1 Derivation of X-Z Plane Pressure Budget Equation

Starting with the 2-D incompressible, non-rotating, inviscid zonal (Eq. 1) and vertical (Eq. 2) momentum equations:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{1}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + B \tag{2}$$

Taking $\frac{\partial}{\partial x}$(Eq. 1) and $\frac{\partial}{\partial z}$(Eq. 2) results in:

$$\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial x \partial z} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x^2} - \frac{\partial \frac{1}{\rho_0}}{\partial x}\frac{\partial p}{\partial x} \tag{3}$$

$$\frac{\partial^2 w}{\partial z \partial t} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + u\frac{\partial^2 w}{\partial z \partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial z} + w\frac{\partial^2 w}{\partial z^2} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z^2} - \frac{\partial \frac{1}{\rho_0}}{\partial z}\frac{\partial p}{\partial z} + \frac{\partial B}{\partial z} \tag{4}$$

Adding Eq. 3 and Eq. 4:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + w\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial z} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} \\ = -\frac{1}{\rho_0}\left(\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p}{\partial x^2}\right) - \frac{\partial \frac{1}{\rho_0}}{\partial z}\frac{\partial p}{\partial z} - \frac{\partial \frac{1}{\rho_0}}{\partial x}\frac{\partial p}{\partial x} + \frac{\partial B}{\partial z}\end{aligned} \tag{5}$$

Using the 2-D continuity equation $\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0$ and with incompressibility, density does not vary in zonal or vertical directions $\left(\frac{\partial \frac{1}{\rho_0}}{\partial z}, \frac{\partial \frac{1}{\rho_0}}{\partial x} = 0\right)$, Eq.5 simplifies to:

$$-\frac{1}{\rho_0}\left(\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p}{\partial x^2}\right) = \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial z} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} - \frac{\partial B}{\partial z} \tag{6}$$

Now we separate p, u, w, B into base state ($\tilde{x}$) and perturbation quantities (x') to arrive at Eq. 7:

$$\begin{aligned}-\frac{1}{\rho_0}\left(\frac{\partial^2 \tilde{p}}{\partial z^2} + \frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2} + \frac{\partial^2 p'}{\partial x^2}\right) = \frac{\partial \tilde{u}}{\partial x}\frac{\partial \tilde{u}}{\partial x} + 2\frac{\partial \tilde{u}}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial \tilde{w}}{\partial z}\frac{\partial \tilde{w}}{\partial z} + 2\frac{\partial \tilde{w}}{\partial z}\frac{\partial w'}{\partial z} + \frac{\partial w'}{\partial z}\frac{\partial w'}{\partial z} \\ + 2\frac{\partial \tilde{w}}{\partial x}\frac{\partial \tilde{u}}{\partial z} + 2\frac{\partial \tilde{w}}{\partial x}\frac{\partial u'}{\partial z} + 2\frac{\partial w'}{\partial x}\frac{\partial \tilde{u}}{\partial z} + 2\frac{\partial w'}{\partial x}\frac{\partial u'}{\partial z} - \frac{\partial \tilde{B}}{\partial z} - \frac{\partial B'}{\partial z}\end{aligned} \tag{7}$$

We now assume that base-state variations of $\tilde{p}, \tilde{w}, \tilde{B}$ are negligible in the zonal and vertical directions, while we retain only the vertical variation of base-state zonal velocity $\frac{\partial \tilde{u}}{\partial z}$ for strongly sheared cases to give us Eq. 8:

$$-\frac{1}{\rho_0}\left(\frac{\partial^2 p'}{\partial z^2} + \frac{\partial^2 p'}{\partial x^2}\right) = \frac{\partial u'}{\partial x}\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\frac{\partial w'}{\partial z} + 2\frac{\partial w'}{\partial x}\frac{\partial u'}{\partial z} + 2\frac{\partial w'}{\partial x}\frac{\partial \tilde{u}}{\partial z} - \frac{\partial B'}{\partial z} \tag{8}$$

Eq. 9 expands $\left(2\frac{\partial w'}{\partial x}\frac{\partial u'}{\partial z}\right)$ into symmetric and antisymmetric parts:

$$2\frac{\partial w'}{\partial x}\frac{\partial u'}{\partial z} = \frac{1}{2}\left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z}\right)^2 - \frac{1}{2}\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right)^2 \tag{9}$$

Substituting Eq. 9 back into Eq. 8 and simplifying slightly:

$$-\frac{1}{\rho_0}\nabla^2_{xz}p' = \left(\frac{\partial u'}{\partial x}\right)^2 + \left(\frac{\partial w'}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z}\right)^2 - \frac{1}{2}\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right)^2 + 2\frac{\partial w'}{\partial x}\frac{\partial \tilde{u}}{\partial z} - \frac{\partial B'}{\partial z} \tag{10}$$

Using the approximation of the Laplacian operator $\left(-\frac{1}{\rho_0}\nabla^2_{xz}p' \propto p'_{xz}\right)$ and with $\left(\xi = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right)$ being the X-Z plane vorticity (about the meridional axis), we come to the final expression of the X-Z plane pressure perturbation budget equation:

$$p'_{xz} \propto \left(\frac{\partial u'}{\partial x}\right)^2 + \left(\frac{\partial w'}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z}\right)^2 - \frac{1}{2}\xi^2 + 2\frac{\partial w'}{\partial x}\frac{\partial \tilde{u}}{\partial z} - \frac{\partial B'}{\partial z} \tag{11}$$

2 Derivation of X-Z Plane Vorticity Budget Equation

Starting with the incompressible, Reynolds-averaged zonal (Eq. 12) and vertical (Eq. 13) momentum equations where $\overline{x}$ is a resolved quantity and $\overline{x'}$ is a sub-grid scale quantity and neglecting Coriolis deflections:

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} + \overline{u'\frac{\partial u'}{\partial x}} + \overline{v'\frac{\partial u'}{\partial y}} + \overline{w'\frac{\partial u'}{\partial z}} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial x} \tag{12}$$

$$\frac{\partial \overline{w}}{\partial t} + \overline{u}\frac{\partial \overline{w}}{\partial x} + \overline{v}\frac{\partial \overline{w}}{\partial y} + \overline{w}\frac{\partial \overline{w}}{\partial z} + \overline{u'\frac{\partial w'}{\partial x}} + \overline{v'\frac{\partial w'}{\partial y}} + \overline{w'\frac{\partial w'}{\partial z}} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z} + \overline{B} \tag{13}$$

Taking $\frac{\partial}{\partial z}$(Eq. 12) and $\frac{\partial}{\partial x}$(Eq. 13) results in:

$$\begin{aligned}\frac{\partial^2 \overline{u}}{\partial t\partial z} + \frac{\partial \overline{u}}{\partial z}\frac{\partial \overline{u}}{\partial x} + \overline{u}\frac{\partial^2 \overline{u}}{\partial z\partial x} + \frac{\partial \overline{v}}{\partial z}\frac{\partial \overline{u}}{\partial y} + \overline{v}\frac{\partial^2 \overline{u}}{\partial z\partial y} + \frac{\partial \overline{w}}{\partial z}\frac{\partial \overline{u}}{\partial z} + \overline{w}\frac{\partial^2 \overline{u}}{\partial z^2} + \overline{\frac{\partial u'}{\partial z}\frac{\partial u'}{\partial x}} + \overline{u'\frac{\partial^2 u'}{\partial z\partial x}} \\ + \overline{\frac{\partial v'}{\partial z}\frac{\partial u'}{\partial y}} + \overline{v'\frac{\partial^2 u'}{\partial z\partial y}} + \overline{\frac{\partial w'}{\partial z}\frac{\partial u'}{\partial z}} + \overline{w'\frac{\partial^2 u'}{\partial z^2}} = -\frac{1}{\rho_0}\frac{\partial^2 \overline{p}}{\partial x\partial z} - \frac{\partial \frac{1}{\rho_0}}{\partial z}\frac{\partial \overline{p}}{\partial x}\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial^2 \overline{w}}{\partial t\partial x} + \frac{\partial \overline{u}}{\partial x}\frac{\partial \overline{w}}{\partial x} + \overline{u}\frac{\partial^2 \overline{w}}{\partial x^2} + \frac{\partial \overline{v}}{\partial x}\frac{\partial \overline{w}}{\partial y} + \overline{v}\frac{\partial^2 \overline{w}}{\partial x\partial y} + \frac{\partial \overline{w}}{\partial x}\frac{\partial \overline{w}}{\partial z} + \overline{w}\frac{\partial^2 \overline{w}}{\partial x\partial z} + \overline{\frac{\partial u'}{\partial x}\frac{\partial w'}{\partial x}} + \overline{u'\frac{\partial^2 w'}{\partial x^2}} \\ + \overline{\frac{\partial v'}{\partial x}\frac{\partial w'}{\partial y}} + \overline{v'\frac{\partial^2 w'}{\partial x\partial y}} + \overline{\frac{\partial w'}{\partial x}\frac{\partial w'}{\partial z}} + \overline{w'\frac{\partial^2 w'}{\partial x\partial z}} = -\frac{1}{\rho_0}\frac{\partial^2 \overline{p}}{\partial x\partial z} - \frac{\partial \frac{1}{\rho_0}}{\partial x}\frac{\partial \overline{p}}{\partial z} + \frac{\partial \overline{B}}{\partial x}\end{aligned} \tag{15}$$

Subtracting Eq. 15 from Eq. 14, noticing the $-\frac{1}{\rho_0(z)}\frac{\partial^2 \overline{p}}{\partial x\partial z}$ terms cancel and $\frac{\partial \frac{1}{\rho_0}}{\partial x}, \frac{\partial \frac{1}{\rho_0}}{\partial z} = 0$ in an incompressible fluid:

$$\frac{\partial}{\partial t}\left(\frac{\partial\overline{u}}{\partial z}-\frac{\partial\overline{w}}{\partial x}\right)+\frac{\partial\overline{u}}{\partial x}\left(\frac{\partial\overline{u}}{\partial z}-\frac{\partial\overline{w}}{\partial x}\right)+\frac{\partial\overline{w}}{\partial z}\left(\frac{\partial\overline{u}}{\partial z}-\frac{\partial\overline{w}}{\partial x}\right)+\overline{\frac{\partial u'}{\partial x}\left(\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x}\right)}+\overline{\frac{\partial w'}{\partial z}\left(\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x}\right)}$$
$$+\frac{\partial\overline{v}}{\partial z}\frac{\partial\overline{u}}{\partial y}-\frac{\partial\overline{v}}{\partial x}\frac{\partial\overline{w}}{\partial y}+\overline{\frac{\partial v'}{\partial z}\frac{\partial u'}{\partial y}}-\overline{\frac{\partial v'}{\partial x}\frac{\partial w'}{\partial y}}+\vec{\overline{V}}\cdot\nabla\left(\frac{\partial\overline{u}}{\partial z}-\frac{\partial\overline{w}}{\partial x}\right)+\overline{\vec{V'}\cdot\nabla\left(\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x}\right)}=-\frac{\partial\overline{B}}{\partial x} \tag{16}$$

Using the definition of X-Z plane resolved vorticity $\left(\overline{\xi}=\frac{\partial\overline{u}}{\partial z}-\frac{\partial\overline{w}}{\partial x}\right)$ and sub-grid vorticity $\left(\overline{\xi'}=\overline{\frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x}},\overline{\xi'}\neq 0\right)$ and absorbing the time tendency and resolved advection term into the total derivative $\left(\frac{D\overline{\xi}}{Dt}=\frac{\partial\overline{\xi}}{\partial t}+\vec{\overline{V}}\cdot\nabla\overline{\xi}\right)$ gives us Eq. 17:

$$\frac{D\overline{\xi}}{Dt}+\left(\frac{\partial\overline{u}}{\partial x}-\frac{\partial\overline{w}}{\partial z}\right)\overline{\xi}+\overline{\left(\frac{\partial u'}{\partial x}-\frac{\partial w'}{\partial z}\right)\xi'}+\frac{\partial\overline{v}}{\partial z}\frac{\partial\overline{u}}{\partial y}-\frac{\partial\overline{v}}{\partial x}\frac{\partial\overline{w}}{\partial y}+\overline{\frac{\partial v'}{\partial z}\frac{\partial u'}{\partial y}}-\overline{\frac{\partial v'}{\partial x}\frac{\partial w'}{\partial y}}+\overline{\vec{V'}\cdot\nabla\xi'}=-\frac{\partial\overline{B}}{\partial x} \tag{17}$$

Expanding the sub-grid advection term $\left(\overline{\vec{V'}\cdot\nabla\xi'}\right)$ using the chain rule:

$$\overline{\vec{V'}\cdot\nabla\xi'}=\overline{u'\frac{\partial\xi'}{\partial x}}+\overline{v'\frac{\partial\xi'}{\partial y}}+\overline{w'\frac{\partial\xi'}{\partial z}}=\frac{\overline{\partial u'\xi'}}{\partial x}+\frac{\overline{\partial v'\xi'}}{\partial y}+\frac{\overline{\partial w'\xi'}}{\partial z}-\overline{\xi'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)} \tag{18}$$

The incompressibility assumption requires the flow to be non-divergent, which results in Eq. 19:

$$\overline{\vec{V'}\cdot\nabla\xi'}=\frac{\overline{\partial u'\xi'}}{\partial x}+\frac{\overline{\partial v'\xi'}}{\partial y}+\frac{\overline{\partial w'\xi'}}{\partial z} \tag{19}$$

The sub-grid stress terms in Eq. 19 can be parameterized using K-theory, relating the turbulent flux to the gradients in resolved vorticity $\left(\overline{u'\xi'}=-K\frac{\partial\overline{\xi}}{\partial x},\overline{v'\xi'}=-K\frac{\partial\overline{\xi}}{\partial y},\overline{w'\xi'}=-K\frac{\partial\overline{\xi}}{\partial z}\right)$. By substituting these relations back into Eq. 19 and assuming that eddy viscosity (K) is approximately constant and can be taken out of the inner derivative, we arrive at the following relation for the sub-grid advection term:

$$\overline{\vec{V'}\cdot\nabla\xi'}\approx -K\nabla^2\overline{\xi} \tag{20}$$

It is unclear how to represent the sub-grid stretching $\overline{\left(\frac{\partial u'}{\partial x}-\frac{\partial w'}{\partial z}\right)\xi'}$ and tilting $\overline{\frac{\partial v'}{\partial z}\frac{\partial u'}{\partial y}}-\overline{\frac{\partial v'}{\partial x}\frac{\partial w'}{\partial y}}$ terms, so we neglect their contributions and substitute the relation in Eq. 20 to arrive at the final X-Z plane vorticity budget:

$$\frac{D\overline{\xi}}{Dt}=-\left(\frac{\partial\overline{u}}{\partial x}-\frac{\partial\overline{w}}{\partial z}\right)\overline{\xi}-\frac{\partial\overline{v}}{\partial z}\frac{\partial\overline{u}}{\partial y}+\frac{\partial\overline{v}}{\partial x}\frac{\partial\overline{w}}{\partial y}-\frac{\partial\overline{B}}{\partial x}+K\nabla^2\overline{\xi} \tag{21}$$