Transfer report

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1 Overview

Numerical schemes for Stochastic Differential Equations (SDEs) and Stochastic Partial Differential Equations (SPDEs) have been widely studied, and even for SDEs and SPDEs with low regularity coefficients. However

~~The works from~~ De Angelis et al. [1]~~,~~ and Flandoli et al. [2] have established the framework for this project. A one dimensional SDE is considered:

$$\begin{cases} dX_t = b(t, X_t)dt + dW_t, & t \in [0, T], \\ X_0 = x_0, \end{cases} \tag{1}$$

where W is a Brownian motion, and $b(t, x)$ is a distribution taking values in a fractional Sobolev space of negative order, namely $H^{-\beta_0}_{\tilde{q}_0, q_0}$.

This type of equations immediately introduce a challenge because the coefficient b can not be evaluated pointwise and it is necessary to give a meaning to the term $\int_0^t b(s, X_s)ds$. This problem is solved in [2], and then in [1] an algorithm for the one dimensional version of the problem is described. The algorithm proposed has two steps for it to produce the numerical solutions:

1. First is performed a process of regularisation of the coefficient b in (1). Since b it is a distribution and cannot be computed pointwise. They use Haar systems, because those are unconditional bases in the space in which the coefficient b exists, producing a sequence

$(b^N)_{N\geq 1}$, which is then submitted to a *randomisation* procedure applying the heat kernel given by

$$P_{\eta_N}\mathbb{1}_{[x_1,x_2)} = \exp(-\eta_N)\left(\Phi\left(\frac{x_2-x}{\sqrt{\eta_N}}\right) - \Phi\left(\frac{x_1-x}{\eta_N}\right)\right) \tag{2}$$

2. And finally applying the Euler-Maruyama scheme for the modified coefficient $P_{\eta_n}b^N$.

2 Literature review

2.1 Background material on SDEs

Let ~~b and σ~~ be Borel measurable functions ~~defined as following~~:

$$b(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d, \tag{3}$$
$$\sigma(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^{d\times n}, \tag{4}$$

let ~~also~~ $W = \{W_t; 0 < t < \infty\}$ be a n-dimensional Brownian motion, ~~then~~ for $T > 0$ and ~~$t \in [0,T]$~~ consider the following equation:

$$\begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t, & t \in [0,T] \\ X_0 = x_0, \end{cases} \tag{5}$$

where $x_0 \in \mathbb{R}^d$.

~~The solution of that equation is defined as in [4].~~ $\mathcal{F}_t$-adapted continuous $(X_t)_{t\in[0,\infty)}$

Definition 1 [4, Def. ...] *A strong solution of equation* (5) *is a* $\mathbb{R}^d$*-valued stochastic process* $\{X_t; 0 \leq t < \infty\}$ *such that*

$$X_t - X_0 = \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s, \tag{6}$$

holds almost surely for every $t \geq 0$.

(it would be good to write it in terms of 1D quantities or formulate everything in 1D.) → $d=n=1$

~~2.1.1 Existence and uniqueness~~

~~For the equation posed and its respective solution as previously defined, it is natural to study the existence and uniqueness of the solutions.~~ The following theorem~~, for which there is a proof in [4, p. 289], illustrates~~ gives/states sufficient conditions for the existence and uniqueness of such a solution:

Theorem 1 [4, Thm.] *Suppose that* ~~$\mathbb{E}\|X_0\|^2 < \infty$ *and that*~~ *there exists a constant* $K > 0$ *such that for all* $x, y \in \mathbb{R}$ *and* $t \geq 0$, ~~*and let the coefficients* $b(t,x)$ *and* $\sigma(t,x)$ *from equation* (5) *satisfy the Lipschitz and linear growth conditions*~~

$$\|b(t,x) - b(t,y)\|_{\mathbb{R}^d} + \|\sigma(t,x) - \sigma(t,y)\|_{\mathbb{R}^{d\times n}} \leq K\|x-y\|_{\mathbb{R}^d}, \tag{7}$$
$$\|b(t,x)\|_{\mathbb{R}^d} + \|\sigma(t,x)\|_{\mathbb{R}^{d\times n}} \leq K(1 + \|x\|_{\mathbb{R}^d}), \tag{8}$$

then there exist a continuous, $(\mathcal{F}_t)$- *adapted process* $X = \{X_t, \cancel{\mathcal{F}_t}; 0 \leq t < \infty\}$ *which is a strong solution to equation* (5)*. This solution is unique up to indistinguishability, i.e: if* $\tilde{X}$ *is also a* strong *solution then* $\mathbb{P}(X_t = \tilde{X}_t;\ \forall 0 \leq t \leq \infty) = 1$.

2.2 Numerical schemes for SDEs

As in the deterministic theory of Differential Equations, most of the equations have no closed form solution ~~(a formula)~~, so that it becomes natural to develop numerical schemes to treat such objects which arise in a variety of problems.

In order to find numerical schemes to solve SDEs, one can use a procedure that is an analogous to the Taylor expansion used ~~on~~ in ODEs. Said procedure is called *Itô-Taylor expansion* and is presented in the following theorem whose derivation can be found in [4, pp. 162–164].

not needed after the change.

~~**Remark 1** *In the following results we focus in coefficients $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ instead of the general setting proposed in equations (3) and (4) this is because the equations for which the numerical scheme has been developed and tested are 1-dimensional. Following as well the presentation of the results in [5].*~~

Theorem 2 *Let $t_0 \geq 0$ and define the operators L^0 and L^1 be*

\begin{align}

$$L^0 = b\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} \tag{9}$$

$$L^1 = \sigma\frac{\partial}{\partial x} \tag{10}$$

before $b = b(t,x)$ $\sigma = \sigma(t,x)$

then the second refinement of the Itô-Taylor expansion for the strong solution of (5) is

$$X_t = X_{t_0} + \cancel{a}^{\,b}(X_{t_0})\int_{t_0}^{t} ds + \cancel{b}^{\,\sigma}(X_{t_0})\int_{t_0}^{t} dW_s + L^1\cancel{b}_{\,\sigma}(X_{t_0})\int_{t_0}^{t}\int_{t_0}^{s} dW_z\, dW_s + R, \tag{11}$$

where the remainder term R is

$$\begin{aligned} R = &\int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z)dz\, ds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 b(X_z)dW_z\, ds + \int_{t_0}^{t}\int_{t_0}^{s} L^0\cancel{b}^{\,\sigma}(X_z)dz\, dW_s \\ &+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^0L^1\cancel{b}^{\,\sigma}(X_u)du\, dW_z\, dW_s + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^1L^1\cancel{b}^{\,\sigma}(X_u)dW_u\, dW_z\, dW_s. \end{aligned} \tag{12}$$

The ~~previous~~ above result illustrates where two of the most widely used numerical schemes for the solution of SDEs, namely the Euler-Maruyama (E-M) and the Milstein scheme, come from. For the E-M scheme, the first three terms on the right hand side of (11) are considered, and for the Milstein scheme the first four. Being the remainder left out, and then it is necessary to prove that leaving that remainder behind allow the schemes to produce an acceptable approximation to the real solution. The derivation of those two schemes in a slightly different fashion can be found in [3, pp. 339–343]. ~~and below they are presented in form of definitions~~.

...maybe I should add all the notation before the definitions in some kind of assumption and refer to it instead of having these repetitive definitions...

Define: discretisation Δt_n and mesh (?) $\max_n \Delta t_n$

Definition 2 (Euler-Maruyama scheme) *For a time interval $[0, T]$ let $\{t_n\}_{n=0}^{N}$ be a discretisation of the interval such that $t_{n+1} > t_n$ for all n and let $\Delta t_n = t_{n+1} - t_n$. Also let X_t be a solution of the SDE and $Y_n := Y_{t_n}$ denote an approximation of X_{t_n}. Denote $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ and define $Y_0 = X_0$. Then an Euler-Maruyama approximation for the solution of (5) at time t_{n+1} is*

$$Y_{n+1} = Y_n + b(\overset{t_{n+1}}{Y_n})\Delta t_n + \sigma(\overset{t_n}{Y_n})\Delta W_n. \tag{13}$$

not here

Furthermore, for numerical computations, let a random vector $Z = (Z_1, ..., Z_{\cancel{m}\,N})$ such that for all n $Z_n \sim \mathcal{N}(0, 1)$, then the n-th iteration of the scheme is

$$Y_{n+1} = Y_n + b(Y_n)\Delta t_n + \sigma(Y_n)\sqrt{\Delta t_n}Z_{n+1} \tag{14}$$

$\Delta W_n = \sqrt{\Delta t_n}\, Z_{n+1}$

Definition 3 (Milstein scheme) *For a time interval $[0, T]$ let $\{t_n\}_{n=0}^{N}$ be a discretisation of the interval such that $t_{n+1} > t_n$ for all n and let $\Delta t_n = t_{n+1} - t_n$. Also let X_t be a solution of the SDE and $Y_n := Y_{t_n}$ denote an approximation of X_{t_n}. Denote $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ and define $Y_0 = X_0$. Then a Milstein approximation for the solution of (5) at time t_{n+1} is*

$$Y_{n+1} = Y_n + b(Y_n)\Delta t_n + \sigma(Y_n)\Delta W_n + \frac{1}{2}\cancel{b}\sigma'(Y_n)\cancel{b}\sigma(Y_n)[(\Delta W_n) - \Delta t_n]. \tag{15}$$

not here

Furthermore, for numerical computations, let a random vector $Z = (Z_1, ..., Z_m)$ such that for all n $Z_n \sim \mathcal{N}(0, 1)$, then the n-th iteration of the scheme is

$$Y_{n+1} = Y_n + b(Y_n)\Delta t_n + \sigma(Y_n)\sqrt{\Delta t_n}Z_{n+1} + \frac{1}{2}\cancel{b}\sigma'(Y_n)\cancel{b}\sigma(Y_n)\Delta t_n(Z_{n+1}^2 - 1) \tag{16}$$

as a comment after definitions.

We call $\gamma \in (0,\infty)$ a strong convergence order of a scheme (Y_n) if for any T there is $K>0$ such that for any discretisation $(t_n)_{n=0}^N$ of $[0,T]$ we have

$$E|X_T - Y_N| \leq K\left(\max_n \Delta t_n\right)^\gamma,$$

where (Y_n) is computed over the discretisation (t_n).

2.2.1 Modes of convergence

Now that reasonable numerical approximations to the solutions of SDEs are given, it is necessary to check whether said approximations will converge to the actual solutions. For this, two forms of convergence exist: strong and weak. And here those are stated as in [6, Section 4].

Definition 4 (Strong convergence) *Let X_t be a solution of equation (5) and Y_n a discrete time approximation of X_t, also let T be a terminal time and $\{t_n\}_{n=0}^N$ a discretisation of the interval $[0,T]$ such that $t_{n+1} > t_n$ for all n, and denote the step size $\Delta t_n = t_{n+1} - t_n$. It is said that Y_n converges to X_t in the strong sense with order $\gamma \in (0,\infty)$ if there exists a constant $K < \infty$ such that*

$$E\|X_T - Y_N\|_{\mathbb{R}^d} \leq K(\Delta t_n)^\gamma \tag{17}$$

for all step sizes Δt_n of any discretisation.

move to numerical section

For strong convergence to be confirmed numerically is necessary to compare a considerable number of paths of both the exact solution of the equation and the approximation with the E-M scheme with respect to the same Brownian motion, and using the law of large numbers to approximate the mean. Then if M sample paths of each are computed and $m \in \{1,...,M\}$ then the approximation of the error will be then given by

$$E|X_T - Y_N| \approx \frac{1}{M}\sum_{m=1}^{M} |X_T^m - Y_N^m|, \tag{18}$$

which by the law of large numbers converge to $E|X_T - Y_N|$ as $n \to \infty$.

In the other hand, there are problems in which this condition can be relaxed because we might only be interested in the approximation of a function f applied on the process X_t. Examples of such functions f that might be of interest are polynomials, in which case one is interested in the moments of the stochastic process that solves the equations considered. When this happens it could be that the value of the function of interest applied to the process is known, for example the mean or the variance of the solution by means of knowing its distribution. For such situations, as mentioned by [6], in which the nature of the problem allows it, it is possible to save computation time by defining another type of convergence in which we can compute only the approximations of the numerical solution and then compare it to a known value that has to be computed only once. For this purpose we define weak convergence.

rewrite as above but M_f depends on T & f.

Definition 5 (Weak convergence) *Let X_t be a solution of equation (5) and Y_n a discrete time approximation of X_t, also let T be a terminal time and $\{t_n\}_{n=0}^N$ a discretization of the interval $[0,T]$ such that $t_{n+1} > t_n$ for all n, and denote the step size $\Delta t_n = t_{n+1} - t_n$, further let f be a polynomial. It is said that Y_n converges to X_t in the weak sense with order $\beta \in (0,\infty)$ if there exists a constant $M_f < \infty$ such that*

$$|E(f(X_T)) - E(f(Y_N))| \leq M_f(\Delta t_n)^\beta \tag{19}$$

for all step sizes Δt_n of any discretisation.

Since the thought of a weak convergence result is usually triggered by the knowledge of $E(f(X_T))$, then numerically this convergence mode is verified using the approximation for the error

$$|E(f(X_T)) - E(f(Y_N))| \approx \left| E(f(X_T)) - \frac{1}{M}\sum_{m=1}^{M} f(Y_N^m)\right|. \tag{20}$$

2.2.2 Theoretical results

If the aim is to approximate numerically the solution of a SDE using the schemes above, it is necessary to prove that said schemes will have an appropriate output, meaning that one must know if the schemes themselves converge to the solution as the discretisation is refined and additionally which the order of the convergence in the two senses that have been discussed earlier. For that objective, Glasserman [3] and Kloeden and Platen [5] state the conditions of the coefficients and the proofs are found in the later.

[5, Thm.]

Theorem 3 (Strong convergence for E-M scheme) *Let the conditions from theorem 1 hold, additionally assume that*

$$\cancel{\mathbb{E}(\|X_0 - \tilde{X}_0\|^2) \leq K\sqrt{\Delta t}} \tag{21}$$

irrelevant to us

~~*and*~~

$$|b(s_0, x) - b(t_0, x)| + |\sigma(s_0, x) - \sigma(t_0, x)| \leq K(1 + |x|)\sqrt{|t_0 - s_0|} \tag{22}$$

then the Euler-Maruyama scheme has a strong order of convergence $\gamma = 1/2$.

~~For a proof of the previous theorem see [5, pp. 342–344]~~.

[5, Thm..]

Theorem 4 (Strong convergence for Milstein scheme) *Let the conditions from Theorems 1 and ?? hold for the coefficients b and σ. Let us use the operators L^0 and L^1 from (9) and (10), additionally consider the following notation*

$$z(t, x) = b(t, x) - \frac{1}{2}(\sigma\sigma')(t, x) \tag{23}$$

$\sigma' = \frac{\partial}{\partial x}\sigma$

Suppose that ~~the~~ (Lipschitz continuity)

$$\begin{aligned} |z(t, x_0) - z(t, y_0)| &\cancel{\leq K_1|x_0 - y_0|}, \\ |\sigma(t, x_0) - \sigma(t, y_0)| &\cancel{\leq K_1|x_0 - y_0|}, \\ \left|L^1\sigma(t, x_0) - L^1\sigma(t, y_0)\right| &\leq K_1|x_0 - y_0|, \end{aligned} \tag{24}$$

~~and~~ *(linear growth)*

$$\begin{aligned} |z(t, x_0)| + |L^1 z(t, x_0)| &\cancel{\leq K_2(1 + |x|)}, \\ |\sigma(t, x_0)| + |L^1\sigma(t, x_0)| &\cancel{\leq K_2(1 + |x|)}, \\ |L^1 L^1\sigma(t, x_0)| &\leq K_2(1 + |x|), \end{aligned} \tag{25}$$

and extended linear growth conditions

$$\begin{aligned} |z(s_0, x) - z(t_0, x)| &\cancel{\leq K_3(1 + |x|)|s_0 - t_0|^{1/2}}, \\ |\sigma(s_0, x) - \sigma(t_0, x)| &\cancel{\leq K_3(1 + |x|)|s_0 - t_0|^{1/2}}, \\ \left|L^1\sigma(s_0, x) - L^1\sigma(t_0, x)\right| &\leq K_3(1 + |x|)|s_0 - t_0|^{1/2}, \end{aligned} \tag{26}$$

hold for all $x, y, x_0, y_0 \in \mathbb{R}$ and $t, s_0, t_0 \in [0, T]$ then the Milstein scheme has a strong rate of convergence $\gamma = 1$.

The previous theorem can be found in [5, pp. 350–351]. Moreover in the same book, one can find Theorem 10.6.3, which is a more robust statement that accounts for the strong approximations of all orders $\gamma = 1/2, 1, 3/2, 2, \ldots$, which means that this result proves both the strong convergence of the E-M and Milstein schemes, mentioned before.

...See the theorem 14.5.2 K&P again for weak convergence...

Theorem 5 (Weak convergence for E-M and Milstein scheme) *Let the conditions from Theorem 1 hold, additionally assume that $b, \sigma \in \mathscr{C}^4$, then the Euler-Maruyama and the Milsten scheme have a weak rate of convergence $\beta = 1$.*

↳ in which variables? b, σ must not depend on t

For the weak convergence, the conditions for both schemes are the same and that applies for the order of convergence. That gives an advantage in terms of computation when we are interested in the weak convergence, since we can use the E-M method without losing accuracy. A proof of this result, in a more general version, is contained in [5, pp. 477–480].

Conclusions: – fix dependence on t in schemes above.
– Add strong 1.5 scheme from the K-P book.
– Study also weak convergence for the distributional drift (new).

2.2.3 Numerical examples

To verify the accuracy of the numerical schemes mentioned earlier, a SDE with a known explicit solution is used. In [5, pp. 117–126] a comprehensive list of such equations is given, and for the present work it will be used a *geometric Brownian motion* (gBm)

$$dX_t = \frac{1}{2} X_t dt + X_t dW_t, \tag{27}$$

whose solution is

$$X_t = X_0 \exp W_t. \tag{28}$$

And for the sake of the verification of weak convergence, the mean is given by

$$E(X_t) = X_0 \exp\left(W_t\right). \tag{29}$$

For this SDE with the notation from definitions 2 and 3, the Euler-Maruyama and Milstein approximations at time t_n are, respectively:

$$Y_{n+1} = Y_n + \frac{1}{2} Y_n \Delta t_n + Y_n \sqrt{\Delta t} Z_{n+1} \tag{30}$$

$$Y_{n+1} = Y_n + \frac{1}{2} Y_n \Delta t_n + Y_n \sqrt{\Delta t_n} Z_{n+1} + \frac{1}{8} \Delta t_n (Z_{n+1}^2 - 1) \tag{31}$$

Running several batches of approximations with a step sizes $\Delta t \in \{1/2^3, \ldots, 1/2^7\}$ the following results for the error of the approximations are found:

2.3 Rough coefficients

The approximation methods for SDEs mentioned above can lead us to good results if the coefficients behave in a certain way, as it was stated in theorems 3 to 5, however for the project in hand we are interested in some particular kind of drift coefficients.

...this definition is completely disconnected because it was somewhere else and I just moved it to this section...

Definition 6 *Let $h_M : \mathbb{R} \to \mathbb{R}$*

$$h_M := \mathbb{1}_{[0,\frac{1}{2})} - \mathbb{1}_{[\frac{1}{2},1)} \tag{32}$$

be the mother Haar wavelet. Then the Haar wavelet system in $\mathbb{R}$ is given by

$$\{h_{j,m} : j \in \mathbb{N} \cup \{-1\}, m \in \mathbb{Z}\} \tag{33}$$

where

$$h_{-1,m}(x) := \sqrt{2} |h_M(x - m)| \tag{34}$$

and

$$h_{j,m}(x) := h_M(2^j x - m) \tag{35}$$

for $j \in \mathbb{N}$ and $m \in \mathbb{Z}$. Alternatively the Haar wavelets can be expressed as

$$h_{j,m}(x) = \mathbb{1}_{[\frac{m}{2^j}, \frac{m+1/2}{2^j})} - \mathbb{1}_{[\frac{m+1/2}{2^j}, \frac{m+1}{2^j})}. \tag{36}$$

2.4 Numerical schemes for SDEs with rough drift

As it was stated above, the foundations of this research project are in the article by De Angelis et al [1], in which it is devised and algorithm to work with drifts that belong to an appropriate fractional Sobolev space. Below the main results of the paper are mentioned, and for this the following notation and assumptions are in order.

Consider the equation (1), the aim is to first make an approximation of the coefficient b with a better behaved coefficient b^N, for this purpose let the following assumptions for b and b^N hold:

Assumption 1 *Let $\beta_0 \in \left(0, \frac{1}{4}\right)$ and $q_0 \in \left(4, \frac{1}{\beta_0}\right)$, fix $\tilde{q}_0 := (1-\beta_0)^{-1}$. Then for some $\kappa \in \left(\frac{1}{2}, 1\right)$ take $b \in \mathscr{C}^{\frac{1}{2}}\left([0,T];\, H^{-\beta_0}_{\tilde{q}_0,q_0}\right)$.*

Assumption 2 *Let $(b^N)_{N\geq 1} \subset \mathscr{C}^{\frac{1}{2}}\left([0,T];\, H^{0}_{\tilde{q}_0,q_0}\right)$ be such that $\lim_{N\to\infty} b^N = b$ in $\mathscr{C}^{\frac{1}{2}}\left([0,T];\, H^{-\beta_0}_{\tilde{q}_0,q_0}\right)$.*

The assumptions above allow us to select the elements of the sequence b^N living in a space of coefficients that is not as rough as the space in which the real coefficient lives, because as is pointed out in [1] if $f \in H^s_{p,q}$ then $f \in H^s_r$ for all $p \wedge q < r < p \vee q$, in particular we could have $H^0_2 = L^2$. Additionally with those spaces we can take advantage of the inclusion property $H^s_r \subset H^t_u$ for all $1 < r \leq u < \infty$ and $-\infty < t \leq s < \infty$ such that $s - 1/r \geq t - 1/u$, for which that assumption of the limit of b^N makes sense as the sequence will belong also to the rough space in which the coefficient lives originally.

Theorem 6 *Let assumptions 1 and 2 hold. Take any (β, q) such that $\beta \in (\beta_0, \frac{1}{2})$ and $q_0 \geq q > \tilde{q} \geq \tilde{q}_0$, where $\tilde{q} := (1-\beta)^{-1}$. Then for any $\frac{1}{2} < \gamma < \gamma_0$ there is a constant $C_\gamma > 0$ such that*

$$\sup_{0\leq t\leq T} \mathbb{E}|X^N_t - X_t| \leq C_\gamma \|P_{\eta_N} b^N - b\|^{2\gamma-1}_{\infty, H^{-\beta}_q} \tag{37}$$

The previous theorem allows us to see the rate of convergence of the approximation X^N to the solution X by finding a bound for the error, and also that said bound is related to the rate of convergence of the approximation for the distributional drift itself.

Since the convergence of the approximation to the solution is dependant on the convergence of the approximation to the distributional drift, it is necessary to see that the approximation of the drift also has a sensible bound. Then is worth to notice as in [1, Remark 3.2] that the semigroup has the following properties:

$$\begin{aligned} \|P_t f\|_{\infty, H^{-s}_q} &\leq \|f\|_{\infty, H^{-s}_q} \\ \|P_t f - f\|_{\infty, H^{-s-\epsilon}_q} &\leq c t^{\epsilon/2} \|f\|_{\infty, H^{-s}_q}, \end{aligned} \tag{38}$$

and hence we have the inequality

$$\|P_{\eta_N} b^N - b\|_{\infty, H^{-\beta}_q} \leq \|b^N - b\|_{\infty, H^{-\beta}_q} + c\eta_N^{\frac{\beta-\beta_0}{2}} \|b\|_{\infty, H^{-\beta_0}_{q_0}}. \tag{39}$$

And then as $\eta_N \to \infty$, if $\beta > \beta_0$ the second term in the inequality goes to zero and then it is necessary to find a bound for the first term, which is achieved in the following theorem.

Theorem 7 *Let assumption 1 hold and let the sequence $(b^N)_{N\geq 1}$ be defined as*

$$b^N(t) := \sum_{j=-1}^{N} \sum_{j=-2^j}^{2^j} \mu_{j,m}(t) 2^{-j(-\beta-\frac{1}{q})} h_{j,m}, \tag{40}$$

which by construction $b^N(t) \in H^0_{\tilde{q}_0,q_0} \subset H^{-\beta_0}_{\tilde{q}_0,q_0}$. Then $(b^N)_{N\geq 1}$ satisfies assumption 2 and for any $\beta \in (\beta_o, \frac{1}{2})$ it holds that

$$\|b^N - b\|_{\infty, H^{-\beta}_2} \leq c 2^{-(N+1)(\beta-\beta_0)} \|b\|_{\infty, H^{-\beta_0}_2}. \tag{41}$$

...comment...

Theorem 8 *Let assumption 1 hold and let $b^N \in \mathscr{C}^{\frac{1}{2}}\left([0,T]; H^0_{\tilde{q}_0,q_0}\right)$ for fixed N. Then as $m \to \infty$*

$$\sup_{0\le t\le T} \mathbb{E}\left|X_t^{N,m} - X_t^N\right| \le C_2(N)m^{-1} + C_3(N)m^{-\frac{1}{2}} \tag{42}$$

where

$$C_2 := c\|P_{\eta_N} b^N\|_{\infty,L^\infty}\left(1 + \|\nabla(P_{\eta_N} b^N)\|_{\infty,L^\infty}\right), \tag{43}$$

$$C_3 := c'\left(\|\nabla(P_{\eta_N} b^N)\|_{\infty,L^\infty} + \left[P_{\eta_N} b^N\right]_{\frac{1}{2},L^\infty}\right), \tag{44}$$

and $c, c' > 0$ are constants independent of (N,m).

Theorem 9 *Let assumption 1 and also b^N defined as in (40) so that assumption 2 holds too, and let $\Theta_* := \frac{1}{2}\left[\frac{3}{4} - \beta_0\left(\gamma_0 - \frac{1}{2}\right)\right]^{-1}$. Then as $m \to \infty$, let $\eta_N = m^{-\Theta_*}$ and $N = 2\Theta_* \log_2 m$ it holds that*

$$\sup_{0\le t\le T} \mathbb{E}\left|X_t^{N,m} - X_t\right| \le c_\epsilon\left(m^{-\Theta_*\left(\frac{1}{2}-\beta_0\right)\left(\gamma_0-\frac{1}{2}\right)-\epsilon}\right) \tag{45}$$

where $\epsilon > 0$ is arbitrarily small and $c_\epsilon > 0$ is a constant depending on ϵ.

2.4.1 Numerical implementations

For the numerical implementation of the scheme described above one must have in mind the considerations that are described next.

When we apply the semigroup to the Haar functions the following form is obtained and it's how is used in the numerical computations:

$$\begin{aligned} P_{\eta_N} h_{j,m} &= P_{\eta_N} \mathbb{1}_{\left[\frac{m}{2^j}, \frac{m+1/2}{2^j}\right)} - P_{\eta_N} \mathbb{1}_{\left[\frac{m+1/2}{2^j}, \frac{m+1}{2^j}\right)} \\ &= \exp(-\eta_N)\left(-\Phi\left(\frac{\frac{m+1}{2^j} - x}{\sqrt{\eta_N}}\right) + 2\Phi\left(\frac{\frac{m+1/2}{2^j} - x}{\sqrt{\eta_N}}\right) - \Phi\left(\frac{\frac{m}{2^j} - x}{\sqrt{\eta_N}}\right)\right), \end{aligned} \tag{46}$$

where N is fixed, η_N is a constant that depends on N, $j = 1, 2, ..., N$ and $m = 1, 2, ..., 2^j$.

3 Contributions

At the moment the same algorithm in the article by De Angelis et al. [1] has been implemented with satisfactory results, confirming the

4 Research plan

5 Training record

- **Measure and Integration Theory**. Collegio Carlo Alberto. Turin, Italy. (Pass 65%).
- **MATH5734M: Advanced Stochastic Calculus and Applications to Finance**. University of Leeds. Spring Semester, 20 credits. (Awaiting for marks).
- Reading of [4] as complement for MATH5734M.
- **Seminar Series on Probability and Financial Mathematics**. University of Leeds. 2020/2021.
- **Stochastic Processes and Their Friends**. University of Leeds. 18-19 March.

- **Conference Beyond the Boundaries**. University of Leeds. 4-7 May.
- **British Early Career Mathematicians' Colloquium**. University of Birmingham. Birmingham, UK. 15-16 Jul.
- **6th Berlin Workshop for Young Researchers on Mathematical Finance**. Humboldt-Universität zu Berlin. Berlin, Germany (online). 23-25 Ago.
- **Bath Mathematical Symposium on PDE and Randomness: Summer School**. University of Bath. Bath, UK (online). 1-3 Sep.

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