On the number of tiles visited by a line segment on a rectangular grid

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Abstract

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Keywords: ***.

1 Introduction

Given $a, b \in \mathbb{R}^+$, consider a grid on $\mathbb{R}^2$ formed by rectangular *tiles* of width a and height b. A line segment of length $\ell \in \mathbb{R}^+$ is located on the plane with arbitrary position and orientation. The segment is said to *visit* a tile if it intersects its interior.[1] This paper deals with the following problems:

- What is the maximum number of tiles that the segment can visit? Conversely, what length should a segment have to visit a given number of tiles?
- What is the average number of tiles visited by a uniformly random segment of a given length? How often does the random segment visit the maximum number of tiles?

As an example of the first question, consider $a = 1.35$, $b = 1$. A segment of unit length can be placed as shown in Figure 1 (left) to make it visit 3 tiles. In fact, this is the maximum number for $\ell = 1$. The figure also illustrates that the solution for length 2.4 is 5 (center), and for 4.7 it is 8 (right).

An equivalent formulation of the problem is obtained allowing segments of length ℓ *or smaller*. The equivalence is clear from the fact that reducing the length cannot increase the number of visited tiles. Either of these formulations will be referred to as the *direct* problem.

[1] The definition uses the interior of the tile, excluding the border, to avoid uninteresting results such as a "zero-length" segment visiting (a vertex of) 4 tiles.

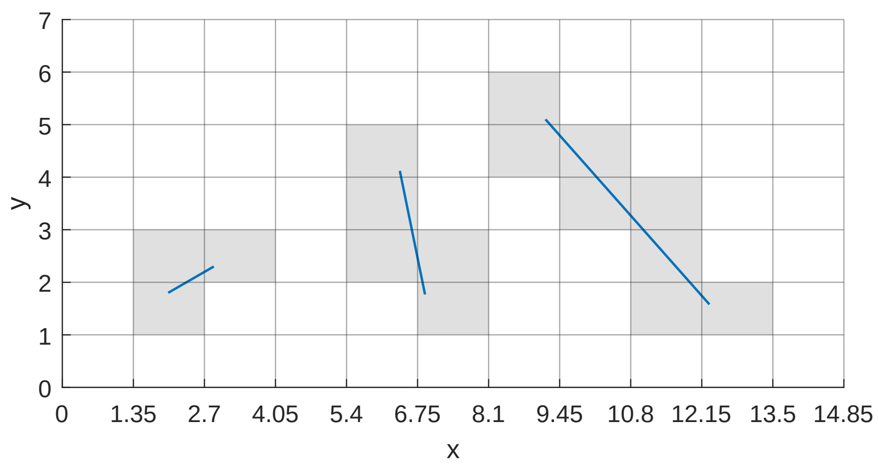


Figure 1: Examples for $a = 1.35$, $b = 1$; $\ell = 1$, $\ell = 2.4$ and $\ell = 4.7$

The *inverse* problem is, given $t \in \mathbb{N}$, to determine the infimum length of all segments that visit at least t tiles. For real-valued lengths this infimum is not a minimum, because given a segment that visits t tiles, its length can always be reduced by some non-zero amount without changing the number of visited tiles. This is a consequence of the interior of each tile being an open set.

The direct and inverse problems are closely related. Namely, if ℓ is the infimum of all lengths that allow visiting at least t tiles (inverse problem), t is the maximum number of tiles that can be visited with lengths slightly greater than ℓ (direct problem).

To address the remaining two questions, the notion of a *random* segment of a given length needs to be precisely defined. This is done as follows. By symmetry, one endpoint of the segment can be assumed to lie in a fixed, reference tile. The position of this endpoint is *uniformly* distributed on the tile. The segment orientation has a *uniform* distribution on the set of all possible directions, and is *independent* of the endpoint position. Solving the problem of how many tiles the segment visits on average, as will be seen, also answers the inverse question (segment length to visit a given number of tiles on average). A natural, related question is with that probability the segment visits the maximum number of tiles.

The problems studied in this paper are related to Buffon's needle problem and Laplace's extension of it. Buffon's original 1733 problem considers tossing a needle of length ℓ onto ruled paper with parallel lines a height b apart, and asks what is the probability of the needle crossing a line if it lands in a uniformly random position and orientation. For a needle of length $\ell = b$, this is $2/\pi$, and hence repeated trials of this experiment can be used to estimate π. This is generalized in [4] to $\frac{2\ell}{\pi b}$ for the expected number of crossings of a needle which might intersect multiple lines.

The Buffon-Laplace needle problem, described in [3], considers a needle tossed onto a grid of rectangular tiles of width a and height b. The number of visited tiles equals one plus the number of crossings with probability 1, as shown in Proposition 1. The probability of the needle staying within a single rectangle, that is visiting only one tile, is computed in [2] for the case where $\ell < \min(a,b)$. This work considers the complementary question of the chance that the needle visits the maximum number of tiles possible for its length, described by a formula in Theorem 7.

The rest of the paper is organized as follows. Fundamental results are presented in §2, which form the basis of the subsequent analysis. The direct and inverse problems for a deterministic segment are considered in §3, first for arbitrary grids and then for a square grid. The analysis for the random segment is carried out in §4. The average number of tiles is computed for arbitrary grids, and the probability that the segment visits the maximum number of tiles is obtained for a square grid.

2 Fundamentals

For a grid with horizontal spacing a and vertical spacing b, lines of the form $x = ka$ or $y = kb$ with $k \in \mathbb{Z}$ will be called *grid lines*. A *tile* is delimited by two pairs of consecutive horizontal and vertical grid lines. The intersection points of horizontal and vertical grid lines will be called *grid points*. These correspond to vertices of the tiles.

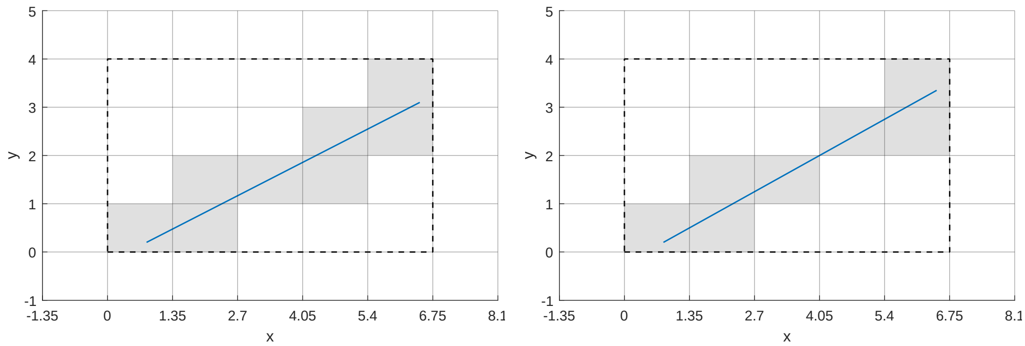
Every segment has an associated *discrete bounding rectangle*, which is the minimum-size rectangle that is formed by grid lines and contains the segment. More specifically, if the segment has endpoints (x_1,y_1), (x_2,y_2), where $x_1,x_2,y_1,y_2 \in \mathbb{R}$, its discrete bounding rectangle has lower-left corner and upper-right corner given as

$$
\begin{aligned}
&(\lfloor \min\{x_1,x_2\}/a \rfloor a, \lfloor \min\{y_1,y_2\}/b \rfloor b),\\
&(\lceil \max\{x_1,x_2\}/a \rceil a, \lceil \max\{y_1,y_2\}/b \rceil b).
\end{aligned}
$$

The dimensions of the discrete bounding rectangle, normalized to the tile width and height respectively, are two integer numbers i, j. Two examples are illustrated in Figure 2, both with $i = 5$, $j = 4$. Clearly, all tiles visited by the segment are contained in the discrete bounding rectangle. Note also that the rectangle can have $i = 0$ or $j = 0$ if the segment coincides with part of a grid line.

Proposition 1. *Consider an arbitrary segment, and let i, j respectively denote the normalized width and height of its discrete bounding rectangle. If $i, j \geq 1$, the number of tiles visited by the segment is at most $i + j - 1$. This bound is attained if and only if the segment does not pass through any grid point in the interior of the rectangle.*

Proof. The segment visits, by definition, two tiles in opposite corners of the discrete bounding rectangle. It can be assumed, without loss of generality, that those tiles are in the lower-left and upper-right corners of the rectangle, as in Figure 2. The visited tiles can be thought of as following a path within the discrete bounding rectangle. Starting at the lower-left tile, the next tile can be the one to the left, the one above, or the one above and to the left. The latter case



(a) The segment does not pass through any interior grid points

(b) The segment passes through some interior grid points

Figure 2: Discrete bounding rectangle and visited tiles

occurs if and only if the segment passes through the grid point between those two tiles.

Since the segment follows a straight line, once it "leaves" a row of tiles in its path from the lower-left to the upper-right corner, it can never visit any more tiles from that row. The same observation applies to the columns.

This implies that the maximum number of visited tiles is $i+j-1$, which is attained if and only if the segment avoids all grid points in the interior of the discrete bounding rectangle, as in Figure 2(a). Note that grid points at the corners of the rectangle do not count for this; and that the segment cannot pass through any other grid points on the rectangle border, because that would imply $i=0$ or $j=0$. Figure 2(b) illustrates a case where the maximum is not attained. □

Proposition 2. *Consider $a, b, \ell \in \mathbb{R}^+$ and $i, j \in \mathbb{N}$, $i, j \geq 2$ arbitrary.*

(i) The following inequalities hold for any segment with length ℓ whose discrete bounding rectangle has normalized dimensions i, j:

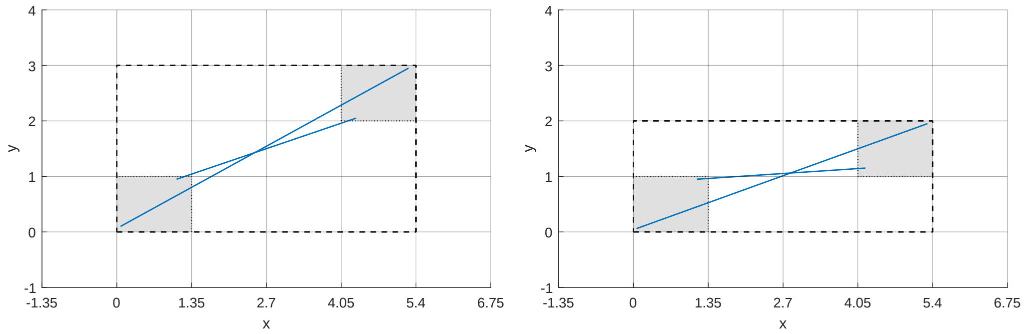
$$\ell > \sqrt{(i-2)^2a^2 + (j-2)^2b^2}, \tag{1}$$

$$\ell \leq \sqrt{i^2a^2 + j^2b^2}. \tag{2}$$

(ii) Conversely, if ℓ, i, j satisfy (1) and (2) there exists a segment of length ℓ whose discrete bounding rectangle has normalized dimensions i and j.

(iii) There is a segment of length not exceeding ℓ that has a discrete bounding rectangle with normalized dimensions i, j if and only if (1) holds.

Proof. (i) The inequalities follow from the fact that the segment endpoints lie in the interiors or on the outer edges of two tiles in opposite corners of the discrete bounding rectangle. This is illustrated in Figures 3(a) and 3(b) for two specific (i, j) pairs respectively. For each (i, j), segments are shown with lengths close to either of the two bounds. Note that (1) is valid even for $i=2$, $j=2$, in which case it reduces to $\ell > 0$.



(a) $i = 4$, $j = 3$

(b) $i = 4$, $j = 2$

Figure 3: Relationship between segment length ℓ and dimensions i, j of the discrete bounding rectangle

(ii) For a, b, ℓ, i, j satisfying the two inequalities, a segment of length ℓ can be found that has its endpoints in the interiors or on the outer edges of the two shaded tiles of a rectangle of normalized dimensions i and j, as in Figure 3, which is thus the discrete bounding rectangle of that segment.

(iii) "(1) $\Rightarrow$ there is a segment...": Assume that (1) holds. It is always possible to choose a length equal to or smaller than ℓ such that both (1) and (2) hold. The result follows, for that length, from part (ii).

"There is a segment... $\Rightarrow$ (1)": Assume that a segment exists with length not exceeding ℓ and with a discrete bounding rectangle with normalized dimensions i, j. From part (i), inequality (1) holds for that segment length, and thus for ℓ. □

According to Proposition 1, in order to maximize the number of visited tiles for a given length, the position and orientation of the segment should be chosen to obtain $i + j - 1$ as large as possible, where i and j are the normalized dimensions of its discrete bounding rectangle. On the other hand, Proposition 2 restricts the i, j values that can be achieved with a given length. A relevant question is: are there any (i, j) pairs that can be disregarded irrespective of the length ℓ? In other words, what is the "smallest" subset of $\mathbb{N}^2$ such that the (i, j) pair that maximizes the number of tiles for any given length can always be found within that subset? A subset that contains a maximizing (i, j) for any length will be called a *sufficient* set. Clearly, this set must contain at least one such pair for each possible value of $i + j - 1$, so that the set can produce that value as solution (for certain lengths). A sufficient set that contains exactly one pair (i, j) for each value of $i + j - 1$ will be called a *minimal sufficient* set.

For instance, it is intuitively clear from Figure 1 that segment orientations near the vertical or horizontal directions (resulting in $i = 1$ with large j, or $j = 1$ with large i) will not maximize the number of visited tiles. The corresponding (i, j) pairs can be left out of the sufficient set. In general, the pairs included in a minimal sufficient set will be different depending on a and b. Two specific examples of minimal sufficient sets will be presented later in this section.

Knowing a minimal sufficient set of (i, j) pairs facilitates the solution of both

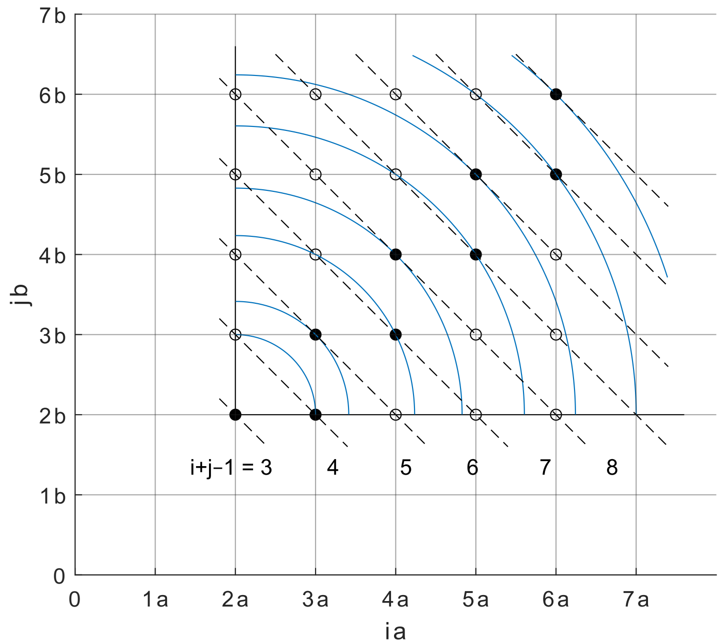


Figure 4: Relationship of segment length and number of visited tiles with the width and height of the discrete bounding rectangle, for $a = b$

the direct an inverse problems, because it reduces the number of pairs that need to be tried. In order to derive a general method to build a minimal sufficient set, it is insightful to consider two specific cases first. It is also convenient to use the equivalent formulation of the direct problem referred to in §1, i.e. finding the maximum number of tiles visited by segments of length ℓ *or less*.

Consider first $a = b = 1$. This is illustrated in Figure 4. Note that in this and in the next figures the axes represent ia and jb (not i and j). Each dashed diagonal line joins (i, j) pairs with the same $i + j - 1$. The radius of each arc represents the lower bound on ℓ given by (1), for certain (i, j) pairs, which are marked with filled circles (these pairs form a minimal sufficient set, as will be seen).

For a given $\ell \in \mathbb{R}^+$, the (i, j) pairs that can be achieved with segments of length not exceeding ℓ are, by Proposition 2.(iii), those whose distance from $(2a, 2b)$ is less than ℓ, i.e. those delimited by the corresponding arc. It is clear from the figure that for $i + j - 1$ odd, if some point (i', j') with $|i' - j'| > 1$ is achievable with the length bound ℓ, then necessarily there is a point (i'', i'') with the same number of visited tiles $i + j - 1$ (same diagonal line) that is also achievable with the same length restriction (because it has smaller distance to $(2a, 2b)$). Similarly, for $i + j - 1$ even, if a point (i', j') with $|i' - j'| > 1$ is achievable, then there is a point $(i'', i'' - 1)$ in the same diagonal that is also achievable with the same length restriction.

Thus for $a = b = 1$, a minimal sufficient set is formed by the pairs (i, j) with $j = i$ or $j = i + 1$. Observe that due to symmetry, any point $(i'', i'' - 1)$ could be replaced by $(i'' - 1, i'')$. This illustrates that the minimal sufficient set is not unique in general.

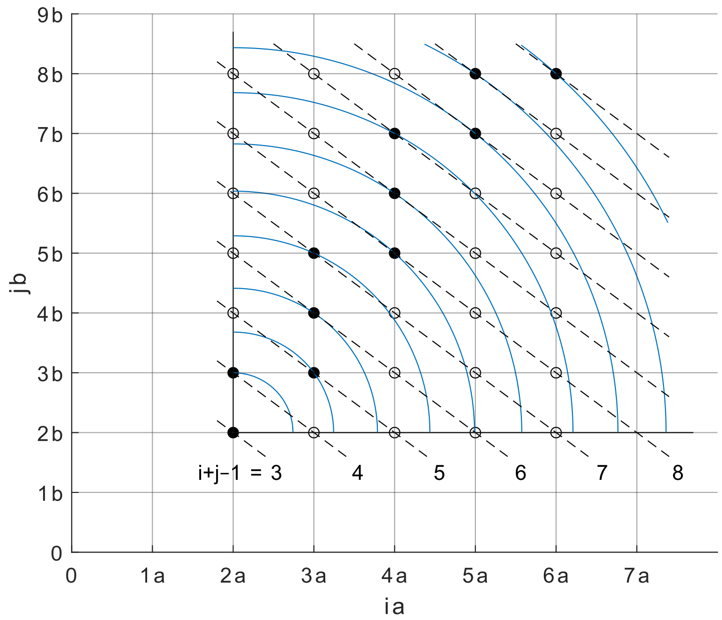


Figure 5: Relationship of segment length and number of visited tiles with the width and height of the discrete bounding rectangle, for $a = 1.35b$

The pairs in the minimal sufficient set are marked with filled circles in Figure 4. Given a length ℓ, the maximum number of visited tiles will be achieved with one of those pairs, namely that in the diagonal line furthest from $(2a, 2b)$ that is within the circle with radius ℓ centered at that point.

As a second example, consider $a = 1.35$, $b = 1$. This is depicted in Figure 5. In this case the minimal sufficient set does not follow a rule as simple as in the previous example.

The following proposition gives an explicit method to obtain a minimal sufficient set, $M = \{(i_3, j_3), (i_4.j_4), \ldots\}$, where the pair (i_t, j_t) corresponds to $i + j - 1 = t$.

Proposition 3. *Given $a, b \in \mathbb{R}^+$ and $t \in \mathbb{N}$, $t \geq 3$, a minimal sufficient set $M = \{(i_3, j_3), (i_4.j_4), \ldots\}$ can be obtained as*

$$i_t = \left\lfloor \frac{(t-3)b^2}{a^2+b^2} + \frac{5}{2} \right\rfloor, \tag{3}$$

$$j_t = \left\lceil \frac{(t-3)a^2}{a^2+b^2} + \frac{3}{2} \right\rceil, \tag{4}$$

where $i_t + j_t - 1 = t$. All pairs (i_t, j_t) are strictly below the line

$$j = \frac{ia^2}{b^2} - \frac{3a^2}{2b^2} + \frac{5}{2}, \tag{5}$$

and above or on the line

$$j = \frac{ia^2}{b^2} - \frac{5a^2}{2b^2} + \frac{3}{2}. \tag{6}$$

Proof. For each t, the pair (i_t, j_t) of the minimal sufficient set should be chosen as that on the line $i+j-1=t$ which minimizes $(i-2)^2a^2+(j-2)^2b^2$. This allows maximizing the sum $i+j-1$, and thus the number of visited tiles, for a given length restriction (direct problem); or minimizing the required lengths for a specified number of visited files (inverse problem).

Consider, for the moment, i, j as if they were real-valued variables, and denote $x = ia$, $y = jb$. The line $i+j-1=t$ then becomes

$$\frac{x}{a}+\frac{y}{b}=t+1, \tag{7}$$

and $(i-2)^2a^2+(j-2)^2b^2$ is expressed as $(x-2a)^2+(y-2b)^2$. The point minimizing this quadratic function along (7) is the intersection of this line with the perpendicular line passing through $(2a, 2b)$, that is,

$$y=\frac{a(x-2a)}{b}+2b. \tag{8}$$

This is depicted for $a=1.35$, $b=1$, $t=7$ in Figure 6, where (8) is shown with solid line. Solving the system of equations (7) and (8) gives

$$\frac{x}{a}-2=\frac{(t-3)b^2}{a^2+b^2}, \tag{9}$$

$$\frac{y}{b}-2=\frac{(t-3)a^2}{a^2+b^2}. \tag{10}$$

In terms of the real-valued variables i, j, the solution (i^+, j^+) is

$$i^+=\frac{(t-3)b^2}{a^2+b^2}+2, \tag{11}$$

$$j^+=\frac{(t-3)a^2}{a^2+b^2}+2. \tag{12}$$

This point is shown in Figure 6 with a square marker.

The variables i, j are actually limited to integer values. The point (i_t, j_t) that minimizes $(i-2)^2a^2+(j-2)^2b^2$ along the line $i+j-1=t$ restricted to integer values is either $(\lceil i^+\rceil, \lfloor j^+\rfloor)$ or $(\lfloor i^+\rfloor, \lceil j^+\rceil)$, whichever is closest to (i^+, j^+), as can be seen in Figure 6. This can be expressed as

$$i_t=\left\lfloor i^+ +\frac{1}{2}\right\rfloor, \tag{13}$$

$$j_t=\left\lceil j^+ -\frac{1}{2}\right\rceil, \tag{14}$$

which in case of tie takes the first of the two points indicated above. Combining (11)–(14) yields (3) and (4).

From (11) and (13),

$$\frac{(t-3)b^2}{a^2+b^2}+\frac{3}{2}<i_t\le\frac{(t-3)b^2}{a^2+b^2}+\frac{5}{2}, \tag{15}$$

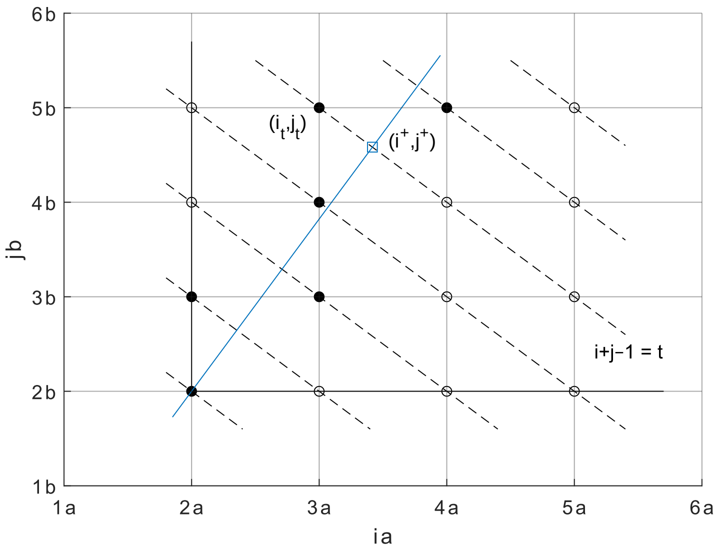


Figure 6: Obtaining (i^+, j^+) and (i_t, j_t) in Proposition 3. Example with $a = 1.35$, $b = 1$, $t = 7$

and similarly, from (12) and (14),

$$\frac{(t-3)a^2}{a^2+b^2} + \frac{3}{2} \leq j_t < \frac{(t-3)a^2}{a^2+b^2} + \frac{5}{2}. \tag{16}$$

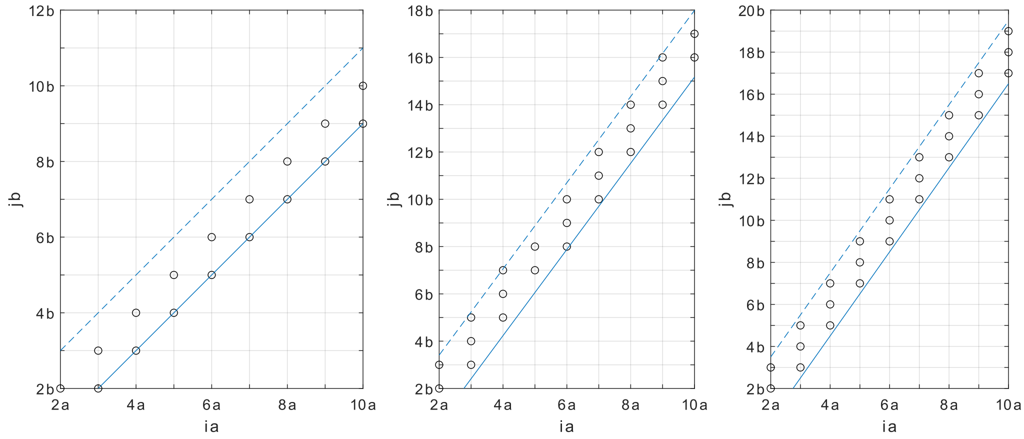
Considering the first inequality in (15) and the second in (16) as equalities and eliminating t gives (5). The point (i_t, j_t) is strictly below that line because the considered inequalities are strict. Similarly, the bounding line (6) results from the second inequality in (15) and the first in (16), and the fact that both inequalities are not strict implies that the bound (6) can actually be attained. □

The bounding lines in Proposition 3 are shown in Figure 7, using three different pairs of grid parameters a, b as examples. Given $(i_t, j_t) \in M$, the next pair (i_{t+1}, j_{t+1}) is obtained by incrementing j if that results in a point below (5). Else i is incremented instead, and the new pair is guaranteed to be above or on (6).

For a^2/b^2 arbitrary, the number of pairs with the same i coordinate, or with the same j, in the set M is in general irregular, because the lines (5) and (6) do not follow a "natural" direction of the grid. This happens for instance in Figure 7(b), where the number of pairs for the first i values equals either 2 or 3 without a clear pattern.[2] On the other hand, a simple pattern arises when a^2/b^2 or b^2/a^2 is a natural number, as seen in Figures 7(a) and 7(c).

A segment whose discrete bounding rectangle has normalized width i and height j is oriented with approximate slope $jb/(ia)$ with respect to the x axis (see Figure 3); and the approximation becomes better for greater segment lengths. From (5) and (6) it can be seen that the positions of the pairs $(i, j) \in M$ have $j/i \approx a^2/b^2$ for large i, j. Therefore the optimal slope for long segments is approximately a/b. This formalizes the intuition that to maximize the number

[2]Strictly, there is a periodic pattern whenever a^2/b^2 is rational, which is the case in Figure 7(b). However, this is not easily discernible unless a^2/b^2 is a ratio of small numbers.



(a) $a = 1, b = 1$ (b) $a = 1.35, b = 1$ (c) $a = \sqrt{2}, b = 1$

Figure 7: Minimal sufficient set M and bounding lines

of visited tiles, the segment should follow a direction along which the perceived "length" of the tile is smaller.

3 Direct and inverse problems for a deterministic segment

The direct and inverse problems defined in §1, considering the segment position and orientation as deterministic, are addressed in this section. The general case for rectangular grids with real-valued segment lengths is analyzed first, in §3.1. The particular case of square grids is addressed in §3.2, as it allows a specialized formula for the direct problem. Lastly, the analysis of a unit square grid with integer-valued lengths is presented in §3.3.

3.1 Arbitrary grid with real-valued lengths

For a grid with parameters $a, b \in \mathbb{R}^+$, the maximum number t of visited tiles with a given real-valued length ℓ can be represented by a function $\tau : \mathbb{R}^+ \to \mathbb{N}$ such that $t = \tau(\ell)$. Similarly, for the inverse problem a function $\lambda : \mathbb{N} \to \mathbb{R}^+$ can be defined such that $\lambda(t)$ gives the infimum length of all segments that visit at least t tiles.

The function τ can be obtained from λ by noting that the maximum number of tiles that can be visited by a segment of length ℓ is the index of the largest term of the sequence $\lambda(t)$ that is less than ℓ:

$$\tau(\ell) = \max\{t \in \mathbb{N} \mid \lambda(t) < \ell\}. \tag{17}$$

Conversely, λ can be obtained from τ as

$$\lambda(t) = \inf\{\ell \in \mathbb{R}^+ \mid \tau(\ell) \geq t\}. \tag{18}$$

For arbitrary $a,b \in \mathbb{R}^+$, the functions τ and λ can be computed using an iterative procedure, which exploits the fact that the pairs $(i_3, j_3), (i_4, j_4), \ldots$ of M are sorted by increasing $i+j-1$, and also by increasing $(i-2)^2a^2+(j-2)^2b^2$. Namely, for τ the following procedure yields the solution: generate successive pairs to find the last one, (i_t, j_t), that satisfies (1); then $\tau(\ell) = t$. For λ the analogous method gives a direct formula. In addition, it is possible to obtain a direct formula also for τ using a different approach. These formulas are given in Theorems 1 and 2.

Theorem 1. *For $a,b \in \mathbb{R}^+$, $a \geq b$ and $\ell \in \mathbb{R}^+$,*

$$\tau(\ell) = i^* + j^* - 1 \tag{19}$$

with

$$i^* = \left\lceil \frac{a + b\,\mathrm{Re}\sqrt{4\ell^2/(a^2+b^2)-1}}{2a} \right\rceil + 1, \tag{20}$$

$$j^* = \left\lceil \frac{\sqrt{\ell^2-(i^*-2)^2a^2}}{b} \right\rceil + 1. \tag{21}$$

The function τ is piecewise constant and left-continuous, with unit-height jumps. A jump occurs at ℓ if and only if $\ell = \lambda(t)$ for some $t \in \mathbb{N}$, $t \geq 4$; and then $\tau(\ell) = t-1$, $\lim_{h\to 0+} \tau(\ell+h) = t$.

Proof. The approach is similar to that used in Proposition 3. Namely, for $i, j \geq 2$ the intersection (i^+, j^+) of the line (6) and the arc centered at $(2a, 2b)$ with radius ℓ is found, and from that the actual integer solution (i^*, j^*) is computed.

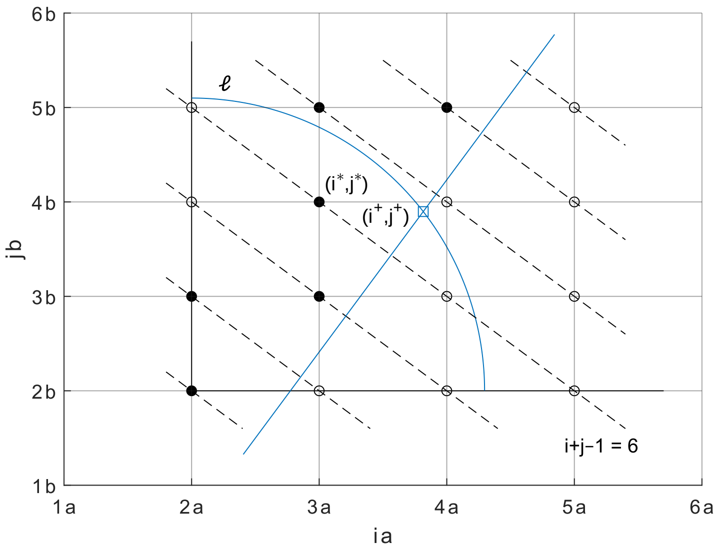
More specifically, assuming that (6) holds, the largest i that can be achieved with lengths not exceeding ℓ is obtained, as given by Proposition 2.(iii). This value, i^*, is computed by rounding down i^+. Then, assuming that $i = i^*$, the largest j allowed by Proposition 2.(iii), j^*, is obtained. As will be seen, in some cases the resulting (i^*, j^*) is in M, and in other cases it is not. However, in either case (i^*, j^*) has the largest $i+j-1$ sum that can be achieved with segments of length up to ℓ. Thus the desired result is i^*+j^*-1.

The two possibilities are illustrated in Figure 8 for $a = 1.35$, $b = 1$. In each case, the arc displayed in the graph is centered at $(2a, 2b)$ and has radius ℓ. The inner region defined by the arc contains all (i, j) pairs that are achievable according to Proposition 2.(iii). As in previous figures, filled circles represent (i, j) pairs that are in M, and empty circles are those that do not belong to M. The solid line is (6). The intersection point (i^+, j^+) is displayed with a square marker.

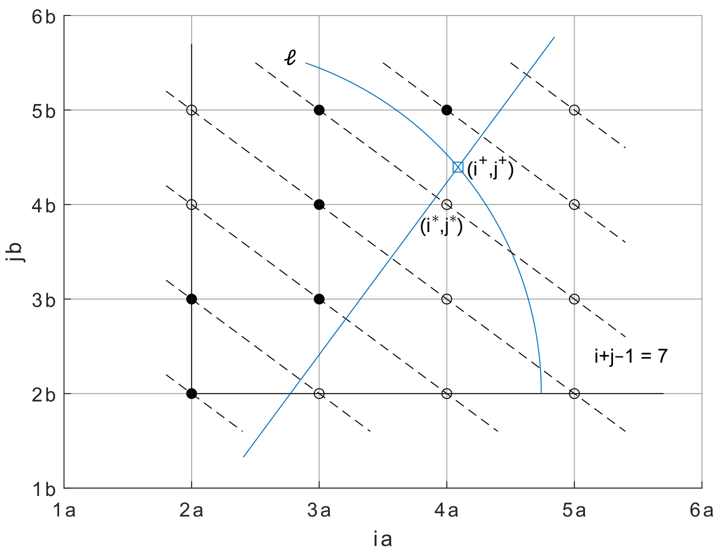
The point (i^+, j^+) can be obtained as a solution of the equation system

$$(i^+-2)^2a^2 + (j^+-2)^2b^2 = \ell^2, \tag{22}$$

$$j^+ = \frac{i^+a^2}{b^2} - \frac{5a^2}{2b^2} + \frac{3}{2}. \tag{23}$$



(a) $\ell = 3.1$: $(i^*, j^*) \in M$



(b) $\ell = 3.7$: $(i^*, j^*) \notin M$

Figure 8: Obtaining (i^*, j^*) in Theorem 1. Examples with $a = 1.35$, $b = 1$

Substituting (23) into (22) yields a quadratic equation for i^+,

$$4a^2(i^+-2)^2-4a^2(i^+-2)+a^2+b^2-\frac{4\ell^2b^2}{a^2+b^2}, \tag{24}$$

which can have zero, one or two real-valued solutions (this respectively means that the line is exterior, tangent or secant to the circle (22)). Only the solution with $i^+, j^+ \geq 2$, if any, is of interest. This solution can only exist as the one with largest i^+ value when there are real-valued solutions, and it is obtained as

$$i^+=\frac{a+b\sqrt{4\ell^2/(a^2+b^2)-1}}{2a}+2. \tag{25}$$

The case with no real-valued solutions to (24) corresponds to $\ell < \sqrt{a^2+b^2}/2$. Since $a \geq b$, this implies that $\ell < a$. Thus any achievable (i, j) pair, if any, will have $i = 2$. Therefore in this case i^* should be set to 2; and then the maximum achievable j^* will be obtained from (23).

If (24) has two real-valued solutions or one real-valued double solution, which is the case when $\ell \geq \sqrt{a^2+b^2}/2$, the largest solution will have $j^+ < 2$ if $i^+ < (b^2/a^2+5)/2$, as can be seen setting $j^+ = 2$ in (23); or equivalently if $\sqrt{a^2+b^2}/2 \leq \ell < (a^2+b^2)/(2a)$, as can be seen substituting $j^+ < 2$ and $i^+ < (b^2/a^2+5)/2$ in (22). Since $i^+ < (b^2/a^2+5)/2 < 3$ for $a \geq b$, only pairs with $i = 2$ are achievable in this case again, and thus the solution i^* should also be 2.

Lastly, for $\ell \geq (a^2+b^2)/(2a)$ the expressions (23) and (25) give $i^+, j^+ \geq 2$, and i^* should be taken as the greatest integer less than i^+, i.e. $\lceil i^+ \rceil - 1$.

The three cases are unified by taking the real part of (25) and computing $i^* = \lceil i^+ \rceil - 1$, as is easily checked. This gives (20). Once i^* is known, j^* is computed, according to (21), as the greatest integer such that (i^*, j^*) is within the arc determined by ℓ. This ensures that (i^*, j^*) is achievable with lengths less than ℓ.

There are two possibilities for the point (i^*, j^*), as stated at the outset. These are illustrated in Figures 8(a) and 8(b) respectively. The first possibility is that $(i^*, j^*) \in M$ (Figure 8(a)). Then, by construction (i^*, j^*) maximizes $i + j - 1$ among all achievable pairs of M, and is therefore optimal.

The second possibility is that $(i^*, j^*) \notin M$ (Figure 8(b)). This happens when the point from M that has $i = i^*$ ($(4,5)$ in the figure) is outside the arc, i.e. it would require a length greater than ℓ. The selected point (i^*, j^*) ($(4,4)$ in the figure), however, has the same $i + j - 1$ sum as the point from M that "should" be used, which is (i^*-1, j^*+1) ($(3,5)$ in the figure); and therefore gives the same result. This is always the case, because $(i^*, j^*+1) \in M$ (it is above or on the bounding line) and $(i^*, j^*) \notin M$ (it is below the line), and due to how M has been constructed, this implies that $(i^*-1, j^*+1) \in M$ and $(i^*-1, j^*+k) \notin M$ for $k = 2, 3, \ldots$. Therefore, (i^*, j^*) is achievable and maximizes $i + j - 1$, which implies that $i^* + j^* - 1$ is the desired solution.

Therefore, whether $(i^*, j^*) \in M$ or $(i^*, j^*) \notin M$, equations (20) and (21) give $i^* + j^* - 1$ equal to $\tau(\ell)$, as claimed.

It should be noted that for the specific case that $a^2/b^2 = 2k-1$, $k \in \mathbb{N}$ the lower bounding line (6) becomes $j = i(2k-1) - 5k + 4$, which gives integer j for integer i. This means that the bound is actually achieved for all pairs in M (see for example Figure 7(a)), and the case $(i^*, j^*) \notin M$ never occurs. Thus, when a^2/b^2 is an odd integer the obtained (i^*, j^*) is guaranteed to be in M, whereas for general a^2/b^2 either (i^*, j^*) or (i^*-1, j^*+1) are in in this set.

As for the properties of τ, it follows from (19)–(21) that this function is piecewise constant and left-continuous. From the procedure described in the previous paragraphs for obtaining (i^*, j^*) it is clear that $i^* + j^* - 1$ increases in steps of 1 when ℓ is increased continuously; that is, τ has jumps of unit height.

Consider an arbitrary ℓ such that for some $t \in \mathbb{N}$, $t \geq 4$

$$\lambda(t) = \ell. \tag{26}$$

To see that τ has a jump at ℓ, assume for the sake of contradiction that τ is continuous, therefore constant, at that point. Then there is $\varepsilon > 0$ for which $\tau(\ell+\varepsilon) = t = \tau(\ell-\varepsilon)$. This means that there exists a segment with length $\ell-\varepsilon$ that visits t tiles, and thus $\lambda(t) \leq \ell - \varepsilon < \ell$, in contradiction with (26). Therefore τ is discontinuous, from the right, at ℓ. By definition of λ, (26) implies that $\tau(\ell) < t$, and that there exists $\varepsilon > 0$ such that $\tau(\ell+h) = t$ for $0 < h < \varepsilon$. This means that

$$\tau(\ell) < t, \tag{27}$$

$$\lim_{h \to 0+} \tau(\ell+h) = t, \tag{28}$$

that is, τ has a jump at ℓ. In addition, since the jump has unit height, it follows from (27) and (28) that $\tau(\ell) = t-1$.

Conversely, assume that (27) and (28) hold for some $\ell \in \mathbb{R}^+$, $t \in \mathbb{N}$. From (27) it follows that $\lambda(t) \geq \ell$. On the other hand, (28) implies that $\lambda(t) \leq \ell$. Thus $\lambda(t) = \ell$. □

Although Theorem 1 is valid for $a \geq b$, the result could be applied for $a < b$ by simply swapping the values of a and b. In other words, (19)–(21) can be used for any grid if a, b are interpreted as the largest and smallest sides of a tile, respectively.

Theorem 2. *For $a, b \in \mathbb{R}^+$ and $t \in \mathbb{N}$,*

$$\lambda(t) = \sqrt{(i^*-2)^2 a^2 + (j^*-2)^2 b^2} \tag{29}$$

with

$$i^+ = \max\left\{\frac{(t-3)b^2}{a^2+b^2}, 0\right\} + 2, \tag{30}$$

$$j^+ = \max\left\{\frac{(t-3)a^2}{a^2+b^2}, 0\right\} + 2, \tag{31}$$

$$i^* = \left\lfloor i^+ + \frac{1}{2} \right\rfloor, \tag{32}$$

$$j^* = \left\lceil j^+ - \frac{1}{2} \right\rceil. \tag{33}$$

Equivalently, for $t \geq 3$*,*

$$\lambda(t) = \sqrt{\frac{(t-3)^2 a^2 b^2}{a^2+b^2} + r^2(a^2+b^2)} \tag{34}$$

with

$$r = |i^* - i^+| = |j^* - j^+|. \tag{35}$$

This function is monotone increasing for $t \geq 3$*.*

Proof. The (i_t, j_t) pair in the minimal sufficient set M corresponds to a maximum of t visited tiles. By construction of this set, any segment that visits t tiles must have length greater than $\sqrt{(i_t-2)^2a^2+(j_t-2)^2b^2}$. For $t \geq 3$ the variables i^*, j^* in (32) and (33) respectively coincide with i_t, j_t as given by (3) and (4). Therefore (29) gives the correct result. For $t \in \{1,2\}$ both (32) and (33) equal 2, and (29) gives 0, which is again the correct result.

For $t \geq 3$, the term $\sqrt{(i_t-2)^2a^2+(j_t-2)^2b^2}$ can be interpreted geometrically as the distance between (i_t, j_t) and $(2a, 2b)$. As can be seen with the help of Figure 6, this distance is the hypotenuse of a right triangle whose other two sides extend from $(2a, 2b)$ to (i^+a, j^+b) and from (i^+a, j^+b) to (i_t, j_t) respectively. Therefore,

$$\begin{aligned}(i_t-2)^2a^2+(j_t-2)^2b^2 = (i^+-2)^2a^2+(j^+-2)^2b^2 \\ +(i_t-i^+)^2a^2+(j_t-j^+)^2b^2.\end{aligned} \tag{36}$$

For $t \geq 3$ it stems from (30) and (31) that

$$(i^+-2)^2a^2+(j^+-2)^2b^2 = \frac{(t-3)^2(b^4a^2+a^2b^4)}{(a^2+b^2)^2} = \frac{(t-3)^2a^2b^2}{a^2+b^2}. \tag{37}$$

The fact that both (i^+, j^+) and (i_t, j_t) are on the line $i+j-1=t$ implies that $i^+ + j^+ = i_t + j_t$, and thus $|i_t - i^+| = |j_t - j^+| = r$. Consequently,

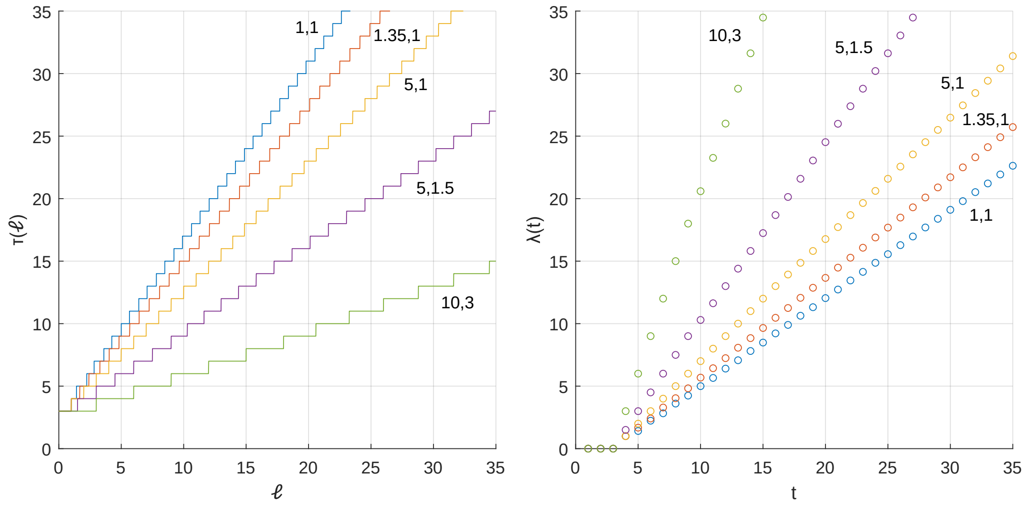
$$(i_t-i^+)^2a^2+(j_t-j^+)^2b^2 = r^2(a^2+b^2). \tag{38}$$

Substituting (37) and (38) into (36) yields (34).

By Theorem 1, τ has unit-height jumps at the values $\lambda(t)$, $t \in \mathbb{N}$, $t \geq 4$. This implies that λ is monotone increasing for $t \geq 3$. □

***Comment on how the new formula has two terms with definite meanings: The first term is the distance-squared to the closest point on the diagonal, The second term is additional distance-squared incurred from rounding (i,j) to integers ***I think we can do without a figure (Figure 6 is inside a proof and uses different notation for the final point).

Theorems 1 and 2 not only give the solutions $\tau(\ell)$ and $\lambda(t)$ to the two problems stated in §1; they also provide a way to actually position a segment of length slightly greater than ℓ or $\lambda(t)$ so that it visits t or $\tau(\ell)$ tiles. Namely, the segment should have its endpoints respectively in the interior of two tiles separated $i^* - 1$ steps horizontally and $j^* - 1$ steps vertically, with its exact position and orientation adjusted to avoid any grid points.



(a) Function τ

(b) Function λ

Figure 9: Functions τ and λ for several pairs a, b

It is interesting to consider the following particular cases: $\ell \gg a,b$; $a \gg b$; and $a = b$. Regarding the first, from (19)–(21) and from (29)–(33) it is seen that for long segments the number of visited tiles and the segment length are approximately proportional, with

$$\lim_{\ell\to\infty} \frac{\tau(\ell)}{\ell} = \lim_{t\to\infty} \frac{t}{\lambda(t)} = \sqrt{1/a^2 + 1/b^2}. \tag{39}$$

As for $a \gg b$, in this case the optimal discrete bounding rectangle has $i^* = 2$, and j^* as large as allowed by ℓ (direct problem) or as required by t (inverse problem), corresponding to an almost vertical segment. Obviously, for $a \gg b$ the length of the segment is best invested in increasing the number of tiles traversed vertically (but the segment should be slightly tilted to cross a vertical edge), and the asymptotic value of (39) is approximately $1/b$.

For $a = b$, either from symmetry considerations or particularizing the formulas in the previous theorems it stems that the optimal orientation of the segment is close to 45°. This case will be dealt with in §3.2, as it lends itself to simplified formulas.

Figure 9 shows τ and λ for several pairs of grid parameters a, b. The graphs illustrate some of the observations of the previous paragraphs. Indeed, the asymptotic slope in Figure 9(a), or the inverse of the asymptotic slope in Figure 9(b), is approximately $\sqrt{2}$ for $a, b = 1$; and it is $1/b$ for the case $a = 5, b = 1$, or even for $a = 5, b = 1.5$ or $a = 10, b = 3$. Comparing the latter two cases it is also seen that scaling a, b and ℓ by the same factor does not alter $\tau(\ell)$, and results in $\lambda(t)$ being scaled by that factor.

3.2 Unit square grid with real-valued lengths

A square grid has $a = b$. For real-valued segment lengths it can be further assumed that $a = 1$. For $a \neq 1$ the results to be obtained apply with ℓ replaced by

ℓ/a.

Particularizing the results in §3.1 to $a = b = 1$ obviously yields simpler formulas.

Corollary 1. *For a unit square grid with $\ell \in \mathbb{R}^+$,*

$$\tau(\ell) = i^* + j^* - 1 \tag{40}$$

with

$$i^* = \left\lceil \frac{1 + \mathrm{Re}\sqrt{2\ell^2 - 1}}{2} \right\rceil + 1, \tag{41}$$

$$j^* = \left\lceil \sqrt{\ell^2 - (i^* - 2)^2} \right\rceil + 1, \tag{42}$$

Corollary 2. *For a unit square grid, and for $t \in \mathbb{N}$,*

$$\lambda(t) = \begin{cases} 0 & \text{for } t = 1, 2 \\ \dfrac{t-3}{\sqrt{2}} & \text{for } t \text{ odd, } t \geq 3 \\ \sqrt{\dfrac{(t-3)^2 + 1}{2}} & \text{for } t \text{ even, } t \geq 4, \end{cases} \tag{43}$$

or equivalently

$$\lambda(t) = \begin{cases} 0 & \text{for } t = 1, 2 \\ \sqrt{\left\lceil \dfrac{(t-3)^2}{2} \right\rceil} & \text{for } t \geq 3. \end{cases} \tag{44}$$

Furthermore, an even simpler formula can be obtained for τ, as the next theorem shows.

Theorem 3. *For a unit square grid with $\ell \in \mathbb{R}^+$,*

$$\tau(\ell) = i^* + j^* - 1 \tag{45}$$

with

$$i^* = \left\lceil \frac{\ell}{\sqrt{2}} \right\rceil + 1, \tag{46}$$

$$j^* = \left\lceil \sqrt{\ell^2 - (i^* - 2)^2} \right\rceil + 1. \tag{47}$$

Equivalently,

$$\tau(\ell) = \left\lfloor \sqrt{2\lceil \ell^2 \rceil - 2} \right\rfloor + 3. \tag{48}$$

Proof. The proof of (45) uses a variation of the minimal sufficient set M defined in Proposition 3 that is more suited to this situation.

For $a = b = 1$, the set M consists of points of the form (i,i) and $(i,i-1)$, as is easily seen from Proposition 3, and as illustrated in Figure 7(a). By symmetry, replacing each point $(i,i-1)$ by $(i-1,i)$ gives a set M' which is also minimal sufficient. For this new set, the lower bounding line (6) can be replaced by the simpler $j = i$. The same approach used in the proof of Theorem 1 can be followed, but using this line. Thus (i^+, j^+) is obtained from

$$(i^+ - 2)^2 + (j^+ - 2)^2 = \ell^2, \tag{49}$$

$$j^+ = i^*, \tag{50}$$

which gives

$$i^+ = \ell/\sqrt{2} + 2. \tag{51}$$

The pair (i^*, j^*) resulting from (51) is in M', and is given in (46) and (47). This establishes (45).

To show (48), it is first noted that for $t \geq 3$ Corollary 2 gives

$$\lambda(t) = \sqrt{\left\lceil \frac{(t-3)^2}{2} \right\rceil}. \tag{52}$$

According to (17), the function $\tau(\ell)$ is given by the largest positive integer t such that

$$\left\lceil \frac{(t-3)^2}{2} \right\rceil < \ell^2. \tag{53}$$

Since the left-hand side of (53) is an integer, the condition of being strictly less than ℓ^2 is equivalent to

$$\left\lceil \frac{(t-3)^2}{2} \right\rceil \leq \lceil \ell^2 \rceil - 1, \tag{54}$$

which in turn is the same as

$$\frac{(t-3)^2}{2} \leq \lceil \ell^2 \rceil - 1. \tag{55}$$

Finally, solving for t gives

$$t \leq \sqrt{2\lceil \ell^2 \rceil - 2} + 3. \tag{56}$$

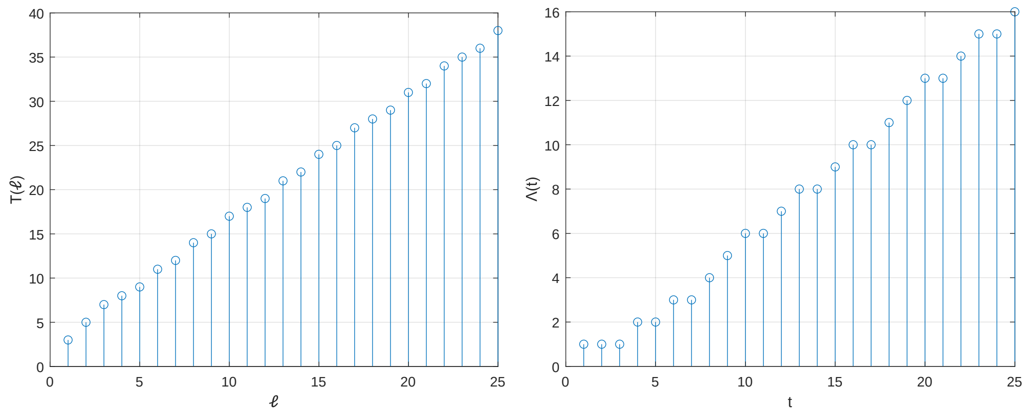
The desired quantity $\tau(\ell)$, that is the largest positive integer t satisfying (56), is thus the right-hand side rounded down, as given by (48). □

From Theorem 3 it is easily seen that odd values of $\tau(\ell)$ correspond to $i^* = j^*$, whereas even values are achieved with $j^* = i^* + 1$. Given $t \geq 3$ with t odd, $\tau(\ell) = t$ if and only

$$\ell \in \left(\frac{t-3}{\sqrt{2}}, \frac{\sqrt{(t-1)^2 + (t-3)^2}}{2} \right]. \tag{57}$$

For $t \geq 4$, t even, $\tau(\ell) = t$ if and only if

$$\ell \in \left(\frac{\sqrt{(t-2)^2 + (t-4)^2}}{2}, \frac{t-2}{\sqrt{2}} \right]. \tag{58}$$



(a) Sequence T (b) Sequence Λ

Figure 10: Sequences T and Λ

3.3 Unit square grid with integer lengths

A natural variation of the first two problems introduced in §1 is to consider $a = b = 1$ with the additional restriction that the segment length can only be a positive integer (equivalently, the square grid has spacing a and the segment length can only be an integer multiple of a).

The *direct problem* in this setting corresponds to the restriction of the function τ to $\mathbb{N}$. This will be denoted as a function $T : \mathbb{N} \to \mathbb{N}$ for greater clarity, although obviously $T(\ell) = \tau(\ell)$ for all $\ell \in \mathbb{N}$. The sequence $T(\ell)$, $\ell \in \mathbb{N}$, which takes values $3, 5, 7, 8, 9, \ldots$, is depicted in Figure 10(a). This is A346232 in the On-Line Encyclopedia of Integer Sequences [5]. For this sequence, the expression in Theorem 3 simplifies in the obvious way, and the following properties hold.

Theorem 4. *For* $\ell \in \mathbb{N}$,

$$T(\ell) = \left\lfloor \sqrt{2\ell^2 - 2} \right\rfloor + 3. \tag{59}$$

In addition,

(i) This sequence is increasing, with $T(\ell+1) - T(\ell) \in \{1, 2\}$.

(ii) There can be no more than 2 *consecutive increments equal to* 1.

(iii) Increments equal to 2 *always appear isolated, except at the initial sequence terms* $3, 5, 7$.

Proof. The equality (59) stems from (48) noting that ℓ is an integer.

In order to prove that $T(\ell+1) - T(\ell) \in \{1, 2\}$, consider the function $q(s) = \sqrt{2s^2 - 2}$ for $s \in \mathbb{R}$, $s > 1$. Its first derivative is

$$q'(s) = \frac{\sqrt{2}\, s}{\sqrt{s^2 - 1}}, \tag{60}$$

and its second derivative is easily seen to be negative. Therefore $q'(s)$ can be bounded for $s \geq 3$ as

$$\lim_{s\to\infty} q'(s) = \sqrt{2} < q'(s) < q'(3) = 3/2. \tag{61}$$

For $\ell \geq 2$, by the mean value theorem [1, section 5.3], when ℓ is increased to $\ell+1$ the term $\sqrt{2\ell^2-2}$ in (59) has an increment that equals $q'(s)$ for some $\ell < s < \ell+1$. Therefore

$$\sqrt{2} < \sqrt{2(\ell+1)^2-2} - \sqrt{2\ell^2-2} < 3/2. \tag{62}$$

Since $1 < \sqrt{2}$ and $3/2 < 2$, (62) implies that $T(\ell+1) - T(\ell)$ can only take the values 1 or 2 for $\ell \geq 3$. In addition, $T(2) - T(1) = T(3) - T(2) = 2$, and thus the result holds for all $\ell \in \mathbb{N}$.

Using the first bound in (62) three times,

$$3\sqrt{2} < \sqrt{2(\ell+3)^2-2} - \sqrt{2\ell^2-2}. \tag{63}$$

Considering that $4 < 3\sqrt{2}$, this implies that $T(\ell+3) - T(\ell) \geq 4$ for $\ell \geq 3$. Therefore at least one of the three increments from $T(\ell)$ to $T(\ell+3)$ is 2. Since $T(2) - T(1) = T(3) - T(2) = 2$, the result holds for all $\ell \in \mathbb{N}$.

Similarly, using the second bound in (62) twice,

$$\sqrt{2(\ell+2)^2-2} - \sqrt{2\ell^2-2} < 3, \tag{64}$$

which implies that $T(\ell+2) - T(\ell) \leq 3$ for $\ell \geq 3$. Therefore the two increments $T(\ell+1) - T(\ell)$ and $T(\ell+2) - T(\ell+1)$ cannot both be 2 for $\ell \geq 3$. □

The *inverse problem* with integer-length segments can be formulated as follows: given $t \in \mathbb{N}$, find the *minimum* integer length that allows visiting at least t tiles. Observe that in this case, unlike with real-valued lengths, there is indeed a minimum length, because every subset of $\mathbb{N}$ has a minimum. This can be expressed as a function $\Lambda : \mathbb{N} \to \mathbb{N}$:

$$\Lambda(t) = \min\{\ell \in \mathbb{N} \mid T(\ell) \geq t\}, \tag{65}$$

which is related to the function λ corresponding to real-valued lengths by

$$\Lambda(t) = \lfloor \lambda(t) \rfloor + 1. \tag{66}$$

The converse to (65) is (compare to (17)):

$$T(\ell) = \max\{t \in \mathbb{N} \mid \Lambda(t) \leq \ell\}. \tag{67}$$

In view of (65) and (67), T and Λ can be considered as "pseudo-inverse" sequences of each other.

The sequence $\Lambda(t)$, $t \in \mathbb{N}$ can be computed using (44) and (66), and has initial values $1,1,1,2,2,3,3,4,5\ldots$, as shown in Figure 10(b). This is A346693 in the On-Line Encyclopedia of Integer Sequences [6]. However, a slightly simpler expression can be obtained from (59) and (65). This is established by the next theorem, which also states some properties of Λ, parallel to those of T.

Theorem 5. *For $t \in \mathbb{N}$,*

$$\Lambda(t) = \begin{cases} 1 & \text{for } t \leq 3 \\ \left\lceil \sqrt{\dfrac{(t-3)^2}{2} + 1} \right\rceil & \text{for } t \geq 4. \end{cases} \tag{68}$$

In addition,

(i) This sequence is non-decreasing. Except for the initial run of 3 equal values, it is formed by runs of 1 or 2 equal values, with an increment of 1 between consecutive runs.

(ii) There can be no more than 3 different consecutive terms.

(iii) A run of 2 equal values always has 2 different terms before and 2 different terms after the run, except for the initial terms $1, 1, 1, 2, 2, 3, 3$.

Proof. Using (59), the inequality $T(\ell) \geq t$ in (65) is written as

$$\left\lfloor \sqrt{2\ell^2 - 2} \right\rfloor + 3 \geq t. \tag{69}$$

Since the right-hand side is an integer, this is equivalent to

$$\sqrt{2\ell^2 - 2} \geq t - 3. \tag{70}$$

For $t \geq 4$, taking squares and rearranging gives

$$\ell \geq \left\lceil \sqrt{\frac{(t-3)^2}{2} + 1} \right\rceil, \tag{71}$$

which combined with (65) yields the second part of (68). The first part results from noting that $t - 3 \leq 0$ for $t \leq 3$, and thus $\ell = 1$ satisfies (70).

The stated properties for Λ follow directly from those of T established by Theorem 4. □

4 Probabilistic characterization for a random segment

Given $\ell \in \mathbb{R}^+$, consider a segment of length ℓ with uniformly random position and orientation. Specifically, the coordinates x_1, y_1 of the first endpoint are independent random variables uniformly distributed on $[0, a)$ and $[0, b)$ respectively. The orientation θ of the segment is uniformly distributed on $[0, 2\pi)$. The variables x_1, y_1 and θ determine the coordinates x_2, y_2 of the second endpoint.

Each realization of the random segment gives rise to a discrete bounding rectangle, whose normalized dimensions i and j are thus random variables, as is the number t of visited tiles. Except for a set of realizations with zero probability, i and j are at least 1, and $t = i + j - 1$ tiles. Note that i and j are not statistically independent.

4.1 Arbitrary grid with real-valued lengths

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined such that $\varphi(\ell)$ gives the average number of tiles visited by a random segment of length ℓ, with the distributions specified in §4. This function can be easily computed using the fact that

$$\varphi(\ell) = \mathrm{E}[t] = \mathrm{E}[i] + \mathrm{E}[j] - 1, \tag{72}$$

and noting that events of probability 0, such as $x_1 = a$ or the segment passing through a grid point, can be disregarded.

Theorem 6. *Given $a, b, \ell \in \mathbb{R}^+$, consider a grid with parameters a, b and a uniformly random segment of length ℓ, as defined above. The average number of tiles visited by the segment, $\varphi(\ell)$, is*

$$\varphi(\ell) = \frac{2\ell}{\pi}\left(\frac{1}{a} + \frac{1}{b}\right) + 1. \tag{73}$$

Proof. First, suppose the grid is simplified to just horizontal lines a height of b apart. This matches the setup of the 1733 Buffon's needle problem described in [3], except that here the length ℓ of the needle may exceed the spacing b, allowing it to cross multiple lines. As shown in [4], the expected (i.e., average) number of lines crossed equals

$$\frac{2\ell}{\pi b}.$$

Where $b < \ell$, this expectation matches the probability of the needle crossing a grid line, since it cannot cross more than one.

Now consider again a grid of width a and height b, as studied by Laplace as an extension to Buffon's original needle problem [2]. The grid crossings decompose into crossings of horizontal and vertical grid lines, respectively. By linearity of expectation, the expected number of crossings is the sum of the respective expectations for parallel lines of width a and b respectively, giving:

$$\frac{2\ell}{\pi}\left(\frac{1}{a} + \frac{1}{b}\right)$$

As noted in Proposition 1, the number of tiles visited by a segment is the count of its grid line crossings plus 1, unless it exactly crosses any grid points, but again that occurs with probability 0 and therefore it does not affect the expectation. Thus, $\varphi(\ell)$ is the above quantity plus 1. □

In view of Theorem 6, the average number of visited tiles as a function of the segment length has a very simple form, namely an affine function. Conversely, for any $t > 1$ it is immediate to compute the length of a random segment that visits t tiles on average, given as $\varphi^{-1}(t)$.

In spite of the dependence between the random variables i and j, their marginal distributions have relatively simple analytic expressions, as established by the next proposition.

For $0 \le s \le 1$, let

$$f(s) = \int_0^s \arccos z \, \mathrm{d}z = s \arccos s - \sqrt{1 - s^2} + 1. \tag{74}$$

Proposition 4. *Given $a, b, \ell \in \mathbb{R}^+$, consider a grid with parameters a, b and a uniformly random segment of length ℓ, as defined above. Let the random variables i, j represent the normalized dimensions of the discrete bounding rectangle. For $n \in \mathbb{N}$,*

$$\Pr[i \geq n] = \begin{cases} 1 & \text{if } n = 1 \\ \dfrac{2\ell}{\pi a}\left(f\left(\dfrac{a(n-1)}{\ell}\right) - f\left(\dfrac{a(n-2)}{\ell}\right)\right) & \text{if } 2 \leq n < \dfrac{\ell}{a}+1 \\ \dfrac{2\ell}{\pi a}\left(1 - f\left(\dfrac{a(n-2)}{\ell}\right)\right) & \text{if } \dfrac{\ell}{a}+1 \leq n < \dfrac{\ell}{a}+2 \\ 0 & \text{if } \dfrac{\ell}{a}+2 \leq n; \end{cases} \tag{75}$$

and $\Pr[j \geq n]$ *is given by the same expressions with a replaced by b.*

Proof. Clearly, $\Pr[i \geq 1] = 1$. In the following it will be assumed that $n \geq 2$. The basic idea is to compute $\Pr[i \geq n]$ conditioned on (x_1, y_1) (or, as will be seen, only on x_1), and then to average over x_1 and y_1 (actually only over x_1).

Given the coordinates (x_1, y_1) of the first endpoint of the segment, with $0 \leq x_1 \leq a$, $0 \leq y_1 \leq b$, the second endpoint (x_2, y_2) lies on a circle with radius ℓ centered at (x_1, y_1), as shown in Figure 11. The segment orientation is a random angle θ uniformly distributed on $[0, 2\pi)$. It is clear from the figure that $i \geq n$ if and only if $x_2 \geq a(n-1)$ or $x_2 \leq -a(n-2)$; and these events are exclusive for $n \geq 2$. Thus

$$\Pr[i \geq n \,|\, x_1, y_1] = \Pr[x_2 \geq a(n-1) \,|\, x_1, y_1] + \Pr[x_2 \leq -a(n-2) \,|\, x_1, y_1]. \tag{76}$$

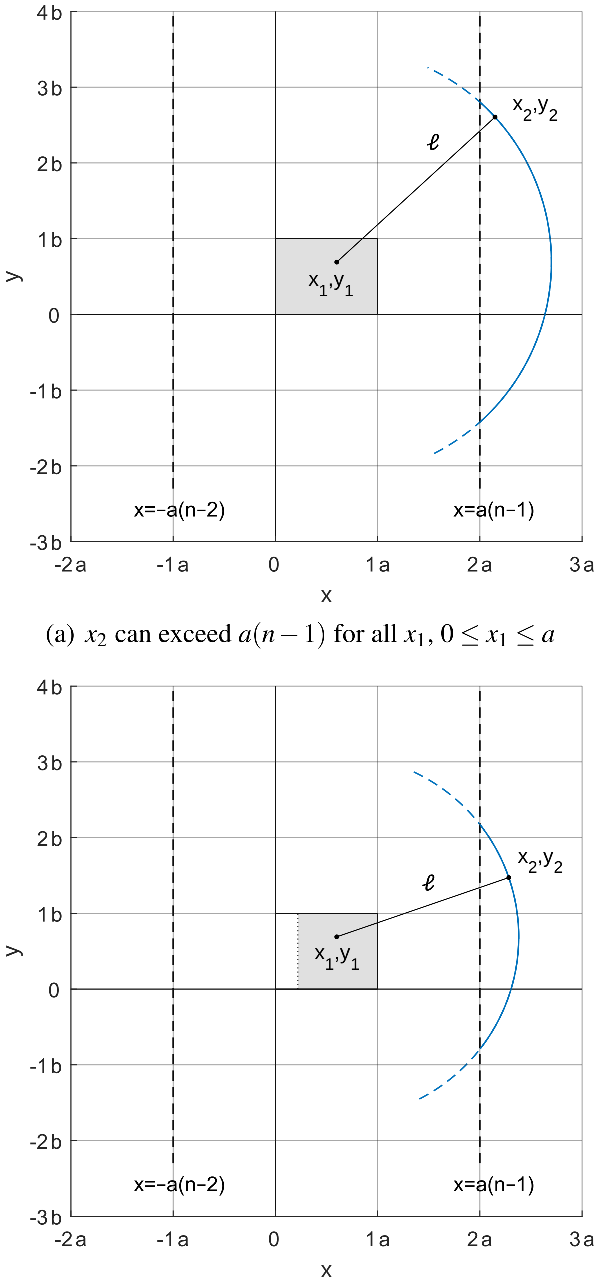
The two conditional probabilities on the right-hand side of (76) are different in general. However, averaging over x_1, y_1 gives, by symmetry, $\Pr[x_2 \geq a(n-1)] = \Pr[x_2 \leq -a(n-2)]$. In addition, the coordinate y_1 does not have any influence on these events, and therefore conditioning on x_1, y_1 is the same as conditioning on x_1. This implies that, for $n \geq 2$,

$$\Pr[i \geq n] = 2\Pr[x_2 \geq a(n-1)] = 2\,\mathrm{E}[\Pr[x_2 \geq a(n-1) \,|\, x_1]]. \tag{77}$$

Consider the event $x_2 \geq a(n-1)$ conditioned on x_1, with $n \geq 2$. There are three possibilities depending on x_1, n and ℓ. If $a(n-1) < \ell$, regardless of x_1 the length ℓ is enough for x_2 to exceed $a(n-1)$ for some angles θ. This is depicted in Figure 11(a), where the section of the arc with solid line represents those angles for which $x_2 > a(n-1)$. If $a(n-2) < \ell \leq a(n-1)$, the length will be enough provided that $x_1 \geq a(n-1) - \ell$, which corresponds to the shaded region in Figure 11(b). Lastly, if $\ell \leq a(n-2)$ it is not possible for x_2 to exceed $a(n-1)$. Note that the coordinate y_1 is irrelevant to this.

In the first two cases above, the probability that $x_2 \geq a(n-1)$, conditioned on x_1, is the length of the arc to the right of the line $x = a(n-1)$ divided by $2\pi\ell$, that is,

$$\Pr[x_2 \geq a(n-1) \,|\, x_1] = \frac{1}{\pi} \arccos \frac{a(n-1) - x_1}{\ell}. \tag{78}$$



(a) x_2 can exceed $a(n-1)$ for all x_1, $0 \leq x_1 \leq a$

(b) x_2 can exceed $a(n-1)$ only if $a(n-1) - \ell \leq x_1 \leq a$

Figure 11: Conditions for the normalized width of the discrete bounding rectangle, i, to be equal or greater than a given n. Example with $a = 1.35$, $b = 1$, $n = 3$

In the first case x_1 has a uniform distribution on $(0,a)$, and $\Pr[x_2 \geq a(n-1)]$ is easily obtained from (78) as

$$\begin{aligned}\Pr[x_2 \geq a(n-1)] &= \frac{1}{\pi a}\int_0^a \arccos\frac{a(n-1)-x_1}{\ell}\,\mathrm{d}x_1 \\ &= \frac{\ell}{\pi a}\left(f\left(\frac{a(n-1)}{\ell}\right)-f\left(\frac{a(n-2)}{\ell}\right)\right),\end{aligned} \tag{79}$$

where the function f is defined in (74). Substituting into (77) yields the result in (75), second line.

The second case is similar, but the integration over x_1 is from $a(n-1)-\ell$ to a. Noting that $f(1)=1$, this gives

$$\Pr[x_2 \geq a(n-1)] = \frac{\ell}{\pi a}\left(1-f\left(\frac{a(n-2)}{\ell}\right)\right), \tag{80}$$

which combined with (77) yields the expression in (75), third line.

The third case obviously gives $\Pr[i \geq n] = 0$, as in (75), fourth line.

The above arguments can be applied to $\Pr[j \geq n]$ if the x and y axes are interchanged. Thus the result is the same with a replaced by b. □

The results in Theorem 6 and Proposition 4 make clear the relationship between the problem considered in this section here and Buffon's needle problem, i.e. the probability that a random segment of fixed length crosses a line in a regular structure of parallel strips [3, section 1.1]. Firstly, since the number π is involved in (73), it is possible to design a simple probabilistic experiment to estimate the value of π, as in Buffon's original experiment. For example, choosing $a=b=\ell=1$ gives $\varphi(\ell)=1+1/\pi$.

Secondly, a grid with $b\to\infty$ corresponds to Buffon's arrangement of parallel strips with spacing a. Thus

$$\lim_{b\to\infty}\varphi(\ell)-1=\frac{2\ell}{\pi a} \tag{81}$$

gives the average number of lines crossed in Buffon's experiment. For $\ell \leq a$ the segment can cross at most one line, and (81) coincides with the probability of crossing [3, section 1.1].

Lastly, $\Pr[i \geq n]$ as computed in Proposition 4 for $n \geq 2$ can be seen as a generalization of the probability of crossing at least one line in Buffon's experiment, when a needle endpoint is only allowed to move in a narrower region of width $n-1$ times smaller than the width of the strips (instead of across the full strip as in Buffon's setting).

***Maybe a graph comparing maximum and average. Or compute/plot their ratio. The asymptotic slopes are $\sqrt(1/a^2+1/b^2)$ for τ and $2/\pi(1/a+1/b)$. It may be interesting to consider the ratio of asymptotic slopes, as a function of a/b. It is maximum for $a=b$, and equals $2\sqrt{2}/\pi = 0.9003$. It tends to $2/\pi$ for $a/b\to\infty$ and for $a/b\to 0$. The minimum number of tiles that can be visited with non-zero probability (corresponding to an n times 1 discrete bounding rectangle if a¿=b) can also be plotted.

The probability of visiting the maximum number of tiles for a rectangular grid is difficult to compute, due to the irregularity of the minimal sufficient set (see §2). In the square case, however, the simplicity of the set makes the problem more tractable.

4.2 Unit square grid with real-valued lengths

Consider a square grid with unit spacing, $a = b = 1$. Again, the results to follow can be applied to a square grid with spacing $a \neq 1$ if ℓ is replaced by ℓ/a.

For a random segment with length $\ell \in \mathbb{R}^+$ on a unit square grid, let the function $\rho : \mathbb{R}^+ \to [0,1]$ be defined such that $\rho(\ell)$ gives the probability that the segment visits the maximum possible number of tiles, $\tau(\ell)$. This function is characterized by the next theorem.

For $u, v \in \mathbb{N}$, $s \in \mathbb{R}^+$, let

$$g(s,u,v) = \int_{\arcsin \frac{u}{2r}}^{\arccos \frac{u}{2r}} \left(s \sin\theta - \frac{u}{2}\right)\left(s\cos\theta - \frac{v}{2}\right) \mathrm{d}\theta. \tag{82}$$

Using the identities $\cos^2\alpha = 1 - \sin^2\alpha$ and $\cos(2\alpha) = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$, the integral in (82) is computed as

$$\begin{aligned} g(s,u,v) = \frac{1}{2\pi}\Bigg(&\left(\arccos\frac{u}{2s} - \arcsin\frac{v}{2s}\right) uv + 2s^2 + \frac{u^2}{2} + \frac{v^2}{2} \\ &- u\sqrt{4s^2 - v^2} - v\sqrt{4s^2 - u^2}\Bigg). \end{aligned} \tag{83}$$

Theorem 7. *Given $\ell \in \mathbb{R}^+$, consider a unit square grid and a uniformly random segment of length ℓ, as previously defined. The probability $\rho(\ell)$ that the segment visits the maximum number of tiles $\tau(\ell)$ is*

$$\begin{aligned} &\rho(\ell) = g(\ell, t-3, t-3) \quad \text{for } t \text{ odd, } \ell \in \left(\tfrac{t-3}{\sqrt{2}}, \tfrac{\sqrt{(t-5)^2+(t-1)^2}}{2}\right]; \\ &\rho(\ell) = g(\ell, t-3, t-3) + 2g(\ell, t-5, t-1) \\ &\qquad \text{for } t \text{ odd, } \ell \in \left(\tfrac{\sqrt{(t-5)^2+(t-1)^2}}{2}, \tfrac{\sqrt{(t-3)^2+(t-1)^2}}{2}\right]; \\ &\rho(\ell) = 2g(\ell, t-4, t-2) \quad \text{for } t \text{ even, } \ell \in \left(\tfrac{\sqrt{(t-4)^2+(t-2)^2}}{2}, \tfrac{\sqrt{(t-6)^2+t^2}}{2}\right]; \\ &\rho(\ell) = 2g(\ell, t-4, t-2) + 2g(\ell, t-6, t) \\ &\qquad \text{for } t \text{ even, } \ell \in \left(\tfrac{\sqrt{(t-6)^2+t^2}}{2}, \tfrac{t-2}{\sqrt{2}}\right]. \end{aligned} \tag{84}$$

Proof. The approach is similar to that used in the proof of Proposition 4, but conditioning on the segment orientation θ instead of on the location of its first endpoint.

The first endpoint of the segment is assumed to be in the reference tile with lower-left corner $(0,0)$. For an arbitrary value t, the segment visits t tiles almost surely if and only if its second endpoint is in a tile with lower-left corner $(i-1, j-1)$, $i, j \geq 1$, $i+j-1=t$, or in a tile symmetrical to this with respect to the line $x = 1/2$ (with $i \leq 0$), or $y = 1/2$ (with $j \leq 0$), or both.

Consider $i, j \geq 1$ for the moment, and let $t = \tau(\ell)$. For a given orientation θ, the segment goes from the reference tile to that with lower-left corner $(i-1, j-1)$ if and only if shifting the segment so that its second endpoint has coordinates $(i-1, j-1)$ results in the first endpoint being in the reference tile. It cannot be the case that the segment "overshoots" past this tile, because that would result in a number of visited tiles exceeding the assumed maximum, t.

There is a range of possible values of θ for which the above condition is satisfied. This range corresponds to the part of the arc with solid line in Figure 12. For convenience, the angle θ is defined clockwise from the downward direction. For each θ in the allowed range, the valid positions for the first endpoint of the (shifted) segment are in the shaded rectangle, whose dimensions depend on θ. The area occupied by this rectangle is the probability $\rho_{i-1,j-1}(\ell, \theta)$ that the segment goes from the reference tile to that with lower-left corner $(i-1, j-1)$ conditioned on θ. The desired probability, $\rho(\ell)$, will be obtained by averaging $\rho_{i-1,j-1}(\ell, \theta)$ over θ, summing over all $i, j \geq 1$ such that the tile with lower-left corner $(i-1, j-1)$ can be reached from the reference tile, and then using symmetry to take into account the tiles with $i \leq 0$ or $j \leq 0$.

The situation is different depending on whether t is odd or even. Consider t odd first. Under this assumption, the range of lengths corresponding to the condition $t = \tau(\ell)$ is given by (57). For any length in this interval, there are some positions within the reference tile from which the tile with lower-left coordinate $((t-1)/2, (t-1)/2)$ can be reached. Equivalently, the shifted segment with one endpoint at $((t-1)/2, (t-1)/2)$ can reach the reference tile. This is illustrated in Figure 12(a). In addition, if

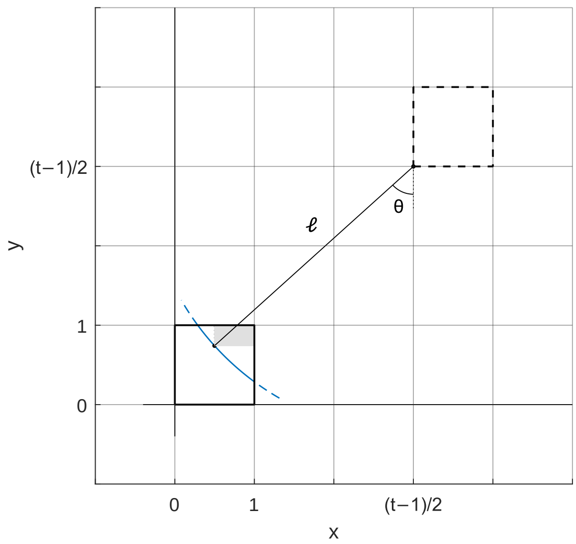
$$\ell > \frac{\sqrt{(t-5)^2 + (t-1)^2}}{2} \tag{85}$$

the two tiles with lower-left coordinates $((t-3)/2, (t+1)/2)$ and $((t+1)/2, (t-3)/2)$ are also reachable from the reference tile; or equivalently the shifted segment can reach the reference tile, as shown in Figure 12(b). This can only happen for $t \geq 5$ (for $t = 3$ the inequality (85) cannot be satisfied for lengths in the range given by (57)). In this case, by symmetry, each of those two tiles contributes the same probability. Other tiles with $i, j \geq 1$ corresponding to the same t do not need to be considered, because they cannot be reached with lengths satisfying (57).

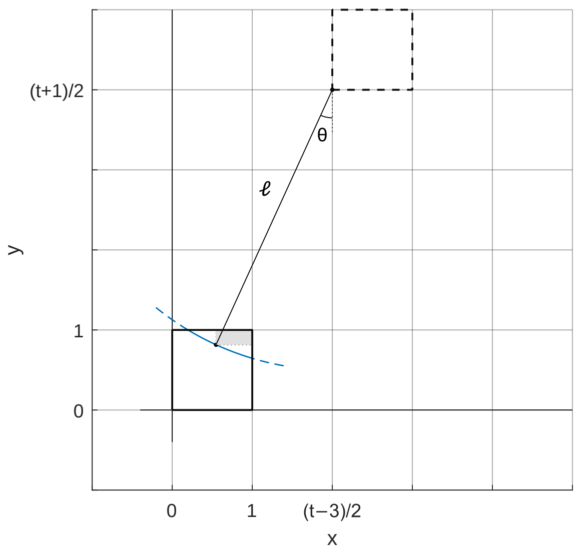
For t even, which corresponds to lengths in the interval (58), the two tiles with lower-left coordinates $(t/2-1, t/2)$ and $(t/2, t/2-1)$ can always be reached from some positions within the reference tile. In addition, if

$$\ell > \frac{\sqrt{((t-6)^2 + t^2}}{2} \tag{86}$$

the two tiles with lower-left coordinates $(t/2-2, t/2+1)$ and $(t/2+1, t/2-2)$ have to be considered too. This can only happen for $t \geq 8$. No other tiles with $i, j \geq 1$ can be reached from the reference tile.



(a) The second endpoint is in the tile with lower-left corner $((t-1)/2,(t-1)/2)$



(b) The second endpoint is in the tile with lower-left corner $((t-3)/2,(t+1)/2)$

Figure 12: Segment orientations and positions of the first endpoint within the reference tile for a given number t of visited tiles. Example for odd t

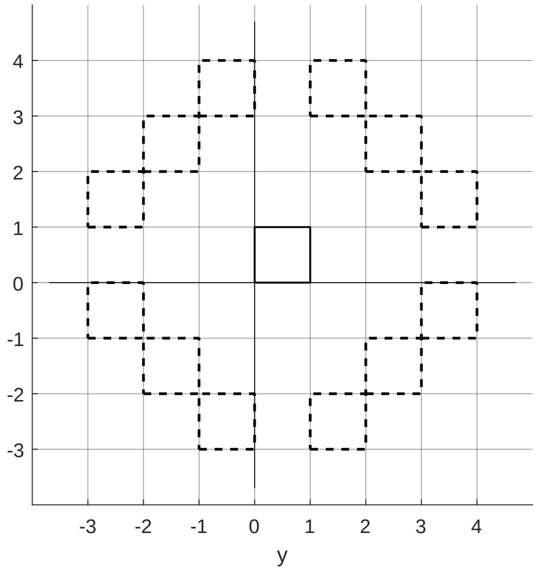


Figure 13: Tiles that can contain the second endpoint of a segment that visits the maximum number of tiles $\tau(\ell)$ in a square grid (example for $\tau(\ell) = 5$)

The above two paragraphs show that, among all possible tiles with lower-left corner $(i-1, j-1)$, $i, j \geq 1$, those whose centers lie on the lines $x = 1/2$ or $y = 1/2$ (i.e. those with $i = 1$ of $j = 1$) are never reachable from the reference tile, in the sense that a segment that touches the maximum number of tiles cannot have its second endpoint in those tiles. This implies that $\rho(\ell)$ can be computed by considering only the tiles with $i, j \geq 1$ (actually $i, j \geq 2$), and then multiplying by 4 to include the symmetrical tiles with respect to $x = 1/2$ and $y = 1/2$. As an example, Figure 13 shows the reference tile (solid line) and the tiles that can contain the second endpoint (dotted lines) for $\tau(\ell) = 5$.

According to the preceding analysis, two cases need to be distinguished for t odd, depending on whether (85) is satisfied or not; and similarly two cases exist for t even, depending on whether (86) holds or not. These are, respectively, the four cases in (84).

In the first case indicated in (84), as argued, t is odd and only the tile with lower-left coordinates $((t-1)/2, (t-1)/2)$ is reachable. For a given θ, the conditional probability $\rho_{(t-1)/2,(t-1)/2}(\ell, \theta)$ is the area of the shaded rectangle in Figure 12(a):

$$\rho_{(t-1)/2,(t-1)/2}(\ell, \theta) = \left(\ell \sin \theta - \frac{t-3}{2}\right) \left(\ell \cos \theta - \frac{t-3}{2}\right). \tag{87}$$

The range of allowed values for θ, as can be seen from the figure, is (θ_0, θ_1) with

$$\theta_0 = \arcsin \frac{t-3}{2\ell}, \tag{88}$$

$$\theta_1 = \arccos \frac{t-3}{2\ell}. \tag{89}$$

Computing the average of (87) over θ, multiplying by 4 to include all quadrants and comparing with (82) gives $\rho(\ell)$ as

$$\begin{aligned}\rho(\ell) &= 4\int_{\theta_0}^{\theta_1} \frac{\rho_{(t-1)/2,(t-1)/2}(\ell,\theta)}{2\pi}\,\mathrm{d}\theta \\ &= \frac{2}{\pi}\int_{\arcsin\frac{t-3}{2\ell}}^{\arccos\frac{t-3}{2\ell}} \left(\ell\sin\theta - \frac{t-3}{2}\right)\left(\ell\cos\theta - \frac{t-3}{2}\right)\mathrm{d}\theta \\ &= g\left(\ell, t-3, t-3\right),\end{aligned} \tag{90}$$

in accordance with (84).

For the second case in (84), $\rho(\ell)$ is obtained as above plus an additional term corresponding to the tile with lower-left corner $((t-3)/2,(t+1)/2)$, multiplied by 2 to account for its symmetrical tile about the line $y=x$, namely that with lower-left corner $((t+1)/2,(t-3)/2)$, and by 4 to include all quadrants. Observing the shaded rectangle in Figure 12(b),

$$\rho_{(t-3)/2,(t+1)/2}(\ell,\theta) = \left(\ell\sin\theta - \frac{t-5}{2}\right)\left(\ell\cos\theta - \frac{t-1}{2}\right). \tag{91}$$

The integration interval (θ_0,θ_1) is in this case

$$\theta_0 = \arcsin\frac{t-5}{2\ell}, \tag{92}$$

$$\theta_1 = \arccos\frac{t-1}{2\ell}. \tag{93}$$

The additional contribution to $\rho(\ell)$, given by

$$8\int_{\theta_0}^{\theta_1} \frac{\rho_{(t-3)/2,(t+1)/2}(\ell,\theta)}{2\pi}\,\mathrm{d}\theta = \frac{4}{\pi}\int_{\arcsin\frac{t-5}{2\ell}}^{\arccos\frac{t-1}{2\ell}} \left(\ell\sin\theta - \frac{t-5}{2}\right)\left(\ell\cos\theta - \frac{t-1}{2}\right)\mathrm{d}\theta, \tag{94}$$

is recognized to be $2g(\ell,t-5,t-1)$. This proves the second equality in (84).

The expressions for the third and fourth cases in (84), corresponding to t even, are established analogously. □

The technique used in the proof of Theorem 7 could in principle be used for obtaining the probability that the number of tiles visited by the segment equals or exceeds any given value $t < \tau(\ell)$. The resulting expressions, however, are cumbersome, because more cases (more subintervals for the length) need to be distinguished depending on how many tiles with lower-left corner $(i-1,j-1)$ such that $i+j-1=t$ can be reached from the reference tile.

Figure 14 shows the probability $\rho(\ell)$ of visiting the maximum number of tiles on a unit square grid, as given by (84). As ℓ grows, $\rho(\ell)$ has a jump when the maximum number of tiles that can be visited increases by 1. This happens

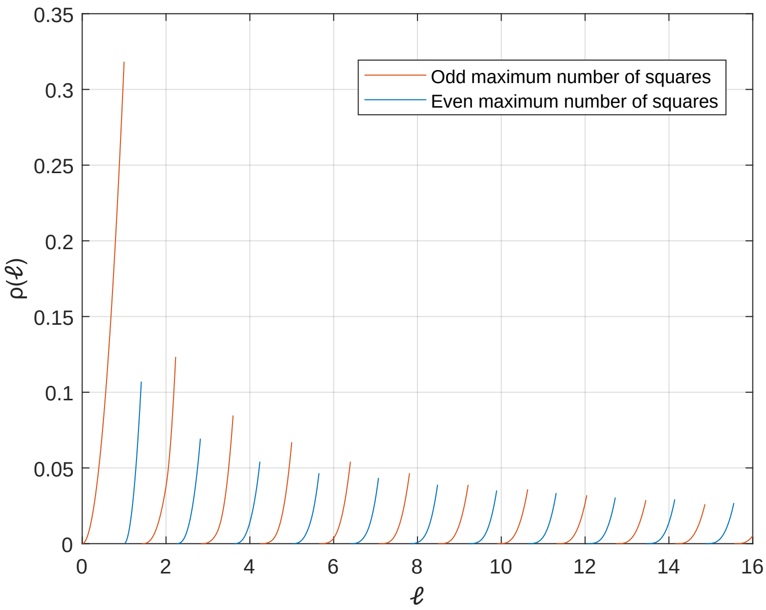


Figure 14: Probability $\rho(\ell)$ that a random segment of length ℓ visits the maximum number of tiles

when ℓ equals the right endpoint of the interval (57) for t odd or of (58) for t even. Let these length values be denoted as ℓ_t:

$$\ell_t = \begin{cases} \dfrac{\sqrt{(t-3)^2+(t-1)^2}}{2} & \text{for } t \text{ odd, } t \geq 3 \\ \dfrac{t-2}{\sqrt{2}} & \text{for } t \text{ odd, } t \geq 4. \end{cases} \tag{95}$$

For example, the first continuous section seen in Figure 14 corresponds to a maximum number of tiles $t = 3$, for lengths in the interval $(0, \ell_3]$, with $\ell_3 = 1$. The second corresponds to $t = 4$, for lengths in $(\ell_3, \ell_4]$, with $\ell_4 = \sqrt{2}$. Within each continuous section the probability monotonically increases from 0 to a maximum value. The heights of the maxima follow a different trend**better word?** for odd and even t, as can be seen in the figure; but in each case they are asymptotically proportional to $1/\ell_t$. This is established by the next result.

Proposition 5. *The function ρ has the following asymptotic behaviour:*

$$\liminf_{\ell \to \infty} \rho(\ell) = 0 \tag{96}$$

$$\lim_{\substack{t \to \infty \\ t \text{ odd}}} t\rho(\ell_t) = \frac{2}{\pi} \tag{97}$$

$$\lim_{\substack{t \to \infty \\ t \text{ even}}} t\rho(\ell_t) = \frac{8}{3\pi} \tag{98}$$

$$\limsup_{\ell \to \infty} \ell\rho(\ell) = \frac{4\sqrt{2}}{3\pi}. \tag{99}$$

Proof. The probability $\rho(\ell)$ becomes 0 whenever the length causes the maximum number of visited tiles to increase by 1. This establishes (96).

For (97) it suffices to note that

$$\lim_{t\to\infty} tg\left(\frac{\sqrt{(t-3)^2+(t-1)^2}}{2}, t-3, t-3\right) = \frac{2}{3\pi} \tag{100}$$

$$\lim_{t\to\infty} tg\left(\frac{\sqrt{(t-3)^2+(t-1)^2}}{2}, t-5, t-1\right) = \frac{2}{3\pi}, \tag{101}$$

as can be checked using L'Hôpital's rule [1, section 5.3]. Combining these expressions with the second equality in (84) yields (97).

Similarly,

$$\lim_{t\to\infty} tg\left(\frac{t-2}{\sqrt{2}}, t-4, t-2\right) = \frac{2}{3\pi} \tag{102}$$

$$\lim_{t\to\infty} tg\left(\frac{t-2}{\sqrt{2}}, t-6, t\right) = \frac{2}{3\pi}, \tag{103}$$

and combining with the fourth equality in (84) gives (98).

For large t, ℓ_t asymptotically approaches $t/\sqrt{2}$. Using this into (98), which is greater than (97), yields (99). □

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