Approximation of Generalised Functions for Numerical Solutions of SDEs

Luis-Mario Chaparro-Jáquez

University of Leeds
School of Mathematics

December 8th 2021

Table of Contents

SDEs with distributional coefficients

Consider the SDE

$$\begin{cases} dX_t = b(t, X_t)dt + dW_t \\ X_0 = x_0, \quad t \in [0, T], \end{cases}$$

where $(W_t)_{t \geq 0}$ is a Brownian motion, and b belongs to a Sobolev space $H_p^{-s}(\mathbb{R})$ of negative and non-integer order, in particular b is a distribution.

Table of Contents

What is an SDE?

Let $(\Omega, \mathcal{F}, \mathbb{F}, W_t)$ be a filtered probability space, a Stochastic Differential Equation (SDE) is a relation

$$X_t = x + \int_0^t b(X_t, t)dt + \int_0^t \sigma(X_t, t)dW_t.$$

What is an SDE?

Let $(\Omega, \mathcal{F}, \mathbb{F}, W_t)$ be a filtered probability space, a Stochastic Differential Equation (SDE) is a relation

$$X_t = x + \int_0^t b(X_t, t)dt + \int_0^t \sigma(X_t, t)dW_t.$$

It is also denoted by

$$\begin{cases} dX_t = X_0 + b(X_t, t)dt + \sigma(X_t, t)dW_t \\ X_0 = x \end{cases}.$$

Solutions of SDEs

Definition (Strong solution of an SDE)

If for a Brownian motion $(W_t)_{t\geq 0}$ there exist a process $(X_t)_{t\geq 0}$ satisfying (3) $\mathbb{P}$-almost surely we say that the SDE has a strong solution.

Existence and uniqueness

Theorem (Karatzas and Shreve 1991)

Suppose that there exists a constant $K \geq 0$ such that for all $x, y \in \mathbb{R}$ and $t \in [0, T]$,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x - y|$$

and

$$|b(t,x)| + |\sigma(t,x)| \leq K(1 - |x|).$$

Then there exist a continuous $\mathbb{F}$-adapted process $(X_t)_{t \geq 0}$ which is a strong solution to , which is unique up to indistinguishability.

Numerical schemes for SDEs

Definition (Euler scheme)

Let $(t_n)_{n=0}^{N}$ be a discretisation of the interval $[0, T]$. Then an Euler approximation for the solution of the SDE at time t_{n+1} is given by

$$Y_{n+1} = b(t, Y_n)\Delta t_n + \sigma(t, Y_n)\Delta W_n$$

Strong convergence rate of Euler scheme

Theorem (Kloeden and Platen 1999)

Suppose conditions from Theorem 2 for coefficients b and σ hold. Additionally assume that for any $0 \leq s < t \leq T$,

$$|b(s,x) - b(t,x)| + |\sigma(s,x) - \sigma(t,x)| \leq K(1 + |x|)\sqrt{|t-s|},$$

holds. Then there exist a constant $M < \infty$ such that

$$\mathbb{E}|X_T - Y_N| \leq M \left(\max_n \Delta t_n \right)^{\frac{1}{2}},$$

where X is the real solution Y the approximated solution and $(t_n)_{n=0}^{N}$ a time grid.

Table of Contents

What is a distribution?

Definition (Distribution)

Let $f \in \mathcal{D}(\mathbb{R}) := C_c^\infty(\mathbb{R})$ then a distribution or generalised function is a linear mapping $\phi \mapsto (f, \phi)$ from $\mathcal{D}(\mathbb{R})$ to $\mathbb{R}$ such that if $\phi_n \to \phi$ in $\mathcal{D}(\mathbb{R})$ then $(f, \phi_n) \to (f, \phi)$ in $\mathbb{R}$ We denote the space of distributions as $\mathcal{D}'(\mathbb{R})$.

What is a distribution?

Definition (Distribution)

Let $f \in \mathcal{D}(\mathbb{R}) := C_c^\infty(\mathbb{R})$ then a distribution or generalised function is a linear mapping $\phi \mapsto (f, \phi)$ from $\mathcal{D}(\mathbb{R})$ to $\mathbb{R}$ such that if $\phi_n \to \phi$ in $\mathcal{D}(\mathbb{R})$ then $(f, \phi_n) \to (f, \phi)$ in $\mathbb{R}$ We denote the space of distributions as $\mathcal{D}'(\mathbb{R})$.

In particular we use a space of distributions

$$I^{-s} L^p(\mathbb{R}) =: H_p^s(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R}),$$

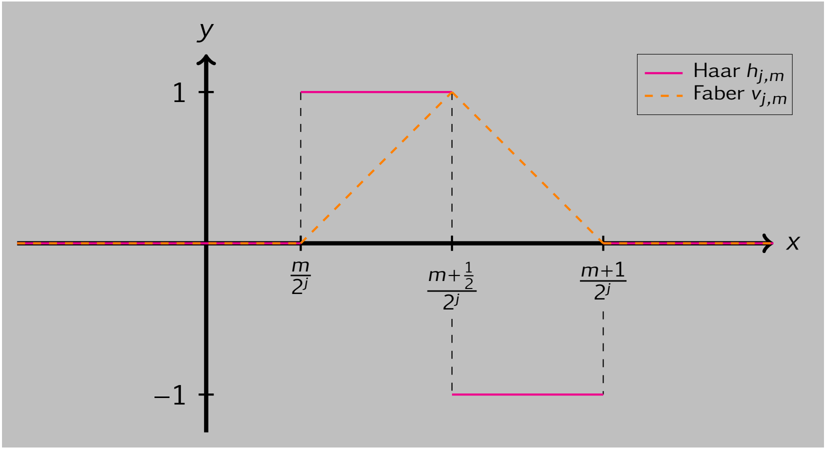
for $s < 0$ and

$$I^s : f \mapsto \mathcal{F}^{-1}\left[\left(1 + |\xi|^2\right)^{\frac{s}{2}} \mathcal{F}[f]\right],$$

and $f \in \mathcal{S}'(\mathbb{R})$.

Table of Contents

A picture of Haar and Faber functions



Approximation with Haar functions

Theorem (Triebel 2010)

Let $I = (0,1)$, $0 \leq p < \infty$, $-\frac{1}{2} < s < \frac{1}{p}$, and $f \in \mathcal{D}'(I)$. Then $f \in H_p^s(I)$ if and only if

$$f = \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{j,m} 2^{-j\left(s-\frac{1}{r}\right)} h_{j,m}.$$

The representation is unique with the coefficients given by

$$\mu_0 := \int_0^1 f(x) h_0(x) dx \quad \text{and} \quad \mu_{j,m} := 2^{j\left(s-\frac{1}{p}+1\right)} \int_0^1 f(x) h_{j,m}(x) dx$$

for $j \in \mathbb{N}$ and $m = 0, \ldots, 2^j - 1$, where the integral above is in the sense of dual pairing.

Faber functions

Theorem (Triebel 2010)

Let $I=(0,1)$, and $f \in H^s_p(I)$ $2 \le p < \infty$, $\frac{1}{2} < s < 1+\frac{1}{p}$, then

$$g = \hat{\mu}_0 v_0 + \hat{\mu}_1 v_1 + \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} \hat{\mu}_{j,m} v_{j,m}$$

The representation is unique with the coefficients given by

$$\begin{cases} \hat{\mu}_{j,m} & = -\frac{1}{2}(\Delta^2_{2^{-j-1}} g)(2^{-j}m) \\ \hat{\mu}_0 & = g(0) \\ \hat{\mu}_1 & = g(1), \end{cases}$$

$(\Delta^2_h g)(x) := g(x+2h) - 2g(x+h) + g(x)$.

How to deal with the dual pairing

Since a Haar function $h_{j,m}$ is the same as the derivative of a Faber function $v_{j,m}$. Thanks to the series representations stated before we can think of a function $f \in H_p^{s-1}$ as $f = \frac{dg}{dx}$ where $g \in H_p^s$ for $2 \leq p < \infty$ and $\frac{1}{2} < s < 1 + \frac{1}{p}$.

How to deal with the dual pairing

Theorem (Triebel 2010)

Let $g \in H_p^s(I)$, then $g' \in H_p^{s-1}(I)$ with $2 \leq p < \infty$, $\frac{1}{2} < s < 1 + \frac{1}{p}$ which can be written as

$$g' = (\hat{\mu}_1 - \hat{\mu}_0)h_0 + \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} 2^{j+1} \hat{\mu}_{j,m} h_{j,m},$$

where

$$\mu_0 = \hat{\mu}_1 - \hat{\mu}_0 \qquad \text{and} \qquad \mu_{j,m} = 2^{j+1} \hat{\mu}_{j,m}.$$

Definition of our coefficient

We select our drift coefficient (which we consider time homogeneous) as the derivative with respect to x of a single path of fractional Brownian motion $B^H(x) \in H^s_p$, i.e:

$$b = \frac{d}{dx} B^H(x) = \mu_0 h_0(x) + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j - 1} \mu_{j,m} h_{j,m}(x),$$

Definition of our coefficient

We select our drift coefficient (which we consider time homogeneous) as the derivative with respect to x of a single path of fractional Brownian motion $B^H(x) \in H^s_p$, i.e:

$$b = \frac{d}{dx} B^H(x) = \mu_0 h_0(x) + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{j,m} h_{j,m}(x),$$

and then our approximation to the coefficient would be

$$b^N = \mu_0 h_0(x) + \sum_{j=0}^{N} \sum_{m=0}^{2^j-1} \mu_{j,m} h_{j,m}(x).$$

Definition of our coefficient

We select our drift coefficient (which we consider time homogeneous) as the derivative with respect to x of a single path of fractional Brownian motion $B^H(x) \in H_p^s$, i.e:

$$b = \frac{d}{dx} B^H(x) = \mu_0 h_0(x) + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{j,m} h_{j,m}(x),$$

and then our approximation to the coefficient would be

$$b^N = \mu_0 h_0(x) + \sum_{j=0}^{N} \sum_{m=0}^{2^j-1} \mu_{j,m} h_{j,m}(x).$$

where

$$\begin{cases} \mu_0 & = B^H(1) - B^H(0) \\ \mu_{j,m} & = -2^j \left(B^H\left(\frac{m+1}{2^j}\right) - 2B^H\left(\frac{m+\frac{1}{2}}{2^j}\right) + B^H\left(\frac{m}{2^j}\right) \right). \end{cases}$$

The heat semigroup

Definition

The heat kernel is a function $\phi(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$ defined as

$$\phi(t,x) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$

To make the coefficient b^N smooth we apply the heat kernel via convolution, denoted as

$$P_{\eta_N} b^N(x) = (\phi * b^N)(x) = \int_{-\infty}^{\infty} \phi(t, x-y) b^N(y) dy.$$

This is called heat semigroup.

The heat semigroup

And as b^N is a sum of Haar functions, the heat semigroup computed over each elemen of that sum will be

$$[P_{\eta_N} h_{j,m}](x) = \left[P_{\eta_N} \mathbb{1}_{\left[\frac{m}{2^j}, \frac{m+1/2}{2^j}\right)}\right](x) - \left[P_{\eta_N} \mathbb{1}_{\left[\frac{m+1/2}{2^j}, \frac{m+1}{2^j}\right)}\right](x)$$
$$= \exp(-\eta_N)\left(-\Phi\left(\frac{x-\frac{m+1}{2^j}}{\sqrt{\eta_N}}\right) + 2\Phi\left(\frac{x-\frac{m+1/2}{2^j}}{\sqrt{\eta_N}}\right) - \Phi\left(\frac{x-\frac{m}{2^j}}{\sqrt{\eta_N}}\right)\right).$$

Table of Contents

A modified Euler scheme

Given that now we have a "nice" approximation of the distributional coefficient we have the SDE

$$\begin{cases} dX_t^N = P_{\eta_N} b^N(X_t^N)dt + dW_t, \quad t \in [0, T] \\ X_0^N = x_0, \end{cases}$$

and the "Euler scheme" will be

$$Y_{n+1}^N = Y_n^N + P_{\eta_N} b^N(Y_n^N)\Delta t_n + \Delta W_n.$$

Order of convergence

- Combining the result that links the previous SDE with the original, and the result for the Euler scheme with the previous equation we will have that the "Euler scheme" to the real solution we will have a convergence rate of

$$\frac{1}{2}\left(\frac{3}{4}-\beta_0\left(\gamma_0-\frac{1}{2}\right)\right)^{-1}\left(\frac{1}{2}-\beta_0\right)\left(\gamma_0-\frac{1}{2}\right)-\epsilon$$

for $b \in H^{-\beta_0}_{q_0,\tilde{q}_0}$ and $\gamma_0 = 1-\beta_0-1/q_0$ where $\beta_0 \in (0, 1/4)$, $q_0 \in (4, 1/\beta_0)$ and $\tilde{q}_0 := (1-\beta_0)^{-1}$ is fixed.

Order of convergence

- Combining the result that links the previous SDE with the original, and the result for the Euler scheme with the previous equation we will have that the "Euler scheme" to the real solution we will have a convergence rate of

$$\frac{1}{2}\left(\frac{3}{4}-\beta_0\left(\gamma_0-\frac{1}{2}\right)\right)^{-1}\left(\frac{1}{2}-\beta_0\right)\left(\gamma_0-\frac{1}{2}\right)-\epsilon$$

  for $b \in H^{-\beta_0}_{q_0,\tilde{q}_0}$ and $\gamma_0 = 1-\beta_0-1/q_0$ where $\beta_0 \in (0,1/4)$, $q_0 \in (4, 1/\beta_0)$ and $\tilde{q}_0 := (1-\beta_0)^{-1}$ is fixed.
- As an example, if $\beta_0 = 0.05$, and $q_0 = 1/\beta_0$ we have a convergence order of approximately 0.12 (De Angelis, Germain, and Issoglio 2020).

Table of Contents

Implementation

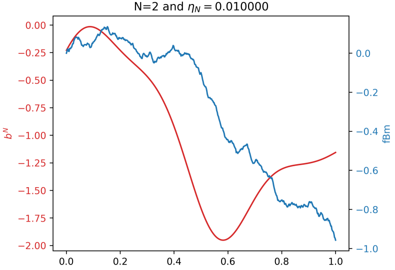


Figure: Coefficient $P_{\eta_N} b^N$ for $N = 2$ and $\eta = 10^{-2}, 10^{-6}$.

Implementation

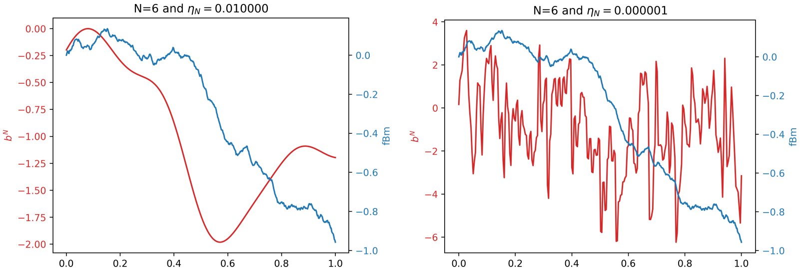


Figure: Coefficient $P_{\eta_N} b^N$ for $N = 6$ and $\eta = 10^{-2}, 10^{-6}$.

Implementation

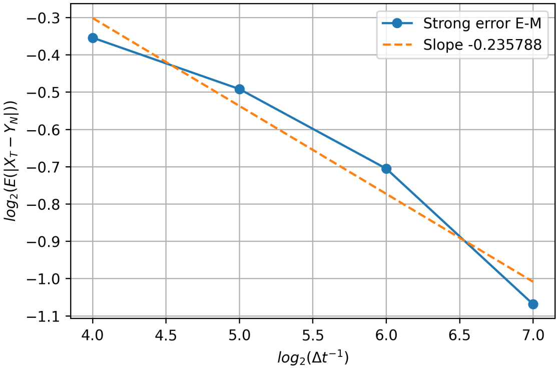


Figure: Error of the Euler scheme for the approximation of the distributional coefficient for real solution with 2^9 time steps and approximations with 2^4 to 2^7 time steps, $b \in H^{-0.05}_{2,q_0}$ and $q_0 > 4$.

References I

- Karatzas, Ioannis and Steven E. Shreve (1991). *Brownian motion and stochastic calculus.* Second edition. Graduate texts in mathematics 113. New York ; Springer-Verlag.
- Kloeden, Peter E. and Eckhard Platen (1999). *Numerical solution of stochastic differential equations.* Corr. 3rd print. Applications of mathematics 23. Berlin ; New York: Springer. 636 pp.
- Triebel, Hans (2010). *Bases in function spaces, sampling, discrepancy, numerical integration.* EMS tracts in mathematics 11. Zürich: European Mathematical Society. 296 pp.

References II

De Angelis, Tiziano, Maximilien Germain, and Elena Issoglio (Oct. 22, 2020). "A Numerical Scheme for Stochastic Differential Equations with Distributional Drift". In: *arXiv:1906.11026 [cs, math]*.