Quantum Mechanics I (141) Problem Set 3

Lyle Kenneth Geraldez

I. Solutions overview

The following problem set seeks to exploit the orthonormality of the eigenstates to calculate relevant quantities in a quantum harmonic oscillator.

II. Harmonic Oscillator

Given that

$$|\Psi(t)\rangle = \frac{1}{\sqrt{5}}|\psi_1(t)\rangle + \frac{2}{\sqrt{5}}|\psi_2(t)\rangle \tag{1}$$

where $|\psi_n(t)\rangle := |\psi_n\rangle e^{-iE_nt/\hbar} := T_n|\psi_n\rangle$. $|\psi_n\rangle$ are normalized and expressed via Hermite polynomials computed using the analytic power series approach:

$$\psi_n(\xi) = \frac{m\omega}{\pi\hbar}^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2}; \quad \xi := \sqrt{\frac{m\omega}{\hbar}}x \tag{2}$$

The associated energies are

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega \tag{3}$$

The position and momentum operators can be expressed in terms of ladder operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a_+}+\hat{a_-}) \tag{4}$$

$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a_+}-\hat{a_-}) \tag{5}$$

which, together with the orthogonality of the stationary states, will be useful in a while. The ladder operators act on the eigenfunctions to either raise or lower the energy level. That is,

$$\hat{a_+}\psi_n = \sqrt{n+1}\psi_{n+1} \tag{6}$$

$$\hat{a_-}\psi_n = \sqrt{n}\psi_{n-1} \tag{7}$$

We are tasked to find $\langle x(t)\rangle$ and $\langle p(t)\rangle$. To find $\langle x(t)\rangle$, we simply "sandwich" it as so

$$\langle x(t)\rangle = \langle\Psi(t)|\,\hat{x}\,|\Psi(t)\rangle \tag{8}$$

Denoting the temporal term as $e^{-iE_nt/\hbar} = T_n$, the wavefunction is

$$|\Psi(t)\rangle = \frac{1}{\sqrt{5}}T_1|\psi_1\rangle + \frac{2}{\sqrt{5}}T_2|\psi_2\rangle \tag{9}$$

Applying the position operator on $|\Psi(t)\rangle$ using Eq. (4),

$$\frac{1}{\sqrt{5}}\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)(T_1|\psi_1\rangle + 2T_2|\psi_2\rangle) \tag{10}$$

Using Eqs. (6) and (7),

$$\frac{1}{\sqrt{5}}\sqrt{\frac{\hbar}{2m\omega}}(T_1\sqrt{2}|\psi_2\rangle + 2\sqrt{3}T_2|\psi_3\rangle + T_1|\psi_0\rangle + 2\sqrt{2}T_2|\psi_1\rangle) \tag{11}$$

Dotting this with $\langle\Psi(t)|$, we foresee that states $|\psi_0\rangle$ and $|\psi_3\rangle$ must vanish due to orthonormality. Simplifying, we have

$$\hat{x}|\Psi(t)\rangle = \frac{1}{\sqrt{5}}\sqrt{\frac{\hbar}{m\omega}}(T_1|\psi_2\rangle + 2T_2|\psi_1\rangle) \tag{12}$$

Multiplying with the dual vector,

$$\left(\frac{1}{\sqrt{5}}(T_1^*\langle\psi_1|) + 2T_2^*\langle\psi_2|\right)\cdot \tag{13}$$

$$\left(\frac{1}{\sqrt{5}}\sqrt{\frac{\hbar}{m\omega}}(T_1|\psi_2\rangle + 2T_2|\psi_1\rangle)\right) \tag{14}$$

Multiplying, observe that $\langle\psi_1|\psi_2\rangle$ and $\langle\psi_1|\psi_2\rangle$ terms must be zero. Hence,

$$\frac{1}{5}\sqrt{\frac{\hbar}{m\omega}}(2T_1^*T_2\langle\psi_1|\psi_1\rangle + 2T_2^*T_1\langle\psi_2|\psi_2\rangle) \tag{15}$$

where normalized vectors lead to unity inner product

$$\frac{2}{5}\sqrt{\frac{\hbar}{m\omega}}(T_1^*T_2 + T_2^*T_1) \tag{16}$$

The energies for n=1 and n=2 are

$$E_1 = \frac{3}{2}\hbar\omega \tag{17}$$

$$E_2 = \frac{5}{2}\hbar\omega \tag{18}$$

Expanding in terms of its exponential form, Eq. (16) is equal to

$$\frac{2}{5}\sqrt{\frac{\hbar}{m\omega}}(e^{i(E_1-E_2)t/\hbar} + e^{i(E_2-E_1)t/\hbar}) \tag{19}$$

Plugging in the energies,

$$\frac{2}{5}\sqrt{\frac{\hbar}{m\omega}}(e^{-i\omega t} + e^{i\omega t}) \tag{20}$$

where one can quickly recognize that this is (twice) the complex definition of the cosine function. Hence,

$$\langle x(t)\rangle = \frac{4}{5}\sqrt{\frac{\hbar}{m\omega}}(\cos(\omega t)) \tag{21}$$

We employ similar procedure to find $\langle p_x(t)\rangle$: use ladders using Eq. (5) and exploit orthonormality. Observe that the only difference in Eqs. (4) and (5) is the sign and the scalar. Hence, we could recycle Eqs. (11) and (12) to find that

$$\hat{p}\,|\Psi(t)\rangle = \frac{i}{\sqrt{5}}\sqrt{\hbar m\omega}(T_1\,|\psi_2\rangle - 2T_2\,|\psi_1\rangle) \tag{22}$$

By similar reasoning of orthogonality earlier, dotting this with the dual gives

$$\frac{i}{5}\sqrt{\hbar m\omega}(2T_1^*T_2\,\langle\psi_1|\psi_1\rangle - 2T_2^*T_1\,\langle\psi_2|\psi_2\rangle) \tag{23}$$

where the inner products are still unity. Using the same exponentials and energies, this simplifies to

$$\frac{2}{5}\sqrt{\hbar m\omega}i(e^{-i\omega t} - e^{i\omega t}) \tag{24}$$

where the exponential term is (twice; negative) the complex definition of the sine function. Hence,

$$\langle p_x(t)\rangle = -\frac{4}{5}\sqrt{\hbar m\omega}\,\sin(\omega t) \tag{25}$$

Notice differentiating Eq. (21) and multiplying by a factor of m yields to the momentum expectation value.

$$m\frac{\mathrm{d}}{\mathrm{d}t}\,\langle x(t)\rangle = m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{5}\sqrt{\frac{\hbar}{m\omega}}(\cos(\omega t))\right) \tag{26}$$

$$-\frac{4}{5}\sqrt{\hbar m\omega}\,\sin(\omega t) \tag{27}$$

Hence, Ehrenfest's theorem holds. Now, we work on a different wavefunction::

$$|\Psi(t)\rangle = \frac{1}{\sqrt{5}}T_1\,|\psi_1\rangle + \frac{2}{\sqrt{5}}T_3\,|\psi_3\rangle \tag{28}$$

Observe that application of $\hat{a}_+$ and $\hat{a}_-$ will yield to the following terms: $|\psi_2\rangle$ and $|\psi_4\rangle$ for $\hat{a}_+$ and $|\psi_0\rangle$ and $|\psi_2\rangle$ for $\hat{a}_-$. Not one of this is one of the eigenstates present in $\Psi(t)$; hence, all inner products will be zero by orthogonality and the position expectation value is then

$$\langle x\rangle = 0 \tag{29}$$

By this argument, we can quickly find the uncertainty of the ground-state wavefunction as follows. The wavefunction is

$$|\Psi(t)\rangle = T_0\,|\psi_0\rangle \tag{30}$$

Again, application of ladder operators will make all resulting terms orthogonal to the original state. (i.e. $\langle\psi_0|\,\hat{a}_\pm\,|\psi_n\rangle$=0, since $n \neq 0$ for all lader operations.) Since $\hat{x}$ and $\hat{p}$ are just linear combinations of the ladder operators, they are equivalent to a successive application of ladder operators where their inner product to $\langle\Psi(t)|$ will always be zero. By this argument, it follows that

$$\langle x\rangle = 0 \iff \langle x\rangle^2 = 0 \tag{31}$$

$$\langle p\rangle = 0 \iff \langle p\rangle^2 = 0 \tag{32}$$

Now, we find $\langle x^2\rangle$ and $\langle p^2\rangle$. Squaring Eqs. (5) and (7), we find that

$$\hat{x}^2 = \frac{\hbar}{2m\omega}(\hat{a}_+\hat{a}_+ + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-\hat{a}_-) \tag{33}$$

$$\hat{p}^2 = -\frac{\hbar m\omega}{2}(\hat{a}_+\hat{a}_+ - \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ - \hat{a}_-\hat{a}_-) \tag{34}$$

The composite operators $\hat{a}_+\hat{a}_+$ and $\hat{a}_-\hat{a}_-$ will turn $|\psi_0\rangle$ into an orthogonal state with respect to $|\psi_0\rangle$. Hence, we can neglect those as we foresee their cancellation later. We proceed to show the effects of these mixed ladder operators to the eigenkets using Eqs. (6) and (7):

$$\hat{a}_-\hat{a}_+\,|\psi_n\rangle = \hat{a}_-\sqrt{n+1}\,|\psi_{n+1}\rangle = n+1\,|\psi_n\rangle \tag{35}$$

$$\hat{a}_+\hat{a}_-\,|\psi_n\rangle = \hat{a}_+\sqrt{n}\,|\psi_{n-1}\rangle = n\,|\psi_n\rangle \tag{36}$$

In the case of $|\psi_0\rangle$,

$$\hat{a}_-\hat{a}_+\,|\psi_n\rangle = |\psi_n\rangle \tag{37}$$

$$\hat{a}_+\hat{a}_-\,|\psi_n\rangle = 0 \tag{38}$$

Applying the operators,

$$\hat{x^2}\,|\Psi(t)\rangle = \frac{\hbar}{2m\omega}T_0(\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+)\,|\psi_0\rangle \tag{39}$$

$$= \frac{\hbar}{2m\omega}T_0\,|\psi_0\rangle \tag{40}$$

$$\hat{p^2}\,|\Psi(t)\rangle = -\frac{\hbar m\omega}{2}T_0(\hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+)\,|\psi_0\rangle \tag{41}$$

$$= \frac{\hbar m\omega}{2}T_0\,|\psi_0\rangle \tag{42}$$

Finally taking the inner products noting that $T_0^*T_0 = 1$,

$$\langle\Psi(t)|\,\hat{x^2}\,|\Psi(t)\rangle = T_0^*T_0\frac{\hbar}{2m\omega}\,\langle\psi_0|\psi_0\rangle \tag{43}$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega} \tag{44}$$

$$\langle\Psi(t)|\,\hat{p^2}\,|\Psi(t)\rangle = T_0^*T_0\frac{\hbar}{2m\omega}\,\langle\psi_0|\psi_0\rangle \tag{45}$$

$$\langle p^2\rangle = \frac{\hbar m\omega}{2} \tag{46}$$

Finally, from Eqs. (31), (32), (44), and (46), and $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$[1]

$$(\Delta x)^2(\Delta p_x)^2 = \left(\frac{\hbar}{2m\omega} - 0\right)\left(\frac{\hbar m\omega}{2} - 0\right) \tag{47}$$

$$= \frac{\hbar^2}{4} \tag{48}$$

[1] Griffith's notation, page 33

The product of the uncertainties ($\Delta x \Delta p_x$) is then

$$\Delta x \Delta p_x = \frac{\hbar}{2} \tag{49}$$