Convergence rate of $X^N - X$ for McKean equations

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April 24, 2022

Todo list

Contents

1 What is this?

An adaptation of [1, Proposition 3.1] for the case of ~~McKean SDEs~~ SDEs with drift in a Besov space of negative order proposed in [2] and [3].

The proof builds on a number of results presented in the sections below.

2 Some useful definitions and results

Here we present some results and definitions to refer on the text.

Definition 1: Local time at zero

For any real-valued continuous semi-martingale, the local time at zero $L^0_t(\bar{Y})$ is defined as

$$L^0_t(\bar{Y}) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_0^t \mathbb{1}_{\{|\bar{Y}|\le\epsilon\}} d\langle \bar{Y}\rangle_s, \ \mathbb{P}\text{-a.s.} \tag{1}$$

For all $t \ge 0$.

The first result, [1, Lemma 5.1], is not necessary to prove for this particular setting since the result holds for any semi-martingale, I include it here for self-containment reasons.

Lemma 1: Bound for local time at zero for a semi-martingale

For any $\epsilon \in (0,1)$ and any real-valued, continuous semi-martingale Z we have

$$\begin{aligned}\mathbb{E}[L^0_t(Z_s)] \le & 4\epsilon - \mathbb{E}\left[\int_0^t \left(\mathbb{1}_{\{Z_s\in(0,\epsilon)\}} + \mathbb{1}_{\{Z_s\ge\epsilon\}} e^{1-Z_s/\epsilon}\right) dZ_s\right] \\ & + \frac{1}{\epsilon}\mathbb{E}\left[\int_0^t \mathbb{1}_{\{Z>\epsilon} e^{1-Z_s/\epsilon\}} d\langle Z\rangle_s\right].\end{aligned}$$

Let us introduce the original and regularised Kolmogorov equations.

Definition 2: Kolmogorov equations

For $\beta \in (0,1/2)$ let $b \in C_T\mathcal{C}^{-\beta}$, $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$, and $b^N \to b$ as $N \to \infty$ in $C_T\mathcal{C}^{-\beta}$ The equations

$$\begin{cases}\partial u_i + \frac{1}{2} b_i \Delta u_i = \lambda u_i - b_i \\ u_i(T) = 0,\end{cases} \tag{2}$$

$$\begin{cases}\partial u_i^N + \frac{1}{2} b_i^N \Delta u_i^N = \lambda u_i^N - b_i^N \\ u_i^N(T) = 0.\end{cases} \tag{3}$$

are called Kolmogorov and regularised Kolmogorov equations. Here written component wise.

Lemma 2: Bounds for $\nabla u, \nabla u^N$

Let

3 Bounds for the difference of solutions to the Kolmogorov equations

We need a bound for $u - u^N$ and $\nabla u - \nabla u^N$ in L_∞ for the case in which $u \in C_T\mathcal{C}^{1+\alpha}$ for some $\alpha > \beta$ which is an adaptation of [1, Lemma 5.2].

The result builds on top of the following result:

Proposition 1: Bound for the ρ-equivalent norm of $u - u^N$

Let u, u^N be (mild) solutions to the Kolmogorov equations from **??** 2 then as $N \to \infty$

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \le \frac{cT^{\frac{1-\beta-\alpha}{2}}\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{1 - c\rho^{\frac{\alpha+\beta-1}{2}}(\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda)} \tag{4}$$

for $\rho \ge \rho_0$, where

$$\rho_0 = 2c(\|b_i\|_{C_T\infty+\alpha} + \lambda)^{\frac{2}{\alpha+\beta+1}} \tag{5}$$

and $\lambda > 0$.

Proof. See that $u^N(T) = u(T) = 0$, and in [2], set g^N, g as b^N, b respectively. See that $b^N \to b$. Then let us reformulate the rest of the aforementioned result for $\lambda \neq 0$.

As u^N, u are mild solutions, we have

$$\begin{aligned}
u_i(t) - u_i^N(t) &= P_{T-t}(u_i(T) - u_i^N(T)) \\
&+ \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N)ds \\
&- \int_t^T P_{s-t}(\lambda u_i + b_i - \lambda u_i^N + b_i^N)ds \\
&= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i^N b_i^N)ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N)ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N)ds \\
&= \int_t^T P_{s-t}[(\nabla u_i b_i - \nabla u_i b_i^N) + (\nabla u_i b_i^N - \nabla u_i^N b_i^N)]ds \\
&- \lambda \int_t^T P_{s-t}(u_i - u_i^N)ds \\
&- \int_t^T P_{s-t}(b_i - b_i^N)ds
\end{aligned}$$

$$= \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds$$
$$+ \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds$$
$$- \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds$$
$$- \int_t^T P_{s-t}(b_i - b_i^N) ds$$

Now let us compute the ρ-equivalent norm of $u - u^N$, for some $\alpha > \beta$

type this result which is lemma 4.2 in the paper

$$\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} &= \sup_{0\le t\le T} e^{-\rho(T-t)} \|u(t) - u^N(t)\|_{1+\alpha} \\
&\le \sup_{0\le t\le T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds\|_{1+\alpha} \\
&+ \sup_{0\le t\le T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(\nabla u_i b_i^N - \nabla u_i^N b_i^N) ds\|_{1+\alpha} \\
&- \sup_{0\le t\le T} e^{-\rho(T-t)} \| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds\|_{1+\alpha} \\
&- \sup_{0\le t\le T} e^{-\rho(T-t)} \| \int_t^T P_{s-t}(b_i - b_i^N) ds\|_{1+\alpha}.
\end{aligned}$$

Let us take each term from the right hand side of the inequality and bound them.

For the first term, using $\gamma + 2\theta = 1 + \alpha$, $\gamma = -\beta$, $\theta = \frac{1+\alpha+\beta}{2}$, $\|P_t f\|_{\gamma+2\theta} \le ct^{-\theta}\|f\|_\gamma\|$ and $\|\nabla g\|_\xi \le c\|g\|_{\xi+1}$

$$\begin{aligned}
&\sup_{0\le t\le T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t}(\nabla u_i b_i - \nabla u_i b_i^N) ds \right\|_{1+\alpha} \\
&\le \sup_{0\le t\le T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\theta} \|\nabla u_i\|_\alpha \|b_i - b_i^N\|_{-\beta} \\
&\le c\|u_i\|_{C_T\mathcal{C}_{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\le t\le T} e^{-\rho(T-t)} (T-t)^{\frac{1-\beta-\alpha}{2}} \\
&\le cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T\mathcal{C}_{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}}
\end{aligned}$$

For the second term, see that for $N \to \infty$, we have $\|b^N\|_{C_T\mathcal{C}^{-\beta}} \le 2\|b\|_{C_T\mathcal{C}^{-\beta}}$

$$\begin{aligned}
&\sup_{0\le t\le T} e^{-\rho(T-t)} \left\| \int_t^T P_{s-t} b_i^N (\nabla u_i - \nabla u_i^N) ds \right\|_{1+\alpha} \\
&\le c \sup_{0\le t\le T} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} 2\|b_i\|_{-\beta} \|\nabla u_i - \nabla u_i^N\|_\alpha ds
\end{aligned}$$

$$
\begin{aligned}
&\leq c\|b_i\|_{C_T\mathcal{C}^{-\beta}}\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \int_t^T (s-t)^{-\theta} e^{-\rho(T-t)} ds \\
&\leq c\|b_i\|_{C_T\mathcal{C}^{-\beta}}\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}
$$

For the third term, which is the one that difers from the proof in [2] we need to use that $\|P_t f\|_{\gamma+2\theta} \leq ct^{-\theta}\|f\|_\gamma$, and in this case we have $\gamma + 2\theta = 1+\alpha$ and $\gamma = 1+\alpha$, so that $\theta = 0$ because $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, so we will have

$$
\begin{aligned}
\sup_{0\leq t\leq T} e^{-\rho(T-t)} &\left\| \lambda \int_t^T P_{s-t}(u_i - u_i^N) ds \right\|_{1+\alpha} \\
&\leq c\lambda \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-0} \|u_i - u_i^N\|_{1+\alpha} ds \\
&= c\lambda \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T e^{-\rho(T-s)} \sup_{0\leq s\leq T} e^{-\rho(T-s)} \|u_i - u_i^N\|_{1+\alpha} ds \\
&= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \int_t^T e^{-\rho(T-s)} e^{-\rho(T-t)} ds \\
&= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \int_t^T e^{-\rho(s-t)} ds \\
&= c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \sup_{0\leq t\leq T} \rho^{-1}[1 - e^{-\rho(T-t)}] \\
&\leq c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{-1} \\
&\leq c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}
$$

And for the last term

$$
\begin{aligned}
\sup_{0\leq t\leq T} e^{-\rho(T-t)} &\left\| \int_t^T P_{T-s}(b_i - b_i^N) ds \right\|_{1+\alpha} \\
&\leq c \sup_{0\leq t\leq T} e^{-\rho(T-t)} \int_t^T (s-t)^{-\frac{\alpha+\beta-1}{2}} \|b_i - b_i^N\|_{-\beta} ds \\
&\leq c\|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \sup_{0\leq t\leq T} e^{-\rho(T-t)} (s-t)^{-\frac{\alpha+\beta-1}{2}} \\
&\leq cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}
\end{aligned}
$$

Putting everything together

$$
\begin{aligned}
\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq\ & cT^{\frac{1-\beta-\alpha}{2}} \|u_i\|_{C_T\mathcal{C}^{1+\alpha}} \|b_i - b_i^N\|_{C_T\mathcal{C}^{-\beta}} \\
&+ c\|b_i\|_{C_T\mathcal{C}^{-\beta}} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \rho^{\frac{\alpha+\beta-1}{2}} \\
&- c\lambda \sup_{0\leq t\leq T} \|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}_{1+\alpha}} \rho^{\frac{\alpha+\beta-1}{2}}
\end{aligned}
$$

$$- cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}},$$

and finally,

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}}(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])$$
$$\leq cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)$$
$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])}$$

As required. □

Note that in the above we can represent the right hand side of the inequality as

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq \frac{cT^{\frac{1-\beta-\alpha}{2}}(\|u_i\|_{C_T\mathcal{C}^{1+\alpha}} - 1)}{(1 - c\rho^{\frac{\alpha+\beta-1}{2}}[\|b\|_{C_T\mathcal{C}^{-\beta}} + \lambda])} \|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{6}$$

$$\|u_i - u_i^N\|^{(\rho)}_{C_T\mathcal{C}^{-\beta}} \leq c(\rho)\|b_i - b_i^N\|_{C^T\mathcal{C}^{-\beta}} \tag{7}$$

this norm is wrong, should be $1+\alpha$ on the lhs and everywhere else

Here is the adaptation of [1, Lemma 5.2].

Proposition 2: Bounds for $\|u - u^N\|_{L_\infty}$ and $\|\nabla u - \nabla u^N\|_{L_\infty}$

Let $\beta \in (0, 1/2)$ and $b \in C_T\mathcal{C}^{-\beta}$. Let $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ be (mild) solutions to the Kolmogorov equations from **??** 2.
Assume, by **??**, that for some $\alpha > \beta$

$$\|u - u^N\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}} \leq c(\rho)\|b - b^N\|_{C_T\mathcal{C}^{-\beta}}. \tag{8}$$

With $c(\rho)$ as in **??** and ρ_0 is large enough such that $c(\rho) > 0$ for all $\rho > \rho_0$.
Then for all $t \in [0, T]$

$$\|u^N(t) - u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{9}$$

$$\|\nabla u^N(t) - \nabla u(t)\|_{L^\infty} \leq \kappa_\rho\|b - b^N\|_{C_T\mathcal{C}^{-\beta}} \tag{10}$$

with $\kappa_\rho = c \cdot c(\rho) \cdot e^{\rho T}$.

Proof. First let us prove Eq. (9).

Let $t \in [0, T]$, and see that since $u, u^N \in C_T\mathcal{C}^{(1+\beta)+}$ there exists $\alpha > \beta$ such that $u, u^N \in C_T\mathcal{C}^{1+\alpha}$, then for any $f \in \mathcal{C}^{1+\alpha}$ we have

$$\|f\|_{C^{1+\alpha}} \leq c \left(\sup_{x\in\mathbb{R}^d} |f(x)| + \sup_{x\neq y\in\mathbb{R}^d} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \right) \tag{11}$$

so we have

$$
\begin{aligned}
\|u^N(t) - u(t)\|_{L^\infty} &= \sup_{x\in\mathbb{R}^d} |u^N(t,x) - u(t,x)| \\
&\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}}
\end{aligned}
\tag{12}
$$

Moreover, using the (ρ)-equivalent norm

$$
\|f\|_{\mathcal{C}^{1+\alpha}} = \sup_{t\in[0,T]} e^{-\rho(T-t)}\|f(t)\|_{\mathcal{C}^{1+\alpha}}, \tag{13}
$$

and Eq. (8) we see that

$$
\begin{aligned}
\left\|u^N - u\right\|_{C_T\mathcal{C}^{1+\alpha}} &= \sup_{t\in[0,T]} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\
&= \sup_{t\in[0,T]} e^{\rho(T-t)} e^{-\rho(T-t)} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\
&\leq e^{\rho T} \sup_{t\in[0,T]} e^{-\rho(T-t)} \left\|u^N - u\right\|_{\mathcal{C}^{1+\alpha}} \\
&= e^{\rho T} \left\|u^N - u\right\|^{(\rho)}_{C_T\mathcal{C}^{1+\alpha}}.
\end{aligned}
\tag{14}
$$

Plugging Eq. (14) into Eq. (12)

$$
\begin{aligned}
\|u^N(t) - u(t)\|_{L^\infty} &\leq c\|u^N(t) - u(t)\|_{\mathcal{C}^{\alpha+1}} \\
&\leq \sup_{t\in[0,T]} c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{\alpha+1}} \\
&= c\left\|u^N - u\right\|_{C_T\mathcal{C}^{\alpha+1}} \\
&\leq c e^{\rho T}\left\|u^N - u\right\|^{(\rho)}_{C_T\mathcal{C}^{\alpha+1}}.
\end{aligned}
\tag{15}
$$

And finally by Eq. (8)

$$
\left\|u^N(t) - u(t)\right\|_{L^\infty} \leq c \cdot c(\rho) \cdot e^{\rho T} \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \tag{16}
$$

which proves Eq. (9).

For Eq. (10) recall that if $f \in \mathcal{C}^{1+\alpha}$ then $\nabla f \in \mathcal{C}^{\alpha}$. Also, by Bernstein inequality [3, Eqn. (9)]

$$
\|\nabla f\|_\alpha \leq c\|f\|_{\infty+\alpha}. \tag{17}
$$

Using the equivalent norm

$$
\|f\|_{C^{1+\alpha}} \leq c\left(\sup_{x\in\mathbb{R}^d}|f(x)| + \sup_{x\in\mathbb{R}^d}|\nabla f(x)| + \sup_{x\neq y\in\mathbb{R}^d}\frac{|\nabla f(x) - \nabla f(y)|}{|x-y|^\alpha}\right) \tag{18}
$$

we can see that

$$
\left\|\nabla u^N(t) - \nabla u(t)\right\|_{L^\infty} \leq c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1+\alpha}}. \tag{19}
$$

And usign the same bounds that we used above for $c\left\|u^N(t) - u(t)\right\|_{\mathcal{C}^{1+\alpha}}$ this point follows.

□

4 Bound for the difference of the auxiliary functions

This is the adaptation of result [1, Lemma 5.3].

Proposition 3: Bound for $|\psi(t,x) - \psi^N(t,x)|$

Take $\rho > \rho_0$ as in **??**, $N \to \infty$, κ_ρ from **??** 2, and $\beta \in (0, 1/2)$, then we have

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \left|\psi(t,x) - \psi^N(t,x)\right| \leq 2\kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}} \tag{20}$$

Proof. Recall the definition of $\psi, \phi \in C_T\mathcal{C}^1$

$$\phi(t,x) := x + u(t,x) \tag{21}$$

$$\psi(t,\cdot) = \phi^{-1}(t,\cdot). \tag{22}$$

Note that

$$u(y) = \int_0^1 \nabla u(\alpha y) y d\alpha + u(0). \tag{23}$$

From there we have

$$u(t,y) - u(t,y') = \int_0^1 \nabla u(t, \alpha(y-y'))(y-y')d\alpha \tag{24}$$

and therefore

$$|u(t,y) - u(t,y')| \geq \left(\int_0^1 |\nabla u(t,\alpha(y-y'))|^2 d\alpha\right)^{1/2} |y-y'|, \tag{25}$$

and by **??** we finally have

$$\begin{aligned} |u(t,y) - u(t,y')| &\leq \left(\frac{1}{4}\int_0^1 d\alpha\right)^{1/2} |y-y'| \\ |u(t,y) - u(t,y')|^2 &\leq \frac{1}{4}|y-y'|^2 \end{aligned} \tag{26}$$

Check this norm

□

5 Bound for the local time at zero of the solution to the SDEs

We need a bound for $\mathbb{E}[L_T^0(Y^N - Y)]$, for Sobolev spaces, this is result [1, Proposition 5.4] we present it here for the solutions to the SDE blonging to the appropriate Besov spaces.

Proposition 4: Bound for local time of solutions to the SDE

Let $b \in C_T\mathcal{C}^{-\beta}$ and $b^N \to b$ in $C_T\mathcal{C}^{-\beta}$ as $N \to \infty$ for $\beta \in (0, \frac{1}{4})$ and for any $\alpha > \beta$

Proof. Recall that Y^N, Y are solutions to the SDEs

$$Y_t = y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1)dW_s \tag{27}$$

and

$$Y_t^N = y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1)dW_s \tag{28}$$

so that the difference $Y^N - Y$ is

$$\begin{aligned} Y^N - Y_t &= (y_0^N + \lambda \int_0^t u^N(s, \psi^N(s, Y_s^N))ds + \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) + 1)dW_s) \\ &\quad - (y_0 + \lambda \int_0^t u(s, \psi(s, Y_s))ds + \int_0^t (\nabla u(s, \psi(s, Y_t)) + 1)dW_s) \\ &= (y_0^N - y_0) + \lambda \int_0^t (u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s)))ds \\ &\quad - \int_0^t (\nabla u^N(s, \psi^N(s, Y_t^N)) - \nabla u(s, \psi(s, Y_t)))dW_s, \end{aligned} \tag{29}$$

and using **??** 1 we have the following bound

$$\begin{aligned} &\mathbb{E}[L_t^0(Y^N - Y)] \\ &\leq 4\epsilon + 1 \\ &- 2(1+\lambda)\mathbb{E}\Big[\int_0^t \Big(\mathbb{1}_{\{Y_s^N - Y_s \in (0,\epsilon)\}} + \mathbb{1}_{\{Y_s^N - Y_s \geq \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}}\Big)\big(u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\big)ds\Big] \\ &+ \frac{1}{\epsilon}\mathbb{E}\Big[\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \big(\nabla u^N(s, \psi^N(s, Y_s^N)) - \nabla u(s, \psi(s, Y_s))\big)^2 ds\Big]. \end{aligned} \tag{30}$$

For the second and third terms of Eq. (30) let us bound the factors involving the differences of u, u^N and $\nabla u, \nabla u^N$.

continue from page three in notes

First, for u, u^N adding and subtracting terms and using triangle inequality we have

$$\begin{aligned} \big|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s))\big| &\leq \big|u^N(s, \psi^N(s, Y_s^N)) - u(s, \psi^N(s, Y_s^N))\big| \\ &\quad + \big|u(s, \psi^N(s, Y_s^N)) - u(s, \psi(s, Y_s^N))\big| \\ &\quad + \big|u(s, \psi(s, Y_s^N)) - u(s, \psi(s, Y_s))\big|. \end{aligned} \tag{31}$$

□

The terms in the right hand side will be bounded as follows:

- For the first term, by **??** 2

$$\left|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi^N(s,Y_s^N))\right| \le \left\|u^N(s)-u(s)\right\|_{L^\infty} \le \kappa_\rho \left\|b-b^N\right\|_{C_T\mathcal{C}^{-\beta}}, \tag{32}$$

- for the second term, observe that u, u^N are $\frac{1}{2}$-Lipschitz and by Section 4 we get

$$\begin{aligned}\left|u(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s^N))\right| &\le \frac{1}{2}\left|\psi^N(s,Y_s^N)-\psi(s,Y_s^N)\right| \\ &\le \kappa_\rho\left\|b^N-b\right\|_{C_T\mathcal{C}^{-\beta}},\end{aligned} \tag{33}$$

- and for the final term, note that ψ, ψ^N are 2-Lipschitz

  so that

$$\left|u(s,\psi(s,Y_s^N)) - u(s,\psi(s,Y_s))\right| \le \frac{1}{2}\left|\psi(s,Y_s^N)-\psi(s,Y_s)\right| \le \left|Y_s^N - Y_s\right|. \tag{34}$$

So that the following bound holds

$$\left|u^N(s,\psi^N(s,Y_s^N)) - u(s,\psi(s,Y_s))\right| \le 2\kappa_\rho\left\|b^N-b\right\|_{C_T\mathcal{C}^{-\beta}}, + \left|Y_s^N - Y^N\right|. \tag{35}$$

Now for the third term in Eq. (30) by adding and subtracting terms and using the triangle inequality

$$\begin{aligned}\left|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right| \le &\left|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi^N(s,Y_s^N))\right| \\ &+ \left|\nabla u(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s^N))\right| \\ &+ \left|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right|\end{aligned}. \tag{36}$$

The terms on the right hand side will be bounded as follows:

- For the first term we use **??** 2 and we have

$$\begin{aligned}\left|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi^N(s,Y_s^N))\right| &\le \left\|\nabla u^N(s) - \nabla u(s)\right\|_{L^\infty} \\ &\le \kappa_\rho\left\|b-b^N\right\|_{C_T\mathcal{C}^{-\beta}},\end{aligned} \tag{37}$$

  for the second term see that $\nabla u, \nabla u^N$ are α-Hölder continuous and using Section 4 we have

$$\begin{aligned}\left|\nabla u(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s^N))\right| &\le \left|\psi^N(s,Y_s^N)-\psi(s,Y_s^N)\right|^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}} \\ &\le (2\kappa_\rho\left\|b^N-b\right\|_{C_T\mathcal{C}^{-\beta}})^\alpha\|u\|_{C_T\mathcal{C}^{1+\alpha}}.\end{aligned} \tag{38}$$

Therefore we get the bound

$$\begin{aligned}\left|\nabla u^N(s,\psi^N(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right| \le{}& \kappa_\rho \left\|b - b^N\right\|_{C_T\mathcal{C}^{-\beta}} \\ &+ 2^\alpha \kappa_\rho^\alpha \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^\alpha \|u\|_{C_T\mathcal{C}^{1+\alpha}} \\ &+ \left|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right| \end{aligned} \quad . \tag{39}$$

Using the bounds for Eq. (35) and Eq. (39) and the inequality

$$(x_1 + \cdots + x_k)^2 \le k(x_1 + \cdots + x_k), \tag{40}$$

for some k, we get

$$\begin{aligned} &\mathbb{E}[L_t^0(Y^N - Y)] \\ &\le 4\epsilon + 4(1+\lambda)\left(2\kappa_\rho \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} t + \mathbb{E}\left[\int_0^t \left|Y_s^N - Y^N\right| ds\right]\right) \\ &+ \frac{1}{\epsilon} 3t \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} \left(\kappa_\rho^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}} + 4(2\kappa_\rho)^{2\alpha} \|u\|_{C_T\mathcal{C}^{1+\alpha}}^2 \left\|b^N - b\right\|_{C_T\mathcal{C}^{-\beta}}^{2\alpha-1}\right) \\ &+ \frac{1}{\epsilon} 3\mathbb{E}\left(\int_0^t \mathbb{1}_{\{Y_s^N - Y_s > \epsilon\}} e^{1 - \frac{Y_s^N - Y_s}{\epsilon}} \left|\nabla u(s,\psi(s,Y_s^N)) - \nabla u(s,\psi(s,Y_s))\right|^2 ds\right) \end{aligned} \tag{41}$$

6 Convergence rate of the solution to the regularised SDE and the original

References

[1] Tiziano De Angelis, Maximilien Germain, and Elena Issoglio. "A Numerical Scheme for Stochastic Differential Equations with Distributional Drift". In: *arXiv:1906.11026 [cs, math]* (Oct. 22, 2020). arXiv: 1906.11026.

[2] Elena Issoglio and Francesco Russo. "A PDE with Drift of Negative Besov Index and Related Martingale Problem". In: (), p. 48.

[3] Elena Issoglio and Francesco Russo. "McKean SDEs with singular coefficients". In: *arXiv:2107.14453 [math]* (July 30, 2021). arXiv: 2107.14453.