An Euler-Maruyama scheme for SDEs with distributional drift

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The SDE

- Consider the SDE

$$X_t = X_0 + \int_0^t b(s, X_s)ds + W_t, \quad t \in [0, T]$$

where the drift b belongs to the space of Schwartz distributions $\mathcal{S}'$.

- In particular $b \in C_T\mathcal{C}^{-\beta}(\mathbb{R}) := C([0, T]; \mathcal{C}^{-\beta}(\mathbb{R}))$, for $\beta \in (0, 1/4)$.

Intuition

Think of the coefficient $-\beta$ as the order of the existing "derivatives" of the functions in said space, or even better, the coefficient for a Hölder space.

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The numerical scheme

As in De Angelis et al. 2022, Section 2.2, for an appropriate sequence $b^N \to b$ in $C_T\mathcal{C}^{-\beta}$ we can define an special Euler-Maruyama (E-M) scheme to approximate numerically the solution of the SDE for any t

$$X^{Nm}_{t_{n+1}} = X^{Nm}_{t_n} + b^N\left(X^{Nm}_{t_n}\right)\Delta t_n + \Delta W_{t_n}.$$

Remark

Note that a normal E-M scheme would only depend on m, whereas this scheme depends on h and N.

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- However, even though this integral is formally well posed by *dual pairing*, in practice we cannot compute it.

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- Therefore, we can select $b := \frac{\partial}{\partial x} B^H(x)$, and since the derivative and convolution commute, the convolution of the previous slide is written as

$$b^N(x) := \left(p_{t(N)} * \frac{\partial}{\partial x} B^H \right)(x) = \left(\frac{\partial}{\partial x} p_{t(N)} * B^H \right)(x)$$

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Some remarks

- The convergence rate of an E-M scheme for an SDE with regular coefficients, i.e: Lipschitz continuous and with linear growth is known to be 1/2 (Kloeden and Platen 1999, Section 10.2).

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- The convergence rate of an E-M scheme for an SDE with regular coefficients, i.e: Lipschitz continuous and with linear growth is known to be 1/2 (Kloeden and Platen 1999, Section 10.2).
- However, **our coefficients are not even functions!**
- Moreover, since our E-M scheme depends on an additional parameter it is natural to expect a different rate, in particular one depending on the extra parameter.

Convergence rate of E-M scheme

Let X_t be the solution to the SDE with drift coefficient $b \in C_T\mathcal{C}^{-\beta}$, for $\beta \in (0, 1/4)$ and X_t^{Nm} be the Euler approximation of the solution with m time steps. Then it holds that

$$\sup_{0 \le t \le T} \mathbb{E}\left[\left|X_t^{Nm} - X_t\right|\right] \le c m^{-\frac{1}{2}+\mu(\beta)+\epsilon},$$

where

$$\mu(\beta) = \frac{1}{2} \cdot \frac{\beta}{(1/2-\beta)(1-2\beta)+\beta},$$

for any $\epsilon > 0$.

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- For $\beta \to \frac{1}{4}$ (distributional drift), we have an exponent

$$-\frac{1}{2} + \underbrace{\frac{1}{2} \cdot \frac{2}{3}}_{\lim_{\beta \to 1/4} \mu(\beta)} + \epsilon = \boxed{-\frac{1}{6} + \epsilon}$$

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