Transfer report

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Contents

1 Overview

...LET THE OVERVIEW TO THE END... Numerical schemes for Stochastic Differential Equations (SDEs) and Stochastic Partial Differential Equations (SPDEs) have been widely studied, and even for SDEs and SPDEs with low regularity coefficients. However

The works from De Angelis et al. [2], and Flandoli et al. [3] have established the framework for this project. A one dimensional SDE is considered:

$$\begin{cases} dX_t = b(t, X_t)dt + dW_t, \quad t \in [0, T], \\ X_0 = x_0, \end{cases} \tag{1}$$

where W is a Brownian motion, and $b(t,x)$ is a distribution taking values in a fractional Sobolev space of negative order, namely $H^{-\beta_0}_{\tilde{q}_0,q_0}$.

This type of equations immediately introduce a challenge because the coefficient b can not be evaluated pointwise and it is necessary to give a meaning to the term $\int_0^t b(s, X_s)ds$. This problem is solved in [3], and then in [2] an algorithm for the one dimensional version of the problem is described. The algorithm proposed has two steps for it to produce the numerical solutions:

1. First is performed a process of regularisation of the coefficient b in (1). Since b it is a distribution and cannot be computed pointwise. They use Haar systems, because those are unconditional bases in the space in which the coefficient b exists, producing a sequence $(b^N)_{N\geq 1}$, which is then submitted to a *randomisation* procedure applying the heat kernel given by
2. And finally applying the Euler-Maruyama scheme for the modified coefficient $P_{\eta_n} b^N$.

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2 Literature review

2.1 Background material on SDEs

Let

$$b(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{R}, \tag{2}$$
$$\sigma(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{R}, \tag{3}$$

be Borel measurable functions and $W = \{W_t; 0 < t < \infty\}$ be a n-dimensional Brownian motion. For $T > 0$ consider the following equation:

for notation we use (4)

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, & t \in [0,T] \\ X_0 = \boxed{x_0} \end{cases} \tag{4}$$

use (5)

for $x_0 \in \mathbb{R}$.

unify notation

Definition 1 (Strong solution) *[5, Definition 5.2.1] Let $(\mathscr{F}_t)$ be a filtration, we call a strong solution of equation (4) (5) ~~to~~ a $\mathscr{F}_t$-adapted, continuous, $\mathbb{R}$-valued stochastic process $(X_t)_{t\in(0,\infty)}$ such that* (5)

$$X_t - \boxed{X_0} = \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{5}$$

holds almost surely for every $t \geq 0$.

Whenever we are talking about a SDE, the expression in (4) is mostly used to shorten notation and make it more familiar, ~~moreover~~ it is always intended to represent the expression (5).

The following theorem states sufficient conditions for the existence and uniqueness for a solution of the SDE:

Theorem 1 *[5, p. 289] Suppose that there exists a constant $K \geq 0$ such that for all $x, y \in \mathbb{R}$ and $t \geq 0$,*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|, \tag{6}$$
$$|b(t,x)| + |\sigma(t,x)| \leq K(1 + |x|). \tag{7}$$

Then there exist a continuous, $\mathscr{F}_t$-adapted process $X = (X_t)_{t\in[0,\infty)}$ which is a strong solution to equation (4). This solution is unique up to indistinguishability, i.e: if $\tilde{X}$ is also a strong solution then $\mathbb{P}(X_t = \tilde{X}_t; \forall 0 \leq t \leq \infty) = 1$.

2.2 Numerical schemes for SDEs

As in the deterministic theory of Differential Equations, most of the equations have no closed form solution, so that it becomes natural to develop numerical schemes to treat such objects which arise in a variety of problems.

In order to construct numerical schemes to solve SDEs, one can use a procedure that is an analogous to the Taylor expansion used in ODEs. Said procedure is called *Itô-Taylor expansion* and is presented in the following theorem whose derivation can be found in [5, pp. 162–164].

To shorten some notation, let us define the operators L^0 and L^1 be

$$L^0 = \frac{\partial}{\partial t} + b\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} \tag{8}$$
$$L^1 = \sigma\frac{\partial}{\partial x}. \tag{9}$$

Theorem 2 *[6, p. 164] Let $t_0 \geq 0$ then the second refinement of the Itô-Taylor expansion for the strong solution of* (4) *is*

$$X_t = X_{t_0} + b(t_0, X_{t_0}) \int_{t_0}^{t} ds + \sigma(t_0, X_{t_0}) \int_{t_0}^{t} dW_s + L^1\sigma(t_0, X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dW_z \, dW_s + R, \tag{10}$$

where the remainder term R is

$$\begin{aligned} R = &\int_{t_0}^{t} \int_{t_0}^{s} L^0 b(X_z) dz \, ds + \int_{t_0}^{t} \int_{t_0}^{s} L^1 b(X_z) dW_z \, ds + \int_{t_0}^{t} \int_{t_0}^{s} L^0 \sigma(X_z) dz \, dW_s \\ &+ \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^0 L^1 b(X_u) du \, dW_z \, dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} L^1 L^1 \sigma(X_u) dW_u \, dW_z \, dW_s \end{aligned} \tag{11}$$

The result above illustrates where two of the most widely used numerical schemes for the solution of SDEs, namely the Euler-Maruyama (E-M) and the Milstein scheme, come from. For the E-M scheme, the first three terms on the right hand side of (10) are considered, and for the Milstein scheme the first four. And then it is necessary to prove that leaving tha remainder R behind allows a scheme to produce an acceptable approximation to the real solution. The derivation of those two schemes in a slightly different fashion can be found in [4, pp. 339–343]. Let us remind that in what follows, for any $f(t, x)$ that is twice differentiable in x, we will write $f' = \frac{\partial f}{\partial x}$ and $f'' = \frac{\partial^2 f}{\partial x^2}$.

Definition 2 *Let $[0, T]$ be a time interval, and $(t_n)_{n=0}^{N}$ be a sequence of elements in $[0, T]$ such that $t_0 < \ldots < t_N$, that is called a discretisation of the interval $[0, T]$. Let also denote $\Delta t_n = t_{n+1} - t_n$ and for a Brownian motion $(W_t)_{t \geq 0}$, $\Delta W_n = W_{t_{n+1}} - W_{t_n}$. We will denote by $Y_n := Y_{t_n}$ a discrete time approximation of X_{t_n} for any n.*

Definition 3 (Euler-Maruyama scheme) *[6, pp. 340–344] Let $(t_n)_{n=0}^{N}$ be a discretisation of the interval $[0, T]$. Then an Euler-Maruyama approximation for the solution of* (4) *at time t_{n+1} is given by*

$$Y_{n+1} = Y_n + b(t_n, Y_n)\Delta t_n + \sigma(t_n, Y_n)\Delta W_n. \tag{12}$$

Definition 4 (Milstein scheme) *[6, pp. 345–351] Let $(t_n)_{n=0}^{N}$ be a discretisation of the interval $[0, T]$. Then a Milstein approximation for the solution of* (4) *at time t_{n+1} is given by*

full stop

$$\hat{Y}_{n+1} = \hat{Y}_n + b(t_n, \hat{Y}_n)\Delta t_n + \sigma(t_n, \hat{Y}_n)\Delta W_n + \frac{1}{2}(\sigma'\sigma)(t_n, \hat{Y}_n)((\Delta W_n)^2 - \Delta t_n). \tag{13}$$

Definition 5 (Order 1.5 strong Taylor scheme) *[6, pp. 351–356] Let $(t_n)_{n=0}^{N}$ be a discretisation of the interval $[0, T]$ Then an order 1.5 strong approximation for the solution of* (4) *at time t_{n+1} is given by*

$$\begin{aligned} \tilde{Y}_{n+1} = &\tilde{Y}_n + b(t_n, \tilde{Y}_n)\Delta t_n + \sigma(t_n, \tilde{Y}_n)\Delta W_n + \frac{1}{2}(\sigma'\sigma)(t_n, \tilde{Y}_n)\left((\Delta W_n)^2 - \Delta t_n\right) \\ &+ (b'\sigma)(t_n, \tilde{Y}_{t_n})\Delta Z_n + \frac{1}{2}\left((bb')(t_n, \tilde{Y}_{t_n}) + \frac{1}{2}(\sigma^2 b'')(t_n, \tilde{Y}_{t_n})\right)(\Delta t_n)^2 \\ &+ \left((b\sigma')(t_n, \tilde{Y}_{t_n}) + \frac{1}{2}(\sigma^2\sigma'')(t_n, \tilde{Y}_{t_n})\right)(\Delta W_n \Delta t_n - \Delta Z_n) \\ &+ \frac{1}{2}\left(\sigma^2\sigma'' + \sigma(\sigma')^2\right)(t_n, \tilde{Y}_{t_n})\left(\frac{1}{3}(\Delta W_n)^2 - \Delta t_n\right)\Delta W_n \end{aligned} \tag{14}$$

where ΔZ_n is given by

$$\Delta Z_n = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} dW_{s_1} \, ds_2 \tag{15}$$

Definition 6 (Order 2.0 weak Taylor scheme) *[6, pp. 464–468] Let $(t_n)_{n=0}^N$ be a discretisation of the interval $[0,T]$ Then an order 2.0 weak approximation for the solution of (4) at time t_{n+1} is given by*

$$\begin{aligned}\tilde{Y}_{n+1} = \tilde{Y}_n &+ b(t_n,\tilde{Y}_n)\Delta t_n + \sigma(t_n,\tilde{Y}_n)\Delta W_n + \frac{1}{2}(\sigma'\sigma)(t_n,\tilde{Y}_n)\left((\Delta W_n)^2 - \Delta t_n\right)\\ &+ (b'\sigma)(t_n,\tilde{Y}_{t_n})\Delta Z_n + \frac{1}{2}\left((bb')(t_n,\tilde{Y}_{t_n}) + \frac{1}{2}(\sigma^2 b'')(t_n,\tilde{Y}_{t_n})\right)(\Delta t_n)^2\\ &+ \left((b\sigma')(t_n,\tilde{Y}_{t_n}) + \frac{1}{2}(\sigma^2\sigma'')(t_n,\tilde{Y}_{t_n})\right)(\Delta W_n\Delta t_n - \Delta Z_n)\end{aligned} \tag{16}$$

where ΔZ_n is given by

$$\Delta Z_n = \int_{\tau_n}^{\tau_{n+1}}\int_{\tau_n}^{s_2} dW_{s_1}\, ds_2 \tag{17}$$

When it comes to numerical computations, one can make use of the fact that for any $0 \leq s < t$ and $Z, \tilde{Z}$ i.i.d. as $\mathscr{N}(0,1)$ we can represent $W_t - W_s$ as $\sqrt{t-s}Z$ and $\int_s^t\int_s^z dW_u\,dz$ as $1/2(t-s)^{3/2}(Z + 1/\sqrt{3}\tilde{Z})$ [6, pp. 351, 352]. Let us denote two random vectors by $Z = (Z_1, ..., Z_N)$ and $\tilde{Z} = (\tilde{Z}_1, ..., \tilde{Z}_N)$ such that for all n, $Z_n, \tilde{Z}_n \sim \mathscr{N}(0,1)$, then the n-th iteration of the schemes above are respectively ~~are~~

I would align the = to help readability

$$Y_{n+1} = Y_n + b(t_n, Y_n)\Delta t_n + \sigma(t_n, Y_n)\sqrt{\Delta t_n}Z_{n+1} \tag{18}$$

$$\hat{Y}_{n+1} = \hat{Y}_n + b(t_n,\hat{Y}_n)\Delta t_n + \sigma(t_n,\hat{Y}_n)\sqrt{\Delta t_n}Z_{n+1} + \frac{1}{2}(\sigma'\sigma)(t_n,\hat{Y}_n)\Delta t_n(Z_{n+1}^2 - 1) \tag{19}$$

$$\begin{aligned}\tilde{Y}_{n+1} = \tilde{Y}_n &+ b(t_n,\tilde{Y}_n)\Delta t_n + \sigma(t_n,\tilde{Y}_n)Z_{n+1}\sqrt{\Delta t_n} + \frac{1}{2}(\sigma'\sigma)(t_n,\tilde{Y}_n)\left(Z_{n+1}^2 - 1\right)\Delta t_n\\ &+ \frac{1}{2}(b'\sigma)(t_n,\tilde{Y}_{t_n})\left(Z_{n+1} + \frac{1}{\sqrt{3}}\tilde{Z}_{n+1}\right)(\Delta t_n)^{3/2}\\ &+ \frac{1}{2}\left((bb')(t_n,\tilde{Y}_{t_n}) + \frac{1}{2}(\sigma^2 b'')(t_n,\tilde{Y}_{t_n})\right)(\Delta t_n)^2\\ &+ \frac{1}{2}\left((b\sigma')(t_n,\tilde{Y}_{t_n}) + \frac{1}{2}(\sigma^2\sigma'')(t_n,\tilde{Y}_{t_n})\right)\left(Z_{n+1}\left(1 - \frac{1}{2}\Delta t_n\right) - \frac{1}{2\sqrt{3}}\tilde{Z}_{n+1}\Delta t_n\right)\sqrt{\Delta t_n}\\ &+ \frac{1}{2}\left(\sigma^2\sigma'' + \sigma(\sigma')^2\right)(t_n,\tilde{Y}_{t_n})\left(\frac{1}{3}Z_{n+1}^3 - Z_{n+1}\right)(\Delta t_n)^{3/2}\end{aligned} \tag{20}$$

$$\begin{aligned}\breve{Y}_{n+1} = \breve{Y}_n &+ b(t_n,\breve{Y}_n)\Delta t_n + \sigma(t_n,\breve{Y}_n)Z_{n+1}\sqrt{\Delta t_n} + \frac{1}{2}(\sigma'\sigma)(t_n,\breve{Y}_n)\left(Z_{n+1}^2 - 1\right)\Delta t_n\\ &+ \frac{1}{2}(b'\sigma)(t_n,\breve{Y}_{t_n})\left(Z_{n+1} + \frac{1}{\sqrt{3}}\breve{Z}_{n+1}\right)(\Delta t_n)^{3/2}\\ &+ \frac{1}{2}\left((bb')(t_n,\breve{Y}_{t_n}) + \frac{1}{2}(\sigma^2 b'')(t_n,\breve{Y}_{t_n})\right)(\Delta t_n)^2\\ &+ \frac{1}{2}\left((b\sigma')(t_n,\breve{Y}_{t_n}) + \frac{1}{2}(\sigma^2\sigma'')(t_n,\breve{Y}_{t_n})\right)\left(Z_{n+1}\left(1 - \frac{1}{2}\Delta t_n\right) - \frac{1}{2\sqrt{3}}\tilde{Z}_{n+1}\Delta t_n\right)\sqrt{\Delta t_n}\end{aligned} \tag{21}$$

Now that reasonable numerical approximations to the solutions of SDEs are given, it is necessary to check whether said approximations will converge to the actual solutions. For this, we will discuss two forms of convergence that are usually of interest for numerical analysis of SDEs: strong and weak convergence.

First, strong convergence will tell us how close the sample paths of the approximations are to the corresponding solutions. Naturally the numerical verification of this mode of convergence requires the simulations of all the paths we want to compare, both from the solution and the approximation.

Definition 7 (Strong convergence) *[8, Section 4] Let (X_t) be a strong solution of equation (4). We call $\gamma \in (0, \infty)$ a strong convergence order of the discrete time approximation (Y_n) if for any T there is $K > 0$ such that for any discretisation $(t_n)_{n=0}^{N}$ of $[0, T]$ we have ~~X~~ there exists a constant $K < \infty$ such that*

that

$$E|X_T - Y_N| \leq K \left(\max_n \Delta t_n \right)^\gamma \tag{22}$$

where (Y_n) is computed over the discretisation (t_n).

not entirely clear.

On the other hand, there are problems in which the condition (22) can be relaxed because we might only be interested in the approximation of a function f applied on the process X_t. Examples of such functions f are polynomials. For example one might be interested in computing moments of the stochastic process that solves the equation considered. For such situations, as mentioned by [8], in which the nature of the problem allows it, it is possible to save computation time by defining another type of convergence in which we can compute only the approximations of the numerical solution and then compare it to a known value that has to be computed only once. For this purpose we define weak convergence.

Definition 8 (Weak convergence) *[8, Section 4] Let (X_t) be a strong solution of equation (4). We call $\gamma \in (0, \infty)$ a strong convergence order of the discrete time approximation (Y_n) if for any T and f there is $M_f > 0$ such that for any discretisation $(t_n)_{n=0}^{N}$ of $[0, T]$ we have*

$$|E(f(X_T)) - E(f(Y_N))| \leq M_f \left(\max_n \Delta t_n \right)^\beta \tag{23}$$

where (Y_n) is computed over the discretisation (t_n).

For the present work we will be using equidistant time increments, of length $h \in \{2^{-3}, ..., 2^{-7}\}$ therefore the conditions for convergence stated above can be seen simply as

$$E|X_T - Y_N| \leq K h^\gamma \qquad \text{and} \qquad |E(f(X_T)) - E(f(Y_N))| \leq M_f h^\beta. \tag{24}$$

Once ~~that~~ we have schemes as above to approximate numerically the solution of an ~~a~~ SDE it is necessary to prove that said schemes will provide an appropriate output, meaning that one must know if the schemes ~~themselves~~ converge, in either of the two senses that have been discussed earlier, to the solution as the discretisation is refined. ~~and a~~ Additionally which is the order of convergence. For that objective, Glasserman [4] and Kloeden and Platen [6] state the conditions on the coefficients and the proofs are found in the latter.

It is desirable to know also

Theorem 3 (Strong convergence for E-M scheme) *[6, Theorem 10.2.2] Let the conditions from Theorem 1 hold, additionally assume that*

$$|b(s_0, x) - b(t_0, x)| + |\sigma(s_0, x) - \sigma(t_0, x)| \leq K(1 + |x|)\sqrt{|t_0 - s_0|} \tag{25}$$

Then the Euler-Maruyama scheme has a strong order of convergence $\gamma = 1/2$.

For a proof of the previous theorem see [6, pp. 342–344]. Also, in the following, the condition on (25) will be called *extended linear growth.*

Theorem 4 (Strong convergence for Milstein scheme) *[6, Theorem 10.3.5] Let the conditions from Theorem 1 and Theorem 3 hold for the coefficients b and σ. Recall that the operators L^0 and L^1 have been defined in (8) and (9), and let us consider the following notation*

$$z(t, x) = b(t, x) - \frac{1}{2}(\sigma\sigma')(t, x) \tag{26}$$

Suppose that the Lipschitz

$$|z(t,x_0) - z(t,y_0)| + |\sigma(t,x_0) - \sigma(t,y_0)| + \left|L^1\sigma(t,x_0) - L^1\sigma(t,y_0)\right| \leq K_1|x_0 - y_0|, \tag{27}$$

linear growth

$$|z(t,x_0)| + |L^1 z(t,x_0)| + |\sigma(t,x_0)| + |L^1\sigma(t,x_0)| + |L^1 L^1\sigma(t,x_0)| \leq K_2(1+|x|), \tag{28}$$

and extended linear growth conditions

$$|z(s_0,x) - z(t_0,x)| + |\sigma(s_0,x) - \sigma(t_0,x)| \left|L^1\sigma(s_0,x) - L^1\sigma(t_0,x)\right| \leq K_3(1-|x|)|s_0 - t_0|^{1/2}, \tag{29}$$

hold for all $x, y, x_0, y_0 \in \mathbb{R}$, $t, s_0, t_0 \in [0,T]$, and constants $K_1, K_2, K_3 > 0$ then the Milstein scheme has a strong rate of convergence $\gamma = 1$.

In general one can have a more robust strong convergence criterion statement that accounts for the strong approximations of all orders, namely $\gamma = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$.

for some order γ where ??

Theorem 5 (Strong convergence of all orders) *Let $Y_n = \{Y_n, t \in [0,T]\}$ be the order $\gamma = 1/2, 1, 3/2, 2, \ldots$, corresponding to an equidistant time discretisation $(t_n)_{n=0}^N$. Suppose that for any $j \geq 0$, Lipschitz continuity*

statement is not clear to me

$$|(L^1)^j\sigma(t,x) - (L^1)^j\sigma(t,y)| \leq K_1|x-y|, \tag{30}$$

and linear growth

$$|(L^1)^j\sigma(t,x)| \leq K_2(1+|x|), \tag{31}$$

hold for all, $t \in [0,T]$ and $x, y \in \mathbb{R}$. Then

same

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|X_t - Y_n|^2\right) \leq K_3(1+|X_0|^2)(\Delta t_n)^{2\gamma} + K_4|X_0 - Y_n(0)|^2 \tag{32}$$

For $K_1, K_2, K_3, K_4 < \infty$ constants independent of the time grid.

Theorem 6 (Weak convergence for all orders) *[6, Theorem 14.5.1]*
Let Y^δ be a weak Taylor approximation of order $\beta = 1, 2, \ldots$ corresponding to the time discretisation $(t_n)_{n=0}^N$. Suppose that the coefficients of the equation are $b, \sigma \in C^{2(\beta+1)}(\mathbb{R})$ and Lipschitz continuous, and that for all $j \geq 0$, linear growth , i.e.

$$|(L^1)^j\sigma(t,x)| \leq K(1+|x|), \tag{33}$$

holds, for $K < \infty$. Then for each $g \in C^{2(\beta+1)}(\mathbb{R})$ there exists a constant C_g, which does not depend on the time discretisation, such that

$$|\mathbb{E}(g(X_T) - g(Y_T^n))| \leq C_g(\Delta t_n)^\beta. \tag{34}$$

Note that for strong convergence we present results for Euler scheme and Milstein scheme separately plus the general statement for all orders, but for weak convergence only the general statement is mentioned. We do this because for weak convergence, Euler and Milstein schemes are both of convergence order 1 and in order to achieve weak convergence of a greater order, it is necessary to include more terms than those the Milstein scheme has. Just as seen above, the following weak convergence order is 2. Said scheme includes two more terms than the Milstein scheme, but one less than the 1.5 strong scheme, so one should be aware that not all strong schemes will have a counterpart in the weak convergence side.

To verify the accuracy of the numerical schemes mentioned earlier, a n SDE with a known explicit solution is used. In [6, pp. 117–126] a comprehensive list of such equations is given, and for the present work we will the following particular case of geometric *Brownian motion* (gBm)

$$\begin{cases} dX_t = \frac{1}{2} X_t dt + X_t dW_t, \\ X_0 = x_0 \end{cases} \tag{35}$$

whose solution is

$$X_t = x_0 \exp W_t \tag{36}$$

Its ~~and which~~ mean is given by

$$E(X_t) = X_0 \exp\left(\frac{1}{2} t\right). \tag{37}$$

For this SDE with the notation from Definition 3 and Definition 4, the Euler-Maruyama, Milstein and 1.5 strong approximations at time t_n are, respectively:

align

$$Y_{n+1} = Y_n + \frac{1}{2} Y_n \Delta t_n + Y_n \sqrt{\Delta t} Z_{n+1} \tag{38}$$

$$\hat{Y}_{n+1} = \hat{Y}_n + \frac{1}{2} \hat{Y}_n \Delta t_n + \hat{Y}_n \sqrt{\Delta t_n} Z_{n+1} + \frac{1}{2} \Delta t_n (Z_{n+1}^2 - 1) \tag{39}$$

$$\begin{aligned} \tilde{Y}_{n+1} = {} & \tilde{Y}_n + \frac{1}{2} \tilde{Y}_n \Delta t_n + \tilde{Y}_n \sqrt{\Delta t_n} Z_{n+1} + \frac{1}{2} \Delta t_n (Z_{n+1}^2 - 1) \\ & + \frac{1}{4} \tilde{Y}_n \left(Z_{n+1} + \frac{1}{\sqrt{3}} \tilde{Z}_{n+1} \right) (\Delta t_n)^{3/2} + \frac{1}{8} \tilde{Y}_n (\Delta t_n)^2 \\ & + \frac{1}{2} \tilde{Y}_n \left(\frac{1}{2} Z_n + \frac{1}{2\sqrt{3}} \tilde{Z}_{n+1} \right) (\Delta t_n)^{3/2} + \frac{1}{2} \left(\frac{1}{3} Z_{n+1}^2 - 1 \right) Z_{n+1} (\Delta t_n)^{3/2} \end{aligned} \tag{40}$$

There are different methods to confirm convergence of numerical methods, in particular we decided to use Monte Carlo methods, i.e: produce a large number or simulations and then computing a deterministic operation over them.

- For strong convergence to be confirmed numerically is necessary to compare a considerable number of paths of both the exact solution of the equation and the approximation with the E-M scheme with respect to the same Brownian motion, and using the law of large numbers to approximate the mean. Then if M sample paths of each are computed and $m \in \{1, ..., M\}$ then the approximation of the strong error will be then given by

$$E|X_T - Y_N| \approx \frac{1}{M} \sum_{m=1}^{M} |X_T^m - Y_N^m|, \tag{41}$$

  which by the law of large numbers converge s to $E|X_T - Y_N|$ as n M $\to \infty$.

- Similarly, ~~Meanwhile~~ for weak convergence is numerically ~~this convergence mode is~~ verified, for the same amount M of sample paths, using the following approximation for the weak error

$$|E(f(X_T)) - E(f(Y_N))| \approx \left| E(f(X_T)) - \frac{1}{M} \sum_{m=1}^{M} f(Y_N^m) \right|, \tag{42}$$

  where we already know $E(f(X_T))$. from (37).

We ran batches of approximations for the gBm and its approximations with the numerical schemes mentioned above, with step sizes $\Delta t \in \{2^{-3}, ..., 2^{-7}\}$, 100000 sample paths on each batch. In Fig. 1 we can see how increasing the amount of points in the approximation to the solution reduces the error. Making a numeric demonstration of the order of convergence for each scheme, which in the graphs is represented by the slope of the dashed orange line..

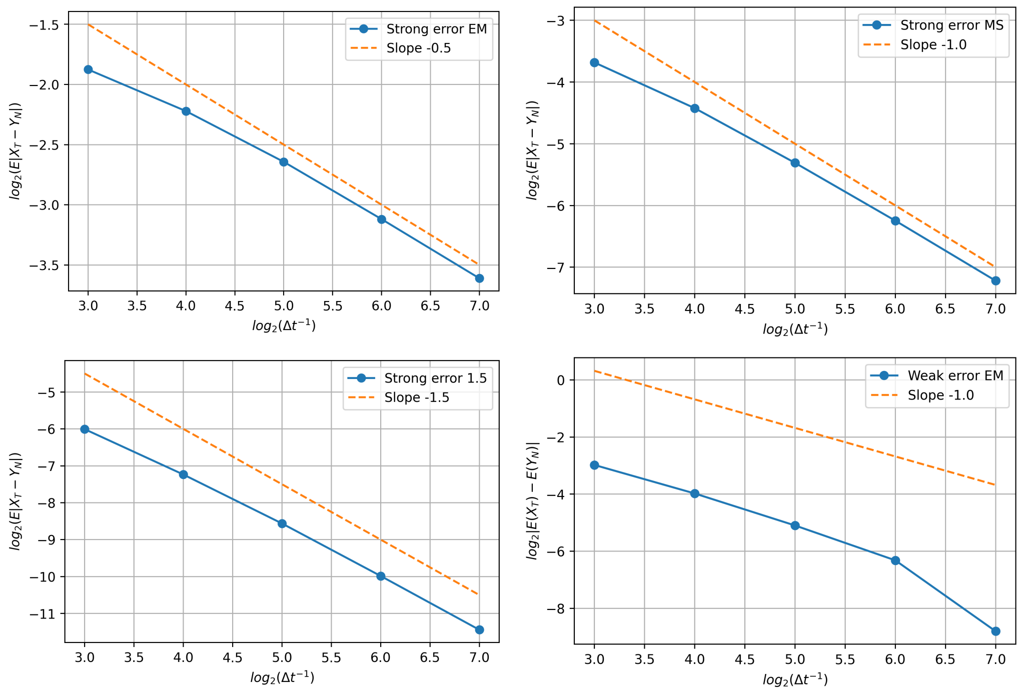


Figure 1: Log-log graphs of the error of approximation in the strong and weak sense for Euler-Maruyama and Milstein schemes. The dashed orange line indicates order of convergence corresponding to each scheme.

2.3 Sobolev spaces and rough coefficients

The approximation methods for SDEs mentioned above can lead us to good results if the coefficients behave in a certain way, as it was stated in Theorems 3, 4 and 6, however for the project in hand we are interested in some particular kind of drift coefficients.

First let us introduce some definitions and notation.

Definition 9 (Fourier transform) *Let $f : \mathbb{R} \to \mathbb{C}$ be in $L^1(\mathbb{R})$. Then the Fourier transform of f is defined as*

$$\hat{f}(\xi) = \mathscr{F}[f](\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x) dx. \tag{43}$$

Definition 10 (Inverse Fourier transform) *Let $f : \mathbb{R} \to \mathbb{R}$ be in $L^1(\mathbb{R})$. Then the inverse Fourier transform of $\hat{f}$ is defined as*

$$f(\xi) = \mathscr{F}^{-1}[\hat{f}](\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \hat{f}(x) dx. \tag{44}$$

Definition 11 (Schwartz functions) *We denote the space of Schwartz functions as*

$$\mathscr{S}(\mathbb{R}) = \{f \in C^\infty : \sup_{x\in\mathbb{R}} |x^\alpha f^{(\beta)}(x)| < \infty, \forall \alpha, \beta \in \mathbb{N}\}, \tag{45}$$

where $f^{(\beta)} = \frac{d^\beta}{dx^\beta} f$.

As the definition points out, Schwartz functions are those whose derivatives are rapidly decreasing. This means that any derivative decays faster than any power of x, and thus the product of those is bounded.

Definition 12 (Tempered distributions) *A tempered distribution on* $\mathbb{R}$ *is a linear mapping* $\phi \to (f, \phi)$ *from* $\mathcal{S}(\mathbb{R})$ *to* $\mathbb{C}$. *The set of all tempered distributions is denoted by* $\mathcal{S}'(\mathbb{R})$.

Definition 13 (Weak derivative) *Suppose* $u, v \in L^1(\mathbb{R})$, *and* $\alpha \in \mathbb{N}$, *we say that* $v = u^{(\alpha)}$ *is the* α*-th weak derivative of* u, *provided*

$$\int_{\mathbb{R}} u\phi^{(\alpha)} dx = (-1)^{\alpha} \int_{\mathbb{R}} v\phi dx \tag{46}$$

for all test functions $\phi \in C_c^{\infty}(\mathbb{R})$.

Definition 14 *[9, Remark 1.2] Let* $f \in \mathcal{S}'(\mathbb{R})$ *or* $f \in \mathcal{S}(\mathbb{R})$, *and let* $\xi, s \in \mathbb{R}$. *The map*

$$I^s : f \mapsto \mathscr{F}^{-1}\left(\left(1+|\xi|^2\right)^{\frac{s}{2}} \mathscr{F} f\right) \tag{47}$$

defines a bijection on $\mathcal{S}(\mathbb{R})$ *and* $\mathcal{S}'(\mathbb{R})$.

if $s>0$
$-s<0$
I^{-s} is an "integral" of order s

if $s<0$
$-s>0$
I^{-s} is a "derivative" of order $-s \in \mathbb{R}$

The definition of fractional Sobolev spaces can be found in different formats by different authors, we have chosen the one in [9, Remark 1.2], and it is presented below.

Definition 15 (Fractional Sobolev spaces) *Let* $s \in \mathbb{R}$, $0 < p < \infty$, *and* I^s *as in* (47) *we call the space*

$$H_r^s := I^{-s} L^{\overset{r}{\cancel{p}}}(\mathbb{R}) \tag{48}$$

a fractional Sobolev space. Further, this is a Banach space if it is equipped with the norm

$$\|f\|_{H_r^s} := \|I^{-s} f\|_{L^r(\mathbb{R})} \tag{49}$$

$I^{-s} L^p =: H^s_p$ is better in terms of regularity in particular they have "derivatives" of order $s \in \mathbb{R}$

$I^{-s} =: H^s_p$ have to be "integrated" s times to become L^p functions

We must add that in the case that $s = k \in \mathbb{N}$ we have the special case of classical Sobolev spaces, denoted usually by $W^{k,r}(\mathbb{R})$.

ADD →

Definition 16 (Sobolev spaces) *Let* $1 < \overset{r}{\cancel{p}} < \infty$ *and* $k \in \mathbb{N}_0$. *Then the space* $W^{k,r}(\mathbb{R})$ *is the collection of all* $f \in \underset{L^r(\mathbb{R})}{\underline{\mathcal{S}'(\mathbb{R})}}$ *such that*

$$\|f\|_{W^{k,r}} := \left(\sum_{\alpha=0}^{k} \|f^{(\alpha)}\|_{L^r}^r\right)^{\frac{1}{r}} < \infty, \tag{50}$$

where $f^{(\alpha)}$ *denotes the weak derivative of order* $\alpha \in \mathbb{N}$.

You should check: $15 \Rightarrow 16$: if $\left((1+|\xi|^2)^{k/2} \mathscr{F} f\right)^{\vee} \in L^r$ then $f^{(\alpha)} \in L^r$, $\forall \alpha \leq k$.
$16 \Rightarrow 15$.

The reason Definition 15 and Definition 16 are equivalent is that under the Fourier transform, the derivative is a multiplier operator, that is

Triebel for reference.

$$\mathscr{F}\left[\frac{d^k}{dx^k} f(x)\right](\xi) = (i\xi)^k [\mathscr{F} f](\xi). \tag{51}$$

So, for an example, let us set $p = 2$ and $k = 1$. For a function $f : \mathbb{R} \to \mathbb{R}$ to belong to the Sobolev space $W^{k,p}$ as in Definition 16 we need that $f, \frac{df}{dx} \in L^2$. And in terms of the Fourier transform we need that $\mathscr{F} f, i\xi \mathscr{F} f \in L^2$. Those last two conditions can be combined as

$$(1 + |\xi|)\mathscr{F} f \in L^2. \tag{52}$$

$i\xi \cdot \mathscr{F} f \in L^2 \iff \int_{\mathbb{R}} |i\xi \mathscr{F} f(\xi)|^2 d\xi < \infty \longleftrightarrow \int_{\mathbb{R}} |\xi \mathscr{F} f(\xi)|^2 d\xi < \infty \iff \int_{\mathbb{R}} |\xi|^2 |\mathscr{F} f(\xi)|^2 d\xi < \infty$

$\Updownarrow$

$|\xi| \cdot \mathscr{F} f \in L^2(\mathbb{R})$

But we can note at least two problems in that condition: first, $(1 + |\xi|)$ is not smooth, and second, it does not account for the *"derivatives of order* $s < k$*"*. But the the term $(1 + |\xi|^2)^{1/2}$ solves both issues. So for a general $k = s \in \mathbb{R}$, Definition 15 is better since the Fourier transform is well defined in Sobolev spaces.

$\hat f, i\xi \hat f, \dots, (i\xi)^k \hat f$ $\qquad$ $(1+|\xi|^2)^{k/2}\hat f \in L^p$ $\qquad$ Def 15

Def 16: $\int |1+i\xi+\dots+(i\xi)^k|^p |\hat f(\xi)|^p d\xi < \infty$ $\Leftrightarrow$ $\int |(1+|\xi|^2)^{k/2}\hat f(\xi)|^p d\xi < \infty$

You need to show that $|1+i\xi+\dots+(i\xi)^k|^p \le \text{const}\, |(1+|\xi|^2)^{k/2}|^p \le \text{cons}\, |1+i\xi+\dots+(i\xi)^k|^p$

$(a+b)^2 \le 2a^2+2b^2 \le 2(a+b)^2$

Sobolev spaces provide us with an appropriate framework to find solutions to Partial Differential Equations (PDEs), relaxing the conditions of differentiability. One simple example of this phenomenon can be seen in the equation

$$\begin{cases} -u''(x) + u(x) = f(x) \text{ on } [a,b], \\ u(a) = u(b) = 0. \end{cases} \tag{53}$$

In the classical sense a solution to this equation is a function in $C^2([a,b])$ that satisfies the equation. However, if we take a function $\phi \in C^1([a,b])$ to multiply Eq. (53), and then integrate by parts we obtain

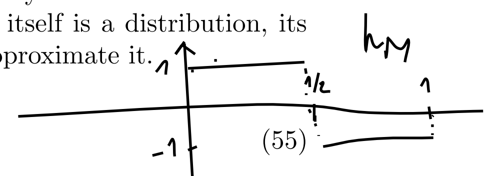
weak derivative of u' $\qquad$ then $u' \in H^1_p$ for some p

$$\boxed{\int_a^b u'\phi' dx + \int_a^b u\phi dx = \int_a^b f\phi dx,} \tag{54}$$

we can now see that the solution u to the equation only needs to belong to $\boxed{C^1}([a,b])$.

why do we care about
- K non integer
- K negative

We are interested in the study of a coefficient b which belongs to an appropriate Sobolev spaces of negative order, acting as the drift for a SDE such as (5). Particularly we want to know how to compute such a coefficient numerically, moreover as the coefficient itself is a distribution, its pointwise evaluation does not make sense and we need some way to approximate it.



Definition 17 *Let $h_M : \mathbb{R} \to \mathbb{R}$*

$$h_M := \mathbb{1}_{[0,\frac{1}{2})} - \mathbb{1}_{[\frac{1}{2},1)} \tag{55}$$

be the mother Haar wavelet. Then the Haar wavelet system in I is given by

$$\{h_0, h_{j,m} : j \in \mathbb{N} \cup \{-1\}, m \in \mathbb{Z}\} \tag{56}$$

where

$$h_0(x) := \mathbb{1}_I(x) \tag{57}$$

and

$$h_{j,m}(x) := h_M(2^j x - m). \tag{58}$$

for $j \in \mathbb{N}$ and $m \in \mathbb{Z}$. Alternatively the Haar wavelets can be expressed as

$$h_{j,m}(x) = \mathbb{1}_{\left[\frac{m}{2^j}, \frac{m+1/2}{2^j}\right)} - \mathbb{1}_{\left[\frac{m+1/2}{2^j}, \frac{m+1}{2^j}\right)}. \tag{59}$$

Haar systems are bases for the spaces in which the drift coefficient lives, thus the following theorem tells us how to represent elements in said space as infinite sums of Haar functions.

Theorem 7 *[9, Theorem 2.9] Let $0 \le r < \infty$, $-\frac{1}{2} < s < \frac{1}{r}$, and let $f \in \mathcal{S}'(I)$. Then $f \in H^s_r(I)$ if and only if*

$$f = \sum_{j=-1}^{\infty} \sum_{m \in Z} \mu_{j,m} 2^{-j(s-\frac{1}{r})} h_{j,m}, \tag{60}$$

with unconditional convergence in $\mathcal{S}'(I)$ and locally in any space $H^\sigma_r(I)$ with $\sigma < s$. The representation is unique with the coefficients given by

$$\mu_{j,m} := 2^{j(s-\frac{1}{r}+1)} \int_{\mathbb{R}} f(x) h_{j,m}(x) dx, \tag{61}$$

where the integral above is in the sense of dual pairing. Moreover the system

$$\left\{2^{-j(s-\frac{1}{r})} h_{j,m} : j \in \mathbb{N} \cup \{-1\}, k \in \mathbb{Z}, m = 0, \dots, 2^j - 1\right\} \tag{62}$$

is an unconditional normalised basis of H^s_r.

There is one more difficulty with that theorem, and is that the integral that defines the coefficients is in the sense of dual pairing, which is not desirable in order to implement a numerical scheme. To overcome this problem we can see that for $g \in H^s_r$ with $2 \le r < \infty$ and $1/2 < s < 1+1/r$. There are systems that serve as basis with coefficients that are more easily computable and are also related to the coefficients of Haar basis, this systems are Faber bases. Just as in [2] for simplicity we just recall here the case in which $I = (0,1)$.

Definition 18 *The Faber system in $(0,1)$ is given by*

$$\{v_0, v_1, v_{j,m} : j \in \mathbb{N}, m = 0, ..., 2^j - 1\} \tag{63}$$

where

$$v_0(x) := \begin{cases} 1-x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad v_1(x) := \begin{cases} x & 0 \le x \le 1 \\ 0 & \text{otherwise,} \end{cases} \tag{64}$$

and the hat functions are defined by

$$v_{j,m} := 2^{j+1} \int_0^x h_{j,m}(y)dy = \begin{cases} 2^{j+1}(x - 2^{-j}m) & x \in \left[\frac{m}{2^j}, \frac{m+1/2}{2^j}\right) \\ 2^{j+1}(2^{-j}(m-1) - x) & x \in \left[\frac{m}{2^j}, \frac{m+1/2}{2^j}\right) \\ 0 & \text{otherwise.} \end{cases} \tag{65}$$

This system allows to represent elements in fractional Sobolev spaces, but in a different space in which the drift b lives, as we can see in the following theorem.

Theorem 8 *[9, Theorem 3.1] Let $g \in H^s_r(0,1)$ for $2 \le r < \infty$ and $1/2 < s < 1 + 1/r$. Then we have the unique Faber representation for g*

$$g = \hat{\mu}_0 v_0 + \hat{\mu}_1 v_1 + \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} \hat{\mu}_{j,m} v_{j,m} \tag{66}$$

with unconditional convergence in $C(0,1)$ and in $H^\sigma_r(0,1)$ with $\sigma < s$. Where the coefficients $\hat{\mu}$ are given by

$$\begin{cases} \hat{\mu}_{j,m} & = -\frac{1}{2}(\Delta^2_{2^{-j-1}} g)(2^{-j}m) \\ \hat{\mu}_0 & = g(0) \\ \hat{\mu}_1 & = g(1), \end{cases} \tag{67}$$

and where $(\Delta^2_h g)(x) := g(x+2h) - 2g(x+h) + g(x)$.

After setting the above theorem it remains to see how those coefficients, that are clearly easier to compute, are related to the coefficients for the Haar representation. As pointed out in [2], if $s < 0$, the space $H^s_r(\mathbb{R})$ contains distributions. We can then define $H^s_{p,q}(\mathbb{R}) := H^s_p(\mathbb{R}) \cap H^s_q(\mathbb{R})$, and by interpolation we have that if $f \in H^s_{p,q}(\mathbb{R})$ then $f \in H^s_r(\mathbb{R})$ for all $p \wedge q < r < p \vee q$. We can see, as is discussed in [2, Remark A.10] and proved in [9, Theorem 3.1] that we have $g' \in H^{s-1}_r(I)$ with $2 \le r < \infty$, $1/2 < s < 1 + 1/r$ which can be written as follows

$$g' = (\hat{\mu}_1 - \hat{\mu}_0)h_0 + \sum_{j=0}^{+\infty} \sum_{m=0}^{2^j-1} 2^{j+1} \hat{\mu}_{j,m} h_{j,m}. \tag{68}$$

the result above is a representation of elements in the space where the distributional coefficient lives as seen in Theorem 7 setting the coefficients to be for the series expansion as

$$\mu_0 = \hat{\mu}_1 - \hat{\mu}_0 \qquad \text{and} \qquad \mu_{j,m} = 2^{j+1} \hat{\mu}_{j,m}. \tag{69}$$

2.4 Numerical schemes for SDEs with rough drift

As it was stated above, the foundations of this research project are in the article by De Angelis et al [2], in which it is devised and algorithm to work with drifts that belong to an appropriate fractional Sobolev space. Below the main results of the paper are mentioned, and for this the following notation and assumptions are in order.

We used previously the notation $C^n(\mathbb{R})$ in the usual way, to denote the space of real-valued functions with continuous n-th derivative. When we need to be more specific we will write $C^n([0,T];B)$, where B is some Banach space. Now if $\alpha \in (0,1)$ the space $C^\alpha()$

Consider the equation (1), the aim is to first make an approximation of the coefficient b with a better behaved coefficient b^N, for this purpose let the following assumptions for b and b^N hold:

Assumption 1 *Let $\beta_0 \in \left(0, \frac{1}{4}\right)$ and $q_0 \in \left(4, \frac{1}{\beta_0}\right)$, fix $\tilde{q}_0 := (1-\beta_0)^{-1}$. Then for some $\kappa \in \left(\frac{1}{2}, 1\right)$ take $b \in C^{\frac{1}{2}}\left([0,T];\, H^{-\beta_0}_{\tilde{q}_0,q_0}\right)$.*

Assumption 2 *Let $(b^N)_{N\geq 1} \subset \mathscr{C}^{\frac{1}{2}}\left([0,T];\, H^{0}_{\tilde{q}_0,q_0}\right)$ be such that*

$$\lim_{N\to\infty} b^N = b \qquad in \qquad \mathscr{C}^{\frac{1}{2}}\left([0,T];\, H^{-\beta_0}_{\tilde{q}_0,q_0}\right). \tag{70}$$

The assumptions above allow us to select the elements of the sequence b^N living in a space of coefficients that is not as rough as the space in which the real coefficient lives, because as is pointed out in [2] if $f \in H^s_{p,q}$ then $f \in H^s_r$ for all $p \wedge q < r < p \vee q$, in particular we could have $H^0_2 = L^2$. Additionally with those spaces we can take advantage of the inclusion property $H^s_r \subset H^t_u$ for all $1 < r \leq u < \infty$ and $-\infty < t \leq s < \infty$ such that $s - 1/r \geq t - 1/u$, for which that assumption of the limit of b^N makes sense as the sequence will belong also to the rough space in which the coefficient lives originally.

$$P_{\eta_N}\mathbb{1}_{[x_1,x_2)} = \exp(-\eta_N)\left(\Phi\left(\frac{x_2-x}{\sqrt{\eta_N}}\right) - \Phi\left(\frac{x_1-x}{\eta_N}\right)\right) \tag{71}$$

Theorem 9 *Let Assumption 1 and 2 hold. Take any (β, q) such that $\beta \in (\beta_0, \frac{1}{2})$ and $q_0 \geq q > \tilde{q} \geq \tilde{q}_0$, where $\tilde{q} := (1-\beta)^{-1}$. Then for any $\frac{1}{2} < \gamma < \gamma_0$ there is a constant $C_\gamma > 0$ such that*

$$\sup_{0\leq t\leq T} \mathbb{E}|X^N_t - X_t| \leq C_\gamma \|P_{\eta_N} b^N - b\|^{2\gamma-1}_{\infty, H^{-\beta}_q} \tag{72}$$

The previous theorem allows us to see the rate of convergence of the approximation X^N to the solution X by finding a bound for the error, and also that said bound is related to the rate of convergence of the approximation for the distributional drift itself.

Since the convergence of the approximation to the solution is dependant on the convergence of the approximation to the distributional drift, it is necessary to see that the approximation of the drift also has a sensible bound. Then is worth to notice as in [2, Remark 3.2] that the semigroup has the following properties:

$$\begin{aligned} \|P_t f\|_{\infty, H^{-s}_q} &\leq \|f\|_{\infty, H^{-s}_q} \\ \|P_t f - f\|_{\infty, H^{-s-\epsilon}_q} &\leq c t^{\epsilon/2} \|f\|_{\infty, H^{-s}_q}, \end{aligned} \tag{73}$$

and hence we have the inequality

$$\|P_{\eta_N} b^N - b\|_{\infty, H^{-\beta}_q} \leq \|b^N - b\|_{\infty, H^{-\beta}_q} + c\eta_N^{\frac{\beta-\beta_0}{2}} \|b\|_{\infty, H^{-\beta_0}_{q_0}}. \tag{74}$$

And then as $\eta_N \to \infty$, if $\beta > \beta_0$ the second term in the inequality goes to zero and then it is necessary to find a bound for the first term.

Note that when we apply the semigroup P_{η_N} to the Haar functions in the Haar series approximation b^N the following form for the numerical computations is obtained:

$$\begin{aligned} P_{\eta_N} h_{j,m} &= P_{\eta_N} \mathbb{1}_{\left[\frac{m}{2^j}, \frac{m+1/2}{2^j}\right)} - P_{\eta_N} \mathbb{1}_{\left[\frac{m+1/2}{2^j}, \frac{m+1}{2^j}\right)} \\ &= \exp(-\eta_N) \left(-\Phi\left(\frac{\frac{m+1}{2^j} - x}{\sqrt{\eta_N}} \right) + 2\Phi\left(\frac{\frac{m+1/2}{2^j} - x}{\sqrt{\eta_N}} \right) - \Phi\left(\frac{\frac{m}{2^j} - x}{\sqrt{\eta_N}} \right) \right), \end{aligned} \tag{75}$$

where N is fixed, η_N is a constant that depends on N, $j = 1, 2, ..., N$ and $m = 1, 2, ..., 2^j$.

Theorem 10 *Let assumption Assumption 1 hold and let the sequence $(b^N)_{N\geq 1}$ be defined as*

$$b^N(t) := \sum_{j=-1}^{N} \sum_{j=-2^j}^{2^j} \mu_{j,m}(t) 2^{-j(-\beta - \frac{1}{q})} h_{j,m}, \tag{76}$$

which by construction $b^N(t) \in H^0_{\tilde{q}_0, q_0} \subset H^{-\beta_0}_{\tilde{q}_0, q_0}$. Then $(b^N)_{N\geq 1}$ satisfies assumption Assumption 2 and for any $\beta \in (\beta_o, \frac{1}{2})$ it holds that

$$\|b^N - b\|_{\infty, H_2^{-\beta}} \leq c 2^{-(N+1)(\beta - \beta_0)} \|b\|_{\infty, H_2^{-\beta_0}}. \tag{77}$$

For the computation of the coefficients $\mu_{j,m}$ above we use the relation between the coefficients of Haar and Faber expansions in the discussion of Theorem 8 in particular with the function g being a fractional Brownian motion (fBm), which is defined as follows.

Definition 19 (Fractional Brownian motion) *A fractional Brownian motion is a Gaussian process $\{B^H(x), x \in \mathbb{R}\}$ with covariance given by*

$$\mathbb{E}[B^H(x) B^H(y)] = \frac{1}{2}(x^{2H} + y^{2H} + |x - y|^{2H}). \tag{78}$$

$B^H \in H^{-\beta_0}_{q_0}(\mathbb{R})$ and as $supp(B^H) \subset I$, $B^H \in H^{-\beta_0}_{q_0}(I)$.

Hence the coefficients have the form

$$\begin{cases} \mu_0 &= B^H(1) - B^H(0) \\ \mu_{j,m} &= -2^j \left(B^H\left(\frac{m+1}{2^j}\right) - 2B^H\left(\frac{m+1/2}{2^j}\right) + B^H\left(\frac{m}{2^j}\right) \right) \end{cases} \tag{79}$$

The fBm is particularly useful because is easy to simulate numerically.

The core of the present problem is the approximation of the distributional coefficient in a way that is manageable numerically. Note that it is first found an approximation to the coefficient b, which we call b^N and with that coefficient we have a new SDE

$$\begin{cases} dX_t^N = b^N(t, X_t^N)dt + \sigma(t, X_t^N)dW_t, \quad t \in [0, T] \\ X_0^N = x_0. \end{cases} \tag{80}$$

We can find a numerical solution $X_t^{N,m}$ for the system above, and the following theorem tells us that the numerical solution coincides with the solution for the new system.

Theorem 11 *Let Assumption 1 hold and let $b^N \in \mathscr{C}^{\frac{1}{2}}\left([0, T]; H^0_{\tilde{q}_0, q_0}\right)$ for fixed N. Then as $m \to \infty$*

$$\sup_{0 \leq t \leq T} \mathbb{E} \left| X_t^{N,m} - X_t^N \right| \leq C_2(N) m^{-1} + C_3(N) m^{-\frac{1}{2}} \tag{81}$$

where

$$C_2 := c\|P_{\eta_N} b^N\|_{\infty,L^\infty} \left(1 + \|\nabla(P_{\eta_N} b^N)\|_{\infty,L^\infty}\right), \tag{82}$$

$$C_3 := c' \left(\|\nabla(P_{\eta_N} b^N)\|_{\infty,L^\infty} + \left[P_{\eta_N} b^N\right]_{\frac{1}{2},L^\infty}\right), \tag{83}$$

and $c, c' > 0$ are constants independent of (N, m).

The second step of the numerical scheme, i.e. approximate the solution after we approximate the drift, is addressed in the Theorem below. This happens because the numerical solution $X_t^{N,m}$ coincides with the approximated solution X_t^N and the later, with the solution X_t. Being the following result essentially a combination of Theorem 9 and 11.

Theorem 12 *Let Assumption 1 and also b^N defined as in (76) so that Assumption 2 holds too, and let $\Theta_* := \frac{1}{2}\left[\frac{3}{4} - \beta_0\left(\gamma_0 - \frac{1}{2}\right)\right]^{-1}$. Then as $m \to \infty$, let $\eta_N = m^{-\Theta_*}$ and $N = 2\Theta_* \log_2 m$ it holds that*

$$\sup_{0 \le t \le T} \mathbb{E}\left|X_t^{N,m} - X_t\right| \le c_\epsilon \left(m^{-\Theta_*\left(\frac{1}{2}-\beta_0\right)\left(\gamma_0 - \frac{1}{2}\right)-\epsilon}\right) \tag{84}$$

where $\epsilon > 0$ is arbitrarily small and $c_\epsilon > 0$ is a constant depending on ϵ.

2.5 BSDEs and FBSDEs

Given $\sigma_s \in L^2(\mathbb{R})$ we know that $M_t := \int_0^t \sigma_s dW_s$ is a square integrable martingale. However, the converse is not always true. Let us consider $\xi \in \mathscr{L}^2(\mathbb{R})$, and $\{\mathscr{F}_t\}_{t\ge 0}$ be the filtration generated by the Wiener process W. Let us define the martingale $Y_t := \mathbb{E}[\xi|\mathscr{F}]$. Then we can see that there exists a unique $Z \in \mathscr{L}^2(\mathbb{R})$ such that

$$Y_t = \xi - \int_t^T Z_s dW_s. \tag{85}$$

That problem is addressed by the theorem below.

Theorem 13 (Martingale representation theorem) *[10, Theorem 2.5.2] Let M be a martingale such that $\mathbb{E}[|M_T|^2] < \infty$, then there exists a unique $\sigma \in L^2(\mathbb{R})$ such that*

$$M_t = M_0 + \int_0^t \sigma_s dW_s \tag{86}$$

The martingale representation theorem shows that the problem posed before has indeed a solution, nevertheless that is just a simple case of the objects that we can define and study. More generally we can define a BSDE as follows:

Definition 20 (Backward SDE) *Let $f : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ be measurable in all variables, $f(t,x,y)$ uniformly Lipschitz continuous in (x,y), and denote $f^0 = f(t,0,0) \in L^2(\mathbb{R})$, finally let $\xi \in L^2(\mathbb{R})$. Then the following is called a Backward SDE:*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \tag{87}$$

for $0 \le t \le T$, $\mathbb{P}$-a.s. where $Y, Z \in L^2(\mathbb{R})$. f is called the generator and ξ the terminal condition of the BSDE.

The conditions established in the definition above are enough to have a unique solution of a BSDE.

Theorem 14 *[10, Theorem 4.3.1] Under the assumptions of Definition 20, the BSDE has a unique solution $(Y, Z) \in L^2(\mathbb{R} \times L^2(\mathbb{R})$.*

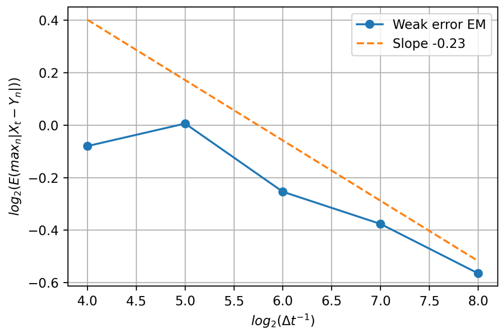


Figure 2: Log-log graphs of the estimation error of the Euler-Maruyama scheme with $N = 9$, $m \in \{2^4, ..., 2^8\}$.

One can go one step further and take the terminal condition ξ to be defined by a SDE. This generalisation is called Forward-Backward SDE (FBSDE).

Definition 21 (Forward-Backward SDE) *Let $f : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ satisfy analogous conditions as in Definition 20 and X defined by* (5)*, the following system is called a Forward-Backward SDE:*

$$\begin{cases} X_t &= X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \\ Y_t &= g(X_T) - \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \end{cases} \tag{88}$$

for $0 \le t \le T$, $\mathbb{P}$-a.s.

Theorem 15 *[7, Theorem 5.1] Under the assumptions of Definition 21 there exists a T_0, such that for any $T \in [0, T_0]$ a FBSDE* (88) *admits a unique solution (X, Y, Z).*

3 Contributions

At the moment the same algorithm in the article by De Angelis et al. [2] has been implemented with satisfactory results, confirming the rate of convergence established in said article. Figure 2 shows the error for the approximation **...elaborate in the estimation of error as in the article...**

It is clear that Milstein scheme is not useful for the problem we have, since we require of the derivative of the diffusion coefficient, but since said coefficient is equals one, the derivative vanishes. That is why in Definition 5 we introduced a different numerical scheme that is useful for this particular problem, some of the terms will vanish again as they are multiplying the derivative of the diffusion term, and also we need the derivative and second derivative of the drift coefficient. Fortunately we have a integral representation of the drift

The 1.5 strong scheme for this problem, i.e: b a distribution and $\sigma = 1$ reads

$$\begin{aligned} \tilde{Y}_{n+1} = \tilde{Y}_n &+ b(t_n, \tilde{Y}_n)\Delta t_n + Z_{n+1}\sqrt{\Delta t_n} \\ &+ \frac{1}{2}(b')(t_n, \tilde{Y}_{t_n}) \left(Z_{n+1} + \frac{1}{\sqrt{3}}\tilde{Z}_{n+1} \right) (\Delta t_n)^{3/2} \\ &+ \frac{1}{2} \left((bb')(t_n, \tilde{Y}_{t_n}) + \frac{1}{2}(b'')(t_n, \tilde{Y}_{t_n}) \right) (\Delta t_n)^2 \\ &+ \frac{1}{4} \left(Z_{n+1} - \frac{1}{\sqrt{3}}\tilde{Z}_{n+1} \right) (\Delta t_n)^{3/2} \end{aligned} \tag{89}$$

As we require the derivative of the distributional coefficient, let us take advantage of the approximation to the distributional coefficient that has been proposed.

$$b^N(x) = \mu_0 h_0(x) + \sum_{j=0}^{N} \sum_{m=1}^{2^j-1} \mu_{j,m} h_{j,m}(x). \tag{90}$$

Applying the semigroup to the object above gives

$$\begin{aligned}
P_{\eta_N} b^N(x) &= \mu_0 P_{\eta_N} h_0(x) + \sum_{j=0}^{N} \sum_{m=1}^{2^j-1} \mu_{j,m} P_{\eta_N} h_{j,m}(x) \\
&= \mu_0 \frac{1}{\sqrt{2\pi\eta_N}} \int_{-\infty}^{\infty} h_0(y) e^{-\frac{(y-x)^2}{2\eta_N}} dy + \sum_{j=0}^{N} \sum_{m=1}^{2^j-1} \mu_{j,m} \frac{1}{\sqrt{2\pi\eta_N}} \int_{-\infty}^{\infty} h_{j,m}(y) e^{-\frac{(y-x)^2}{2\eta_N}} dy \\
&= \mu_0 \frac{1}{\sqrt{2\pi\eta_N}} \int_0^1 e^{-\frac{(y-x)^2}{2\eta_N}} dy \\
&+ \sum_{j=0}^{N} \sum_{m=1}^{2^j-1} \mu_{j,m} \frac{1}{\sqrt{2\pi\eta_N}} \left(\int_{\frac{m}{2^j}}^{\frac{m+1/2}{2^j}} e^{-\frac{(y-x)^2}{2\eta_N}} dy - \int_{\frac{m+1/2}{2^j}}^{\frac{m+1}{2^j}} e^{-\frac{(y-x)^2}{2\eta_N}} dy \right).
\end{aligned} \tag{91}$$

The derivatives can get inside the integral sign thanks to the Leibniz rule, which then allows us to compute the semigroup over the derivatives of the Haar functions multiplied by the heat kernel as it is shown in the following.

$$\begin{aligned}
\frac{\partial}{\partial x}(P_{\eta_N} b^N)(x) &= \mu_0 \frac{1}{\eta_N \sqrt{2\pi\eta_N}} \int_0^1 (y-x) e^{-\frac{(y-x)^2}{2\eta_N}} dy \\
&+ \sum_{j=0}^{N} \sum_{m=1}^{2^j-1} \mu_{j,m} \frac{1}{\eta_N \sqrt{2\pi\eta_N}} \left(\int_{\frac{m}{2^j}}^{\frac{m+1/2}{2^j}} (y-x) e^{-\frac{(y-x)^2}{2\eta_N}} dy \right. \\
&\qquad \left. - \int_{\frac{m+1/2}{2^j}}^{\frac{m+1}{2^j}} (y-x) e^{-\frac{(y-x)^2}{2\eta_N}} dy \right) \\
&= \mu_0 \frac{1}{\sqrt{2\pi\eta_N}} \left(\exp\left(-\frac{(1-x)^2}{2\eta_N}\right) - \exp\left(-\frac{x^2}{2\eta_N}\right) \right) \\
&+ \sum_{j=0}^{N} \sum_{m=1}^{2^j-1} \mu_{j,m} \frac{1}{\sqrt{2\pi\eta_N}} \left(\exp\left(-\frac{\left(\frac{m}{2^j}-x\right)^2}{2\eta_N}\right) \right. \\
&\qquad -2\exp\left(-\frac{\left(\frac{m+1/2}{2^j}-x\right)^2}{2\eta_N}\right) \\
&\qquad \left. +\exp\left(-\frac{\left(\frac{m+1}{2^j}-x\right)^2}{2\eta_N}\right) \right)
\end{aligned} \tag{92}$$

Since b^N is the finite sum of Haar functions (times some coefficients) we can rewrite $\left(\frac{\partial}{\partial x} P_\eta b^N\right)(x)$ as the sum of terms of the form $\left(\frac{\partial}{\partial x} P_\eta \mathbb{1}_{[a,b]}\right)(x)$. Afterwards you will plug in values for a, b which are the interval bounds for

Haar functions partition (e.g. $[\frac{m}{2^j}, \frac{m+1/2}{2^j}]$)

Same for

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2}(P_{\eta_N} b^N)(x) &= \mu_0 \frac{1}{\sqrt{2\pi\eta_N}} \int_0^1 \frac{1}{\eta_N}\left(\frac{(y-x)^2}{\eta_N} - 1\right) e^{-\frac{(y-x)^2}{2\eta_N}} dy \\
&+ \sum_{j=0}^{N}\sum_{m=1}^{2^j-1} \mu_{j,m} \frac{1}{\sqrt{2\pi\eta_N}} \left(\int_{\frac{m}{2^j}}^{\frac{m+1/2}{2^j}} \frac{1}{\eta_N}\left(\frac{(y-x)^2}{\eta_N} - 1\right) e^{-\frac{(y-x)^2}{2\eta_N}} dy \right. \\
&\qquad \left. - \int_{\frac{m+1/2}{2^j}}^{\frac{m+1}{2^j}} \frac{1}{\eta_N}\left(\frac{(y-x)^2}{\eta_N} - 1\right) e^{-\frac{(y-x)^2}{2\eta_N}} dy \right) \\
&= -\mu_0 \frac{1}{\sqrt{2\pi\eta_N}} \left((1-x)\exp\left(-\frac{(1-x)^2}{2\eta_N}\right) - x\exp\left(-\frac{x^2}{2\eta_N}\right) \right. \\
&\qquad \left. + \Phi_{\mathcal{N}(0,1)}\left(\frac{1-x}{\sqrt{\eta_N}}\right) - \Phi_{\mathcal{N}(0,1)}\left(\frac{-x}{\sqrt{\eta_N}}\right) \right) \\
&+ \sum_{j=0}^{N}\sum_{m=1}^{2^j-1} \mu_{j,m} \frac{1}{\sqrt{2\pi\eta_N}} \left(\left(\frac{m+1}{2^j} - x\right) \exp\left(-\frac{\left(\frac{m+1}{2^j} - x\right)^2}{2\eta_N}\right) \right. \\
&\qquad - \left(\frac{m+1/2}{2^j} - x\right) \exp\left(-\frac{\left(\frac{m+1/2}{2^j} - x\right)^2}{2\eta_N}\right) \\
&\qquad \left. + \left(\frac{m}{2^j} - x\right) \exp\left(-\frac{\left(\frac{m}{2^j} - x\right)^2}{2\eta_N}\right) + (\exp(\eta_N) - 1) P_{\eta_N} h_{j,m} \right)
\end{aligned} \tag{93}
$$

CHANGE INTO $x-y$!!! ...

4 Research plan

Following the foundations that have been established so far the following steps are to develop numerical schemes to solve FBSDEs with a rough drift in the SDE associated. The equations that we are going to attack are of the form

$$
\begin{cases}
X_t &= X_0 + \int_0^t b(s, X_s) ds + \int_0^t dW_s, \\
Y_t &= g(X_T) - \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.
\end{cases} \tag{94}
$$

We can see that the forward element of the equation above is exactly the same equation with distributional for which the numerical schemes have already been implemented. With the first equation solved we can proceed to solve the BSDE remaining by plugging into f and in the terminal condition. Since the distributional coefficient only appears in the forward equation, we can use known numerical methods for BSDEs such as the ones discussed in [1, Section 3]. It is possible to study first the special case for $f = 0$ which leaves us with an equation with the same form as (85) and then the component Y will be exactly the conditional expectation of the terminal condition ξ.

Once the numerical methods for such equations are studied, the following task is to prove the convergence of the scheme. Let us remember that the convergence for numerical schemes for the forward equation is dependant on the convergence for the approximation of the distributional coefficient as proved in [2], then we expect to see a dependency in a similar fashion for the convergence rate of the solution for the BSDE.

Further, we can have a system in which the BSDE contains a rough driver, the system would be

$$
\begin{cases}
X_t &= X_0 + \int_0^t dW_s, \\
Y_t, &= g(X_T) + \int_t^T b(s, X_s) Z_s ds + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.
\end{cases} \tag{95}
$$

This is a completely new problem, and we have to come up with schemes to solve these rough BSDEs. The first approach could be to use the techniques from [2] to deal with the coefficient b and then use known methods for BSDEs. Numerically we can always try to adapt the schemes that are already known, however the theoretical results of convergence might be more challenging, and that should be the next big step for the project.

Another problem that is of interest for the present project is the case of a SDE with a rough coefficient just as the one for which the numerical methods have been implemented, but with the difference of if being driven by a different process than a regular Brownian motion. Namely, we can have an equation driven by a fractional Brownian motion, that is different to the one used to approximate the distributional coefficient. Such equation reads

$$
X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t dW_s^H \tag{96}
$$

5 Training record

- **Measure and Integration Theory**. Collegio Carlo Alberto. Turin, Italy *(Online)*. (Pass 65%).
- **MATH5734M: Advanced Stochastic Calculus and Applications to Finance**. University of Leeds. Spring Semester, 20 credits. (Pass 72%).
- Reading of [5, Sections 3.2, 3.3, 5.2] as complement for MATH5734M.
- **Seminar Series on Probability and Financial Mathematics**. University of Leeds. Leeds, UK *(Online)*. 2020/2021.
- **Stochastic Processes and Their Friends**. University of Leeds. Leeds, UK *(Online)*. 18-19 March 2021.
- **Conference Beyond the Boundaries**. University of Leeds. Leeds, UK *(Online)*. 4-7 May 2021.
- **British Early Career Mathematicians' Colloquium**. University of Birmingham. Birmingham, UK *(Online)*. 15-16 July 2021.
- **6th Berlin Workshop for Young Researchers on Mathematical Finance**. Humboldt-Universität zu Berlin. Berlin, Germany *(Online)*. 23-25 August 2021.
- **Bath Mathematical Symposium on PDE and Randomness: Summer School**. University of Bath. Bath, UK *(Online)*. 1-3 September 2021.

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