1 Problem Set 4

1. **Partial Differential Equations**

Completely work out the differential equations that result from an application of the method of separation of variables to the 3-dimensional Laplace equations in cylindrical coordinates:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0 \tag{1}$$

Use $V = S(s)\Phi(\phi)Z(z)$ for the separation ansatz. You do not need to solve the resulting differential equations.

Plugging the separation ansatz into the Laplace equation 1,

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$
$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}(S(s)\Phi(\phi)Z(z))\right)+\frac{1}{s^2}\frac{\partial^2}{\partial \phi^2}(S(s)\Phi(\phi)Z(z))+\frac{\partial^2}{\partial z^2}(S(s)\Phi(\phi)Z(z))=0$$

We can drop the parameter of the individual functions for brevity. Clearing the constant terms outside the derivatives,

$$\frac{\Phi Z}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)+\frac{SZ}{s^2}\frac{\partial^2 \Phi}{\partial \phi^2}+S\Phi\frac{\partial^2 Z}{\partial z^2}=0$$

Dividing by $S\Phi Z$ and separating the Z equation,

$$\frac{1}{sS}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)+\frac{1}{s^2\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}+\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=0$$
$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=-\left(\frac{1}{sS}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)+\frac{1}{s^2\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}\right)$$

Observe the the left-hand side and the right-hand side are independent equations. Hence, they are constant and we can write them as

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=-\left(\frac{1}{sS}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)+\frac{1}{s^2\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}\right)\equiv C_1 \tag{2}$$

from which we get our first differential equation:

$$\boxed{\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2}=C_1 Z} \tag{3}$$

Now, let us separate the right-hand side. From equation 2,

$$\left(\frac{1}{sS}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)+\frac{1}{s^2\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}\right)=-C_1$$
$$\frac{1}{sS}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right)=-\frac{1}{s^2\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}-C_1$$

Multiplying by s^2 and rearranging,

$$\frac{s}{S}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + s^2C_1 = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}$$

Again, the left-hand side and the right-hand side expressions are independent. Hence, they are constant and we can write them as

$$\frac{s}{S}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + s^2C_1 = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} \equiv C_2 \tag{4}$$

from which we get our second differential equation - the ϕ equation:

$$\boxed{\frac{\mathrm{d}^2\Phi}{\mathrm{d}\Phi^2} = -\Phi C_2} \tag{5}$$

The third differential equation - the s equation - can also be readily extracted from equation 4

$$\frac{s}{S}\frac{\mathrm{d}}{\mathrm{d}s}\left(s\frac{\mathrm{d}S}{\mathrm{d}s}\right) + s^2C_1 - C_2 = 0$$

Dividing by $\frac{s}{S}$, executing a product rule on the first term,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(s\frac{\mathrm{d}S}{\mathrm{d}s}\right) + sSC_1 - \frac{S}{s}C_2 = 0$$
$$s\frac{\mathrm{d}^2S}{\mathrm{d}s^2} + \frac{\mathrm{d}S}{\mathrm{d}s} + sSC_1 - \frac{S}{s}C_2 = 0$$

From which we get our s equation,

$$\boxed{s^2\frac{\mathrm{d}^2S}{\mathrm{d}s^2} + s\frac{\mathrm{d}S}{\mathrm{d}s} + S(s^2C_1 - C_2) = 0} \tag{6}$$

Observe that this is a Bessel differential equation and the solution is the Bessel function.

The generating function for the Ultraspherical polynomial is

$$g(x,t) = (1-2xt+t^2)^{-\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x)t^n \text{ where } 0 \geq \alpha \geq 1$$

A) Evaluate $C_n^\alpha(-1)$, $C_n^\alpha(0)$, and $C_n^\alpha(1)$

B) Derive a recursion relation that arises from taking the partial derivative with respect x of the generating function.

C) Derive a recursion relation that arises from taking the partial derivative with respect t of the generating function.

A. Evaluations

The ultraspherical polynomials are defined in terms of its generating function

$$g(x,t) = (1-2xt+t^2)^{-\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x)t^n \tag{7}$$

Finding $C_n^\alpha(-1)$,

$$g(-1,t) = (1+t)^{-2\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(-1)t^n \tag{8}$$

Using binomial theorem,

$$(1+t)^{-2\alpha} = \sum_{k=0}^{\infty} \binom{-2\alpha}{k}(t)^k = \sum_{k=0}^{\infty} \binom{-2\alpha}{k}(t^k)$$

Letting k=n,

$$(1+t)^{-2\alpha} = \sum_{n=0}^{\infty} \binom{-2\alpha}{n}(t^n)$$

From equation 8,

$$\sum_{n=0}^{\infty} C_n^{(\alpha)}(-1)t^n = \sum_{n=0}^{\infty} \binom{-2\alpha}{n}t^n$$

By comparing coefficients,

$$C_n^{(\alpha)}(-1) = \binom{-2\alpha}{n} = \frac{(-2\alpha)!}{n!(-2\alpha-n)!} \qquad \text{for n=1,2,3,...}$$

Finding $C_n^\alpha(0)$,

$$g(0,t) = (1+t^2)^{-\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x)t^n \tag{9}$$

Using binomial theorem,

$$(1+t^2)^{-\alpha} = \sum_{k=0}^{\infty} \binom{-\alpha}{k} (t^2)^k = \sum_{k=0}^{\infty} \binom{-\alpha}{k} (t^{2k})$$

Letting 2k=n,

$$(1+t^2)^{-\alpha} = \sum_{n=0}^{\infty} \binom{-\alpha}{\frac{n}{2}} (t^n)$$

From equation 9,

$$\sum_{n=0}^{\infty} C_n^{(\alpha)}(0) t^n = \sum_{n=0}^{\infty} \binom{-\alpha}{\frac{n}{2}} t^n$$

By comparing coefficients,

$$C_n^{(\alpha)}(x) = \binom{-\alpha}{\frac{n}{2}} = \frac{(-\alpha)!}{(\frac{n}{2})!(-\alpha-\frac{n}{2})!} \qquad \text{for n=2,4,6,...}$$

Finding $C_n^{\alpha}(1)$,

$$g(1,t) = (1-t)^{-2\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(1) t^n \tag{10}$$

Using binomial theorem,

$$(1-t)^{-2\alpha} = \sum_{k=0}^{\infty} \binom{-2\alpha}{k} (-t)^k = \sum_{k=0}^{\infty} \binom{-2\alpha}{k} (-1)^k (t^k)$$

Letting k=n,

$$(1-t)^{-2\alpha} = \sum_{n=0}^{\infty} \binom{-2\alpha}{n} (-1)^n (t^n)$$

From equation 10,

$$\sum_{n=0}^{\infty} C_n^{(\alpha)}(1) t^n = \sum_{n=0}^{\infty} \binom{-2\alpha}{n} (-1)^n t^n$$

By comparing coefficients,

$$C_n^{(\alpha)}(-1) = \binom{-2\alpha}{n} (-1)^n = \frac{(-1)^n(-2\alpha)!}{n!(-2\alpha-n)!} \qquad \text{for n=1,2,3,...}$$

B. $\frac{\partial}{\partial x}$ recurrence relation

Differentiating equation 7 with respect to x and letting $\frac{\partial C_n^{(\alpha)}}{\partial x} = C_n^{'(\alpha)}$

$$\frac{\partial}{\partial x}(1-2xt+t^2)^{-\alpha} = \frac{\partial}{\partial x}\sum_{n=0}^{\infty} C_n^{(\alpha)}(x)t^n$$
$$2t\alpha(1-2xt+t^2)^{-\alpha-1} = \sum_{n=0}^{\infty} C_n^{'(\alpha)}(x)t^n$$

Multiplying both sides by $(1-2xt+t^2)$,

$$2t\alpha(1-2xt+t^2)^{-\alpha} = (1-2xt+t^2)\sum_{n=0}^{\infty} C_n^{'(\alpha)}(x)t^n$$
$$2t\alpha\sum_{n=0}^{\infty} C_n^{(\alpha)}(x)t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} C_n^{'(\alpha)}(x)t^n$$

Expanding the terms to an entirely summations equation

$$\sum_{n=0}^{\infty} 2\alpha C_n^{(\alpha)}(x)t^{n+1} = \sum_{n=0}^{\infty} C_n^{'(\alpha)}(x)t^n - \sum_{n=0}^{\infty} 2xC_n^{'(\alpha)}(x)t^{n+1} + \sum_{n=0}^{\infty} C_n^{'(\alpha)}(x)t^{n+2}$$

Collecting like terms,

$$\sum_{n=0}^{\infty}(2\alpha C_n^{(\alpha)}(x) + 2xC_n^{'(\alpha)}(x))t^{n+1} = \sum_{n=0}^{\infty} C_n^{'(\alpha)}(x)t^n + \sum_{n=0}^{\infty} C_n^{'(\alpha)}(x)t^{n+2} \tag{11}$$

To be able to compare coefficients, we should convert the exponents of t into $n+1$ by manipulating indices. Revising the second term by letting n=r+1 and converting back to the standard dummy index n,

$$\sum_{n=0}^{\infty} C_n^{'(\alpha)}(x)t^n = \sum_{r=-1}^{\infty} C_{r+1}^{'(\alpha)}(x)t^{r+1} = \sum_{n=-1}^{\infty} C_{n+1}^{'(\alpha)}(x)t^{n+1}$$

Applying a similar process to the third term, this time letting n=r-1,

$$\sum_{n=0}^{\infty} C_n^{'(\alpha)}(x)t^{n+2} = \sum_{r=1}^{\infty} C_{r-1}^{'(\alpha)}(x)t^{r+1} = \sum_{n=1}^{\infty} C_{n-1}^{'(\alpha)}(x)t^{n+1}$$

Collecting these modifications to equation 11,

$$\sum_{n=0}^{\infty}(2\alpha C_n^{(\alpha)}(x) + 2xC_n^{'(\alpha)}(x))t^{n+1} = \sum_{n=-1}^{\infty} C_{n+1}^{'(\alpha)}(x)t^{n+1} + \sum_{n=1}^{\infty} C_{n-1}^{'(\alpha)}(x)t^{n+1} \tag{12}$$

From the uniqueness of Taylor's series, we can equate the coefficients to arrive at the recurrence relation for $n \geq 1$:

$$C_{n+1}^{'(\alpha)}(x) + C_{n-1}^{'(\alpha)}(x) = 2xC_{n+1}^{'(\alpha)}(x) + 2\alpha C_n^{(\alpha)}(x) \tag{13}$$

C. $\frac{\partial}{\partial t}$ relation

The methods here are somewhat similar with deriving the first recurrence relation equation 13. Differentiating equation 7 with respect to t,

$$\frac{\partial}{\partial t}(1-2xt+t^2)^{-\alpha} = \frac{\partial}{\partial x}\sum_{n=0}^{\infty} C_n^{(\alpha)}(x)t^n$$
$$2\alpha(x-t)(1-2xt+t^2)^{-\alpha-1} = \sum_{n=0}^{\infty} nC_n^{(\alpha)}(x)t^{n-1}$$

Multiplying both sides by $(1-2xt+t^2)$,

$$2\alpha(x-t)(1-2xt+t^2)^{-\alpha} = (1-2xt+t^2)\sum_{n=0}^{\infty} nC_n^{(\alpha)}(x)t^{n-1}$$
$$2\alpha(x-t)\sum_{n=0}^{\infty} C_n^{(\alpha)}(x)t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} nC_n^{(\alpha)}(x)t^{n-1}$$

Again, expanding all terms as to form an entire summations equation,

$$\sum_{n=0}^{\infty} 2\alpha x C_n^{(\alpha)}(x)t^n - \sum_{n=0}^{\infty} 2\alpha C_n^{(\alpha)}(x)t^{n+1} = \sum_{n=0}^{\infty} nC_n^{(\alpha)}(x)t^{n-1} - \sum_{n=0}^{\infty} 2nxC_n^{(\alpha)}(x)t^n + \sum_{n=0}^{\infty} nC_n^{(\alpha)}(x)t^{n+1}$$

Collecting like terms,

$$\sum_{n=0}^{\infty}(2n+2\alpha)xC_n^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty} nC_n^{(\alpha)}(x)t^{n-1} + \sum_{n=0}^{\infty}(n+2\alpha)C_n^{(\alpha)}(x)t^{n+1} \tag{14}$$

Likewise, to equate the coefficients, we shall re-index such that the exponents are equal. First, observe that n=0 has no contribution to the summation and we can move the index like so

$$\sum_{n=0}^{\infty} nC_n^{(\alpha)}(x)t^{n-1} = \sum_{n=1}^{\infty} nC_n^{(\alpha)}(x)t^{n-1}$$

Now, letting r=n-1 and ultimately converting back to the standard index n,

$$\sum_{n=1}^{\infty} nC_n^{(\alpha)}(x)t^{n-1} = \sum_{r=0}^{\infty}(r+1)C_{r+1}^{(\alpha)}(x)t^r = \sum_{n=0}^{\infty}(n+1)C_{n+1}^{(\alpha)}(x)t^n$$

Revising the third term by letting r=n+1,

$$\sum_{n=0}^{\infty}(n+2\alpha)C_n^{(\alpha)}(x)t^{n+1} = \sum_{r=1}^{\infty}(r-1+2\alpha)C_{r-1}^{(\alpha)}(x)t^r = \sum_{n=1}^{\infty}(n-1+2\alpha)C_{n-1}^{(\alpha)}(x)t^n$$

Plugging these revisions into equation 14,

$$\sum_{n=0}^{\infty}(2n+2\alpha)xC_n^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty}(n+1)C_{n+1}^{(\alpha)}(x)t^n + \sum_{n=1}^{\infty}(n-1+2\alpha)C_{n-1}^{(\alpha)}(x)t^n \tag{15}$$

From Taylor's series uniqueness theorem, we can equate the coefficients to arrive at the second recurrence relation

$$(2n+2\alpha)xC_n(x)^{(\alpha)} = (n+1)C_{n+1}(x)^{(\alpha)} + (n-1+2\alpha)C_{n-1}^{(\alpha)}(x) \tag{16}$$

2 Problem Set 3

2. **Initial Value Problems - Method of Integral Transforms**

Solve for $x(t)$

$$\frac{\mathrm{d}^4 x}{\mathrm{d}t^4} + \alpha^4 x = \beta \sin(\omega t) \tag{17}$$

where $x(0) = A$, $\dot{x}(0) = 0$, $\ddot{x}(0) = 0$, and $\dddot{x}(0) = 0$

This is a fourth order nonhomogenous ordinary differential equation with vanishing initial conditions. Hence, this suggests a Laplace transform approach.

Transforming both sides,

$$\mathcal{L}\left(\frac{\mathrm{d}^4 x}{\mathrm{d}t^4} + \alpha^4 x = \beta \sin(\omega t)\right)$$

$$s^4 X(s) - s^3 x(0) - s^2 \dot{x}(0) - s\ddot{x}(0) - \dddot{x}(0) + \alpha^4 X(s) = \beta \frac{\omega}{\omega^2 + s^2}$$

Invoking initial conditions,

$$s^4 X(s) - s^3 A + \alpha^4 X(s) = \beta \frac{\omega}{\omega^s + s^2}$$

Rearranging,

$$X(s)\left(s^4 + \alpha^4\right) = \beta \frac{\omega}{\omega^2 + s^2} + s^3 A \tag{18}$$

$$X(s) = \frac{\beta\omega}{(\omega^2 + s^2)(\alpha^4 + s^4)} + \frac{As^3}{(\alpha^4 + s^4)} \tag{19}$$

To find x(t), we simply invert X(s) starting off by defining some convenient quantities:

$$b \equiv \omega^2 \qquad u \equiv s^2 \qquad a \equiv \alpha^2$$

Using these definitions,

$$X(s) = \beta\omega \frac{1}{(u + b)(u^2 + a^2)} + \frac{As^3}{u^2 + a^2}$$

We can do a partial fraction decomposition on the first term (omitting the $\beta\omega$ first for convenience)

$$\frac{1}{(u + b)(u^2 + a^2)} \equiv \frac{C_1}{u + b} + \frac{C_2 u + C_3}{u^2 + a^2}$$

Comparing coefficients,

$$1 = u^2(C_1 + C_2) + u(bC_2 + C_3) + (C_1 a^2 + C_3 b)$$

$$\therefore C_1 = -C_2 \qquad \therefore C_2 b = -C_3 \qquad \therefore C_1 a^2 + C_2 b = 1$$

Using the three equations to find the three C_i, we find that

$$\therefore C_1 = \frac{1}{a^2+b^2} \qquad \therefore C_2 = -\frac{1}{a^2+b^2} \qquad \therefore C_3 = \frac{b}{a^2+b^2}$$

Updating equation 18,

$$\begin{aligned} X(s) &= \frac{\beta\omega}{a^2+b^2}\left(\left(\frac{1}{u+b}\right)+\left(\frac{1-u}{u^2+a^2}\right)\right)+\frac{As^3}{s^4+\alpha^4} \\ &= \frac{\beta\omega}{a^2+b^2}\left(\frac{1}{s^2+\omega^2}+\frac{1}{s^4+\alpha^4}-\frac{s^2}{s^4+\alpha^4}\right)+\frac{As^3}{s^4+\alpha^4} \end{aligned} \tag{20}$$

Inverting back,

$$x(t) = \frac{\beta\omega}{a^2+b^2}\left(\mathcal{L}^{-1}\frac{1}{s^2+\omega^2}+\mathcal{L}^{-1}\frac{1}{s^4+\alpha^4}-\mathcal{L}^{-1}\frac{s^2}{s^4+\alpha^4}\right)+A\left(\mathcal{L}^{-1}\frac{s^3}{s^4+\alpha^4}\right) \tag{21}$$

We have four forms of rational expressions that we have to invert. From left to right, the first inverse is:

$$\mathcal{L}^{-1}\frac{1}{s^2+\omega^2} = \frac{1}{\omega}\sin(\omega t)$$

For the remaining terms, we can evaluate another partial fraction decomposition of form[1]

$$\frac{s^n}{s^4+\alpha^4} \qquad \text{for n=0,2,3}$$

Completing the squares of the denominator and invoking difference of two squares identity,

$$\frac{s^n}{(s^2+\alpha^2)^2-(\sqrt{2}s\alpha)^2} = \frac{s^n}{(s^2+\alpha^2+\sqrt{2}s\alpha)(s^2+\alpha^2-\sqrt{2}s\alpha)}$$

Decomposing into partial fractions[2],

$$\begin{aligned} \frac{s^n}{(s^2+\alpha^2+\sqrt{2}s\alpha)(s^2+\alpha^2-\sqrt{2}s\alpha)} &= \frac{C_{4n}s+C_{5n}}{s^2+\alpha^2+\sqrt{2}s\alpha}+\frac{C_{6n}s+C_{7n}}{s^2+\alpha^2-\sqrt{2}s\alpha} \\ &= \frac{C_{4n}s+C_{5n}}{(s+\frac{\alpha}{\sqrt{2}})^2+\frac{\alpha^2}{2}}+\frac{C_{6n}s+C_{7n}}{(s-\frac{\alpha}{\sqrt{2}})^2+\frac{\alpha^2}{2}} \end{aligned}$$

We can form our solution ansatz first by doing an inverse Laplace transform and then finding the coefficients later. Defining another parameter for convenience,

$$k \equiv \frac{\alpha}{\sqrt{2}}$$

$$\begin{aligned} \frac{s^n}{(s^2+\alpha^2+\sqrt{2}s\alpha)(s^2+\alpha^2-\sqrt{2}s\alpha)} &= \frac{C_{4n}(s+k)-C_{4n}k+C_{5n}}{(s+k)^2+k^2}+\frac{C_{6n}(s-k)+C_{6n}k+C_{7n}}{(s-k)^2+k^2} \\ &= \frac{C_{4n}(s+k)}{(s+k)^2+k^2}+\frac{-C_{4n}k+C_{5n}}{(s+k)^2+k^2}+\frac{C_{6n}(s-k)}{(s-k)^2+k^2}+\frac{C_{6n}k+C_{7n}}{(s-k)^2+k^2} \end{aligned}$$

[1] We did this to simultaneously solve three inverse Laplace transforms in a single go.
[2] The constants indices' were chosen from 4 to 7 for continuity from our first 3 constants. The further affix n represents the nth case.

The expressions are now ready aboard the inverse Laplace train.

$$\begin{aligned}\mathcal{L}^{-1}\frac{s^n}{s^4+\alpha^4} &= C_{4n}\mathcal{L}^{-1}\frac{(s+k)}{(s+k)^2+k^2}+(-C_{4n}k+C_{5n})\mathcal{L}^{-1}\frac{1}{(s+k)^2+k^2}\\ &+ C_{6n}\mathcal{L}^{-1}\frac{(s-k)}{(s-k)^2+k^2}+(C_{6n}k+C_{7n})\mathcal{L}^{-1}\frac{1}{(s-k)^2+k^2}\end{aligned}$$

From the basic Laplace transforms and applying shifting theorems,

$$\begin{aligned}\mathcal{L}^{-1}\frac{s^n}{s^4+\alpha^4} &= C_{6n}e^{-kt}\cos kt+\frac{-C_{4n}k+C_{5n}}{k}e^{-kt}\sin kt\\ &+ C_{6n}e^{kt}\cos kt+\frac{C_{6n}k+C_{7n}}{k}e^{kt}\sin kt\end{aligned}$$

From the definition of k,

$$\begin{aligned}\mathcal{L}^{-1}\frac{s^n}{s^4+\alpha^4} &= C_{6n}e^{-\frac{\alpha}{\sqrt{2}}t}\cos\frac{\alpha}{\sqrt{2}}t+\frac{-C_{4n}\frac{\alpha}{\sqrt{2}}+C_{5n}}{\frac{\alpha}{\sqrt{2}}}e^{-\frac{\alpha}{\sqrt{2}}t}\sin\frac{\alpha}{\sqrt{2}}t\\ &+ C_{6n}e^{\frac{\alpha}{\sqrt{2}}t}\cos\frac{\alpha}{\sqrt{2}}t+\frac{C_{6n}\frac{\alpha}{\sqrt{2}}+C_{7n}}{\frac{\alpha}{\sqrt{2}}}e^{\frac{\alpha}{\sqrt{2}}t}\sin\frac{\alpha}{\sqrt{2}}t\\ &\equiv D_n(t)\end{aligned} \tag{22}$$

Updating equation 21,

$$x(t)=\frac{\beta\omega}{a^2+b^2}\left(\frac{1}{\omega}\sin\omega t+D_0(t)-D_2(t)\right)+A\left(D_3(t)\right) \tag{23}$$

Our last task is to find the coefficients

$$\begin{aligned}s^n &= (C_{4n}s+C_{5n})(s^2+\alpha^2-\sqrt{2}s\alpha)+(C_{6n}s+C_{7n})(s^2+\alpha^2+\sqrt{2}s\alpha)\\ &= s^3(C_{4n}+C_{6n})+s^2(-C_{4n}\sqrt{2}\alpha+C_{6n}\sqrt{2}\alpha+C_{5n}+C_{7n})\\ &+ s^1(C_{4n}\alpha^2+C_{6n}\alpha^2-C_{5n}\sqrt{2}\alpha+C_{7n}\sqrt{2}\alpha)+s^0(C_{5n}\alpha^2+C_{7n}\alpha^2)\end{aligned}$$

This gives us four equations to solve four unknowns. For generality, we can use a Kronecker-Delta symbol. Then, for a given n, where n = 0,2,3,

$$\begin{aligned}\delta_{3n} &= C_{4n}+C_{6n}\\ \delta_{2n} &= -C_{4n}\sqrt{2}\alpha+C_{6n}\sqrt{2}\alpha+C_{5n}+C_{7n}\\ \delta_{1n} &= C_{4n}\alpha^2+C_{6n}\alpha^2-C_{5n}\sqrt{2}\alpha+C_{7n}\sqrt{2}\alpha\\ \delta_{0n} &= C_{5n}\alpha^2+C_{7n}\alpha^2\end{aligned}$$

We can express this in matrix equation for brevity with form $RX_n=T_n$,

$$\begin{bmatrix}1 & 0 & 1 & 0\\ -\sqrt{2}\alpha & 1 & \sqrt{2}\alpha & 1\\ \alpha^2 & -\sqrt{2}\alpha & \alpha^2 & \sqrt{2}\alpha\\ 0 & \alpha^2 & 0 & \alpha^2\end{bmatrix}\begin{bmatrix}C_{4n}\\ C_{5n}\\ C_{6n}\\ C_{7n}\end{bmatrix}=\begin{bmatrix}\delta_{3n}\\ \delta_{2n}\\ \delta_{1n}\\ \delta_{0n}\end{bmatrix}$$

X_n contain the coefficients. This can be found by an elaborate usage of Gaussian elimination. For n=0,2,3,

$$X_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}\alpha}{4} \\ \frac{1}{2} \\ \frac{\sqrt{2}\alpha}{4} \end{bmatrix} \qquad X_2 = \begin{bmatrix} -\frac{\sqrt{2}}{4\alpha} \\ 0 \\ \frac{\sqrt{2}}{4\alpha} \\ 0 \end{bmatrix} \qquad X_0 = \begin{bmatrix} \frac{\sqrt{2}}{4\alpha^3} \\ \frac{1}{2a^2} \\ -\frac{\sqrt{2}}{4\alpha^3} \\ \frac{1}{2a^2} \end{bmatrix}$$

Writing the coefficients explicitly,

$$\begin{array}{cccc} C_{43} = \frac{1}{2} & C_{53} = \frac{\sqrt{2}\alpha}{4} & C_{63} = \frac{1}{2} & C_{73} = \frac{\sqrt{2}\alpha}{4} \\ C_{42} = -\frac{\sqrt{2}}{4\alpha} & C_{52} = 0 & C_{62} = \frac{\sqrt{2}}{4\alpha} & C_{72} = 0 \\ C_{40} = \frac{\sqrt{2}}{4\alpha^3} & C_{50} = \frac{1}{2\alpha^2} & c_{60} = -\frac{\sqrt{2}}{4\alpha^3} & C_{70} = \frac{1}{2a^2} \end{array}$$

Plugging this coefficients into $D_n(t)$ at equation 22,

$$\begin{aligned} D_3 &= \frac{1}{2}e^{-\frac{\alpha}{\sqrt{2}}t}\cos\frac{\alpha}{\sqrt{2}}t + \frac{-\frac{1}{2}\frac{\alpha}{\sqrt{2}} + \frac{\sqrt{2}\alpha}{4}}{\frac{\alpha}{\sqrt{2}}}e^{-\frac{\alpha}{\sqrt{2}}t}\sin\frac{\alpha}{\sqrt{2}}t \\ &+ \frac{1}{2}e^{\frac{\alpha}{\sqrt{2}}t}\cos\frac{\alpha}{\sqrt{2}}t + \frac{\frac{\alpha}{2\sqrt{2}} + \frac{\sqrt{2}\alpha}{4}}{\frac{\alpha}{\sqrt{2}}}e^{\frac{\alpha}{\sqrt{2}}t}\sin\frac{\alpha}{\sqrt{2}}t \end{aligned}$$

$$\begin{aligned} D_2 &= \frac{\sqrt{2}}{4a}e^{-\frac{\alpha}{\sqrt{2}}t}\cos\frac{\alpha}{\sqrt{2}}t + \frac{-\frac{\sqrt{2}}{4a}\frac{\alpha}{\sqrt{2}}}{\frac{\alpha}{\sqrt{2}}}e^{-\frac{\alpha}{\sqrt{2}}t}\sin\frac{\alpha}{\sqrt{2}}t \\ &+ \frac{\sqrt{2}}{4a}e^{\frac{\alpha}{\sqrt{2}}t}\cos\frac{\alpha}{\sqrt{2}}t + \frac{\sqrt{2}}{4\alpha}e^{\frac{\alpha}{\sqrt{2}}t}\sin\frac{\alpha}{\sqrt{2}}t \end{aligned}$$

$$\begin{aligned} D_0 &= -\frac{\sqrt{2}}{4\alpha^3}e^{-\frac{\alpha}{\sqrt{2}}t}\cos\frac{\alpha}{\sqrt{2}}t + \frac{-\frac{\sqrt{2}}{4\alpha^3}\frac{\alpha}{\sqrt{2}} + \frac{1}{2a^2}}{\frac{\alpha}{\sqrt{2}}}e^{-\frac{\alpha}{\sqrt{2}}t}\sin\frac{\alpha}{\sqrt{2}}t \\ &+ -\frac{\sqrt{2}}{4\alpha^3}e^{\frac{\alpha}{\sqrt{2}}t}\cos\frac{\alpha}{\sqrt{2}}t + \frac{\frac{1}{4a^2} + \frac{1}{2a^2}}{\frac{\alpha}{\sqrt{2}}}e^{\frac{\alpha}{\sqrt{2}}t}\sin\frac{\alpha}{\sqrt{2}}t \end{aligned}$$

Hence, using the above coefficients D_n, we have our solution for the differential equation from equation 23,

$$\begin{aligned} x(t) &= \frac{\beta\omega}{a^2+b^2}\left(\frac{1}{\omega}\sin\omega t + D_0(t) - D_2(t)\right) + A\left(D_3(t)\right) \\ &= \frac{\beta\sin\omega t}{a^2+b^2} + \sum_{i=0}^{3} D_i(t)\left(\delta_{i0}\frac{\beta\omega}{a^2+b^2} - \delta_{i2}\frac{\beta\omega}{a^2+b^2} + \delta_{i3}A\right) \end{aligned}$$

Solve for $u(x,t)$ for $x > 0$, $t > 0$, $\alpha > 0$ if

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} = v^2 \frac{\partial^2 u}{\partial x^2}$$

given the boundary conditions u(0,t)=0 when $t \geq 0$, u(x,0)=f(x), and $\frac{\partial u}{\partial t}|_{t=0} = 0$

We can convert the partial differential equation into an ordinary differential equation via a Fourier transform by defining the transform as

$$\mathcal{FT}u(x,t) = U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)e^{ikx}\mathrm{d}x$$

Transforming both sides, we get

$$\frac{\partial^2 U(k,t)}{\partial t^2} + \alpha \frac{\partial U(k,t)}{\partial t} + v^2 k^2 U(k,t) = 0$$

This is a second order homogeneous ODE in t with characteristic equation

$$\lambda_t^2 + \alpha\lambda_t + v^2k^2 = 0$$

Giving us

$$\lambda_t = \frac{-\alpha \pm i\sqrt{4v^2k^2 - \alpha^2}}{2} \equiv -\beta \pm i\omega(k) \tag{24}$$

where

$$\beta \equiv \frac{\alpha}{2} \qquad\qquad \omega(k) \equiv \frac{\sqrt{4v^2k^2 - \alpha^2}}{2} \tag{25}$$

Using equation 24, we can form our solution in the k domain

$$U(k,t) = C_+(k)e^{-\beta t}e^{i\omega(k)t} + C_-(k)e^{-\beta t}e^{-i\omega(k)t} \tag{26}$$

and simply inverting, we can form our solution in the x domain

$$u(x,t) = \frac{1}{\sqrt{2\pi}}e^{-\beta t} \int_{-\infty}^{\infty} \left(C_+(k)e^{i\omega(k)t} + C_-(k)e^{-i\omega(k)t}\right) e^{-ikx}\mathrm{d}k \tag{27}$$

We can interpret equation 27 as a superposition of decaying rightwards and leftwards moving waves. We proceed to find the constants $C_+(k)$ and $C_-(k)$. Since $u(x,0) = 0$, from equation 27,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(C_+(k) + C_-(k)\right) e^{-ikx}\mathrm{d}k = f(x)$$

We find that, if F(k) is the Fourier transform of f(x),

$$C_+(k) + C_-(k) = F(k) \tag{28}$$

Invoking another initial condition and transforming it,

$$\frac{\partial u(x,t)}{\partial t}|_{t=0} = 0 \iff \frac{\partial U(k,t)}{\partial t}|_{t=0} = 0$$

Differentiating equation 26 and evaluating t=0,

$$\frac{\partial U(k,t)}{\partial t}|_{t=0} = 0 \iff (-\beta + i\omega(k))C_+(k) + (-\beta - i\omega(k))C_-(k) = 0 \tag{29}$$

Simultaneously solving for equations 28 and 29, we find that

$$C_+ = F(k)\left(\frac{1}{2} + \frac{\beta}{2i\omega(k)}\right) \tag{30}$$

$$C_- = F(k)\left(\frac{1}{2} - \frac{\beta}{2i\omega(k)}\right) \tag{31}$$

Updating equation 27,

$$\begin{aligned} u(x,t) &= e^{-\beta t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\left(\frac{1}{2} + \frac{\beta}{2i\omega(k)}\right) e^{i\omega(k)t}e^{-ikx}\mathrm{d}k \\ &+ e^{-\beta t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\left(\frac{1}{2} - \frac{\beta}{2i\omega(k)}\right) e^{-i\omega(k)t}e^{-ikx}\mathrm{d}k \end{aligned} \tag{32}$$

Before proceeding, it is interesting to note that when $\alpha = 0$, from the definitions on equation 25, $\beta = 0$ and $\omega = vk$. Then equation 32 greatly reduces to a modified Fourier transform discussed on class which has a solution

$$u(x,t) = \frac{1}{2}f(x - vt) + \frac{1}{2}f(x + vt)$$

Going back to our general case, to derive an analytic solution, we can combine the exponentials by linearizing $\omega(k)$ via a second degree Taylor expansion about k_0 and invoking the convolution theorem for the remaining factors

$$\begin{aligned} \omega(k) &= \omega(k_0) + \frac{\mathrm{d}\omega}{\mathrm{d}t}(k_0)(k - k_0) \equiv \omega(k_0) + v(k_0)(k - k_0) \\ &= [(\omega(k_0) - k_0 v(k_0)) + v(k_0)k] \end{aligned}$$

$$\begin{aligned} u(x,t) &= e^{-\beta t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\left(\frac{1}{2} + \frac{\beta}{2i\omega(k)}\right) e^{i[(\omega(k_0)-k_0v(k_0))]t}e^{-ik(x-v(k_0)t)}\mathrm{d}k \\ &+ e^{-\beta t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\left(\frac{1}{2} - \frac{\beta}{2i\omega(k)}\right) e^{-i[(\omega(k_0)-k_0v(k_0))]t}e^{-ik(x+v(k_0)t)}\mathrm{d}k \end{aligned} \tag{33}$$

Then, yeeting the time-dependent factors out the integral,

$$\begin{aligned} u(x,t) &= e^{-\beta t}e^{i[(\omega(k_0)-k_0v(k_0))]t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\left(\frac{1}{2} + \frac{\beta}{2i\omega(k)}\right) e^{-ik(x-v(k_0)t)}\mathrm{d}k \\ &+ e^{-\beta t}e^{-i[(\omega(k_0)-k_0v(k_0))]t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\left(\frac{1}{2} - \frac{\beta}{2i\omega(k)}\right) e^{-ik(x+v(k_0)t)}\mathrm{d}k \end{aligned} \tag{34}$$

Defining the second terms as a Fourier transform

$$G(k) \equiv \frac{1}{2} + \frac{\beta}{2i\omega(k)} = \mathcal{FT}(g(x)) \qquad H(k) \equiv \frac{1}{2} - \frac{\beta}{2i\omega(k)} = \mathcal{FT}(h(x)) \tag{35}$$

or

$$g(x) = \mathcal{FT}^{-1}\left(\frac{1}{2} + \frac{\beta}{2i\omega(k)}\right) \qquad h(x) = \mathcal{FT}^{-1}\left(\frac{1}{2} - \frac{\beta}{2i\omega(k)}\right) \tag{36}$$

These can be evaluated[3] as

$$g(x) = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(1 + \frac{\alpha}{i\sqrt{4v^2k^2 - \alpha^2}}e^{-ikx}\mathrm{d}k\right) \qquad h(x) = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(1 - \frac{\alpha}{i\sqrt{4v^2k^2 - \alpha^2}}e^{-ikx}\mathrm{d}k\right) \tag{37}$$

We can rewrite equation 34 as

$$\begin{aligned} u(x,t) &= e^{-\beta t}e^{i[(\omega(k_0)-k_0v(k_0))]t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} FT(f(x))\mathcal{FT}(g(x))e^{-ik(x-v(k_0)t)}\mathrm{d}k \\ &+ e^{-\beta t}e^{-i[(\omega(k_0)-k_0v(k_0))]t}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} FT(f(x))FT(h(x))e^{-ik(x+v(k_0)t)}\mathrm{d}k \end{aligned} \tag{38}$$

Notice that since f(x) is the initial shape of the medium, the condition $u(0,t) = 0$ suggests that there are no displacements at coordinate $x = 0$. Hence, $f(0) = 0$. This can be easily confirmed by plugging in $f(0) = 0$ at equation 38. $x = 0$ acts as a node.

Finally, by convolution theorem,

$$\begin{aligned} u(x,t) &= e^{-\beta t}e^{i[(\omega(k_0)-k_0v(k_0))]t}[(f * g)(x - v(k_0)t)] \\ &+ e^{-\beta t}e^{-i[(\omega(k_0)-k_0v(k_0))]t}[(f * h)(x + v(k_0)t)] \end{aligned} \tag{39}$$

Observe that the solution has some dispersive dependence on the choice spatial frequency k_0. That is, waves with different k travels at different speed v. We can derive a more explicit analytic solution of u(x,t) when were given a specific f(x) so we'll stop here.

[3]I am not well-versed in the complex domain

3 Problem Set 5

3. **Special Functions**

A. Calculate the Green's function G(x,x') when $\frac{d^2x}{dx^2}G + \frac{dG}{dx} = \delta(x - x')$ where 0<x'<x, and the boundary conditions are $\frac{dG}{dx} = 0$ when x=0 and when x=L.

B. Use the results above to solve for $y(x) : \frac{d^2y}{dx^2} + \frac{dy}{dx} = f(x)$ where the boundary conditions are $\frac{dy}{dx} = 0$ when x=0 and when x=L

This problem was given as a bonus.

The orthogonality relations for Hermite polynomials are given by

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{x^2}dx = 2^n n!\sqrt{\pi}\delta_{m,n} \tag{40}$$

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}) \tag{41}$$

A. Determine the coefficients a_n in Hermite series for the Dirac delta function $\delta(x-x') = \sum_{n=0}^{\infty} a_n H_n(x)$

B. Determine the coefficients b_0, b_1, b_2 in the Hermite series $e^{-ax^2} = \sum_{n=0}^{\infty} b_n H_n(x)$, where a>0

We can expand Dirac delta function into an orthogonal series expansion using Hermite polynomials by applying the following transformation

$$\delta(x-x') = \sum_{n=0}^{\infty} a_n H_n(x)$$

$$\int_{-\infty}^{\infty} \delta(x-x')H_m(x)e^{-x^2}\mathrm{d}x = \sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\mathrm{d}x$$

From the orthogonality relation at equation 40,

$$\int_{-\infty}^{\infty} \delta(x-x')H_m(x)e^{-x^2}\mathrm{d}x = \sum_{n=0}^{\infty} a_n 2^n n!\sqrt{\pi}\delta_{m,n}$$

From the definition of the Dirac delta function,

$$H_m(x')e^{-x'^2} = \sum_{n=0}^{\infty} a_n 2^n n!\sqrt{\pi}\delta_{m,n}$$

Observe that the summation on the right hand side is only nonzero when m=n. That is,

$$H_m(x')e^{-x'^2} = a_m 2^m m!\sqrt{\pi}$$

Reverting the dummy index back to n and solving for a_n,

$$a_n = \frac{H_n(x')e^{-x'^2}}{2^n n!\sqrt{\pi}}$$

Finally, from Rodrigue's formula at equation 41,

$$a_n = \frac{(-1)^n \frac{d^n}{dx^n}(e^{-x^2})|_{x=x'}}{2^n n!\sqrt{\pi}}$$

Hence, the orthogonal series expansion of the Dirac delta function is

$$\delta(x-x') = \sum_{n=0}^{\infty} \frac{(-1)^n \frac{d^n}{dx^n}(e^{-x^2})|_{x=x'}}{2^n n!\sqrt{\pi}} H_n(x)$$

Now, we are tasked to expand the exponential e^{-x^2} using Hermite polynomials. Again, we start off by transforming the equation as follows

$$e^{-\alpha x^2} = \sum_{b=0}^{\infty} b_n H_n(x)$$
$$\int_{-\infty}^{\infty} e^{-\alpha x^2} H_m(x) e^{-x^2} \mathrm{d}x = \sum_{n=0}^{\infty} b_n \int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} \mathrm{d}x$$

Using the orthogonality relation at equation 40 on the right hand side,

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} H_m(x) e^{-x^2} \mathrm{d}x = \sum_{n=0}^{\infty} b_n 2^n n! \sqrt{\pi} \delta_{m,n}$$

Again, right hand side is only nonzero when m=n,

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} H_m(x) e^{-x^2} \mathrm{d}x = b_m 2^m m! \sqrt{\pi}$$

Reverting back to dummy index n and solving for b_n,

$$b_n = \frac{\int_{-\infty}^{\infty} e^{-\alpha x^2} H_n(x) e^{-x^2} \mathrm{d}x}{2^n n! \sqrt{\pi}} \tag{42}$$

From Rodrigue's formula at equation 41,

$$b_n = \frac{\int_{-\infty}^{\infty} e^{-\alpha x^2} (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) e^{-x^2} \mathrm{d}x}{2^n n! \sqrt{\pi}} = \frac{(-1)^n \int_{-\infty}^{\infty} e^{-\alpha x^2} \frac{d^n}{dx^n} (e^{-x^2}) \mathrm{d}x}{2^n n! \sqrt{\pi}}$$

[4]For n=0,

$$b_0 = \frac{\int_{-\infty}^{\infty} e^{-x^2(\alpha+1)} \mathrm{d}x}{\sqrt{\pi}} = \sqrt{\frac{1}{\alpha+1}}$$

For n=1,

$$b_1 = \frac{-\int_{-\infty}^{\infty} e^{-\alpha x^2} \frac{d}{dx} (e^{-x^2}) \mathrm{d}x}{2\sqrt{\pi}} = \frac{-\int_{-\infty}^{\infty} e^{-\alpha x^2} (-2x e^{-x^2}) \mathrm{d}x}{2\sqrt{\pi}} = \frac{\int_{-\infty}^{\infty} x e^{-x^2(\alpha+1)} \mathrm{d}x}{\sqrt{\pi}} = 0$$

For n=2,

$$b_2 = \frac{\int_{-\infty}^{\infty} e^{-\alpha x^2} \frac{d^2}{dx^2} (e^{-x^2}) \mathrm{d}x}{8\sqrt{\pi}} = \frac{\int_{-\infty}^{\infty} e^{-x^2(\alpha+1)} (1 - 2x^2) \mathrm{d}x}{8\sqrt{\pi}} = \frac{\int_{-\infty}^{\infty} e^{-x^2(\alpha+1)} \mathrm{d}x}{8\sqrt{\pi}} - \frac{\int_{-\infty}^{\infty} x^2 e^{-x^2(\alpha+1)} \mathrm{d}x}{4\sqrt{\pi}}$$
$$= \frac{1}{8}\sqrt{\frac{1}{\alpha+1}} - \frac{1}{8}\sqrt{\frac{1}{(\alpha+1)^3}}$$

[4]In these integrals, I am using derived relations of Gaussian integrals from the earlier topics in Physics 117

4 Problem Set 6

4. **Integral Equations**

Solve for $y(x)$ using Laplace transform in the integral equation

$$y(x) = x + \int_0^x (x-t)^2 y(t)\mathrm{d}t \tag{43}$$

Applying a Laplace transform to equation 43,

$$\mathcal{L}(y(x)) = \mathcal{L}(x) + \left(\mathcal{L}\int_0^x (x-t)^2 y(t)\mathrm{d}t\right) \tag{44}$$

Note that

$$\int_0^x (x-t)^2 y(t)\mathrm{d}t = x^2 * y(t)$$

Hence, equation 44 becomes,

$$\mathcal{L}(y(x)) = \mathcal{L}x + \mathcal{L}(x^2 * y(t)) = \mathcal{L}x + \mathcal{L}(x^2)\mathcal{L}(y(t))$$

Letting $Y(s) = \mathcal{L}(y(x)) = \mathcal{L}(y(t))$,

$$Y(s) = \frac{1}{s^2} + \frac{2}{s^3}Y(s)$$

Solving for Y(s),

$$Y(s) = \frac{s}{s^3 - 2}$$

Factoring out the denominator,

$$Y(s) = \frac{s}{(s-c_1)(s-c_2)(s-c_3)}$$

where

$$\begin{aligned} c_1 &= 2^{\frac{1}{3}} \\ c_2 &= -\frac{2^{\frac{1}{3}}}{2} + i\frac{2^{\frac{1}{3}}3^{\frac{1}{2}}}{2} \\ c_3 &= -\frac{2^{\frac{1}{3}}}{2} - i\frac{2^{\frac{1}{3}}3^{\frac{1}{2}}}{2} \end{aligned}$$

Using Residue theorem

$$\mathrm{Res}_i = \frac{1}{(n-1)!}\lim_{x\to c_i}\frac{d^{n-1}}{dz^{n-1}}[(z-c_i)^n f(z)e^{zt}] \tag{45}$$

For simple poles, n=1 and equation 45 reduces to

$$\text{Res}_i = \lim_{s \to c_i} [(s - c_i) Y(s) e^{st}] \tag{46}$$

where x(t) is the sum of the residues

$$y(t) = \sum_i \text{Res}_i$$

Finding the individual residues using equation 46,

$$\text{Res } (Y(s), c_1) = \lim_{s \to 2^{\frac{1}{3}}} (s - 2^{\frac{1}{3}}) \frac{s}{s^3 - 2} e^{st} = \frac{1}{(3)2^{(1/3)}} e^{2^{(1/3)} t}$$

$$\text{Res } (Y(s), c_2) = \lim_{s \to -\frac{2^{\frac{1}{3}}}{2} + i\frac{2^{\frac{1}{3}} 3^{\frac{1}{2}}}{2}} (s + \frac{2^{\frac{1}{3}}}{2} - i\frac{2^{\frac{1}{3}} 3^{\frac{1}{2}}}{2}) \frac{s}{s^3 - 2} e^{st} = \frac{2i2^{\frac{1}{3}}\sqrt{3}(\frac{-1}{2^{\frac{2}{3}}} + i2^{1/3}\sqrt{3}) e^{-(1/2^{2/3} + i2^{1/3}\sqrt{3})} t}{-2 + (-\frac{1}{2^{2/3}} + i2^{1/3}\sqrt{3})^3}$$

$$\text{Res } (Y(s), c_3) = \lim_{s \to -\frac{2^{\frac{1}{3}}}{2} - i\frac{2^{\frac{1}{3}} 3^{\frac{1}{2}}}{2}} (s + \frac{2^{\frac{1}{3}}}{2} + i\frac{2^{\frac{1}{3}} 3^{\frac{1}{2}}}{2}) \frac{s}{s^3 - 2} e^{st} = \frac{-2i2^{\frac{1}{3}}\sqrt{3}(\frac{-1}{2^{\frac{2}{3}}} - i2^{1/3}\sqrt{3}) e^{(-1/2^{2/3} - i2^{1/3}\sqrt{3}t)}}{-2 + (-\frac{1}{2^{2/3}} - i2^{1/3}\sqrt{3})^3}$$

Summing up the residues, we have our solution

$$y(t) = \frac{1}{(3)2^{(1/3)}} e^{2^{(1/3)} t} + \frac{2i2^{\frac{1}{3}}\sqrt{3}(\frac{-1}{2^{\frac{2}{3}}} + i2^{1/3}\sqrt{3}) e^{-(1/2^{2/3} + i2^{1/3}\sqrt{3}t)}}{-2 + (-\frac{1}{2^{2/3}} + i2^{1/3}\sqrt{3})^3} + \frac{-2i2^{\frac{1}{3}}\sqrt{3}(\frac{-1}{2^{\frac{2}{3}}} - i2^{1/3}\sqrt{3}) e^{-(1/2^{2/3} - i2^{1/3}\sqrt{3})t}}{-2 + (-\frac{1}{2^{2/3}} - i2^{1/3}\sqrt{3})^3}$$

Calculate the eigenvalues and corresponding eigenfunctions for the integral equation

$$\phi(x) = \lambda \int_0^\pi [x^2 \sin t + t^2 \cos x]\phi(t)\mathrm{d}t$$

We can solve this by separating the kernel, applying an integral transform to the integral equation using the temporal parts[5] of the kernel as the transform kernel[6], thereby generating systems of linear equations with $\int_0^\pi N_i(x)\phi(x)\mathrm{d}x = c_i$ as unknowns, casting it into matrix eigenvalue problem, and solving for the corresponding eigenfunction for each respective eigenvalue gives us the solution.

Decomposing the kernel,

$$M_1 = x^2 \qquad M_2 = \cos x \qquad N_1 = \sin t \qquad N_2 = t^2 \tag{47}$$

We can directly cast it in matrix form as follows

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \lambda \begin{bmatrix}\int_0^\pi N_1(x)M_1(x) & \int_0^\pi N_1(x)M_2(x)\\ \int_0^\pi N_2(x)M_1(x) & \int_0^\pi N_2(x)M_2(x)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

Plugging definitions on equation 47,

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \lambda \begin{bmatrix}\int_0^\pi \sin(x)x^2 & \int_0^\pi \sin(x)\cos(x)\\ \int_0^\pi x^2x^2 & \int_0^\pi x^2\cos(x)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

Using DI method[7] for integrating the diagonal terms, one can easily solve the anti-derivatives. The anti-diagonal integrals are trivial. Evaluating them,

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \lambda \begin{bmatrix}(2-x^2)\cos(x) + 2x\sin(x)|_0^\pi & -\frac{1}{2}\cos^2(x)|_0^\pi\\ \frac{x^5}{5}|_0^\pi & (x^2-2)\sin(x) + 2x\cos(x)|_0^\pi\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

Evaluating the limits,

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \lambda \begin{bmatrix}\pi^2-4 & 0\\ \frac{\pi^5}{5} & -2\pi\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

$$\begin{bmatrix}\lambda(4-\pi^2)+1 & 0\\ \lambda\frac{\pi^5}{5} & 2\lambda\pi+1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

To solve, recall that the square matrix must have a zero determinant. That is,

$$\begin{gathered}(\lambda(4-\pi^2)+1)(2\lambda\pi+1) = 0\\ (\lambda(4-\pi^2))(2\lambda\pi) + (\lambda(4-\pi^2)) + (2\lambda\pi) + 1 = 0\\ \lambda^2(-2\pi^3+8\pi) + \lambda(-\pi^2+2\pi+4) + 1 = 0\end{gathered}$$

Using the quadratic formula, we can find the eigenvalues

$$\lambda_- = -\frac{1}{2\pi} \qquad \lambda_+ = \frac{1}{\pi^2-4}$$

The associated eigenfunction for λ_+ can be calculated by plugging it into the matrix and solving for c_1 and c_2,

$$\begin{bmatrix}\frac{1}{\pi^2-4}(4-\pi^2)+1 & 0\\ \frac{1}{\pi^2-4}\frac{\pi^5}{5} & 2\frac{1}{\pi^2-4}\pi+1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

[5] N_i

[6] $\int_0^\pi N_i(x)[...]\mathrm{d}x$

[7] Also known as tabular method - basically integration by parts but the formatting made it solvable off the top

Executing a matrix multiplication on the second row,

$$\frac{\pi^5/5}{\pi^2-4}c_1 + \frac{2(\pi+1)}{\pi^2-4}c_2 = 0$$
$$\frac{-\pi}{5}c_1 = 2(\pi+1)c_2$$

Choosing $c_1 = 5$, we get $c_2 = \frac{-\pi}{2\pi+1}$
And we can derive our solution

$$\phi_+(x) = \lambda_+(c_1 M_1(x) + c_2 M_2(x))$$
$$\phi_+(x) = \frac{1}{\pi^2-4}(5x^2 + \frac{-\pi\cos(x)}{2\pi+1})$$

For the associated eigenfunction for λ_-,

$$\begin{bmatrix} -\frac{1}{2\pi}(4-\pi^2)+1 & 0 \\ -\frac{1}{2\pi}\frac{\pi^5}{5} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which, unfortunately, could not contain any nontrivial eigenfunction.