Transfer report

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Contents

1 Overview

project

Numerical schemes for Stochastic Differential Equations (SDEs) and Stochastic Partial Differential Equations (SPDEs) have been widely studied, and even for SDEs and SPDEs with low regularity coefficients. However

The works from De Angelis et al. [1], Flandoli et al. [2] have established the framework for this project. A one dimensional SDE is considered:

$$\begin{cases} dX_t = b(t, X_t)dt + dW_t, & t \in [0, T] \\ X_0 = x. \end{cases} \tag{1}$$

the indentation

Where W_t is a Brownian motion, and $b(t, X_t)$ is a distribution taking values in a fractional Sobolev space of negative order, namely $H^{-\beta_0}_{\tilde{q}_0, q_0}$.

This type of equations immediately introduce a challenge because is not possible to evaluate pointwise the coefficient b and it is necessary to give a meaning to the term $\int_0^t b(s, X_s)ds$, problem that is solved in [2].

In [1] an algorithm is described in order to find the solutions to such equations, this is a two step procedure:

1. Regularisation of the coefficient b in (1), since it is a distribution and cannot be computed pointwise for the sake of a numerical method.

2. Application of the well known Euler-Maruyama scheme with the regularised coefficient.

This chain of actions leads to have two sources of error for the algorithm, nevertheless it is also shown in the same paper that **...mention the rates of convergence...**

It is worth of mention that, as is mentioned in [2], the coefficient can belong to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ and a number of results will hold. **...double check this statement...** However the belonging of said object to an appropiate Sobolev space

2 Literature review

2.1 Background material on SDEs

Let b and σ be Borel measurable functions defined as following:

gap

$$
\begin{aligned}
b(t,x) &: [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d \\
\sigma(t,x) &: [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times n}
\end{aligned}
$$

remove

multiple
lim

let also $\{W_t; 0 < t < \infty\}$ be a n-dimensional Brownian motion and ~~$\{X_t; 0 \leq t < \infty\}$ a d-dimensional~~ stochastic process such that the following equation holds:

consider the SDE

$$
\begin{cases}
dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t, \quad t \in [0,T] \\
X_0 = x
\end{cases}
\tag{2}
$$

Given also an $\mathbb{R}^d$ valued random vector $\xi =: X_0$ that will work as an initial contidion for the equation, the SDE has an integral version which incidentaly gives the definition of a strong solution for the SDE:

DEF

$$
X_t - X_0 = \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s. \tag{3}
$$

2.1.1 Existence and uniqueness

...it is natural to study

~~Noticing that such equations exist~~, the questions of existence and uniqueness of solutions are the key step to be able to study SDEs.

strong
sol'ns

~~In the previous section, the way to construct a solution was mentioned, namely integrate the equation.~~ For this solution to exist and further be unique, the coefficients must satisfy the Lipschitz condition and also have linear growth, i.e.

suf cond for existence

norma def

$$\|b(t,x) - b(t,y)\|_d + \|\sigma(t,x) - \sigma(t,y)\|_{d\times n} \leq K\|x - y\|_d \tag{4}$$

$$\|b(t,x)\|_d + \|\sigma(t,x)\|_{d\times n} \leq K(1 + \|x\|_d) \tag{5}$$

remove

2.1.2 Singular SDEs

2.2 Numerical schemes for SDEs

As in the deterministic theory of Differential Equations, most of the equations have no closed form solution (a formula), so that it becomes natural to develop numerical schemes to treat such objects which arise in a number of problems in different areas.

Just as in the case of deterministic differential equations, one would like to find some kind of Taylor expansion for SDEs and then construct numerical schemes from there.

As it is stated in any book of numerical analysis, such as [3] and [4] the Taylor expansion for an ODE would be:

Definition 1 (Taylor expansion) *Let f and X_t be real valued functions, and further $f \in \mathscr{C}^{r+1}$. One can have the following initial value problem (IVP):*

$$\begin{cases} \frac{d}{dt}X_t = a(X_t), \quad t \in [t_0, T], \quad 0 \leq t_0 \leq T, \\ X_{t_0} = x \end{cases} \tag{6}$$

deterministic maybe drop it

Then by the chain rule

$$\frac{d}{dt}f(X_t) = a(X_t)\frac{\partial}{\partial x}f(X_t). \tag{7}$$

Define an operator

$$L = a\frac{\partial}{\partial x}. \tag{8}$$

The Taylor expansion for the solution is

$$X_t = X_{t_0} + \sum_{\ell=1}^{r} \frac{(t-t_0)^\ell}{\ell!} L^\ell X_{t_0} + \int_{t_0}^{t} \cdots \int_{t_0}^{s_2} L^{r+1} X_{s_1} ds_1 \ldots ds_{r+1} \tag{9}$$

For a Taylor expansion of a SDE, also called Itô-Taylor expansions, take the integral form of the generic SDE (3), and follow a similar procedure applying Itô's formula instead of the chain rule. Then the following definition states the simplest Itô-Taylor expansion:

consider X_t given by

THEOREM reference

Definition 2 (Itô-Taylor expansion) *For equation (3) setting $\boxed{\cancel{s} = t_0}$ and a function $f : \mathbb{R} \mapsto \mathbb{R}$. Let the operators L^0 and L^1 be*

for any t_0

$$L^0 = b\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial^2}{\partial x^2} \tag{10}$$

$$L^1 = b\frac{\partial}{\partial x}. \tag{11}$$

Then the Itô-Taylor expansion is

$$\begin{aligned} X_t &= X_{t_0} + a(X_{t_0})(t-t_0) + b(X_{t_0})(W_t - W_{t_0}) + R \\ R &= \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(X_z)dzds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(X_z)dW_z ds \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z)dz dW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^1 b(X_z)dW_z dW_s \end{aligned} \tag{12}$$

Now that a general form of the approximation is given it is only necessary to identify a way to use that approximation in a numerical method, and that is how the Euler-Maruyama method is specified:

Definition 3 (Euler-Maruyama approximation) *For a time interval* $[0, T]$ *let* $\{\tau_n\}_{n=0}^{N}$ *be a discretization of the interval such that* $\Delta = T/N = \tau_{n+1} - \tau_n$ *for all* n*. Also let* $Y_0 = X_0$ *and* $\Delta W_n = W_{\tau_n+1} - W_{\tau_n}$ *Then an Euler-Maruyama approximation for the solution of* (2) *is*

t_n

$$Y_{n+1} = Y_n + b(Y_n)\Delta t + b(Y_n)\Delta W_n \tag{13}$$

Y_n approx X_t

Note that the Euler-Maruyama approximation is essentially the Taylor expansion but dropping the remainder R of equation (12). rewrite

milstein scheme

2.2.1 Modes of convergence

Now that reasonable numerical approximations to the solutions of SDEs are given, it is necessary to check whether said solutions will converge to the actual solutions.

For this, two forms of convergence exist: strong and weak. And here those are stated as in [5].

what is N and t?

Definition 4 (Strong convergence) *Let* X *be a solution of equation* (2) *and* Y *a discrete time approximation of* X*. We say that* Y *converges to* X *in the strong sense with order* $\gamma \in (0, \infty)$ *if there exists a constant* $K < \infty$ *such that*

$$E\|X_t - Y_N\| \geq K\Delta^\gamma \tag{14}$$

t_N $\leq$

for all step sizes $\Delta \in (0, 1)$.

~~However~~ there are problems in which this condition can be relaxed because is likely ~~that what is needed~~ to approximate is the expectation of a function f applied on the process X_t. Examples of such functions f that might be of interest are the moments of the process, for which another type of convergence exists and is stated as following.

polynomials

Definition 5 (Weak convergence) *Let X be a solution of equation* (2) *and Y a discrete time approximation of X. We say that Y converges to X in the weak sense with order $\beta \in (0, \infty)$ if, for any polynomial f, there exists a constant $M_f < \infty$ such that*

$$\|E(f(X_{t_N})) - E(f(Y_N))\| \leq M_f \Delta^\beta \tag{15}$$

for all step sizes $\Delta \in (0, 1)$.

t and N ??

2.2.2 Theoretical results

convergence of E-M scheme (???)
Milstein

2.2.3 Numerical examples

processes and plots

3 ~~Mathematics~~ undertaken to date

CONTRIBUTIONS
Implementation

4 Research plan

5 Training record

- **Measure and Integration Theory**. Collegio Carlo Alberto. Turin, Italy. (Pass 65%).
- **MATH5734M: Advanced Stochastic Calculus and Applications to Finance**. University of Leeds. Spring Semester, 20 credits. (Awaiting for marks).
- Reading of [6] as complement for MATH5734M.
- **Seminar Series on Probability and Financial Mathematics**. University of Leeds. 2020/2021.
- **Stochastic Processes and Their Friends**. University of Leeds. 18-19 March.
- **Conference Beyond the Boundaries**. University of Leeds. 4-7 May.
- **British Early Career Mathematicians' Colloquium**. University of Birmingham. Birmingham, UK. 15-16 Jul.
- **6th Berlin Workshop for Young Researchers on Mathematical Finance**. Humboldt-Universität zu Berlin. Berlin, Germany (online). 23-25 Ago.
- **Bath Mathematical Symposium on "PDE and Randomness": Summer School**. University of Bath. Bath, UK (online). 1-3 Sep.

References

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[3] A. Iserles, *A first course in the numerical analysis of differential equations*, ser. Cambridge texts in applied mathematics. Cambridge ; New York: Cambridge University Press, 1996, 378 pp., ISBN: 978-0-521-55376-6 978-0-521-55655-2.

[4] P. E. Kloeden and E. Platen, *Numerical solution of stochastic differential equations*, Corr. 3rd print, ser. Applications of mathematics 23. Berlin ; New York: Springer, 1999, 636 pp., ISBN: 978-3-540-54062-5.

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