On the number of tiles visited by a line segment on a rectangular grid

Luis Mendo[1] and Alex Arkhipov[2]

[1]Universidad Politécnica de Madrid. luis.mendo@upm.es
[2]**Affiliation, e-mail**

August 23, 2021

Abstract

***

Keywords: ***.

1 Introduction

Given $a, b \in \mathbb{R}^+$, consider a grid on $\mathbb{R}^2$ formed by rectangular *tiles* of width a and height b. A line segment of length $\ell \in \mathbb{R}^+$ is located on the plane with arbitrary position and orientation. The segment is said to *visit* a tile if it intersects its interior.[1] This paper deals with the following problems:

- What is the maximum number of tiles that the segment can visit?
- What length should a segment have to visit a given number of tiles?
- What is the average number of tiles visited by a uniformly random segment of a given length?

As an example of the first problem, consider $a = 1.35$, $b = 1$. A segment of unit length can be placed as shown in Figure 1 (left) to make it visit 3 tiles. In fact, the solution for $\ell = 1$ is 3 tiles. The figure also illustrates that the solution for length 2.4 is 5 (center), and for 4.7 it is 8 (right).

An equivalent formulation of the problem is obtained allowing segments of length ℓ *or smaller*. The equivalence is clear from the fact that reducing the length cannot increase the number of visited tiles. Either of these formulations will be referred to as the *direct* problem.

The *inverse* problem is, given $t \in \mathbb{N}$, to determine the infimum length of all segments that visit at least t tiles. For real-valued lengths this infimum is not a

[1]The definition uses the interior of the tile, excluding the border, to avoid uninteresting results such as a "zero-length" segment visiting (a vertex of) 4 tiles.

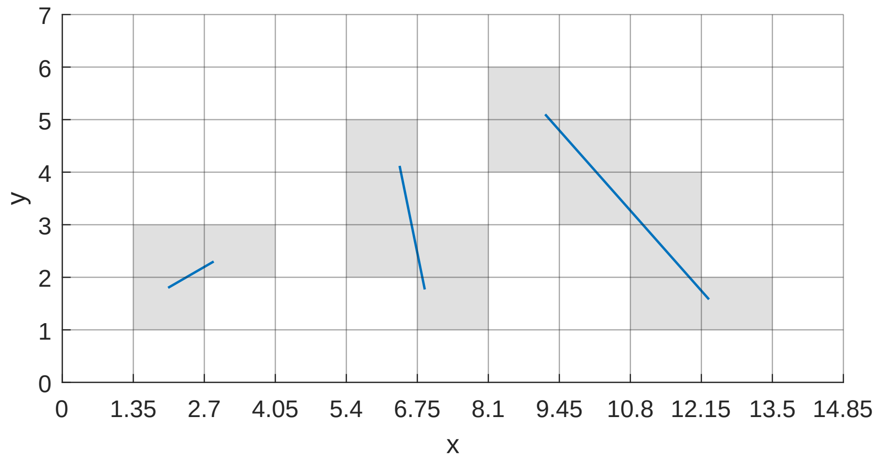


Figure 1: Examples for $a = 1.35, b = 1$; $\ell = 1$, $\ell = 2.4$ and $\ell = 4.7$

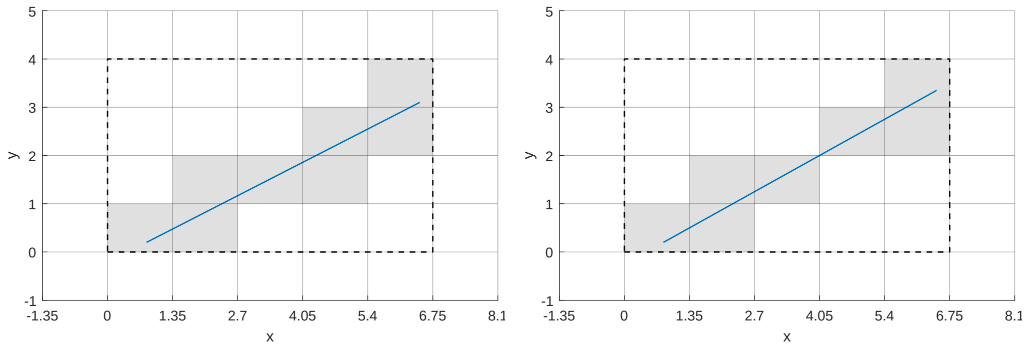
minimum, because given a segment that visits t tiles, its length can always be reduced by some non-zero amount without changing the number of visited tiles. This is a consequence of the interior of each tile being an open set.

The direct and inverse problems are closely related. Namely, if ℓ is the infimum of all lengths that allow visiting at least t tiles (inverse problem), t is the maximum number of tiles that can be visited with lengths slightly greater than ℓ (direct problem).

The problem of how many tiles a *random* segment of a given length visits on average is precisely defined as follows. By symmetry, one endpoint of the segment can be assumed to lie in a fixed, reference tile. The position of this endpoint is *uniformly* distributed on the tile. The segment orientation has a *uniform* distribution on the set of all possible directions, and is *independent* of the endpoint position. As will be seen, solving this problem also answers the inverse question (segment length to visit a given number of tiles on average), which thus need not be considered separately.

The problems studied in this paper are related to ... ***maybe a short paragraph here mentioning related problems: Buffon's needle, Buffon-Laplace problem. Some references: "Buffon's Noodle Problem", J.F. Ramaley, 1969. "On Laplace's Extension of the Buffon Needle Problem", B.J. Arnow, 1994. *An Introduction to Geometrical Probability*, A.M Mathai, section 1.1.
https://www.jstor.org/stable/2317945
https://www.jstor.org/stable/2687085

The rest of the paper is organized as follows. Fundamental results are presented in §2, which form the basis of the subsequent analysis. The direct and inverse problems for a deterministic segment are considered in §3, first for real-valued lengths, and then for integer-valued lengths on a unit square grid. The problem of computing the average number of tiles visited by a random segment is analyzed in §4.



(a) The segment does not pass through any interior grid points

(b) The segment passes through some interior grid points

Figure 2: Canonical rectangle and visited tiles

2 Fundamentals

For a grid with horizontal spacing a and vertical spacing b, lines of the form $x = ka$ or $y = kb$ with $k \in \mathbb{Z}$ will be called *grid lines*. A *tile* is delimited by two pairs of consecutive horizontal an vertical grid lines. The intersection points of horizontal and vertical grid lines will be called *grid points*. These correspond to vertices of the tiles.

Every segment has an associated *canonical rectangle*, which is the minimum-size rectangle that is formed by grid lines and contains the segment. More specifically, if the segment has endpoints (x_1, y_1), (x_2, y_2), where $x_1, x_2, y_1, y_2 \in \mathbb{R}$, its canonical rectangle has lower-left corner and upper-right corner given as

$$(\lfloor \min\{x_1, x_2\}/a \rfloor a, \lfloor \min\{y_1, y_2\}/b \rfloor b),$$
$$(\lceil \max\{x_1, x_2\}/a \rceil a, \lceil \max\{y_1, y_2\}/b \rceil b).$$

The dimensions of the canonical rectangle, normalized to the tile width and height respectively, are two integer numbers i, j. Two examples are illustrated in Figure 2, both with $i = 5$, $j = 4$. Clearly, all tiles visited by the segment are contained in the canonical rectangle. Note also that the canonical rectangle can have $i = 0$ or $j = 0$ if the segment coincides with part of a grid line.

Proposition 1. *Consider an arbitrary segment, and let i, j respectively denote the normalized width and height of its canonical rectangle. If $i, j \geq 1$, the number of tiles visited by the segment is at most $i + j - 1$. This bound is attained if and only if the segment does not pass through any grid point in the interior of the rectangle.*

Proof. The segment visits, by definition, two tiles in opposite corners of the canonical rectangle. It can be assumed, without loss of generality, that those tiles are in the lower-left and upper-right corners of the rectangle, as in Figure 2. The visited tiles can be thought of as following a path within the canonical rectangle. Starting at the lower-left tile, the next tile can be the one to the left, the one above, or the one above and to the left. The latter case occurs if and only if the segment passes through the grid point between those two tiles.

Since the segment follows a straight line, once it "leaves" a row of tiles in its path from the lower-left to the upper-right corner, it can never visit any more tiles from that row. The same observation applies for the columns.

This implies that the maximum number of visited tiles is $i+j-1$, which is attained if and only if the segment avoids all grid points in the interior of the canonical rectangle, as in Figure 2(a). Note that grid points at the corners of the rectangle do not count for this; and that the segment cannot pass through any other grid points on the rectangle border, because that would imply $i=0$ or $j=0$. Figure 2(b) illustrates a case where the maximum is not attained. $\square$

Proposition 2. *Consider $a,b,\ell \in \mathbb{R}^+$ and $i,j \in \mathbb{N}$, $i,j \geq 2$ arbitrary.*

(i) The following inequalities hold for any segment with length ℓ whose canonical rectangle has normalized dimensions i,j:

$$\ell > \sqrt{(i-2)^2a^2+(j-2)^2b^2}, \tag{1}$$
$$\ell \leq \sqrt{i^2a^2+j^2b^2}. \tag{2}$$

(ii) Conversely, if ℓ, i, j satisfy (1) and (2) there exists a segment of length ℓ whose canonical rectangle has normalized dimensions i and j.

(iii) There is a segment of length not exceeding ℓ that has a canonical rectangle with normalized dimensions i,j if and only if (1) holds.

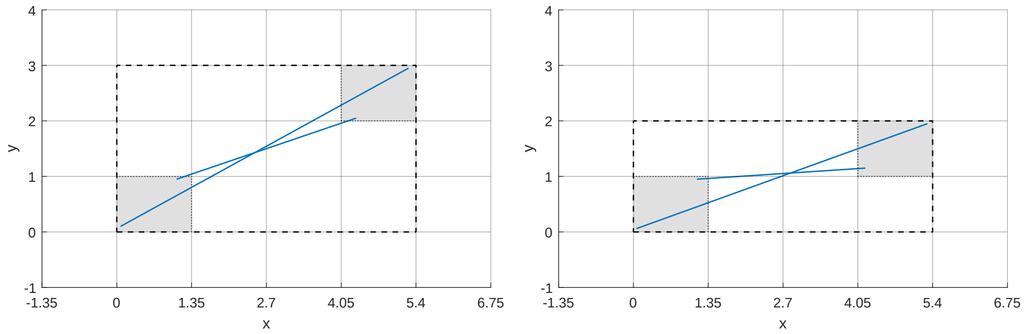
Proof. (i) The inequalities follow from the fact that the segment endpoints lie in the interiors or on the outer edges of two tiles in opposite corners of the canonical rectangle. This is illustrated in Figures 3(a) and 3(b) for two specific (i,j) pairs respectively. For each (i,j), segments are shown with lengths close to either of the two bounds. Note that (1) is valid even for $i=2$, $j=2$, in which case it reduces to $\ell > 0$.

(ii) For a, b, ℓ, i, j satisfying the two inequalities, a segment of length ℓ can be found that has its endpoints in the interiors or on the outer edges of the two shaded tiles of a rectangle of normalized dimensions i and j, as in Figure 3. Therefore this segment has the given rectangle as canonical.

(iii) "(1) $\Rightarrow$ there is a segment…". Assume that (1) holds. It is always possible to choose a length equal to or smaller than ℓ such that both (1) and (2) hold. The result follows, for that length, from part (ii).

"There is a segment… $\Rightarrow$ (1) ". Assume that a segment exists with length not exceeding ℓ and with a canonical rectangle with normalized dimensions i, j. From part (i), (1) holds for that segment length, and thus for ℓ. $\square$

According to Proposition 1, in order to maximize the number of visited tiles for a given length, the position and orientation of the segment should be chosen to obtain $i+j-1$ as large as possible, where i and j are the normalized dimensions of its canonical rectangle. On the other hand, Proposition 2 restricts the i,j values that can be achieved with a given length. A relevant question is: are there any (i,j) pairs that can be disregarded irrespective of the length ℓ? In



(a) $i = 4$, $j = 3$ (b) $i = 4$, $j = 2$

Figure 3: Relationship between segment length ℓ and dimensions i, j of the canonical rectangle

other words, what is the "smallest" subset of $\mathbb{N}^2$ such that the (i, j) pair that maximizes the number of tiles for any length can always be found within that subset? A subset that contains a maximizing (i, j) for any length will be called a *sufficient* set. Clearly, this set must contain at least one such pair for each possible value of $i + j - 1$, so that the set can produce that value as solution (for the corresponding lengths). A sufficient set that contains exactly one pair (i, j) for each value of $i + j - 1$ will be termed a *minimal sufficient* set.

For instance, it is intuitively clear from Figure 1 that segment orientations near the vertical or horizontal directions (resulting in $i = 1$ with large j, or $j = 1$ with large i) will not maximize the number of visited tiles. The corresponding (i, j) pairs can be left out of the sufficient set. In general, the pairs included in a minimal sufficient set will be different depending on a and b. Two specific examples of minimal sufficient sets will be presented later in this section.

Knowing a minimal sufficient set of (i, j) pairs facilitates the solution of both the direct an inverse problems, because it reduces the number of pairs that need to be tried. In order to derive a general method to build a minimal sufficient set, it is insightful to consider two specific cases first, as will be presented in the following. It is also helpful to use the equivalent formulation of the direct problem referred to in §1, i.e. finding the maximum number of tiles visited by segments of length ℓ *or less*.

Thus each choice of (i, j) implies a maximum of $i + j - 1$ visited tiles on one hand; and sets a lower bound on the required length ℓ on the other hand.

Consider first the case $a = b = 1$. This is illustrated in Figure 4. Note that in this and in the next figures the axes represent ia and jb (not i and j). Each dashed diagonal line joins (i, j) pairs with the same $i + j - 1$. The radius of each arc represents the lower bound on ℓ given by (1), for certain (i, j) pairs, which are marked with filled circles (these pairs form a minimal sufficient set, as will be seen).

For a given $\ell \in \mathbb{R}^+$, the (i, j) pairs that can be achieved with segments of length not exceeding ℓ are, by Proposition 2.(iii), those whose distance from $(2a, 2b)$ is less than ℓ, i.e. those delimited by the corresponding arc. It is clear from the figure that for $i + j - 1$ odd, if some point (i_1, j_1) with $|i_1 - j_1| > 1$

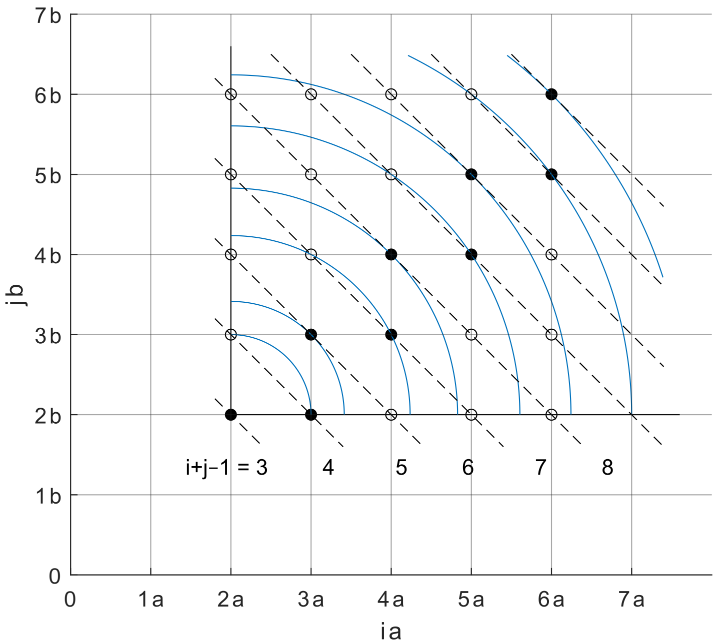


Figure 4: Relationship between segment length and number of visited tiles with the width and height of the canonical rectangle, for $a = b$

is achievable with the length bound ℓ, then necessarily there is a point (i_0, i_0) with the same number of visited tiles $i+j-1$ (same diagonal line) that is also achievable with the same length restriction (because it has smaller distance to $(2a, 2b)$). Similarly, for $i+j-1$ even, if a point (i_1, j_1) with $|i_1 - j_1| > 1$ is achievable, then there is a point (i_0, i_0+1) in the same diagonal that is also achievable with the same length restriction.

Thus for $a = b = 1$, a minimal sufficient set is formed by the pairs (i, j) with $j = i$ or $j = i+1$. Observe that due to symmetry, any point (i_0, i_0+1) could be replaced by (i_0+1, i_0). This illustrates that the minimal sufficient set is not unique in general.

The pairs in the minimal sufficient set are marked with filled circles in Figure 4. Given a length, the maximum number of visited tiles will be achieved with one of those pairs, namely that in the diagonal line furthest from $(2a, 2b)$ that is within the circle with radius ℓ centered at that point.

As a second example, consider $a = 1.35$, $b = 1$. This is depicted in Figure 5. In this case the minimal sufficient set does not follow a rule as simple as in the previous example.

In general, a systematic way to obtain a minimal sufficient set is to gradually increase ℓ starting from 0, which gives an arc of increasing radius in the figure; and to take note of the first (i, j) pair that is reached for each line of constant $i+j-1$. If two such pairs are reached at the same time, any of them can be used. In Figure 5 this rule produces the following pairs, in order: $(2,2)$, $(2,3)$, $(3,3)$, $(3,4)$, $(3,5)$, $(4,5)$, $\ldots$, as indicated by the filled circles.

Based on the above, an explicit procedure that produces a minimal sufficient set can be given. This set will be denoted as $M = \{(i_3, j_3), (i_4.j_4), \ldots\}$.

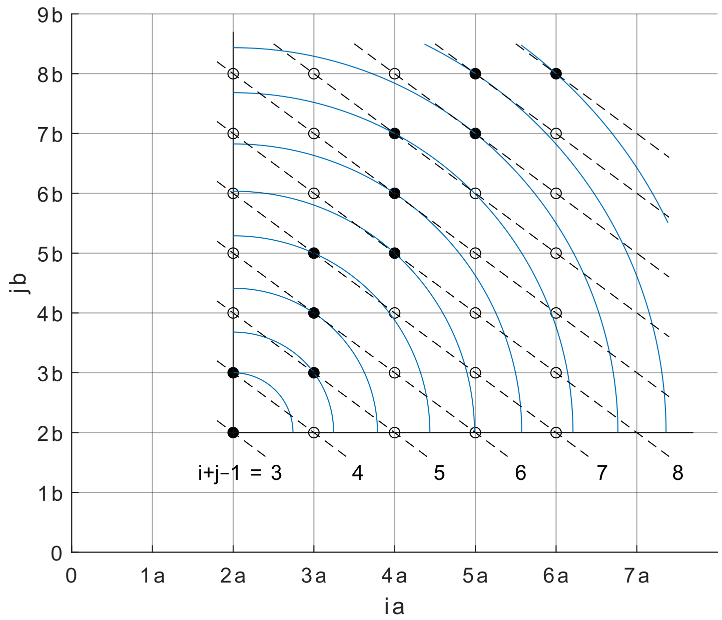


Figure 5: Relationship between segment length and number of visited tiles with the width and height of the canonical rectangle, for $a = 1.35b$

Proposition 3. *Given grid parameters $a, b \in \mathbb{R}^+$, a minimal sufficient set M can be obtained as follows. The chosen pair for $i + j - 1 = 3$ is $(2,2)$. For each $t \in \mathbb{N}, t \geq 3$, let (i_t, j_t) denote the pair chosen for $i + j - 1 = t$. Then (i_{t+1}, j_{t+1}) is either $(i_t + 1, j_t)$ or $(i_t, j_t + 1)$, whichever minimizes $(i_{t+1} - 2)^2 a^2 + (j_{t+1} - 2)^2 b^2$; if both are minimizers $(i_t + 1, j_t)$ is taken. Equivalently, (i_{t+1}, j_{t+1}) is $(i_t, j_t + 1)$ if*

$$j_t < \frac{i_t a^2}{b^2} - \frac{3a^2}{2b^2} + \frac{3}{2}, \tag{3}$$

or $(i_t + 1, j_t)$ otherwise.

Proof. For each t, the pair (i_t, j_t) of the minimal sufficient set should be chosen as that on the line $i + j - 1 = t$ which minimizes $(i-2)^2 a^2 + (j-2)^2 b^2$. This allows maximizing the sum $i + j - 1$, and thus the number of visited tiles, for a given length restriction (direct problem); or minimizing the required lengths for a specified number of visited files (inverse problem).

The fact that (i_t, j_t) minimizes $(i-2)^2 a^2 + (j-2)^2 b^2$ among all pairs with $i + j - 1 = t$ can be expressed as follows: for any $k \in \mathbb{N}$,

$$(i_t - 2)^2 a^2 + (j_t - 2)^2 b^2 \leq (i_t - 2 - k)^2 a^2 + (j_t - 2 + k)^2 b^2, \tag{4}$$
$$(i_t - 2)^2 a^2 + (j_t - 2)^2 b^2 \leq (i_t - 2 + k)^2 a^2 + (j_t - 2 - k)^2 b^2. \tag{5}$$

To see that (i_{t+1}, j_{t+1}) is either $(i_t, j_t + 1)$ or $(i_t + 1, j_t)$ it suffices to prove that, for any $k \in \mathbb{N}$,

$$(i_t - 2)^2 a^2 + (j_t - 1)^2 b^2 \leq (i_t - 2 - k)^2 a^2 + (j_t - 1 + k)^2 b^2, \tag{6}$$
$$(i_t - 1)^2 a^2 + (j_t - 2)^2 b^2 \leq (i_t - 1 + k)^2 a^2 + (j_t - 2 - k)^2 b^2. \tag{7}$$

With reference to Figure 5, consider for instance $(i_7, j_7) = (3,5)$. Graphically it can be seen that (i_8, j_8) could only be $(3,6)$ or $(4,5)$, because for example $(2a,7b)$ is further from $(2a,2b)$ than $(3a,6b)$ is (this corresponds to (6) with $k = 1$), and $(5a,4b)$ is also further from $(2a,2b)$ than $(4a,5b)$ is (this corresponds to (7) with $k = 1$).

The inequality (6) can be rewritten as

$$\begin{aligned}(i_t-2)^2a^2+(j_t-2)^2b^2+2(j_t-2)b^2+b^2 \\ \leq (i_t-2-k)^2a^2+(j_t-2+k)^2b^2+2(j_t-2+k)b^2+b^2.\end{aligned} \tag{8}$$

Clearly,

$$2(j_{t+1}-2)b^2+b^2 < 2(j_{t+1}-2+k)b^2+b^2. \tag{9}$$

Adding (4) and (9) yields (8). Therefore (6) holds. A similar procedure can be used to establish (7).

The condition for choosing (i_t, j_t+1) over (i_t+1, j_t) is that

$$(i_t-2)^2a^2+(j_t-1)^2b^2 < (i_t-1)^2a^2+(j_t-2)^2b^2. \tag{10}$$

This can be expressed as

$$2(j_t-2)b^2+b^2 < 2(i_t-2)a^2+a^2. \tag{11}$$

Rearranging terms, (11) is seen to be the same as (3). □

The condition (3) for incrementing j_t in Proposition 3 can be expressed in terms of two bounding lines,

$$j = \frac{ia^2}{b^2} - \frac{3a^2}{2b^2} + \frac{5}{2}, \tag{12}$$

$$j = \frac{ia^2}{b^2} - \frac{5a^2}{2b^2} + \frac{3}{2}. \tag{13}$$

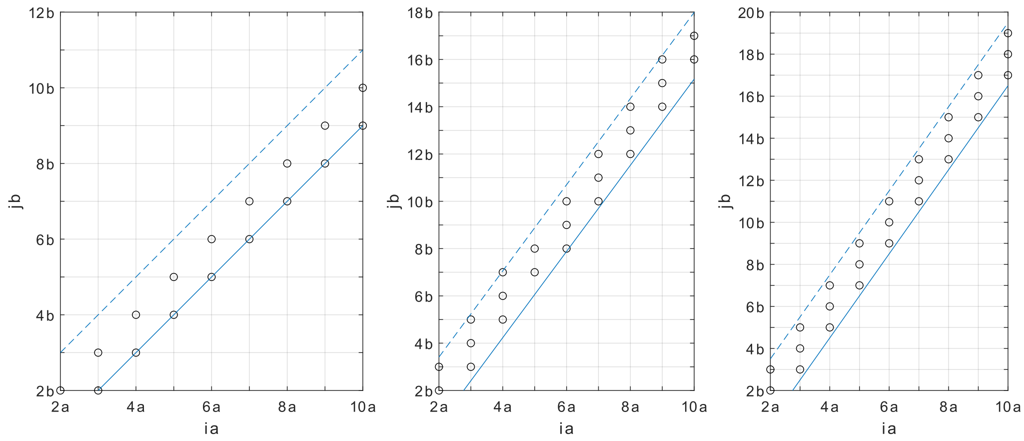
All pairs (i, j) in M are strictly below the upper line (12) and above or on the lower line (13). This is shown in Figure 6, using three different pairs of grid parameters a, b as examples. Given $(i_t, j_t) \in M$, the next pair (i_{t+1}, j_{t+1}) is obtained by incrementing j if that results in a point below (12). Else j cannot be increased and i is incremented instead, and the new pair is guaranteed to be above or on (13).

The construction of M in Proposition 3 computes the pairs (i_t, i_t) *sequentially*: $(i_3, i_3), \ldots, (i_{t-1}, i_{t-1})$ need to be previously computed in order to obtain (i_t, i_t). Proposition 4 avoids this by providing a direct formula. Its derivation is based on finding the intersection point of (13) with $i + j - 1 = t$ and appropriately rounding to integer coordinates in order to satisfy (12).

Proposition 4. *Given $a, b \in \mathbb{R}^+$ and $t \in \mathbb{N}, t \geq 3$, the pair (i_t, j_t) from the set M defined in Proposition 3 can be computed as*

$$i_t = \left\lfloor \frac{(2t-1)b^2+5a^2}{2(a^2+b^2)} \right\rfloor, \tag{14}$$

$$j_t = \left\lceil \frac{2ta^2+3(b^2-a^2)}{2(a^2+b^2)} \right\rceil. \tag{15}$$



(a) $a = 1, b = 1$ (b) $a = 1.35, b = 1$ (c) $a = \sqrt{2}, b = 1$

Figure 6: Minimal sufficient set M and bounding lines

Proof. Let (i^+, j^+) denote the intersection of (13) and the line $i + j - 1 = t$. This is easily seen to be

$$i^+ = \frac{(2t-1)b^2 + 5a^2}{2(a^2+b^2)}, \tag{16}$$

$$j^+ = \frac{2ta^2 + 3(b^2-a^2)}{2(a^2+b^2)}. \tag{17}$$

This point is shown with a square marker in Figure 7, which also shows the elements of M as filled circles, as well as the two bounding lines. The line (12) is (13) shifted one step left in i and one step up in j, i.e. it is shifted by a diagonal step. Since all points of M are between these two lines or on the lower line, the only possibility for (i_t, j_t), as illustrated in the figure, is to be the point with integer coordinates closest to (i^+, j^+) along the line $i + j - 1 = t$ in the left, upwards direction. This gives (14) and (15). $\square$

For a^2/b^2 arbitrary, the number of pairs with the same i coordinate in the set M is in general irregular, because the line (13) does not follow a "natural" direction of the grid. This happens for instance in Figure 6(b), where the number of pairs for the first i values equals either 2 or 3 without a clear pattern.[2] On the other hand, a simple pattern arises for $a^2/b^2 \in \mathbb{N}$, as seen in Figures 6(a) and 6(c).

A segment whose canonical rectangle has normalized width i and height j is oriented with approximate slope $jb/(ia)$ with respect to the x axis (See Figure 3); and the approximation becomes better for greater segment lengths. From (12) and (13) it can be seen that the positions of the pairs $(i, j) \in M$ have $j/i \approx a^2/b^2$ for large i, j. Therefore the optimal slope for long segments is approximately a/b. This formalizes the intuition that to maximize the number

[2]Strictly, there is a periodic pattern whenever a^2/b^2 is rational, which is the case in Figure 6(b). However, this is not easily discernible unless a^2/b^2 is a ratio of small numbers.

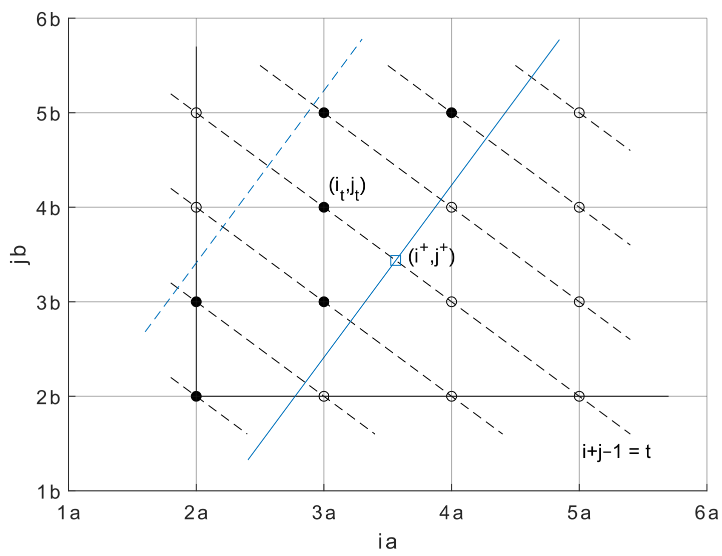


Figure 7: Obtaining (i^*, j^*) in Proposition 4. Example with $a = 1.35, b = 1$

of visited tiles, the segment should follow a direction along which the perceived "length" of the tile is smaller.

3 Direct and inverse problems for a deterministic segment

The direct and inverse problems defined in §1, considering the segment position and orientation as deterministic, are addressed in this section. The general case for rectangular grids with real-valued segment lengths is analyzed first, in §3.1. The particular case of square grids is addressed in §3.2, as it allows a specialized formula for the direct problem. Lastly, the analysis of a unit square grid with integer-valued lengths is given in §3.3.

3.1 Arbitrary grid with real-valued lengths

For a grid with parameters $a, b \in \mathbb{R}^+$, the maximum number t of visited tiles with a given real-valued length ℓ can be represented by a function $\tau : \mathbb{R}^+ \to \mathbb{N}$ such that $t = \tau(\ell)$. Similarly, for the inverse problem a function $\lambda : \mathbb{N} \to \mathbb{R}^+$ can be defined such that $\lambda(t)$ gives the infimum length of all segments that visit at least t tiles.

The function τ can be obtained from λ by noting that the maximum number of tiles that can be visited by a segment of length ℓ is the index of the largest term of the sequence $\lambda(t)$ that is less than ℓ:

$$\tau(\ell) = \max\{t \in \mathbb{N} \mid \lambda(t) < \ell\}. \tag{18}$$

Conversely, λ can be obtained from τ as

$$\lambda(t) = \inf\{\ell \in \mathbb{R}^+ \mid \tau(\ell) \geq t\}. \tag{19}$$

For arbitrary $a, b \in \mathbb{R}^+$, the functions τ and λ can be computed using an iterative procedure, which exploits the fact that the pairs $(i_3, j_3), (i_4, j_4), \ldots$ of M are sorted by increasing $i+j-1$, and also by increasing $(i-2)^2a^2+(j-2)^2b^2$. Namely, for τ the following procedure yields the solution: generate successive pairs to find the last one, (i_t, j_t), that satisfies (1); then $\tau(\ell) = t$. For λ the analogous method gives a direct formula. In addition, it is possible to obtain a direct formula also for τ using an approach similar to that used in Proposition 4. These formulas are given in Theorems 1 and 2.

Theorem 1. *For $a, b \in \mathbb{R}^+$, $a \geq b$ and $\ell \in \mathbb{R}^+$,*

$$\tau(\ell) = i^* + j^* - 1 \tag{20}$$

with

$$i^* = \left\lfloor \frac{a + b\,\mathrm{Re}\sqrt{4\ell^2/(a^2+b^2)-1}}{2a} \right\rfloor + 1, \tag{21}$$

$$j^* = \left\lfloor \frac{\sqrt{\ell^2-(i^*-2)^2a^2}}{b} \right\rfloor + 1. \tag{22}$$

The function τ is piecewise constant and left-continuous, with unit-height jumps. A jump occurs at ℓ if and only if $\ell = \lambda(t)$ for some $t \in \mathbb{N}, t \geq 4$; and then $\tau(\ell) = t-1$, $\lim_{h\to 0+}\tau(\ell+h) = t$.

Proof. The general idea is similar to that used in Proposition 4: find the intersection (i^+, j^+) of the line (13) and the arc centered at $(2a, 2b)$ with radius ℓ for $i, j \geq 2$, and from that obtain the actual pair (i^*, j^*) of integer numbers which yields the solution.

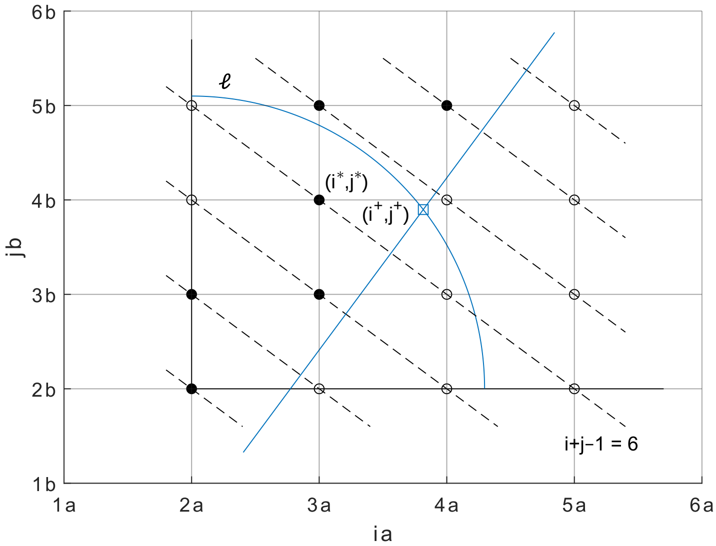
Specifically, assuming first that (13) holds, the largest i that can be achieved with lengths not exceeding ℓ is obtained, as given by Proposition 2.(iii). This value, i^*, is computed by rounding down i^+. Then, assuming that $i = i^*$, the largest j allowed by Proposition 2.(iii), j^*, is obtained. As will be seen, in some cases the resulting (i^*, j^*) is in M, and in other cases it is not. However, in either case (i^*, j^*) has the largest $i+j-1$ sum that can be achieved with segments of length up to ℓ. Thus the desired result is i^*+j^*-1.

The two possibilities are illustrated in Figure 8 for $a = 1.35, b = 1$. In each case, the arc displayed in the graph is centered at $(2a, 2b)$ and has radius ℓ. The inner region defined by the arc contains all (i, j) pairs that are achievable according to Proposition 2.(iii). As in previous figures, filled circles represent (i, j) pairs that are in M, and empty circles are those that do not belong to M. The solid line is (13). The intersection point (i^+, j^+) is displayed with a square marker.

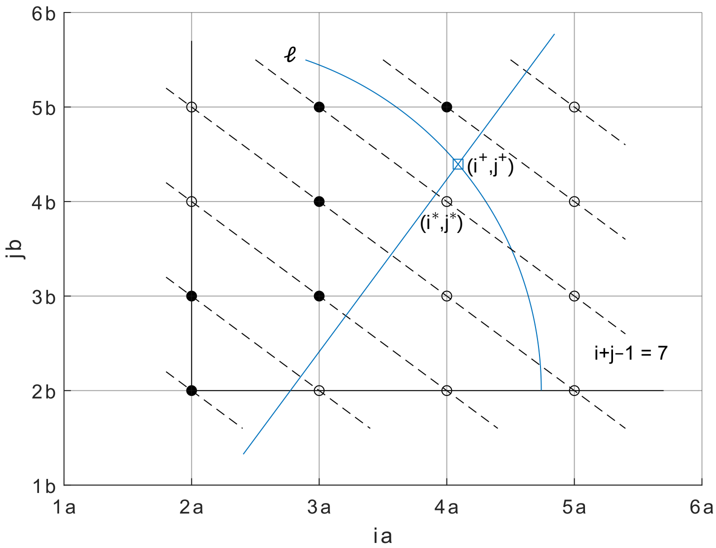
The point (i^+, j^+) can be obtained as a solution of the equation system

$$(i^+-2)^2a^2 + (j^+-2)^2b^2 = \ell^2, \tag{23}$$

$$j^+ = \frac{i^+a^2}{b^2} - \frac{5a^2}{2b^2} + \frac{3}{2}. \tag{24}$$



(a) $\ell = 3.1$: $(i^*, j^*) \in M$



(b) $\ell = 3.7$: $(i^*, j^*) \notin M$

Figure 8: Obtaining (i^*, j^*) in Theorem 1. Examples with $a = 1.35, b = 1$

Substituting (24) into (23) yields a quadratic equation for i^+,

$$4a^2(i^+-2)^2 - 4a^2(i^+-2) + a^2 + b^2 - \frac{4\ell^2 b^2}{a^2+b^2}, \tag{25}$$

which can have zero, one or two real-valued solutions (this respectively means that the line is exterior, tangent or secant to the circle (23)). Only the solution with $i^+, j^+ \geq 2$, if any, is of interest. This solution can only exist as the one with largest i^+ value when there are real-valued solutions, and it is obtained as

$$i^+ = \frac{a + b\sqrt{4\ell^2/(a^2+b^2)-1}}{2a} + 2. \tag{26}$$

The case with no real-valued solutions to (25) corresponds to $\ell < \sqrt{a^2+b^2}/2$. Since $a \geq b$, this implies that $\ell < a$. Thus any achievable (i,j) pair, if any, will have $i = 2$. Therefore in this case i^* should be set to 2; and then the maximum achievable j^* will be obtained from (24).

If (25) has two real-valued solutions or one real-valued double solution, which is the case when $\ell \geq \sqrt{a^2+b^2}/2$, the largest solution will have $j^+ < 2$ if $i^+ < (b^2/a^2+5)/2$, as can be seen setting $j^+ = 2$ in (24); or equivalently if $\sqrt{a^2+b^2}/2 \leq \ell < (a^2+b^2)/(2a)$, as can be seen substituting $j^+ < 2$ and $i^+ < (b^2/a^2+5)/2$ in (23). Since $i^+ < (b^2/a^2+5)/2 < 3$ for $a \geq b$, only pairs with $i = 2$ are achievable in this case again, and thus the solution i^* should also be 2.

Lastly, for $\ell \geq (a^2+b^2)/(2a)$ the expressions (24) and (26) give $i^+, j^+ \geq 2$, and i^* should be taken as the greatest integer less than i^+, i.e. $\lceil i^+ \rceil - 1$.

The three cases are unified by taking the real part of (26) and computing $i^* = \lceil i^+ \rceil - 1$, as is easily checked. This gives (21). Once i^* is known, j^* is computed, according to (22), as the greatest integer such that (i^*, j^*) is within the arc determined by ℓ. This ensures that (i^*, j^*) is achievable with lengths less than ℓ.

There are two possibilities for the point (i^*, j^*), as stated at the outset. These are illustrated in Figures 8(a) and 8(b) respectively. The first possibility is that $(i^*, j^*) \in M$ (Figure 8(a)). Then, by construction (i^*, j^*) maximizes $i + j - 1$ among all achievable pairs of M, and is therefore optimal.

The second possibility is that $(i^*, j^*) \notin M$ (Figure 8(b)). This happens when the point from M that has $i = i^*$ ($(4,5)$ in the figure) is outside the arc, i.e. it would require a length greater than ℓ. The selected point (i^*, j^*) ($(4,4)$ in the figure), however, has the same $i + j - 1$ sum as the point from M that "should" be used, which is (i^*-1, j^*+1) ($(3,5)$ in the figure); and therefore gives the same result. This is always the case, because $(i^*, j^*+1) \in M$ (it is above or on the bounding line) and $(i^*, j^*) \notin M$ (it is below the line), and due to how M has been constructed, this implies that $(i^*-1, j^*+1) \in M$ and $(i^*-1, j^*+k) \notin M$ for $k = 2, 3, \ldots$. Therefore, (i^*, j^*) is achievable and maximizes $i + j - 1$, which implies that $i^* + j^* - 1$ is the desired solution.

Therefore, whether $(i^*, j^*) \in M$ or $(i^*, j^*) \notin M$, equations (21) and (22) give $i^* + j^* - 1$ equal to $\tau(\ell)$, as claimed.

It should be noted that for the specific case that $a^2/b^2 = 2k-1$, $k \in \mathbb{N}$ the lower bounding line (13) becomes $j = i(2k-1) - 5k + 4$, which gives integer j for integer i. This means that the bound is actually achieved for all pairs in M (see for example Figure 6(a)), and the case $(i^*, j^*) \notin M$ never occurs. Thus, when a^2/b^2 is an odd integer the obtained (i^*, j^*) is guaranteed to be in M, whereas for general a^2/b^2 either (i^*, j^*) or (i^*-1, j^*+1) are in in this set.

As for the properties of τ, it follows from (20)–(22) that it is piecewise constant and left-continuous. From the procedure described in the previous paragraphs for obtaining (i^*, j^*) it is clear that $i^* + j^* - 1$ increases in steps of 1 when ℓ is increased continuously; that is, τ has jumps of unit value.

Consider an arbitrary ℓ such that for some $t \in \mathbb{N}, t \geq 4$

$$\lambda(t) = \ell. \tag{27}$$

To see that τ has a jump at ℓ, assume for the sake of contradiction that τ is continuous, therefore constant, at that point. Then there is $\varepsilon > 0$ for which $\tau(\ell+\varepsilon) = t = \tau(\ell-\varepsilon)$. This means that there exists a segment with length $\ell-\varepsilon$ that visits t tiles, and thus $\lambda(t) \leq \ell-\varepsilon < \ell$, in contradiction with (27). Therefore τ is discontinuous, from the right, at ℓ. By definition of λ, (27) implies that $\tau(\ell) < t$, and that there exists $\varepsilon > 0$ such that $\tau(\ell+h) = t$ for $0 < h < \varepsilon$. This means that

$$\tau(\ell) < t, \tag{28}$$

$$\lim_{h\to 0+} \tau(\ell+h) = t, \tag{29}$$

that is, τ has a jump at ℓ. In addition, since the jump has unit height, it follows from (28) and (29) that $\tau(\ell) = t-1$.

Conversely, assume that (28) and (29) hold for some $\ell \in \mathbb{R}^+$, $t \in \mathbb{N}$. From (28) it follows that $\lambda(t) \geq \ell$. On the other hand, (29) implies that $\lambda(t) \leq \ell$. Thus $\lambda(t) = \ell$. □

Although Theorem 1 is valid for $a \geq b$, the result could also be applied for $a < b$ by simply swapping the values of a and b. In other words, (20)–(22) can be used for any grid if a, b are interpreted as the largest and smallest sides of a tile, respectively.

Theorem 2. *For $a, b \in \mathbb{R}^+$ and $t \in \mathbb{N}$,*

$$\lambda(t) = \sqrt{(i^*-2)^2 a^2 + (j^*-2)^2 b^2} \tag{30}$$

with

$$i^* = \max\left\{ \left\lfloor \frac{(2t-1)b^2 + 5a^2}{2(a^2+b^2)} \right\rfloor, 2 \right\}, \tag{31}$$

$$j^* = \max\left\{ \left\lceil \frac{2ta^2 + 3(b^2-a^2)}{2(a^2+b^2)} \right\rceil, 2 \right\}. \tag{32}$$

This function is monotone increasing for $t \geq 3$.

Proof. The (i_t, j_t) pair in the minimal sufficient set M corresponds to a maximum of t visited tiles. By construction of this set, any segment that visits t tiles must have length greater than $\sqrt{(i_t-2)^2a^2+(j_t-2)^2b^2}$.

For $t \geq 3$ the expressions (31) and (32) coincide with (14) and (15). Therefore (30) gives the desired result.

For $t \in \{1,2\}$ both (31) and (32) equal 2, and (30) gives 0, which is the correct result.

By Theorem 1, τ has unit-height jumps at the values $\lambda(t)$, $t \in \mathbb{N}, t \geq 4$. This implies that λ is monotone increasing for $t \geq 3$. $\square$

Theorems 1 and 2 not only give the solutions $\tau(\ell)$ and $\lambda(t)$ to the two problems stated in §1; they also provide a way to actually position a segment of length slightly greater than ℓ or $\lambda(t)$ so that it visits t or $\tau(\ell)$ tiles. Namely, the segment should have its endpoints respectively in the interior of two tiles separated i^*-1 steps horizontally and j^*-1 steps vertically, with its exact position and orientation adjusted to avoid any grid points.

It is interesting to consider the following particular cases: $\ell \gg a,b$; $a \gg b$; and $a=b$. Regarding the first, from (20)–(22) and from (30)–(32) it is seen that for long segments the number of visited tiles and the segment length are approximately proportional, with

$$\lim_{\ell\to\infty}\frac{\tau(\ell)}{\ell}=\lim_{t\to\infty}\frac{t}{\lambda(t)}=\sqrt{1/a^2+1/b^2}. \tag{33}$$

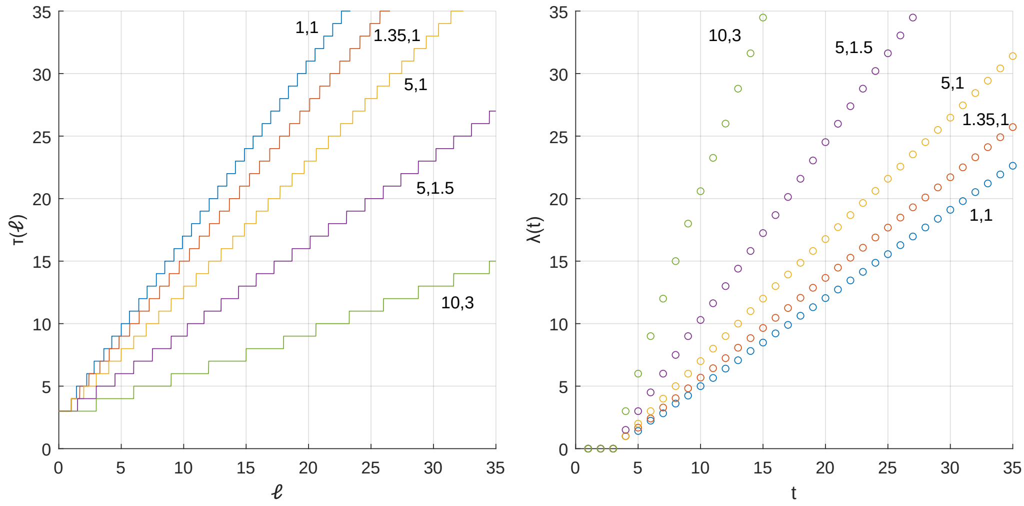
As for $a \gg b$, in this case the optimal canonical rectangle has $i^*=2$, and j^* as large as allowed by ℓ (direct problem) or as required by t (inverse problem), corresponding to an almost vertical segment. Obviously, for $a \gg b$ the length of the segment is best invested in increasing the number of tiles traversed vertically (but the segment should be slightly tilted to cross a vertical edge), and the asymptotic value of (33) is approximately $1/b$.

For $a=b$, either from symmetry considerations or particularizing the formulas in the previous theorems it stems that the optimal orientation of the segment is close to 45°. This case will be dealt with in §3.2, as it lends itself to somewhat simplified formulas.

Figure 9 shows τ and λ for several pairs of grid parameters a, b. The graphs illustrate some of the observations of the previous paragraphs. Indeed, the asymptotic slope in Figure 9(a), or the inverse of the asymptotic slope in Figure 9(b), is approximately $\sqrt{2}$ for $a,b=1$, and $1/b$ for the case $a=5, b=1$, and even for $a=5, b=1.5$ and $a=10, b=3$. Comparing the latter two cases it is also seen that scaling a, b and ℓ by the same factor does not alter $\tau(\ell)$; or scaling a, b results in $\lambda(t)$ being scaled by the same factor.

3.2 Square grid with real-valued lengths

Particularizing the results in §3.1 to a square grid, $a=b$, obviously yields simpler formulas.



(a) Function τ (b) Function λ

Figure 9: Functions τ and λ for several pairs a, b

Corollary 1. *For a square grid with $a \in \mathbb{R}^+$, and for $\ell \in \mathbb{R}^+$,*

$$\tau(\ell) = i^* + j^* - 1 \tag{34}$$

with

$$i^* = \left\lceil \frac{1 + \operatorname{Re}\sqrt{2\ell^2/a^2 - 1}}{2} \right\rceil + 1, \tag{35}$$

$$j^* = \left\lceil \sqrt{\ell^2/a^2 - (i^* - 2)^2} \right\rceil + 1, \tag{36}$$

Corollary 2. *For a square grid with $a \in \mathbb{R}^+$, and for $t \in \mathbb{N}$,*

$$\frac{\lambda(t)}{a} = \begin{cases} 0 & \textit{for } t = 1, 2 \\ \dfrac{t-3}{\sqrt{2}} & \textit{for } t \textit{ odd, } t \geq 3 \\ \sqrt{\dfrac{(t-3)^2 + 1}{2}} & \textit{for } t \textit{ even, } t \geq 3. \end{cases} \tag{37}$$

Furthermore, an even simpler formula can be obtained for τ. Its derivation uses a variation of the minimal sufficient set M which is more suited to this case.

Theorem 3. *For a square grid with $a \in \mathbb{R}^+$, and for $\ell \in \mathbb{R}^+$,*

$$\tau(\ell) = i^* + j^* - 1 \tag{38}$$

with

$$i^* = \left\lceil \frac{\ell}{a\sqrt{2}} \right\rceil + 1, \tag{39}$$

$$j^* = \left\lceil \sqrt{\ell^2/a^2 - (i^* - 2)^2} \right\rceil + 1. \tag{40}$$

Proof. For $a = b$ the set M from Proposition 3 consists of points of the form (i,i) and $(i+1,i)$, as is easily seen from Proposition 4, and as illustrated in Figure 6(a). By symmetry, replacing each point $(i+1,i)$ by $(i,i+1)$ gives a set M' which is also minimal sufficient. For this new set, the lower bounding line (13) can be replaced by the simpler $j = i$. The same approach used in the proof of Theorem 1 can be used, but using this line. Thus (i^+, j^+) is obtained from

$$(i^+ - 2)^2 a^2 + (j^+ - 2)^2 a^2 = \ell^2, \tag{41}$$

$$j^+ = i^*, \tag{42}$$

which gives

$$i^+ = \frac{\ell}{a\sqrt{2}} + 2. \tag{43}$$

The pair (i^*, j^*) resulting from (43) is in M', and is given in (39) and (40). $\square$

From Theorem 3 it is easily seen that odd values of $\tau(\ell)$ correspond to $i^* = j^*$, whereas even values are achieved with $j^* = i^* + 1$. Given $t \geq 3$ with t odd, $\tau(\ell) = t$ if and only

$$\frac{\ell}{a} \in \left(\frac{t-3}{\sqrt{2}}, \frac{\sqrt{(t-1)^2 + (t-3)^2}}{2} \right]. \tag{44}$$

For $t \geq 4$, t even, $\tau(\ell) = t$ if and only if

$$\frac{\ell}{a} \in \left(\frac{\sqrt{(t-2)^2 + (t-4)^2}}{2}, \frac{t-2}{\sqrt{2}} \right]. \tag{45}$$

3.3 Unit square grid with integer lengths

A natural variation of the two problems introduced in §1 is to consider $a = b = 1$, with the additional restriction that the segment length can only be a positive integer (equivalently, a square grid is considered with step a and the segment length can only be an integer multiple of a).

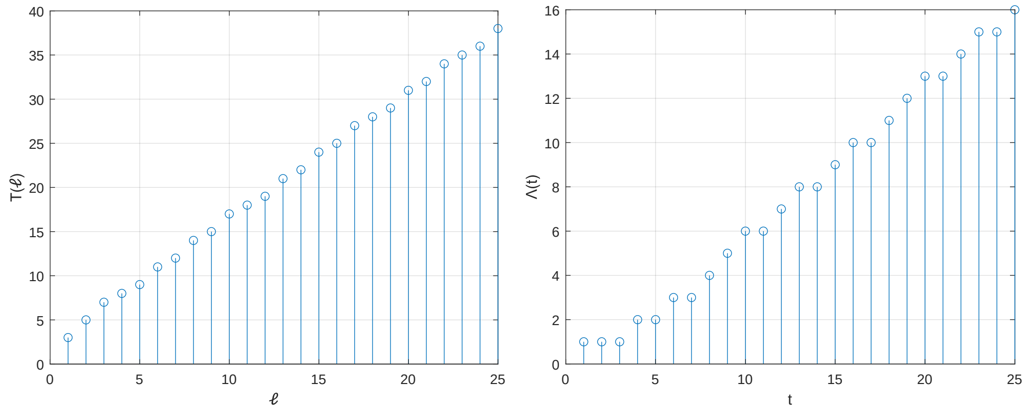
The *direct problem* in this setting corresponds to the restriction of the function τ to $\mathbb{N}$. This will be denoted as a function $T : \mathbb{N} \to \mathbb{N}$ for greater clarity, although obviously $T(\ell) = \tau(\ell)$ for all $\ell \in \mathbb{N}$. The sequence $T(\ell)$, $\ell \in \mathbb{N}$ takes values $3, 5, 7, 8, 9, \ldots$, and is depicted in Figure 10(a). This is A346232 in the On-Line Encyclopedia of Integer Sequences [3]. For this sequence, a simpler expression can be obtained than those resulting from §3.2 with $a = 1$, and the following properties hold.

Theorem 4. *For $\ell \in \mathbb{N}$,*

$$T(\ell) = \left\lfloor \sqrt{2\ell^2 - 2} \right\rfloor + 3. \tag{46}$$

In addition,

(i) This sequence is increasing, with $T(\ell + 1) - T(\ell) \in \{1, 2\}$.



(a) Sequence T

(b) Sequence Λ

Figure 10: Sequences T and Λ

(ii) There can be no more than 2 *consecutive increments equal to* 1.

(iii) Increments equal to 2 *always appear isolated, except at the initial sequence terms* $3,5,7$.

Proof. ***The derivation of (46) in CGCC can probably be adapted. i, j there are $i-2$, $j-2$ here. (It cannot be expressed with $\lceil\cdot\rceil$ instead of $\lfloor\cdot\rfloor$, which would be more "aesthetically aligned" with previous expressions, because for example $\ell = 3$ makes $2\ell^2-2$ a square).

In order to prove that $T(\ell+1)-T(\ell) \in \{1,2\}$, consider the function $q(r) = \sqrt{2r^2-2}$ for $r \in \mathbb{R}$, $r > 1$. Its first derivative is

$$q'(r) = \frac{\sqrt{2}r}{\sqrt{r^2-1}}, \tag{47}$$

and its second derivative is easily seen to be negative. Therefore $q'(r)$ can be bounded for $r \geq 3$ as

$$\lim_{r\to\infty} q'(r) = \sqrt{2} < q'(r) < q'(3) = 3/2. \tag{48}$$

For $\ell \geq 2$, by the mean value theorem [1, section 5.3], when ℓ is increased to $\ell+1$ the term $\sqrt{2\ell^2-2}$ in (46) has an increment that equals $q'(r)$ for some $\ell < r < \ell+1$. Therefore

$$\sqrt{2} < \sqrt{2(\ell+1)^2-2} - \sqrt{2\ell^2-2} < 3/2. \tag{49}$$

Since $1 < \sqrt{2}$ and $3/2 < 2$, (49) implies that $T(\ell+1)-T(\ell)$ can only take the values 1 or 2 for $\ell \geq 3$. In addition, $T(2)-T(1) = T(3)-T(2) = 2$, and thus the result holds for all $\ell \in \mathbb{N}$.

Using the first bound in (49) three times,

$$3\sqrt{2} < \sqrt{2(\ell+3)^2-2} - \sqrt{2\ell^2-2}. \tag{50}$$

Considering that $4 < 3\sqrt{2}$, this implies that $T(\ell+3) - T(\ell) \geq 4$ for $\ell \geq 3$. Therefore at least one of the three increments from $T(\ell)$ to $T(\ell+3)$ is 2. Since $T(2) - T(1) = T(3) - T(2) = 2$, the result holds for all $\ell \in \mathbb{N}$.

Similarly, using the second bound in (49) twice,

$$\sqrt{2(\ell+2)^2 - 2} - \sqrt{2\ell^2 - 2} < 3, \tag{51}$$

which implies that $T(\ell+2) - T(\ell) \leq 3$ for $\ell \geq 3$. Therefore the two increments $T(\ell+1) - T(\ell)$ and $T(\ell+2) - T(\ell+1)$ cannot both be 2 for $\ell \geq 3$. □

The *inverse problem* with integer-length segments can be formulated as follows: given $t \in \mathbb{N}$, find the *minimum* integer length that allows visiting at least t tiles. Observe that in this case, unlike with real-valued lengths, there is indeed a minimum length, because every subset of $\mathbb{N}$ has a minimum. This can be expressed as a function $\Lambda : \mathbb{N} \to \mathbb{N}$:

$$\Lambda(t) = \min\{\ell \in \mathbb{N} \mid T(\ell) \geq t\}, \tag{52}$$

which is related to the function λ corresponding to real-valued lengths by

$$\Lambda(t) = \lfloor \lambda(t) \rfloor + 1. \tag{53}$$

The converse to (52) is (compare to (18)):

$$T(\ell) = \max\{t \in \mathbb{N} \mid \Lambda(t) \leq \ell\}. \tag{54}$$

In view of (52) and (54), T and Λ can be considered as "pseudo-inverse" sequences of each other.

The sequence $\Lambda(t)$, $t \in \mathbb{N}$ can be computed using (37) and (53), and has initial values $1,1,1,2,2,3,3,4,5\ldots$, as shown in Figure 10(b). This is A**** in the On-Line Encyclopedia of Integer Sequences [**?**]. However, a simpler expression can be obtained from (46) and (52). This is established by the next theorem, which also states some properties of Λ, parallel to those of T.

Theorem 5. *For* $t \in \mathbb{N}$,

$$\Lambda(t) = \begin{cases} 1 & \text{for } t \leq 3 \\ \left\lceil \sqrt{\dfrac{(t-3)^2}{2} + 1} \right\rceil & \text{for } t \geq 4. \end{cases} \tag{55}$$

In addition,

(i) This sequence is non-decreasing. Except for the initial run of 3 *equal values, it is formed by runs of* 1 *or* 2 *equal values, with an increment of* 1 *between consecutive runs.*

(ii) There can be no more than 3 *different consecutive terms.*

(iii) A run of 2 *equal values always has* 2 *different terms before and* 2 *different terms after the run, except for the initial terms* $1,1,1,2,2,3,3$.

Proof. Using (46), the inequality $T(\ell) \geq t$ in (52) is written as

$$\left\lfloor \sqrt{2\ell^2 - 2} \right\rfloor + 3 \geq t. \tag{56}$$

Since the right-hand side is an integer, this is equivalent to

$$\sqrt{2\ell^2 - 2} \geq t - 3. \tag{57}$$

For $t \geq 4$, taking squares and rearranging gives

$$\ell \geq \left\lceil \sqrt{\frac{(t-3)^2}{2} + 1} \right\rceil, \tag{58}$$

which combined with (52) yields the second part of (55). The first part results from noting that $t - 3 \leq 0$ for $t \leq 3$, and thus $\ell = 1$ satisfies (57).

The stated properties for Λ follow directly from those of T established by Theorem 4. □

4 Average number of tiles visited by a random segment

Given $\ell \in \mathbb{R}^+$, consider a segment of length ℓ with random position and orientation. Specifically, the coordinates x_1, y_1 of the first endpoint are independent random variables uniformly distributed on $[0, a)$ and $[0, b)$ respectively. The orientation θ of the segment is uniformly distributed on $[0, 2\pi)$. The variables x_1, y_1 and θ determine the coordinates x_2, y_2 of the second endpoint. Note that events of probability 0, such as $x_1 = a$ or the segment passing through a grid point, can be disregarded for computing the average number of tiles visited by the segment.

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined such that $\varphi(\ell)$ gives the average number of tiles visited by a random segment of length ℓ, with the above distributions.

Each realization of the random segment gives rise to a canonical rectangle. The normalized dimensions i and j of the canonical rectangle are thus random variables. Except for a set of realizations with zero probability, i and j are at least 1, and the segment visits $t = i + j - 1$ tiles.

The random variables i and j are not independent. As a consequence, while the marginal distributions of i and j are easy to compute, the distribution of $i + j$ is difficult. However, knowing the marginal distributions is sufficient to compute the average number of visited tiles:

$$\varphi(\ell) = \mathrm{E}[t] = \mathrm{E}[i] + \mathrm{E}[j] - 1. \tag{59}$$

For $0 \leq r \leq 1$, let

$$f(r) = \int_0^r \arccos z \, \mathrm{d}z = r \arccos r - \sqrt{1 - r^2} + 1. \tag{60}$$

Proposition 5. *Given $a, b, \ell \in \mathbb{R}^+$, consider a grid with parameters a, b and a uniformly random segment of length ℓ, as defined above. Let the random variables i, j represent the normalized dimensions of the canonical rectangle. For $n \in \mathbb{N}$,*

$$\Pr[i \geq n] = \begin{cases} 1 & \text{if } n = 1 \\ \dfrac{2\ell}{\pi a}\left(f\left(\dfrac{a(n-1)}{\ell}\right) - f\left(\dfrac{a(n-2)}{\ell}\right)\right) & \text{if } 2 \leq n < \dfrac{\ell}{a} + 1 \\ \dfrac{2\ell}{\pi a}\left(1 - f\left(\dfrac{a(n-2)}{\ell}\right)\right) & \text{if } \dfrac{\ell}{a} + 1 \leq n < \dfrac{\ell}{a} + 2 \\ 0 & \text{if } \dfrac{\ell}{a} + 2 \leq n; \end{cases} \tag{61}$$

and $\Pr[j \geq n]$ *is given by the same expressions with a replaced by b.*

Proof. Clearly, $\Pr[i \geq 1] = 1$. In the following it will be assumed that $n \geq 2$. The basic idea is to compute $\Pr[i \geq n]$ conditioned on (x_1, y_1) (or, as will be seen, only on x_1), and then to average over x_1 and y_1 (actually only over x_1).

Given the coordinates (x_1, y_1) of the first endpoint of the segment, with $0 \leq x_1 \leq a$, $0 \leq y_1 \leq b$, the second endpoint (x_2, y_2) lies on a circle with radius ℓ centered at (x_1, y_1), as shown in Figure 11. The segment orientation is a random angle θ uniformly distributed on $[0, 2\pi)$. It is clear from the figure that $i \geq n$ if and only if $x_2 \geq a(n-1)$ or $x_2 \leq -a(n-2)$; and these events are exclusive for $n \geq 2$. Thus

$$\Pr[i \geq n \,|\, x_1, y_1] = \Pr[x_2 \geq a(n-1) \,|\, x_1, y_1] + \Pr[x_2 \leq -a(n-2) \,|\, x_1, y_1]. \tag{62}$$

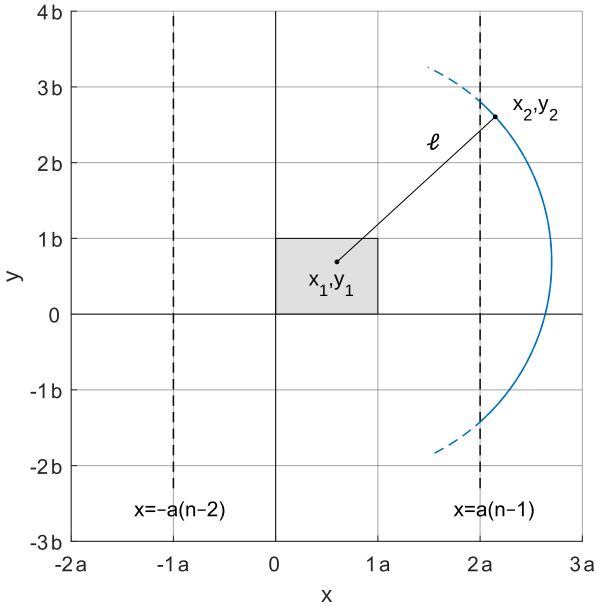
The two conditional probabilities on the right-hand side of (62) are different in general. However, averaging over x_1, y_1 gives, by symmetry, $\Pr[x_2 \geq a(n-1)] = \Pr[x_2 \leq -a(n-2)]$. In addition, the coordinate y_1 does not have any influence on these events, and therefore conditioning on x_1, y_1 is the same as conditioning on x_1. This implies that, for $n \geq 2$,

$$\Pr[i \geq n] = 2\Pr[x_2 \geq a(n-1)] = 2\,\mathrm{E}[\Pr[x_2 \geq a(n-1) \,|\, x_1]]. \tag{63}$$

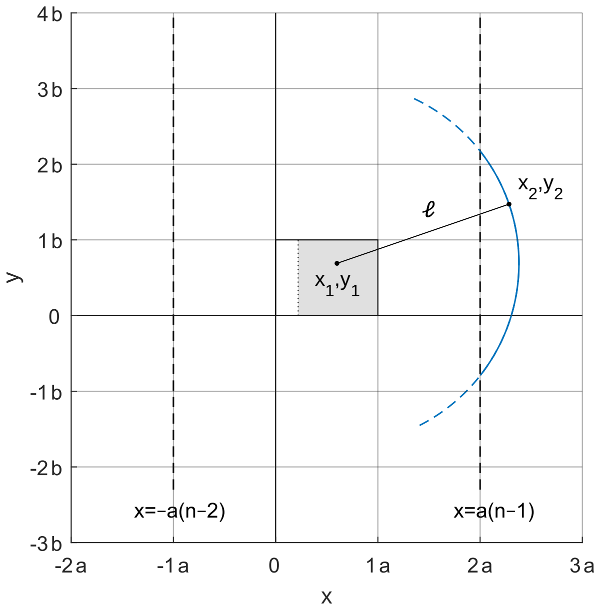
Consider the event $x_2 \geq a(n-1)$ conditioned on x_1, with $n \geq 2$. There are three possibilities depending on x_1, n and ℓ. If $a(n-1) < \ell$, regardless of x_1 the length ℓ is enough for x_2 to exceed $a(n-1)$ for some angles θ. This is depicted in Figure 11(a), where the section of the arc with solid line represents those angles for which $x_2 > a(n-1)$. If $a(n-2) < \ell \leq a(n-1)$, the length will be enough provided that $x_1 \geq a(n-1) - \ell$, which corresponds to the shaded region in Figure 11(b). Lastly, if $\ell \leq a(n-2)$ it is not possible for x_2 to exceed $a(n-1)$. Note that the coordinate y_1 is irrelevant to this.

In the first two cases above, the probability that $x_2 \geq a(n-1)$, conditioned on x_1, is the length of the arc to the right of the line $x = a(n-1)$ divided by $2\pi\ell$, that is,

$$\Pr[x_2 \geq a(n-1) \,|\, x_1] = \frac{1}{\pi} \arccos \frac{a(n-1) - x_1}{\ell}. \tag{64}$$



(a) x_2 can exceed $a(n-1)$ for all x_1, $0 \leq x_1 \leq a$



(b) x_2 can exceed $a(n-1)$ only if $a(n-1)-\ell \leq x_1 \leq a$

Figure 11: Conditions for the normalized width of the canonical rectangle, i, to be equal or greater than a given n. Example with $a = 1.35$, $b = 1$, $n = 3$

In the first case x_1 has a uniform distribution on $(0,a)$, and $\Pr[x_2 \geq a(n-1)]$ is easily obtained from (64) as

$$\begin{aligned}\Pr[x_2 \geq a(n-1)] &= \frac{1}{\pi a}\int_0^a \arccos\frac{a(n-1)-x_1}{\ell}\,\mathrm{d}x_1 \\ &= \frac{\ell}{\pi a}\left(f\left(\frac{a(n-1)}{\ell}\right) - f\left(\frac{a(n-2)}{\ell}\right)\right),\end{aligned} \tag{65}$$

where the function f is defined in (60). Substituting into (63) yields the result in (61), second line.

The second case is similar, but the integration over x_1 is from $a(n-1)-\ell$ to a. Noting that $f(1) = 1$, this gives

$$\Pr[x_2 \geq a(n-1)] = \frac{\ell}{\pi a}\left(1 - f\left(\frac{a(n-2)}{\ell}\right)\right), \tag{66}$$

which combined with (63) yields the expression in (61), third line.

The third case obviously gives $\Pr[i \geq n] = 0$, as in (61), fourth line.

The analysis above can be analogously applied to $\Pr[j \geq n]$ if the x and y axes are interchanged. Thus the result is the same with a replaced by b. □

Theorem 6. *Given $a,b,\ell \in \mathbb{R}^+$, consider a grid with parameters a,b and a uniformly random segment of length ℓ, as defined above. The average number of tiles visited by the segment, $\varphi(\ell)$, is*

$$\varphi(\ell) = \frac{2\ell}{\pi}\left(\frac{1}{a} + \frac{1}{b}\right) + 1. \tag{67}$$

Proof. The expected value of i can be obtained as

$$\mathrm{E}[i] = \sum_{n=1}^{\infty} \Pr[i \geq n]. \tag{68}$$

According to (61), this sum only has a finite number of non-zero terms: 1 for $n=1$, plus $\lceil \ell/a \rceil - 1$ terms given by the second line of (61), plus one term given by the third line, corresponding to $n = \lceil \ell/a \rceil + 1$:

$$\begin{aligned}\mathrm{E}[i] = 1 + \sum_{n=2}^{\lceil \ell/a \rceil} \frac{2\ell}{\pi a}\left(f\left(\frac{a(n-1)}{\ell}\right) - f\left(\frac{a(n-2)}{\ell}\right)\right) + \\ \frac{2\ell}{\pi a}\left(1 - f\left(\frac{a(\lceil \ell/a \rceil - 1)}{\ell}\right)\right).\end{aligned} \tag{69}$$

Most terms involving the function f in (69) cancel, with the result

$$\mathrm{E}[i] = 1 + \frac{2\ell}{\pi a}(1 - f(0)) = 1 + \frac{2\ell}{\pi a}. \tag{70}$$

Similarly,

$$\mathrm{E}[j] = 1 + \frac{2\ell}{\pi b}. \tag{71}$$

Substituting (70) and (71) into (59) gives (67). □

In view of Theorem 6, the average number of visited tiles has a remarkably simple dependence on the segment length, namely an affine function. Conversely, for any $t > 1$ it is immediate to compute the length of a random segment that visits t tiles on average, given as $\varphi^{-1}(t)$.

The results in Proposition 5 and Theorem 6 make clear the relationship between the problem considered here and Buffon's needle problem, i.e. the probability that a random segment of fixed length crosses a line in a regular structure of parallel strips [2, section 1.1]. Firstly, since the number π is involved in (67), it is possible to design a simple probabilistic experiment to estimate the value of π, as in Buffon's original experiment. For example, choosing $a = b = \ell = 1$ gives $\varphi(\ell) = 1 + 1/\pi$.

Secondly, a grid with $b \to \infty$ corresponds to Buffon's arrangement of parallel strips with spacing a. Thus

$$\lim_{b\to\infty} \varphi(\ell) - 1 = \frac{2\ell}{\pi a} \tag{72}$$

gives the average number of lines crossed in Buffon's experiment. For $\ell \leq a$ the segment can cross at most one line, and (72) coincides with the probability of crossing [2, section 1.1].

Lastly, $\Pr[i \geq n]$ as computed in Proposition 5 for $n \geq 2$ can be seen as a generalization of the probability of crossing at least one line in Buffon's experiment, when a needle endpoint is only allowed to move in a narrower region of width $n-1$ times smaller than the width of the strips (instead of across the full strip as in Buffon's setting).

***Maybe a graph comparing maximum and average. Or compute/plot their ratio. The asymptotic slopes are $\sqrt{(}1/a^2 + 1/b^2)$ for τ and $2/\pi(1/a + 1/b)$. It may be interesting to consider the ratio of asymptotic slopes, as a function of a/b. It is maximum for $a = b$, and equals $2\sqrt{2}/\pi = 0.9003$. It tends to $2/\pi$ for $a/b \to \infty$ and for $a/b \to 0$. The minimum number of tiles that can be visited with non-zero probability (corresponding to an n times 1 canonical rectangle if a¿=b) can also be plotted.

References

[1] Stephen Abbott. *Undergraduate Analysis*. Springer, second edition, 2015.

[2] A. M. Mathai. *An Introduction to Geometrical Probability*. Gordon and Breach, 1999.

[3] Neil J. A. Sloane and The OEIS Foundation Inc. The on-line encyclopedia of integer sequences. https://oeis.org/A346232, 2021.