpha this is possible and refer to the remark we have to add above

$$\begin{aligned}
\|\nabla u(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s^N)\|) &\leq \|\psi^N(s,Y_s^N) - \psi(s,Y_s^N)\|^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\leq (2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}})^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}}.
\end{aligned} \tag{52}$$

Therefore we get the bound

$$\begin{aligned}\|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s)\|) \le \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}} \\ + \alpha\kappa_\rho^\alpha \left\|b^N - b\right\|^\alpha_{C_T\mathcal{C}^{-\beta}} \|u\|_{C_T\mathcal{C}^{1+\alpha}} \\ + \|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|.\end{aligned} \tag{53}$$

{eq:bound

Here we can also notice that $\mathbb{1}_{\{Y_s^N - Y_s<\epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} < 1$, then using (53) and the inequality

$$(x_1 + \cdots + x_k)^2 \le k(x_1^2 + \cdots + x_k^2), \tag{54}$$

for $k = 3$, we can get the bound

$$\begin{aligned}(47) &\le \frac{1}{\epsilon}\mathbb{E}\int_0^t \left(3\kappa_\rho^2\left\|b - b^N\right\|^2_{C_T\mathcal{C}^{-\beta}} + 3\cdot 2^{2\alpha}\kappa_\rho^{2\alpha}\left\|b^N - b\right\|^{2\alpha}_{C_T\mathcal{C}^{-\beta}}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}}\right)ds \\ &+ \frac{1}{\epsilon}\mathbb{E}\int_0^t 3\mathbb{1}_{\{Y_s^N - Y_s>\epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}}\|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|^2 ds \\ &\le \frac{1}{\epsilon}3t\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}\left(\kappa_\rho^2\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + (2\kappa_\rho)^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}}\left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}\right) \\ &+ \frac{1}{\epsilon}3\mathbb{E}\left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s>\epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}}\left|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right|^2 ds\right)\end{aligned} \tag{55}$$

{eq:bound

Now let us denote the last term in (55) by $I_t^{N,\epsilon}$. Pick $\zeta \in (0,1)$ such that $\alpha\zeta > \frac{1}{2}$, and since $\epsilon \in (0,1)$ we have $\epsilon^\zeta > \epsilon$. Then split the indicator function $\mathbb{1}_{\{Y_s^N - Y_s>\epsilon\}}$ into $\mathbb{1}_{\{\epsilon<Y_s^N - Y_s\le\epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s>\epsilon^\zeta\}}$. Leading to the integral

$$\begin{aligned}I_t^{N,\epsilon} = \frac{1}{\epsilon}3\mathbb{E}\Bigg(\int_0^t \left(\mathbb{1}_{\{\epsilon<Y_s^N - Y_s\le\epsilon^\zeta\}} + \mathbb{1}_{\{Y_s^N - Y_s>\epsilon^\zeta\}}\right) e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \\ \left|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right|^2 ds\Bigg)\end{aligned} \tag{56}$$

{eq:INeps

For the first term of (56) we use the fact that ∇u is α-Hölder continuous uniformly in $s \in [0,T]$ with constant $\|u\|_{C_T\mathcal{C}^{1+\alpha}}$ and that ψ is 2-Lipschitz

$$\begin{aligned}\|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|^2 &\le \left\|\|\psi(s,Y_s^N) - \psi(s,Y_s)\|^\alpha\|u\|_{C_T\mathcal{C}^{1+\alpha}}\right\|^2 \\ &\le \left\|2^\alpha\|Y_s^N - Y_s\|^\alpha\|u\|_{C_T\mathcal{C}^{1+\alpha}}\right\|^2 \\ &= 2^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}}\|Y_s^N - Y_s\|^{2\alpha}\end{aligned} \tag{57}$$

For the other term we need another way to bound it, because even though the event when $\|Y^N - Y\| > \epsilon^\zeta$ is small, we can potentially have a quantity that blows up for the bound. EI: the explanation needs adjusting - speak to Elena In order to solve this problem, we can use the fact that ∇u is uniformly bounded by $1/2$ thanks to Lemma 2, and then we can bound the difference of the gradients as follows:

$$\begin{aligned}\|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\|^2 &\le \|\nabla u(s,\psi(s,Y_s^N)) + \nabla u(s,\psi(s,Y_s))\|^2 \\ &\le \sup_{(s,x)\in[0,T]\times\mathbb{R}}\|\nabla u(s,\psi(s,Y_s^N)) + \nabla u(s,\psi(s,Y_s))\|^2 \\ &= \|2\nabla u\|^2_{L_\infty}.\end{aligned} \tag{58}$$

Therefore we have that for all $t \in [0,T]$ LM: check where else I need to say this

$$
\begin{aligned}
I_t^{N,\epsilon} &\le \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \left(\mathbb{1}_{\{\epsilon < Y_s^N - Y_s \le \epsilon^\zeta\}} \right) e^{1-\frac{Y_s^N - Y_s}{\epsilon}} 2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \|Y_s^N - Y_s\|^{2\alpha} ds \right) \\
&\quad + \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon^\zeta\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \|2\nabla u\|^2_{L_\infty} ds \right) \\
&\le \frac{1}{\epsilon} 3\mathbb{E}\left(2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \|\epsilon^\zeta\|^{2\alpha} t \right) + \frac{1}{\epsilon} 3\mathbb{E}\left(4 e^{1-\epsilon^{\zeta-1}} \|\nabla u\|^2_{L_\infty} t \right) \\
&\le \sup_{t\in[0,T]} \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \epsilon^{2\alpha\zeta} + 4 e^{1-\epsilon^{\zeta-1}} \|\nabla u\|^2_{L_\infty} \right) t \\
&= \frac{3}{\epsilon} \left(2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \epsilon^{2\alpha\zeta} + 4 e^{1-\epsilon^{\zeta-1}} \|\nabla u\|^2_{L_\infty} \right) T.
\end{aligned} \tag{59}
$$

{eq:INeps

Now by combining (49), (55) and (59), and taking the sup over $[0,T]$ we will get

$$
\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] \le 4\epsilon& \\
&+ 4\lambda 2\kappa_\rho T \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ 4\lambda \mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\| ds \right] \\
&+ \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T\kappa_\rho^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T(2\kappa_\rho)^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}} \\
&+ \frac{3}{\epsilon} 2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} T \epsilon^{2\alpha\zeta} \\
&+ \frac{3}{\epsilon} 4 \|\nabla u\|^2_{L_\infty} T e^{1-\epsilon^{\zeta-1}}
\end{aligned} \tag{60}
$$

then we take $\epsilon = \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}$ and we get

$$
\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] \le 4\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}& \\
&+ 4\lambda 2\kappa_\rho T \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ 4\lambda \mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\| ds \right] \\
&+ 3T\kappa_\rho^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ 3T(2\kappa_\rho)^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}} \\
&+ 2^{2\alpha} \|u\|^2_{C_T\mathcal{C}^{1+\alpha}} T \left\|b^N - b\right\|^{2\alpha\zeta-1}_{C_T\mathcal{C}^{-\beta}} \\
&+ 4 \|\nabla u\|^2_{L_\infty} T \left\|b^N - b\right\|^{-1}_{C_T\mathcal{C}^{-\beta}} \exp\left(1 - \left\|b^N - b\right\|^{\zeta-1}_{C_T\mathcal{C}^{-\beta}}\right)
\end{aligned} \tag{61}
$$

which can be written as

$$
\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] &\le c_1 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + c_2 \mathbb{E}\left[\int_0^t \|Y_s^N - Y^N\| ds \right] \\
&\quad + c_3 \left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}} + c_4 \left\|b^N - b\right\|^{2\alpha\zeta-1}_{C_T\mathcal{C}^{-\beta}} \\
&\quad + c_5 \exp\left(1 - \left\|b^N - b\right\|^{\zeta-1}_{C_T\mathcal{C}^{-\beta}}\right)
\end{aligned} \tag{62}
$$

{eq:bound

where

$$
\begin{aligned}
c_1 &= 4 + 4\lambda 2\kappa_\rho T + 3\kappa_\rho^2 T \\
c_2 &= 4\lambda \\
c_3 &= 3(2\kappa_\rho)^{2\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}}^2 T \\
c_4 &= 2^{2\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}}^2 T \\
c_5 &= 4 \|\nabla u\|_{L_\infty}^2 \|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{-1} T
\end{aligned} \tag{63}
$$

{eq:const

Finally, observe that since $\zeta \in (0,1)$, the term $\exp\left(1 - \|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{\zeta-1}\right)$ decays faster than any polynomial, thus controlling c_5, and the last term in (62) goes to zero. Also $\alpha\zeta$ is arbitrarily close to α, and $\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}$ controls $\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}$ therefore we can create the bound (42)

eq:bound_constants eq:local_time_YNY_bound

Question: is this clear enough? Am I making sense if I am taking α fixed? EI: no if α was fixed you could not do this. But $\alpha > \beta$ in your statement, hence it works. You need to explain the details however. Maybe at this stage you could introduce $\alpha' = \alpha\zeta$ to explain, that the result works for α' but since ζ can be chosen arbitrarily close to 1 then α' is arbitrarily close to α and α was chosen such that $\alpha > \beta$ which means the result is valid for all $\alpha' > \beta$. For simplicity we write α in place of α' in the statement.
Also it is better to explain the meaning of $o()$ and what terms go in there.

$$
\mathbb{E}[L_t^0(Y^N - Y)] \le o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) + c_2 \mathbb{E}\left(\int_0^t \|Y^N - Y\| ds\right) + c_4 \|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \tag{64}
$$

□

6 Convergence rate of the solution to the regularised SDE and the original

In this section we present a bound for $\mathbb{E}[X^N - X]$ in terms of $\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}$.

_original **Proposition 5.** *Let assumptions 1 2 hold, then for any $\alpha \in (1/2, 1-\beta)$ there is a constant C_α such that*

as:bN_converges_in_ctcb

$$
\mathbb{E}[X^N - X] \le C_\alpha \|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}, \tag{65}
$$

{eq:EXN-X

as $N \to \infty$.

Proof. Note that by definition of ψ, ψ^N we have

$$
\begin{aligned}
|X_t^N - X_t| &= |\psi^N(t, \phi^N(t, X_t^N)) - \psi(t, \phi(t, X_t))| \\
&= |\psi^N(t, Y_t^N) - \psi(t, Y_t)|,
\end{aligned} \tag{66}
$$

then adding and subtracting, and using the triangle inequality we get

$$
|X_t^N - X_t| \le |\psi^N(t, Y_t^N) - \psi(t, Y_t^N)| + |\psi(t, Y_t^N) - \psi(t, Y_t)|. \tag{67}
$$

Where the first term is bounded by $2\kappa\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}$ (Proposition 3) and since ψ is 2-Lipschitz uniformly in $t \in [0,T]$ the second term is bounded by $2|Y^N - Y|$, therefore

prop:bound_psi-psiN

$$
|X^N - X| \le 2\kappa \|b^N - b\|_{C_T\mathcal{C}^{-\beta}} + 2|Y^N - Y|. \tag{68}
$$

{eq:XN-X}

By assumption the first term above goes to zero as $N \to \infty$, then we only need a bound for the second term.

By Itô-Tanaka's formula

$$
|Y^N - Y| = |y_0^N - y_0| + \frac{1}{2} L_t^0(Y^N - Y) + \int_0^t \operatorname{sgn}(Y^N - Y) d(Y^N - Y), \tag{69}
$$

{eq:YNY_i

by taking expectation and using the definitions of Y^N, Y we have

$$
\begin{aligned}
\mathbb{E}|Y^N - Y| &= \mathbb{E}|y_0^N - y_0| + \mathbb{E}\frac{1}{2} L_t^0(Y^N - Y) \\
&\quad + \lambda \mathbb{E} \int_0^t \operatorname{sgn}(Y^N - Y)(u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))) ds,
\end{aligned} \tag{70}
$$

{eq:EYNY}

then observe that the first term above is a constant, for the second we have a bound in Proposition 4 and for the third we use Lemma 5, and the fact that $\operatorname{sgn}(x) \le 1$ therefore

$$\begin{aligned}
\mathbb{E}|Y^N - Y| \le & |u^N(0,x) - u(0,x)| \\
& + \frac{1}{2}\left[o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds \right) + B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right] \\
& + \mathbb{E}\left[\int_0^t \left(2\kappa_\rho \|b^N - b\|_{C_T\mathcal{C}^{-\beta}} + \|Y_s^N - Y^N\| \right) ds \right] \\
\le & |u^N(0,x) - u(0,x)| \\
& + \frac{1}{2}\left[o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right) + A\mathbb{E}\left(\int_0^t \|Y^N - Y\| ds \right) + B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right] \\
& + \lambda 2\kappa_\rho \|b^N - b\|_{C_T\mathcal{C}^{-\beta}} t + \lambda\mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\| ds \right),
\end{aligned} \tag{71}$$

Note that the terms in orange are controlled by $o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right)$, so after merging those terms and the two involving $\mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\| ds\right)$ we get

$$\mathbb{E}|Y^N - Y| \le B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} + (A+\lambda)\mathbb{E}\left(\int_0^t \|Y_s^N - Y^N\| ds \right) + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right). \tag{72}$$

From there, using Gronwall's lemma we get the following bound

$$\mathbb{E}|Y^N - Y| \le B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} T e^{(A+\lambda)T} + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right). \tag{73}$$

Now we use (73) to bound (68), and as the *small-o* term controls the second term in (68) we obtain

$$\mathbb{E}\left[|X^N - X|\right] \le B\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} T e^{(A+\lambda)T} + o\left(\|b^N - b\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \right) \tag{74}$$

□

7 Small comment about convergence rate of Euler scheme to regularized equation

It works just like in [1] LM: make all the comment

8 Convergence rate of Euler scheme

LM: check assumptions, maybe put them into the assumptions above or smth

Theorem 2. *Let X_t be the solution to the SDE (1) with drift coefficient $b \in C_T\mathcal{C}^{-\beta}$, and X_t^{Nm} be the Euler approximation of the solution with m time steps. Let also $\beta \in (0, 1/4)$, then it holds that*

$$\sup_{0\le t\le T} \mathbb{E}\left[|X_t^{Nm} - X_t|\right] \le c m^{-\frac{1}{2}+\mu+\epsilon}, \tag{75}$$

where

$$\mu = \frac{1}{2} \cdot \frac{\beta}{(1/2-\beta)(1-2\beta)+\beta}, \tag{76}$$

for any $\epsilon > 0$.

Proof. First, by triangle inequality we have

$$\circledast := \sup_{0\le t\le T} \mathbb{E}\left[|X_t^{Nm} - X_t|\right] \le \mathbb{E}\left[|X_t^{Nm} - X_t^N|\right] + \mathbb{E}\left[|X_t^N - X_t|\right], \tag{77}$$

the first term in the right hand side is bounded by [1][Proposition 3.4] and the second one by Proposition 5, so that putting those results together we get

$$
\begin{aligned}
\circledast &\leq A_N m^{-1} + B_N m^{-1/2} + c\left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}} \\
&\leq c\Big[\left\|b^N\right\|_{\infty,L^\infty}\left(1+\left\|\nabla b^N\right\|_{\infty,L^\infty}\right)m^{-1} + \left(\left\|\nabla b^N\right\|_{\infty,L^\infty} + \left[b^N\right]_{1/2,L^\infty}\right)m^{-1/2} + \left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\hat\beta}}\Big],
\end{aligned}
\tag{78}
$$

{eq:er_co

for any $\alpha \in (1/2, 1-\beta)$ and $\hat\beta \in (0,1/2)$.

We require to find some bounds for the L^∞ (semi) norms, so that we use Schauder estimates (10) and Bernstein inequality (12) , this is possible thanks to the definition of $b^N := p_{f_m} * b$, where $f_m \to 0$ when $m \to \infty$, and also consider the definition of the norm

$$\|g\|_{C_T\mathcal{C}^\delta} = \left\|b^N\right\|_{L^\infty} + \sup_{x\neq y}\frac{|b^N(x) - b^N(y)|}{|x-y|^\delta},$$

and the seminorm

$$[g]_{1/2,L^\infty} = \sup_{t\neq st,s\in[0,T]}\frac{\|g(t)-g(s)\|_{L^\infty}}{|t-s|^{1/2}}.$$

We have the following bounds:

$$\left\|b^N\right\|_{L^\infty} \leq \left\|b^N\right\|_{C_T\mathcal{C}^\epsilon} \leq c f_m^{-\frac{\epsilon+\beta}{2}}\|b\|_{C_T\mathcal{C}^{-\beta}}, \tag{79}$$

{eq:er02}

$$\left\|\nabla b^N\right\|_{L^\infty} \leq \left\|\nabla b^N\right\|_{C_T\mathcal{C}^\epsilon} \leq c\left\|b^N\right\|_{C_T\mathcal{C}^{\epsilon+1}} \leq c f_m^{-\frac{\epsilon+\beta+1}{2}}\|b\|_{C_T\mathcal{C}^{-\beta}}, \tag{80}$$

{eq:er03}

$$
\begin{aligned}
\left[b^N\right]_{1/2,L^\infty} &\leq \sup_{t\neq s}\frac{\|b^N(t)-b^N(s)\|_{C_T\mathcal{C}^\epsilon}}{|t-s|^{1/2}} \\
&\leq \sup_{t\neq s} c f_m^{-\frac{\epsilon+\beta}{2}}\frac{\|b(t)-b(s)\|_{C_T\mathcal{C}^{-\beta}}}{|t-s|^{1/2}} \\
&= c f_m^{-\frac{\epsilon+\beta}{2}}[b]_{1/2,C_T\mathcal{C}^{-\beta}}.
\end{aligned}
\tag{81}
$$

{eq:er04}

Plugging that into (78) we get

$$
\begin{aligned}
\circledast &\leq c\left[\left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\hat\beta}} + \left\|b^N\right\|_{\infty,L^\infty}\left(1+\left\|\nabla b^N\right\|_{\infty,L^\infty}\right)m^{-1} + \left(\left\|\nabla b^N\right\|_{\infty,L^\infty} + \left[b^N\right]_{1/2,L^\infty}\right)m^{-1/2}\right] \\
&\leq c\left[\left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\hat\beta}} + f_m^{-\frac{\epsilon+\beta}{2}}\|b\|_{C_T\mathcal{C}^{-\beta}}\left(1 + f_m^{-\frac{\epsilon+\beta+1}{2}}\|b\|_{C_T\mathcal{C}^{-\beta}}\right)m^{-1} + \left(f_m^{-\frac{\epsilon+\beta+1}{2}}\|b\|_{C_T\mathcal{C}^{-\beta}} + f_m^{-\frac{\epsilon+\beta}{2}}[b]_{1/2,C_T\mathcal{C}^{-\beta}}\right)m^{-1/2}\right] \\
&\leq c\left[\left(f_m^{\nu/2}\|b\|_{C_T\mathcal{C}^{-\hat\beta+\nu}}\right)^{2\alpha-1} + f_m^{-\frac{\epsilon+\beta}{2}}\|b\|_{C_T\mathcal{C}^{-\beta}}\left(1 + f_m^{-\frac{\epsilon+\beta+1}{2}}\|b\|_{C_T\mathcal{C}^{-\beta}}\right)m^{-1} + \left(f_m^{-\frac{\epsilon+\beta+1}{2}}\|b\|_{C_T\mathcal{C}^{-\beta}} + f_m^{-\frac{\epsilon+\beta}{2}}[b]_{1/2,C_T\mathcal{C}^{-\beta}}\right)m^{-1/2}\right]
\end{aligned}
\tag{82}
$$

{eq:er_bo

Where the first term in the last inequality comes from Schauder estimates (11) by taking $\nu := \hat\beta - \beta$, Also, since the norms are finite they can be absorbed by a constant and we have:

$$\circledast \leq c\left[f_m^{\frac{\nu}{2}(2\alpha-1)} + \left(f_m^{-\frac{\epsilon+\beta}{2}} + f_m^{-\frac{2\epsilon+2\beta+1}{2}}\right)m^{-1} + \left(f_m^{-\frac{\epsilon+\beta+1}{2}} + f_m^{-\frac{\epsilon+\beta}{2}}\right)m^{-1/2}\right]. \tag{83}$$

{eq:er_al

Considering only the slowest terms in (83) and substituting ν we get

$$\circledast \leq f_m^{\frac{\hat\beta-\beta}{2}(2\alpha-1)} + m^{-1/2}f_m^{-\frac{\epsilon+\beta}{2}} \tag{84}$$

{eq:er_sl

The optimal of this quantity is when the two terms on the left hand side are equal, this is

$$
\begin{aligned}
m^{-1/2} &= f_m^{\frac{(2\alpha-1)(\hat\beta-\beta)+\epsilon+\beta}{2}} \\
f_m &= m^{-\frac{1}{(2\alpha-1)(\hat\beta-\beta)+\epsilon+\beta}}
\end{aligned}
\tag{85}
$$

{eq:er_eq

So, if we plug it into the rate (any of the two terms in (eq:er_slowest 84)), we get

$$\circledast \leq c m^{-\frac{1}{2}+\frac{1}{2}\frac{\beta+\epsilon}{(2\alpha-1)(\beta-\beta)+\beta+\epsilon}} \tag{86}$$

{eq:er_ra

Note that in order to attain the best rate we requier to minimize the second term of the exponent of (eq:er_rate_raw 86), i. e:

$$\inf_{\substack{\alpha\in(1/2,1-\beta_0)\\ \beta\in(\beta_0,1/2)\\ \epsilon>0}} \left\{ \frac{\beta+\epsilon}{\beta+\epsilon+(2\alpha-1)(\beta-\beta)} \right\} = \frac{\beta_0}{(1/2-\beta_0)(1-2\beta_0)+\beta_0+\epsilon}.$$

Therefore, we have

$$\mathbb{E}\left[\left|X_t^{Nm}-X_t\right|\right] \leq m^{-\frac{1}{2}+\frac{1}{2}\cdot\frac{\beta_0}{(1/2-\beta_0)(1-2\beta_0)+\beta_0}+\epsilon}, \tag{87}$$

{eq:er_fi

as required. □

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