Transfer report

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Contents

1 Overview

Numerical schemes for Stochastic Differential Equations (SDEs) and Stochastic Partial Differential Equations (SPDEs) have been widely studied, and even for SDEs and SPDEs with low regularity coefficients. However

The works from De Angelis et al. [1], Flandoli et al. [2] have established the framework for this project. A one dimensional SDE is considered:

$$\begin{cases} dX_t = b(t, X_t)dt + dW_t, \quad t \in [0, T] \\ X_0 = x_0, \end{cases} \tag{1}$$

where W_t is a Brownian motion, and $b(t, X_t)$ is a distribution taking values in a fractional Sobolev space of negative order, namely $H^{-\beta_0}_{\tilde{q}_0, q_0}$.

This type of equations immediately introduce a challenge because is not possible to evaluate pointwise the coefficient b and it is necessary to give a meaning to the term $\int_0^t b(s, X_s)ds$, problem that is solved in [2].

In [1] an algorithm is described in order to find the solutions to such equations, this is a two step procedure:

1. Regularisation of the coefficient b in (1), since it is a distribution and cannot be computed pointwise for the sake of a numerical method.

2. Application of the well known Euler-Maruyama scheme with the regularised coefficient.

This chain of actions leads to have two sources of error for the algorithm, nevertheless it is also shown in the same paper that **...mention the rates of convergence...**

It is worth of mention that, as is mentioned in [2], the coefficient can belong to the space of tempered distributions $\mathscr{S}'(\mathbb{R}^d)$ and a number of results will hold. **...double check this statement...** However the belonging of said object to an appropiate Sobolev space

2 Literature review

2.1 Background material on SDEs

Let b and σ be Borel measurable functions defined as following:

$$\begin{aligned} b(t,x) &: [0,\infty] \times \mathbb{R} \to \mathbb{R} \\ \sigma(t,x) &: [0,\infty] \times \mathbb{R} \to \mathbb{R}^n, \end{aligned}$$

let also $\{W_t; 0 < t < \infty\}$ be a n-dimensional Brownian motion then for $T > 0$ and $t \in [0,T]$ consider the following equation:

$$\begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t, \quad t \in [0,T] \\ X_0 = x_0. \end{cases} \tag{2}$$

Definition 1 *As in [3] a strong solution of equation* (2) *is a real-valued stochastic process* $\{X_t; 0 \leq t < \infty\}$ *such that*

$$X_t - X_0 = \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s, \tag{3}$$

holds almost surely.

2.1.1 Existence and uniqueness

For the equation posed and its respective solution as previously defined, it is natural to study the existence and uniqueness of the solutions. The following theorem by [3, p. 289] illustrates sufficient conditions for the existence of such a solution:

Theorem 1 *Let* $K \geq 0$ *be a constant such that for all* $x, y \in \mathbb{R}$ *and* $t \geq 0$, *and let the coefficients* $b(t,x)$ *and* $\sigma(t,x)$ *from equation* (2) *satisfy the Lipschitz and linear growth conditions*

$$\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\|_n \leq K\|x - y\|, \tag{4}$$

$$\|b(t,x)\| + \|\sigma(t,x)\|_n \leq K^2(1 + \|x\|^2), \tag{5}$$

then there exist a continuous, adapted process $X = \{X_t, \mathscr{F}; 0 \geq < \infty\}$ *which is a strong solution to equation* (2).

2.2 Numerical schemes for SDEs

As in the deterministic theory of Differential Equations, most of the equations have no closed form solution (a formula), so that it becomes natural to develop numerical schemes to treat such objects which arise in a number of problems in different areas.

In order to find numerical schemes to solve SDEs, one can use a procedure that is an analogous to the Taylor expansion used on ODEs. This will be called an *Itô-Taylor expansion* and is presented in the following theorem whose derivation can be found in [3, pp. 162–164].

Theorem 2 *Let* $t_0 \geq 0$ *and define the operators* L^0 *and* L^1 *be*

$$L^0 = b\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial^2}{\partial x^2} \tag{6}$$

$$L^1 = b\frac{\partial}{\partial x}. \tag{7}$$

then the Itô-Taylor expansion of second order for the strong solution of (2) *is*

$$X_t = X_{t_0} + a(X_{t_0})\int_{t_0}^{t} ds + b(X_{t_0})\int_{t_0}^{t} dW_s + L^1 b(X_{t_0})\int_{t_0}^{t}\int_{t_0}^{s} dW_z\, dW_s + R, \tag{8}$$

where the remainder term R is

$$\begin{aligned} R = &\int_{t_0}^{t}\int_{t_0}^{s} L^0 a(X_z) dz\, ds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(X_z) dW_z\, ds + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(X_z) dz\, dW_s \\ &+ \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^0 L^1 b(X_u) du\, dW_z\, dW_s + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{z} L^1 L^1 b(X_u) dW_u\, dW_z\, dW_s \end{aligned} \tag{9}$$

The previous result illustrates where two of the most widely used numerical schemes for the solution of SDEs, namely the Euler-Maruyama (E-M) and the Milstein scheme, come from. For the E-M scheme, the first three terms on the right hand side of (8) are considered, and for the Milstein scheme the first four. Being the remainder left out, and then it remains to prove that leaving that remainder behind allow the schemes an acceptable level of accuracy to approximate the solutions of SDEs.

Definition 2 (Euler-Maruyama scheme) *For a time interval $[0, T]$ let $\{t_n\}_{n=0}^{N}$ be a discretisation of the interval such that $t_{n+1} > t_n$ for all n and let $\Delta t_n = t_{n+1} - t_n$. Also let X_t be a solution of the SDE and $Y_n := Y_{t_n}$ denote an approximation of X_{t_n}. Denote $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ and define $Y_0 = X_0$. Then an Euler-Maruyama approximation for the solution of* (2) *at time t_{n+1} is*

$$Y_{n+1} = Y_n + b(Y_n)\Delta t_n + b(Y_n)\Delta W_n. \tag{10}$$

Further for numerical computations, let a random variable $Z \sim \mathcal{N}(0, 1)$, then the n-th iteration of the scheme is

$$Y_{n+1} = Y_n + b(Y_n)\Delta t_n + b(Y_n)\sqrt{\Delta t_n}Z \tag{11}$$

Definition 3 (Milstein scheme) *For a time interval $[0, T]$ let $\{t_n\}_{n=0}^{N}$ be a discretisation of the interval such that $t_{n+1} > t_n$ for all n and let $\Delta t_n = t_{n+1} - t_n$. Also let X_t be a solution of the SDE and $Y_n := Y_{t_n}$ denote an approximation of X_{t_n}. Denote $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ and define $Y_0 = X_0$. Then a Milstein approximation for the solution of* (2) *at time t_{n+1} is*

$$Y_{n+1} = Y_n + b(Y_n)\Delta t_n + b(Y_n)\Delta W_n + \frac{1}{2}b'(Y_n)b(Y_n)[(\Delta W_n) - \Delta t_n]. \tag{12}$$

Further for numerical computations, let a random variable $Z \sim \mathcal{N}(0, 1)$, then the n-th iteration of the scheme is

$$Y_{n+1} = Y_n + b(Y_n)\Delta t_n + b(Y_n)\sqrt{\Delta t_n}Z + \frac{1}{2}b'(Y_n)b(Y_n)\Delta t_n(Z^2 - 1) \tag{13}$$

2.2.1 Modes of convergence

Now that reasonable numerical approximations to the solutions of SDEs are given, it is necessary to check whether said approximations will converge to the actual solutions. For this, two forms of convergence exist: strong and weak. And here those are stated as in [6].

Definition 4 (Strong convergence) *Let X_t be a solution of equation* (2) *and Y_n a discrete time approximation of X_t, also let T be a terminal time and $\{t_n\}_{n=0}^{N}$ a discretization of the interval $[0, T]$ such that $t_{n+1} > t_n$ for all n, and denote the step size $\Delta t_n = t_{n+1} - t_n$. It is said that Y_n converges to X_t in the strong sense with order $\gamma \in (0, \infty)$ if there exists a constant $K < \infty$ such that*

$$E\|X_T - Y_N\| \le K(\Delta t_n)^{\gamma} \tag{14}$$

for all step sizes Δt_n of any discretisation.

For strong convergence to be confirmed numerically is necessary to compare a considerable number of paths of both the exact solution of the equation and the approximation with the E-M scheme with respect to the same Brownian motion, and using the law of large numbers to approximate the mean. Then if M sample paths of each are computed and $m \in \{1, ..., M\}$ approximation will be then given as follows:

$$E|X_T - Y_N| \approx \frac{1}{M} \sum_{m=1}^{M} |X_T^m - Y_N^m|, \tag{15}$$

which by the law of large numbers converge to $E|X_T - Y_N|$ as $n \to \infty$.

In the other hand, there are problems in which this condition can be relaxed since what is needed to know is the approximation of a function f applied on the process X_t. Examples of such functions f that might be of interest are polynomials, which give the moments of the process. When this happens it could be that the value of the function of interest applied to the process is known, for example the mean or the variance of the solution by means of knowing its distribution. For such situations, as mentioned by [6], in which the nature of the problem allows it it is possible to save computation time by defining another type of convergence in which one can compute only the approximations of the numerical solution and then compare it to a known value that has to be computed only once.

Definition 5 (Weak convergence) *Let X_t be a solution of equation* (2) *and Y_n a discrete time approximation of X_t, also let T be a terminal time and $\{t_n\}_{n=0}^{N}$ a discretization of the interval $[0, T]$ such that $t_{n+1} > t_n$ for all n, and denote the step size $\Delta t_n = t_{n+1} - t_n$, further let f be a polynomial. It is said that Y_n converges to X_t in the weak sense with order $\beta \in (0, \infty)$ if there exists a constant $M_f < \infty$ such that*

$$|E(f(X_T)) - E(f(Y_N))| \leq M_f(\Delta t_n)^{\beta} \tag{16}$$

for all step sizes Δt_n of any discretisation.

Since the thought of a weak convergence result is usually triggered by the knowledge of $E(f(X_T))$, then numerically this convergence mode is verified using the approximation

$$|E(f(X_T)) - E(f(Y_N))| \approx \left| E(f(X_T)) - \frac{1}{M} \sum_{m=1}^{M} f(Y_N^m)) \right| \tag{17}$$

...note about the meaning of the convergence rate, found in lecture notes but not yet in proper reference...

2.2.2 Theoretical results

If the aim is to approximate numerically the solution of a SDE with the schemes above, it is necessary to prove that said schemes will have an appropriate level of accuracy, meaning that one must know if the schemes themselves converge to the solution as the discretisation is refined and furthermore which the order of the convergence in the two senses that has been discussed earlier.

...type the proofs and show to Elena...

assumptions for each

Theorem 3 *The Euler-Maruyama scheme has a strong rate of convergence $\gamma = 1/2$.*

Theorem 4 *The Euler-Maruyama scheme has a weak rate of convergence $\beta = 1$.*

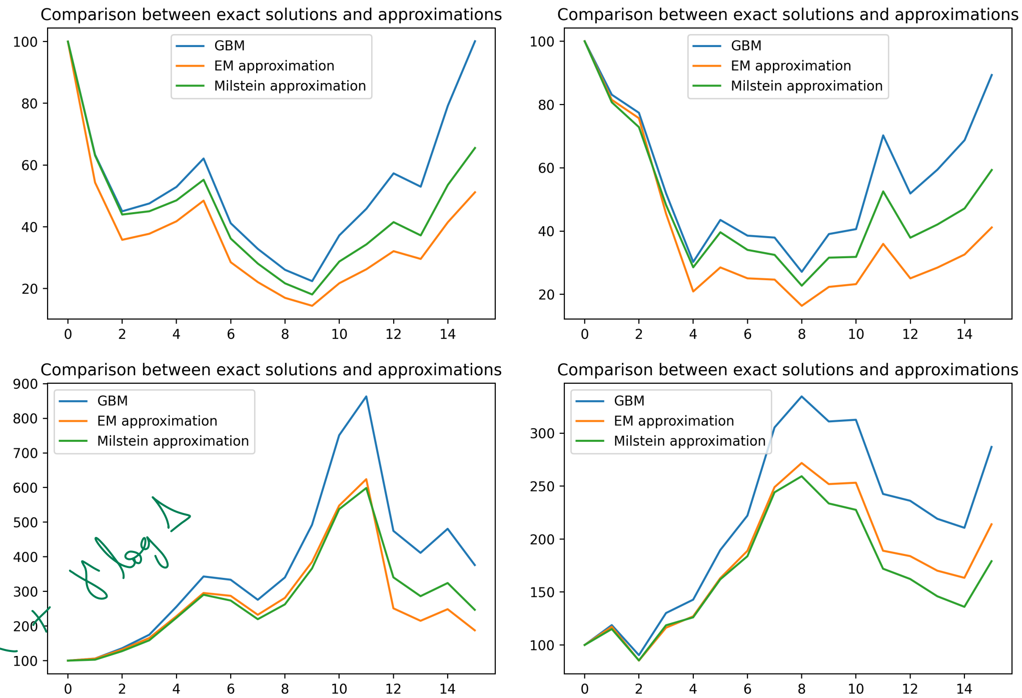
references not proofs

Theorem 5 *The Milstein scheme has a strong rate of convergence $\gamma = 1$.*

Theorem 6 *The Milstein scheme has a weak rate of convergence $\beta = 1$.*

• command of simulation
• rough SDES
• Milstein scheme (contrib)

remove and add better the graphs of error

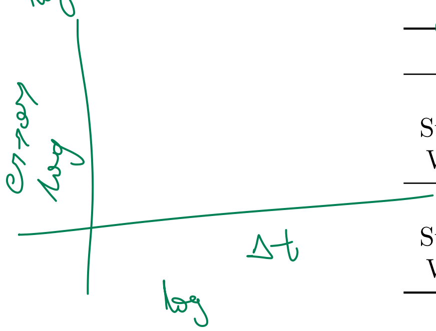


error = $K\Delta^\gamma$

log(error) = log K + γ log Δ

Figure 1

name it by Δ that is used



	Batch 1	Batch 2	Batch 3	Batch 4
	Euler-Maruyama scheme			
Strong convergence				
Weak convergence				
	Milstein scheme			
Strong convergence				
Weak convergence				

plot the errors (log) and see γ as the slope

2.2.3 Numerical examples

To verify the precision of the numerical schemes mentioned earlier, a SDE with a known explicit solution is used. In [5, pp. 117–126] a comprehensive list of such equations is given, and for the present work it will be used a *geometric Brownian motion* (gBm)

$$dX_t = \frac{1}{2}X_t dt + X_t dW_t, \tag{18}$$

whose solution is

$$X_t = X_0 \exp W_t, \tag{19}$$

And for the sake of the verification of weak convergence, the mean is given by

$$E(X_t) = X_0 \exp\left(\frac{1}{2}W_t\right). \tag{20}$$

Running several batches of approximations with a step size $\Delta t = \frac{1}{8}$ the following results for the error of the approximations are found:

3 Mathematics undertaken to date

4 Research plan

5 Training record

- **Measure and Integration Theory**. Collegio Carlo Alberto. Turin, Italy. (Pass 65%).
- **MATH5734M: Advanced Stochastic Calculus and Applications to Finance**. University of Leeds. Spring Semester, 20 credits. (Awaiting for marks).
- Reading of [3] as complement for MATH5734M.
- **Seminar Series on Probability and Financial Mathematics**. University of Leeds. 2020/2021.
- **Stochastic Processes and Their Friends**. University of Leeds. 18-19 March.
- **Conference Beyond the Boundaries**. University of Leeds. 4-7 May.
- **British Early Career Mathematicians' Colloquium**. University of Birmingham. Birmingham, UK. 15-16 Jul.
- **6th Berlin Workshop for Young Researchers on Mathematical Finance**. Humboldt-Universität zu Berlin. Berlin, Germany (online). 23-25 Ago.
- **Bath Mathematical Symposium on "PDE and Randomness": Summer School**. University of Bath. Bath, UK (online). 1-3 Sep.

References

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[3] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, Second edition., ser. Graduate texts in mathematics 113. New York ; Springer-Verlag, 1991, ISBN: 0-387-97655-8.

[4] A. Iserles, *A first course in the numerical analysis of differential equations*, ser. Cambridge texts in applied mathematics. Cambridge ; New York: Cambridge University Press, 1996, 378 pp., ISBN: 978-0-521-55376-6 978-0-521-55655-2.

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[6] E. Platen, "An introduction to numerical methods for stochastic differential equations," *Acta Numerica*, vol. 8, pp. 197–246, Jan. 1999, Publisher: Cambridge University Press, ISSN: 1474-0508, 0962-4929. DOI: 10.1017/S0962492900002920. [Online]. Available: https://www.cambridge.org/core/journals/acta-numerica/article/an-introduction-to-numerical-methods-for-stochastic-differential-equations/34AEA7B7D62931AE332FD168CDA3B8AB (visited on 09/11/2021).