Convergence rate of $X^N - X$ for McKean equations

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Contents

1 What is this?

An adaptation of [1, Proposition 3.1] for the case of ~~McKean SDEs~~ SDEs with drift in a Besov space of negative order proposed in [2] and [3].

The proof builds on a number of results presented in the sections below.

2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

def:local_time_zero

Definition 1. *Local time at zero For any real-valued continuous semi-martingale Z, the local time at zero $L_t^0(\bar{Y})$ is defined as*

$$L_t^0(Z) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t \mathbb{1}_{\{|Z| \le \epsilon\}} d\langle Z \rangle_s, \ \mathbb{P}\text{-}a.s. \tag{1}$$

For all $t \ge 0$.

The first result, [1, Lemma 5.1], is not necessary to prove for this particular setting since the result holds for any semi-martingale, I include it here for self-containment reasons. EI: Instead of this sentence you should write something like 'The lemma below is from [1] and its proof can be found in [1, Lemma 5.1]. We include the statement here for ease of reading'

lemma:local-time-at-0

Lemma 1. *Bound for local time at zero for a semi-martingale For any $\epsilon \in (0,1)$ and any real-valued, continuous semi-martingale Z we have*

$$\begin{aligned}\mathbb{E}[L_t^0(Z_s)] \le & 4\epsilon - 2\mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Z_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Z_s \ge \epsilon\}} e^{1 - Z_s/\epsilon}\right) dZ_s\right] \\ & + \frac{1}{\epsilon}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z > \epsilon\}} e^{1 - Z_s/\epsilon} d\langle Z \rangle_s\right].\end{aligned}$$

Let us introduce the original and regularised Kolmogorov equations.

def:kolmogorov_eqns **Definition 2.** *Kolmogorov equations*

For $\beta \in (0, 1/2)$ let $b \in C_T\mathcal{C}^{-\beta}$, $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$, and $b^N \to b$ as $N \to \infty$ in $C_T\mathcal{C}^{-\beta}$ The equations

$$\begin{cases} \partial_t u_i + \frac{1}{2}\Delta u_i + b_i \nabla u_i = \lambda u_i - b_i \\ u_i(T) = 0, \end{cases} \tag{2}$$

$$\begin{cases} \partial u_i^N + \frac{1}{2}\Delta u_i^N + b_i^N \nabla u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0. \end{cases} \tag{3}$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

emma:bounds_gradients **Lemma 2.** *Bounds for $\nabla u, \nabla u^N$ Let LM: type this result which is lemma 4.2 in the paper*

...

3 Bounds for the difference of solutions to the Kolmogorov equations

We need a bound for $u - u^N$ and $\nabla u - \nabla u^N$ in L_∞ for the case in which $u \in C_T\mathcal{C}^{1+\alpha}$ for some $\alpha > \beta$ which is an adaptation of de_angelis_numerical_2020 [1, Lemma 5.2].

The result builds on top of the following result:

prop:diff_u_uN **Proposition 1.** *Bound for the ρ-equivalent norm of $u - u^N$*

Let u, u^N be (mild) solutions to the Kolmogorov equations from Definition def:kolmogorov_eqns *?? then as $N \to \infty$*

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} (\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}} (\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda)} \tag{4}$$

for $\rho \geq \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_T\infty+\alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \tag{5}$$

and $\lambda > 0$.

Proof. See that $u^N(T) = u(T) = 0$, and in issoglio_pde_nodate [2], set g^N, g as b^N, b respectively. See that $b^N \to b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$.

As u^N, u are mild solutions, we have

$$\begin{aligned} u_i(t) - u_i^N(t) &= P_{T-t}(u_i(T) - u_i^N(T)) \\ &\quad + \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \\ &\quad - \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N) ds \\ &= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N) ds \end{aligned}$$

$$
\begin{aligned}
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N)ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N)ds \\
&= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)]ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N)ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N)ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N)ds \\
&+ \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N)ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N)ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N)ds
\end{aligned}
$$

Now let us compute the ρ-equivalent norm of $u - u^N$, for some $\alpha > \beta$
LM: this norm is wrong, should be $1 + \alpha$ on the lhs and everywhere else

$$
\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T \mathcal{C}^{-\beta}} &= \sup_{0 \le t \le T} e^{-\rho(T-t)} \|u(t) - u^N(t)\|_{1+\alpha} \\
&\le \sup_{0 \le t \le T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N)ds\|_{1+\alpha} \\
&+ \sup_{0 \le t \le T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N)ds\|_{1+\alpha} \\
&- \sup_{0 \le t \le T} e^{-\rho(T-t)} \|\lambda \int_t^T P_{s-t}(u_i - u_i^N)ds\|_{1+\alpha} \\
&- \sup_{0 \le t \le T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(b_i - b_i^N)ds\|_{1+\alpha}.
\end{aligned}
$$

Let us take each term from the right hand side of the inequality and bound them.

For the first term, using $\gamma + 2\theta = 1 + \alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \le ct^{-\theta}\|f\|_\gamma\|$ and $\|\nabla g\|_\xi \le c\|g\|_{\xi+1}$

$$
\begin{aligned}
&\sup_{0 \le t \le T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N)ds \right\|_{1+\alpha} \\
&\qquad \le \sup_{0 \le t \le T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_\alpha \|b_i - b_i^N\|_{-\beta}
\end{aligned}
$$

$$
\begin{aligned}
&\le c\|u_i\|_{C_T\mathcal{C}_{1+\alpha}}\|b_i-b_i^N\|_{C_T\mathcal{C}^{-\beta}}\sup_{0\le t\le T}e^{-\rho(T-t)}(T-t)^{\frac{1-\beta-\alpha}{2}}\\
&\le cT^{\frac{1-\beta-\alpha}{2}}\|u_i\|_{C_T\mathcal{C}_{1+\alpha}}\|b_i-b_i^N\|_{C_T\mathcal{C}^{-\beta}}
\end{aligned}
$$

For the second term, see that for $N\to\infty$, we have $\|b^N\|_{C_T\mathcal{C}^{-\beta}}\le 2\|b\|_{C_T\mathcal{C}^{-\beta}}$

$$
\begin{aligned}
&\sup_{0\le t\le T}e^{-\rho(T-t)}\left\|\int_t^T P_{s-t}b_i^N(\nabla u_i-\nabla u_i^N)ds\right\|_{1+\alpha}\\
&\qquad\le c\sup_{0\le t\le T}\int_t^T(s-t)^{-\theta}e^{-\rho(T-t)}2\|b_i\|_{-\beta}\|\nabla u_i-\nabla u_i^N\|_\alpha ds\\
&\qquad\le c\|b_i\|_{C_T\mathcal{C}^{-\beta}}\|u_i-u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\int_t^T(s-t)^{-\theta}e^{-\rho(T-t)}ds\\
&\qquad\le c\|b_i\|_{C_T\mathcal{C}^{-\beta}}\|u_i-u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}
$$

For the third term, which is the one that difers from the proof in issoglio_pde_nodate [2] we need to use that $\|P_tf\|_{\gamma+2\theta}\le ct^{-\theta}\|f\|_\gamma$, and in this case we have $\gamma+2\theta=1+\alpha$ and $\gamma=1+\alpha$, so that $\theta=0$ because $u,u^N\in C_T\mathcal{C}^{1+\alpha}$, so we will have

$$
\begin{aligned}
&\sup_{0\le t\le T}e^{-\rho(T-t)}\left\|\lambda\int_t^T P_{s-t}(u_i-u_i^N)ds\right\|_{1+\alpha}\\
&\qquad\le c\lambda\sup_{0\le t\le T}e^{-\rho(T-t)}\int_t^T(s-t)^{-0}\|u_i-u_i^N\|_{1+\alpha}ds\\
&\qquad= c\lambda\sup_{0\le t\le T}e^{-\rho(T-t)}\int_t^T e^{-\rho(T-s)}\sup_{0\le s\le T}e^{-\rho(T-s)}\|u_i-u_i^N\|_{1+\alpha}ds\\
&\qquad= c\lambda\sup_{0\le t\le T}\|u_i-u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}}\int_t^T e^{-\rho(T-s)}e^{-\rho(T-t)}ds\\
&\qquad= c\lambda\sup_{0\le t\le T}\|u_i-u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}}\int_t^T e^{-\rho(s-t)}ds\\
&\qquad= c\lambda\sup_{0\le t\le T}\|u_i-u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}}\sup_{0\le t\le T}\rho^{-1}[1-e^{-\rho(T-t)}]\\
&\qquad\le c\lambda\sup_{0\le t\le T}\|u_i-u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}}\rho^{-1}\\
&\qquad\le c\lambda\sup_{0\le t\le T}\|u_i-u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}}\rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}
$$

And for the last term

$$
\begin{aligned}
&\sup_{0\le t\le T}e^{-\rho(T-t)}\left\|\int_t^T P_{T-s}(b_i-b_i^N)ds\right\|_{1+\alpha}\\
&\qquad\le c\sup_{0\le t\le T}e^{-\rho(T-t)}\int_t^T(s-t)^{-\frac{\alpha+\beta-1}{2}}\|b_i-b_i^N\|_{-\beta}ds
\end{aligned}
$$

$$\leq c\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\leq t\leq T} e^{-\rho(T-t)}(s-t)^{-\frac{\alpha+\beta-1}{2}}$$
$$\leq cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}$$

Putting everything together

$$\begin{aligned}\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq\ & cT^{\frac{1-\beta-\alpha}{2}}\|u_i\|_{C_T\mathcal{C}^{1+\alpha}}\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \\ &+ c\|b_i\|_{C_T\mathcal{C}^{-\beta}}\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}\rho^{\frac{\alpha+\beta-1}{2}} \\ &- c\lambda \sup_{0\leq t\leq T}\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}}\rho^{\frac{\alpha+\beta-1}{2}} \\ &- cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}},\end{aligned}$$

and finally,

$$\begin{aligned}\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda]) \\ \leq cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1) \\ \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}\end{aligned}$$

As required. □

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{6}$$

LM: Check this norm

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq c(\rho)\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{7}$$

Here is the adaptation of [de_angelis_numerical_2020 1, Lemma 5.2].

prop:diff_uN_graduN **Proposition 2.** *Bounds for $\|u - u^N\|_{L_\infty}$ and $\|\nabla u - \nabla u^N\|_{L_\infty}$*

Let $\beta \in (0, 1/2)$ and $b \in C_T\mathcal{C}^{-\beta}$. Let $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ be (mild) solutions to the Kolmogorov equations from Definition def:kolmogorov_eqns ??.

Assume, by Proposition prop:diff_u_uN *??, that for some $\alpha > \beta$*

$$\|u - u^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq c(\rho)\|b - b^N\|_{C_T\mathcal{C}^{-\beta}}. \tag{8}$$ {eq:u-uNb-bN}

With $c(\rho)$ as in Proposition prop:diff_u_uN *?? and ρ_0 is large enough such that $c(\rho) > 0$ for all $\rho > \rho_0$. Then for all $t \in [0, T]$*

$$\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{9}$$ {eq:uNu_bounded_by_bN

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{10}$$ {eq:graduNu_bounded_b

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$.

Proof. First let us prove (eq:uNu_bounded_by_bNb ??).

Let $t \in [0,T]$, and see that since $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ there exists $\alpha > \beta$ such that $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, then for any $f \in \mathcal{C}^{1+\alpha}$ we have

$$\|f\|_{C^{1+\alpha}} \leq c \left(\sup_{x\in\mathbb{R}^d} |f(x)| + \sup_{x\neq y\in\mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha} \right) \tag{11}$$

so we have

$$\begin{aligned} \|u^N(t) - u(t)\|_{L^\infty} &= \sup_{x\in\mathbb{R}^d} |u^N(t,x) - u(t,x)| \\ &\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \end{aligned} \tag{12}$$

{eq:u-uN_in_Linfinity

Moreover, using the (ρ)-equivalent norm

$$\|f\|_{\mathcal{C}^{1+\alpha}} = \sup_{t\in[0,T]} e^{-\rho(T-t)} \|f(t)\|_{\mathcal{C}^{1+\alpha}}, \tag{13}$$

and (eq:u-uNb-bN ??) we see that

$$\begin{aligned} \left\|u^N - u\right\|_{C_T\mathcal{C}^{1+\alpha}} &= \sup_{t\in[0,T]} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\ &= \sup_{t\in[0,T]} e^{\rho(T-t)} e^{-\rho(T-t)} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\ &\leq e^{\rho T} \sup_{t\in[0,T]} e^{-\rho(T-t)} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\ &= e^{\rho T} \left\|u^N - u\right\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}}. \end{aligned} \tag{14}$$

{eq:norm_bounded_by_r

Plugging (eq:norm_bo ??) into (…inity_bounded_by_u-uN_in_Calpha1 ??)

$$\begin{aligned} \|u^N(t) - u(t)\|_{L^\infty} &\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \\ &\leq \sup_{t\in[0,T]} c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{\alpha+1}} \\ &= c\left\|u^N - u\right\|_{C_T\mathcal{C}^{\alpha+1}} \\ &\leq ce^{\rho T}\left\|u^N - u\right\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}. \end{aligned} \tag{15}$$

And finally by (eq:u-uNb-bN ??)

$$\left\|u^N(t) - u(t)\right\|_{L^\infty} \leq c \cdot c(\rho) \cdot e^{\rho T} \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \tag{16}$$

which proves (eq:uNu_bounded_by_bNb ??).

For (eq:graduNu_bounded_by_bNb ??) recall that if $f \in \mathcal{C}^{1+\alpha}$ then $\nabla f \in \mathcal{C}^\alpha$. Also, by Bernstein inequality [issoglio_mckean_2021 3, Eqn. (9)]

$$\|\nabla f\|_\alpha \leq c\|f\|_{\infty+\alpha}. \tag{17}$$

Using the equivalent norm

$$\|f\|_{C^{1+\alpha}} \leq c \left(\sup_{x\in\mathbb{R}^d} |f(x)| + \sup_{x\in\mathbb{R}^d} |\nabla f(x)| + \sup_{x\neq y\in\mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha} \right) \tag{18}$$

we can see that

$$\left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \leq c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1+\alpha}}. \tag{19}$$

And usign the same bounds that we used above for $c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1+\alpha}}$ this point follows.

□

4 Bound for the difference of the auxiliary functions

This is the adaptation of result [de_angelis_numerical_2020 1, Lemma 5.3].

pproppbounddpaixppsiN

Proposition 3. *Bound for* $\left|\psi(t,x) - \psi^N(t,x)\right|$

Take $\rho > \rho_0$ *as in Proposition* prop:diff_u_uN **??**, $N \to \infty$, κ_ρ *from Proposition* prop:diff_uN_graduN **??**, *and* $\beta \in (0, 1/2)$, *then we have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \left|\psi(t,x) - \psi^N(t,x)\right| \leq 2\kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}} \tag{20}$$

Proof. Recall the definition of $\psi, \phi \in C_T\mathcal{C}^1$

$$\phi(t,x) := x + u(t,x) \tag{21}$$

$$\psi(t,\cdot) = \phi^{-1}(t,\cdot). \tag{22}$$

Note that

$$u(y) = \int_0^1 \nabla u(\alpha y) y d\alpha + u(0). \tag{23}$$

From there we have

$$u(t,y) - u(t,y') = \int_0^1 \nabla u(t, \alpha(y-y'))(y-y')d\alpha \tag{24}$$

and therefore

$$|u(t,y) - u(t,y')| \geq \left(\int_0^1 \left|\nabla u(t,\alpha(y-y'))\right|^2 d\alpha\right)^{1/2} |y-y'|, \tag{25}$$

and by Lemma lemma:bounds_gradients **??** we finally have

$$\begin{aligned} |u(t,y) - u(t,y')| &\leq \left(\frac{1}{4}\int_0^1 d\alpha\right)^{1/2} |y-y'| \\ |u(t,y) - u(t,y')|^2 &\leq \frac{1}{4}|y-y'|^2 \end{aligned} \tag{26}$$

LM: continue from page three in notes

□

5 Bound for the local time at zero of the solution to the SDEs

We need a bound for $\mathbb{E}[L^0_T(Y^N - Y)]$, for Sobolev spaces, this is result [de_angelis_numerical_2020 1, Proposition 5.4] we present it here for the solutions to the SDE blonging to the appropriate Besov spaces.

:bound_local_time_sde **Proposition 4.** *Let A, B be constants, $b \in C_T\mathcal{C}^{-\beta}$ and $b^N \to b$ in $C_T\mathcal{C}^{-\beta}$ as $N \to \infty$ for $\beta \in (0, \frac{1}{4})$ and for any $\alpha > \beta$*

$$\mathbb{E}[L^0_t(Y^N - Y)] \le o(\|b^N - b\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}) + A\mathbb{E}\Big(\int_0^t |Y^N - Y| ds\Big) + B\|b^N - b\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}. \tag{27}$$

{eq:local_time_YNY_bo

Proof. Recall that Y^N, Y are solutions to the SDEs

$$Y_t = y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1)dW_s \tag{28}$$

and

$$Y^N_t = y^N_0 + \lambda \int_0^t u^N(s, \psi^N(s, Y^N_s))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y^N_t)) + 1)dW_s \tag{29}$$

so that the difference $Y^N - Y$ is

$$\begin{aligned} Y^N - Y_t &= (y^N_0 + \lambda \int_0^t u^N(s, \psi^N(s, Y^N_s))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y^N_t)) + 1)dW_s) \\ &- (y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1)dW_s) \\ &= (y^N_0 - y_0) + \lambda \int_0^t (u^N(s, \psi^N(s, Y^N_s)) - u(s, \psi(s, Y_s)))ds \\ &- \int_0^t (\nabla u^N(s, \psi^N(s, Y^N_t)) - \nabla u(s, \psi(s, Y_t)))dW_s, \end{aligned} \tag{30}$$

and using Lemma lemma:local-time-at-0 ?? we have the following bound

$$\begin{aligned} &\mathbb{E}[L^0_t(Y^N - Y)] \\ &\le 4\epsilon \\ &- 2\lambda\mathbb{E}\Big[\int_0^t \Big(\mathbb{1}_{\{Y^N_s - Y_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Y^N_s - Y_s \ge \epsilon\}} e^{1 - \frac{Y^N_s - Y_s}{\epsilon}}\Big)\big(u^N(s, \psi^N(s, Y^N_s)) - u(s, \psi(s, Y_s))\big)ds\Big] \end{aligned} \tag{31}$$

{eq:local_time_diff_u

$$+ \frac{1}{\epsilon}\mathbb{E}\Big[\int_0^t \mathbb{1}_{\{Y^N_s - Y_s > \epsilon\}} e^{1 - \frac{Y^N_s - Y_s}{\epsilon}} \big(\nabla u^N(s, \psi^N(s, Y^N_s)) - \nabla u(s, \psi(s, Y_s))\big)^2 ds\Big]. \tag{32}$$

{eq:local_time_diff_g

LM: add the explanation of why to drop the difussion term

For (??) and (??) let us bound the factors involving the differences of u, u^N and $\nabla u, \nabla u^N$, noting also that for any a, b, we have $a - b \leq |a - b|$

First, for (??) adding and subtracting terms and using triangle inequality we have

$$\begin{aligned}\left|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s))\right| \leq{}& \left|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi^N(s,Y_s^N))\right| \\ &+ \left|u(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s^N))\right| \\ &+ \left|u(s,\psi(s,Y_s^N)) - u(s,\psi(s,Y_s))\right|.\end{aligned} \tag{33}$$

The terms in the right hand side will be bounded as follows:

- For the first term, by Proposition ??

$$\left|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi^N(s,Y_s^N))\right| \leq \left\|u^N(s) - u(s)\right\|_{L^\infty} \leq \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}}, \tag{34}$$

- for the second term, observe that u, u^N are $\frac{1}{2}$-Lipschitz and by Proposition ?? we get

$$\begin{aligned}\left|u(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s^N))\right| &\leq \frac{1}{2}\left|\psi^N(s,Y_s^N) - \psi(s,Y_s^N)\right| \\ &\leq \kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}},\end{aligned} \tag{35}$$

- and for the final term, note that ψ, ψ^N are 2-Lipschitz

  so that

$$\left|u(s,\psi(s,Y_s^N)) - u(s,\psi(s,Y_s))\right| \leq \frac{1}{2}\left|\psi(s,Y_s^N) - \psi(s,Y_s)\right| \leq \left|Y_s^N - Y_s\right|. \tag{36}$$

So that the following bound holds

$$\left|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s))\right| \leq 2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + \left|Y_s^N - Y^N\right|. \tag{37}$$

{eq:bound_u_abs}

Now we need to bound the result of the local time of the difference $Y_s^N - Y_s$. First notice that $Y_s^N - Y_s \geq \epsilon$, then $e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \leq 1$, also it is clear that $\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}}$ and $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}}$ are bounded by 1, therefore $\mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1-\frac{Y_s^N - Y_s}{\epsilon}} \leq 1$. Using the previous arguments and (??) leads to have

$$\begin{aligned}(??) &\leq 2\lambda \mathbb{E}\left[\int_0^t 2\left(2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + \left|Y_s^N - Y^N\right|\right) ds\right] \\ &\leq 4\lambda \left(2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} t + \mathbb{E}\left[\int_0^t \left|Y_s^N - Y^N\right| ds\right]\right)\end{aligned} \tag{38}$$

{eq:bound_integral_uN

Now for (??) by adding and subtracting terms and using the triangle inequality

$$
\begin{aligned}
\left|\nabla u^N(s,\psi^N(s,Y^N_s)) - \nabla u(s,\psi(s,Y_s))\right| \le{}& \left|\nabla u^N(s,\psi^N(s,Y^N_s)) - \nabla u(s,\psi^N(s,Y^N_s))\right| \\
&+ \left|\nabla u(s,\psi^N(s,Y^N_s)) - \nabla u(s,\psi(s,Y^N_s))\right| \\
&+ \left|\nabla u(s,\psi(s,Y^N_s)) - \nabla u(s,\psi(s,Y_s))\right|
\end{aligned}
\quad . \tag{39}
$$

The terms on the right hand side will be bounded as follows:

- For the first term we use Proposition prop:diff_uN_graduN ?? and we have

$$
\begin{aligned}
\left|\nabla u^N(s,\psi^N(s,Y^N_s)) - \nabla u(s,\psi^N(s,Y^N_s))\right| &\le \left\|\nabla u^N(s) - \nabla u(s)\right\|_{L^\infty} \\
&\le \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}},
\end{aligned} \tag{40}
$$

for the second term see that $\nabla u, \nabla u^N$ are α-Hölder continuous and using Proposition prop:bound_psi-psiN ?? we have

$$
\begin{aligned}
\left|\nabla u(s,\psi^N(s,Y^N_s)) - \nabla u(s,\psi(s,Y^N_s))\right| &\le \left|\psi^N(s,Y^N_s) - \psi(s,Y^N_s)\right|^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}} \\
&\le (2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}})^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}}.
\end{aligned} \tag{41}
$$

Therefore we get the bound

$$
\begin{aligned}
\left|\nabla u^N(s,\psi^N(s,Y^N_s)) - \nabla u(s,\psi(s,Y_s))\right| \le{}& \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ \alpha\kappa_\rho^\alpha \left\|b^N - b\right\|^\alpha_{C_T\mathcal{C}^{-\beta}} \|u\|_{C_T\mathcal{C}^{1+\alpha}} \\
&+ \left|\nabla u(s,\psi(s,Y^N_s)) - \nabla u(s,\psi(s,Y_s))\right|.
\end{aligned} \tag{42}
$$

{eq:bound_gradu_abs}

Here we can also notice that $\mathbb{1}_{\{Y^N_s - Y_s < \epsilon\}} e^{1 - \frac{Y^N_s - Y_s}{\epsilon}} < 1$, then using (eq:bound_gradu_abs ??) and the inequality

$$
(x_1 + \cdots + x_k)^2 \le k(x_1^2 + \cdots + x_k^2), \tag{43}
$$

for $k = 3$, we can get the bound

$$
\begin{aligned}
(\text{eq:local\_time\_diff\_gradu ??}) \le{}& \frac{1}{\epsilon}\mathbb{E}\int_0^t \Big(3\kappa_\rho^2 \left\|b - b^N\right\|^2_{C_T\mathcal{C}^{-\beta}} + 3\cdot 2^{2\alpha}\kappa_\rho^{2\alpha}\left\|b^N - b\right\|^{2\alpha}_{C_T\mathcal{C}^{-\beta}}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}}\Big) ds \\
&+ \frac{1}{\epsilon}\mathbb{E}\int_0^t 3\mathbb{1}_{\{Y^N_s - Y_s > \epsilon\}} e^{1-\frac{Y^N_s - Y_s}{\epsilon}} \left|\nabla u(s,\psi(s,Y^N_s)) - \nabla u(s,\psi(s,Y_s))\right|^2 ds \\
\le{}& \frac{1}{\epsilon} 3t \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \Big(\kappa_\rho^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + (2\kappa_\rho)^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}} \left\|b^N - b\right\|^{2\alpha-1}_{C_T\mathcal{C}^{-\beta}}\Big) \\
&+ \frac{1}{\epsilon} 3\mathbb{E}\Big(\int_0^t \mathbb{1}_{\{Y^N_s - Y_s > \epsilon\}} e^{1-\frac{Y^N_s - Y_s}{\epsilon}} \left|\nabla u(s,\psi(s,Y^N_s)) - \nabla u(s,\psi(s,Y_s))\right|^2 ds\Big)
\end{aligned} \tag{44}
$$

{eq:bound_integral_gr

Now let us denote the last term in (eq:bound_integral_graduNu ??) by $I_t^{N,\epsilon}$. Pick $\zeta \in (0,1)$ such that $\alpha\zeta > \frac{1}{2}$, and since $\epsilon \in (0,1)$ we have $\epsilon^\zeta > \epsilon$. Then split the indicator function $\mathbb{1}_{\{Y^N_s - Y_s > \epsilon\}}$ into $\mathbb{1}_{\{\epsilon < Y^N_s - Y_s \le \epsilon^\zeta\}} + \mathbb{1}_{\{Y^N_s - Y_s > \epsilon^\zeta\}}$. Leading to the integral

$$I_t^{N,\epsilon} = \frac{1}{\epsilon}3\mathbb{E}\Big(\int_0^t \big(\mathbb{1}_{\{\epsilon<Y_s^N-Y_s\le\epsilon\zeta\}} + \mathbb{1}_{\{Y_s^N-Y_s>\epsilon\zeta\}}\big)e^{1-\frac{Y_s^N-Y_s}{\epsilon}}\big|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\big|^2 ds\Big) \tag{45}$$

{eq:INepsilon}

For the first term of (??) we use the fact that ∇u is α-Holder continuous uniformly in $s\in[0,T]$ with constant $\|u\|_{C_T\mathcal{C}^{1+\alpha}}$ and that ψ is 2-Lipschitz

$$\begin{aligned}\big|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\big|^2 &\le \Big|\big|\psi(s,Y_s^N)-\psi(s,Y_s)\big|^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}}\Big|^2 \\ &\le \Big|2^\alpha\big|Y_s^N-Y_s\big|^\alpha\|u\|_{C_T\mathcal{C}^{1+\alpha}}\Big|^2 \\ &= 2^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}}\big|Y_s^N-Y_s\big|^{2\alpha}\end{aligned} \tag{46}$$

For the other term see that ∇u is uniformly bounded by 1/2 thanks to Lemma ?? therefore,

$$\begin{aligned}\big|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\big|^2 &\le \big|\nabla u(s,\psi(s,Y_s^N)) + \nabla u(s,\psi(s,Y_s))\big|^2 \\ &\le \sup_{(s,x)\in[0,T]\times\mathbb{R}}\big|\nabla u(s,\psi(s,Y_s^N)) + \nabla u(s,\psi(s,Y_s))\big|^2 \\ &= \|2\nabla u\|^2_{L_\infty}\end{aligned} \tag{47}$$

Therefore we have that for all $t\in[0,T]$

$$\begin{aligned} I_t^{N,\epsilon} &\le \frac{1}{\epsilon}3\mathbb{E}\Big(\int_0^t \big(\mathbb{1}_{\{\epsilon<Y_s^N-Y_s\le\epsilon\zeta\}}\big)e^{1-\frac{Y_s^N-Y_s}{\epsilon}}2^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}}\big|Y_s^N-Y_s\big|^{2\alpha}ds\Big) \\ &\quad+\frac{1}{\epsilon}3\mathbb{E}\Big(\int_0^t \mathbb{1}_{\{Y_s^N-Y_s>\epsilon\zeta\}}e^{1-\frac{Y_s^N-Y_s}{\epsilon}}\|2\nabla u\|^2_{L_\infty}ds\Big) \\ &\le \frac{1}{\epsilon}3\mathbb{E}\Big(2^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}}\big|\epsilon^\zeta\big|^{2\alpha}t\Big) + \frac{1}{\epsilon}3\mathbb{E}\Big(4e^{1-\epsilon^{\zeta-1}}\|\nabla u\|^2_{L_\infty}t\Big) \\ &\le \sup_{t\in[0,T]}\frac{3}{\epsilon}\Big(2^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}}\epsilon^{2\alpha\zeta} + 4e^{1-\epsilon^{\zeta-1}}\|\nabla u\|^2_{L_\infty}\Big)t \\ &= \frac{3}{\epsilon}\Big(2^{2\alpha}\|u\|^2_{C_T\mathcal{C}^{1+\alpha}}\epsilon^{2\alpha\zeta} + 4e^{1-\epsilon^{\zeta-1}}\|\nabla u\|^2_{L_\infty}\Big)T\end{aligned} \tag{48}$$

{eq:INepsilon_bound}

Now by combining (??), (??) and (??), and taking the sup over $[0,T]$ we will get

Question: Should I take that sup? It makes sense to me in order to have constants instead of something depending on t, but then the integral of the difference $Y^N - Y$ is from 0 to t in the paper

$$
\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] \leq 4\epsilon \\
&+ 4\lambda 2\kappa_\rho T \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ 4\lambda \mathbb{E}\left[\int_0^t \left|Y_s^N - Y^N\right| ds\right] \\
&+ \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T\kappa_\rho^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \frac{1}{\epsilon} 3T(2\kappa_\rho)^{2\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}}^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \\
&+ \frac{3}{\epsilon} 2^{2\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}}^2 T \epsilon^{2\alpha\zeta} \\
&+ \frac{3}{\epsilon} 4 \|\nabla u\|_{L_\infty}^2 T e^{1-\epsilon^{\zeta-1}}
\end{aligned} \tag{49}
$$

then we take $\epsilon = \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}$ and we get

$$
\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] \leq 4\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ 4\lambda 2\kappa_\rho T \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ 4\lambda \mathbb{E}\left[\int_0^t \left|Y_s^N - Y^N\right| ds\right] \\
&+ 3T\kappa_\rho^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \\
&+ 3T(2\kappa_\rho)^{2\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}}^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \\
&+ 2^{2\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}}^2 T \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha\zeta-1} \\
&+ 4 \|\nabla u\|_{L_\infty}^2 T \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{-1} \exp\left(1 - \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{\zeta-1}\right)
\end{aligned} \tag{50}
$$

which can be written as

$$
\begin{aligned}
\mathbb{E}[L_t^0(Y^N - Y)] \leq c_1 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + c_2 \mathbb{E}\left[\int_0^t \left|Y_s^N - Y^N\right| ds\right] \\
+ c_3 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} + c_4 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha\zeta-1} \\
+ c_5 \exp\left(1 - \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{\zeta-1}\right)
\end{aligned} \tag{51}
$$

{eq:bound_constants}

where

$$
\begin{aligned}
c_1 &= 4 + 4\lambda 2\kappa_\rho T + 3\kappa_\rho^2 T \\
c_2 &= 4\lambda \\
c_3 &= 3(2\kappa_\rho)^{2\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}}^2 T \\
c_4 &= 2^{2\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}}^2 T \\
c_5 &= 4 \|\nabla u\|_{L_\infty}^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{-1} T
\end{aligned} \tag{52}
$$

{eq:constants_c}

Finally, observe that since $\zeta \in (0,1)$, the term $\exp\left(1 - \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{\zeta-1}\right)$ decays faster than any polynomial, thus controlling c_5, and the last term in (**??** eq:bound_constants) goes to zero. Also

$\alpha\zeta$ is arbitrarily close to α, and $\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}$ controls $\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}$ therefore we can create the bound (eq:local_time_YNY_bound ??)

Question: is this clear enough? Am I making sense if I am taking α fixed?

$$\mathbb{E}[L_t^0(Y^N - Y)] \leq o(\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}) + c_2\mathbb{E}\left(\int_0^t \left|Y^N - Y\right| ds\right) + c_4\left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1} \tag{53}$$

□

6 Convergence rate of the solution to the regularised SDE and the original

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