1 Problem Set 2: Electrostatics

1. **Potential, Field, and Energy of a Charged Disk**

A two-dimensional disk of radius R carries a uniform charge per unit area $\sigma > 0$.

a) Calculate the potential at any point on the symmetry axis of the disk.
b) Calculate the potential at any point on the rim of the disk.
c) Sketch the electric field pattern everywhere in the plane of the disk.
d) Calculate the work needed to assemble the disk.

To analyze the system, we can use cylindrical coordinates shown below.

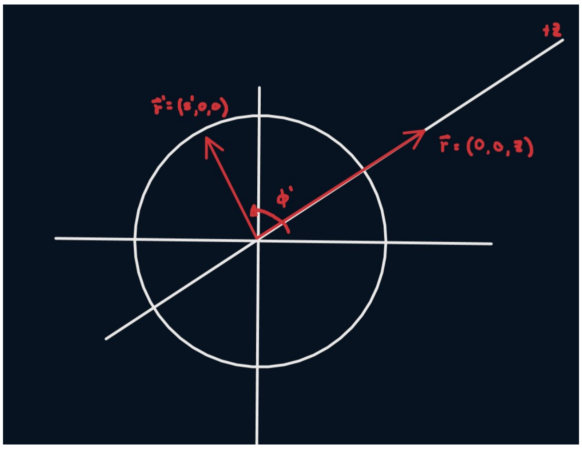

Figure 1: Cylindrical coordinate system was used to analyze the system.

The first task is to find the potential at any point on the symmetry axis of the disk. Constraining the potential function allows us to exploit a simplifying symmetry as we will see shortly. Recall that the potential of a surface with the reference point set at infinity is given as

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(\vec{r})}{|\vec{r} - \vec{r}'|} \, \mathrm{d}a \tag{1}$$

We will try to derive a general expression first and see what's convenient for the problem (i.e. if it makes our lives significantly better to simply analyze the special case). This way, we can recycle results for future problems. The charge is uniform so

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int \frac{1}{|\vec{r} - \vec{r}'|} \, \mathrm{d}a \tag{2}$$

To evaluate the norm of the difference of two vectors, we use the cosine law. From the discussion in vector analysis, recall that dotting a vector difference with itself gives us $(\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = A^2 + B^2 - 2AB \cos\gamma$ where γ is the angle between $\vec{A}$ and $\vec{B}$. Hence, the denominator of the integrand gives us

$$|\vec{r} - \vec{r}'| = (r^2 + r'^2 - 2rr'\cos\gamma)^{1/2} \tag{3}$$

Updating the integral,

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int \frac{1}{(r^2 + r'^2 - 2rr'\cos\gamma)^{1/2}} \, \mathrm{d}a \tag{4}$$

Applying a cylindrical Jacobian gives us

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int_S \frac{1}{(r^2 + r'^2 - 2rr'\cos\gamma)^{1/2}} s' \, \mathrm{d}s' \, \mathrm{d}\phi' \tag{5}$$

where S is the disk. Now comes the role of symmetry. The first task is only to account the potentials at the z-axis - perpendicular to the disk. Hence, the position vector of the field point $\vec{r}$ must be perpendicular to the position vector of the source point $\vec{r'}$ for all points on the disk. Since $\vec{r} \perp \vec{r'}$, $\cos\gamma = 0$, reducing the integral into

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int_S \frac{1}{(r^2 + r'^2)^{1/2}} s' \; \mathrm{d}s' \; \mathrm{d}\phi' \tag{6}$$

Moreover, expressing the position vectors in cylindrical basis gives us $\vec{r} = (s, 0, z)$ and $\vec{r'} = (s', 0, z')$. The problem demands that source points are constrained at $z' = 0, \forall s'$ while field points are constrained at $s = 0, \forall z$. Hence, $\vec{r} = (0, 0, z)$ and $\vec{r'} = (s', 0, 0)$. This gives us the norms readily as $r^2 = z^2$ and $r'^2 = s'^2$. The integral is then

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int_S \frac{1}{(z^2 + s'^2)^{1/2}} s' \; \mathrm{d}s' \; \mathrm{d}\phi' \tag{7}$$

Separating into iterated integrals and integrating over the circular disk,

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{R} \frac{s'}{(z^2 + s'^2)^{1/2}} \; \mathrm{d}s' \; \mathrm{d}\phi' \tag{8}$$

One can immediately recognize that the integral demands trigonometric substitution of form [1] $s' = z \tan\theta$, $\mathrm{d}s' = z \sec^2\theta \; \mathrm{d}\theta$, $(s'^2 + z^2)^{1/2} = z \sec\theta$. The limits of integration runs from $\theta = 0$ to $\theta = \tan^{-1}(R/z)$. It is important to note that we are assuming $z \neq 0$. The integral becomes

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{\tan^{-1}(R/z)} \frac{z \tan\theta}{z \sec\theta} z \sec^2\theta \; \mathrm{d}\theta \; \mathrm{d}\phi' \tag{9}$$

Simplifying,

$$V(\vec{r}) = \frac{\sigma z}{4\pi\varepsilon_0} \int_0^{2\pi} \mathrm{d}\phi' \int_0^{\tan^{-1}(R/z)} \sec\theta \tan\theta \; \mathrm{d}\theta \tag{10}$$

where the integral now is thankfully elementary. Completing the integration,

$$\begin{aligned} V(\vec{r}) =& \frac{\sigma z}{4\pi\varepsilon_0} \int_0^{2\pi} \mathrm{d}\phi' \int_0^{\tan^{-1}(R/z)} \sec\theta \tan\theta \; \mathrm{d}\theta \qquad (11) \\ =& \frac{\sigma z}{4\pi\varepsilon_0} (2\pi) (\sec\theta|_0^{\tan^{-1}(R/z)}) \qquad (12) \end{aligned}$$

From previous notes in Math 21, the trigonometric term evaluates to $\sec(\tan^{-1}(x)) = \sqrt{x^2 + 1}$. Finally, we have the potential along the symmetry line of the disc expressed as

$$\begin{aligned} V(z) &= \frac{\sigma z}{4\pi\varepsilon_0} (2\pi) \left(\left(\frac{R^2}{z^2} + 1 \right)^{1/2} - 1 \right) \qquad (13) \\ &= \frac{\sigma z}{2\varepsilon_0} \left(\left(\frac{R^2}{z^2} + 1 \right)^{1/2} - 1 \right) \qquad (14) \end{aligned}$$

Hence

$$\boxed{V(z) = \frac{\sigma z}{2\varepsilon_0} \left(\left(\frac{R^2}{z^2} + 1 \right)^{1/2} - 1 \right)} \tag{15}$$

The next task is to calculate the potential at the rim. In this case, the position vectors are given as $\vec{r} = (s, 0, 0)$ and $\vec{r'} = (s', 0, 0)$. It can be shown that placing a cylindrical coordinate system centered at

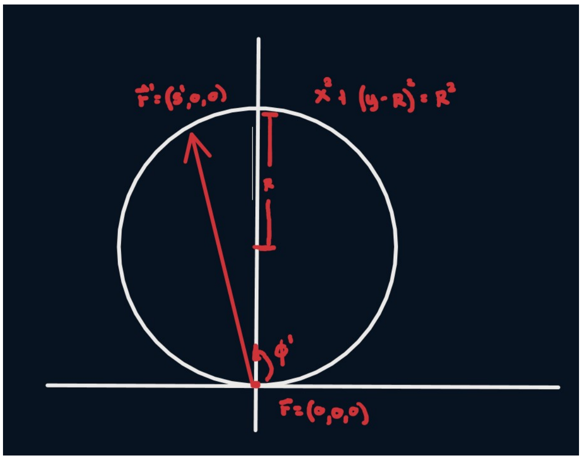

Figure 2: The disk has been offset by an amount R in the +y direction

the disk's center would give us a diverging integral. [2]. A way to put up with this is to shift our coordinate system such that the origin lies on the rim of the disk shown at Figure 2.

To identify the integration limits in this new coordinate system, we will first describe the equation of the disk in Cartesian coordinate system. The change in coordinates shifts the circle upwards by an amount R. Hence, the circle's equation is now $x^2 + (y - R)^2 = R^2$ or $s = 2R\ \sin\phi$ in polar coordinates. Unlike the previous task, the two position vectors are not perpendicular and so $\cos\gamma \neq 0$. To evaluate the integral, we should introduce ϕ dependence on the separation vector. Using these, the potential can be expressed as

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{\pi} \int_0^{2R\sin\phi'} \frac{s'}{(r^2 + r'^2 - 2rr'\cos\gamma)^{1/2}}\ \mathrm{d}s'\ \mathrm{d}\phi' \tag{16}$$

Since we want to analyze the potential at the rim, we would end up in a ϕ-dependent s. However, we could simply evaluate the field at the origin $s = 0$ and argue by symmetry that this holds for all points in the rim. Hence, our position vectors are simply $\vec{r} = (0, 0, 0)$ and $\vec{r}' = (s', 0, 0)$ with norms $r = 0$ and $r' = s'$, respectively. This vastly simplifies the integral into

$$V(\vec{0}) = \frac{\sigma}{4\pi\varepsilon_0} \int_0^{\pi} \int_0^{2R\sin\phi'} \mathrm{d}s'\ \mathrm{d}\phi' \tag{17}$$

$$= \frac{\sigma}{4\pi\varepsilon_0} \int_0^{\pi} 2R\sin\phi'\ \mathrm{d}\phi' \tag{18}$$

$$= \frac{\sigma R}{2\pi\varepsilon_0}(-\cos|_0^{\pi}) = \frac{\sigma R}{\pi\varepsilon_0} \tag{19}$$

Hence, the potential at the rim of the disk is

$$\boxed{V(0) = \frac{\sigma R}{\pi\varepsilon_0}}$$

The next task is to sketch the electric field in the plane of the disk. This means finding $\vec{E}$ for $\vec{r} = (s, 0, 0)$. We can first solve for the potential and simply take the gradient afterwards. The potential integral is expressed as

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int \frac{s'}{(s^2 + s'^2 - 2ss'\cos\gamma)^{1/2}}\ \mathrm{d}a \tag{20}$$

[1] The θ here is merely a mathematical object which may or may not have any physical significance. I did not intend to confuse the reader with the polar angle θ

[2] I didn't include the derivation due to time limitations. The hint greatly reduced the time taken to devise a new coordinate system

What we can do is to place $\vec{r}$ at the x-axis such that $\gamma = \phi$ and simply argue that the field must be azimuthally symmetric such that

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0}\int_0^R\int_0^{2\pi}\frac{s'}{(s^2+s'^2-2ss'\cos\phi)^{1/2}}\,\mathrm{d}s\,\mathrm{d}\phi' \tag{21}$$

Observe that there is no way, no symmetry to exploit, to simplify the expression and we are forced to completely evaluate the integral to find the behavior of the electric field in the disk's plane. I am not entirely sure if I lack the mathematical prerequisites, such as using special functions or using other coordinate systems, to probe this problem. Instead, I'll assume that only the z-direction has non-zero contribution. Then, we can take the gradient of Eq. (15) to find an expression for the field. In cartesian coordinates, this reduces to an ordinary derivative

$$\vec{E} = -\frac{\mathrm{d}}{\mathrm{d}z}\frac{\sigma z}{2\varepsilon_0}\left(\left(\frac{R^2}{z^2}+1\right)^{1/2}-1\right)\hat{z} \tag{22}$$

$$= \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{(z^2+R^2)^{1/2}}\right)\hat{z} \tag{23}$$

Hence, the electric field, in this case, is

$$\boxed{\vec{E} = \left(\frac{\sigma}{2\varepsilon_0}-\frac{\sigma}{2\varepsilon_0}\frac{z}{(z^2+R^2)^{1/2}}\right)\hat{z}} \tag{24}$$

Observe that at $z = 0$, the electric field is similar to that of an infinite plane and diminishes due to the second term. Moreover, this diminish has a positive derivative and is strictly increasing. The perpendicular field at the plane can be sketched as:

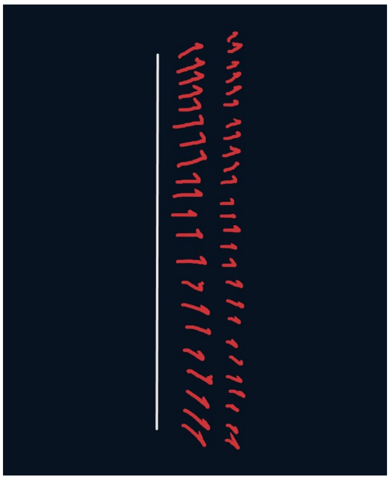

Figure 3: Field lines directed on the positive z-axis are perpendicular to the disk at the center and monotonically diminishes at z increases. Away from the center, it gradually picks up radial components.

Lastly, to find the work needed to assemble the disk, we need to evaluate an integral given by

$$W = \frac{1}{2}\int \sigma V \mathrm{d}a \tag{25}$$

Observe that we needed an expression for V that we are desperately seeking earlier. It turns out that a paper by Cifta used elliptic integrals and Bessel functions to find a closed form expression for V on the disk's surface. According to the paper, the electrostatic energy of a disk is given by

$$U = \frac{4}{3}\sigma V_0 R^2 \tag{26}$$

where V_0 is the potential at the origin. From Eq. (15),

$$V_0 = \frac{\sigma R}{2\varepsilon_0} \tag{27}$$

Plugging this in, we have the work required to assemble the disk as

$$\boxed{W = \frac{2\sigma^2 R^3}{3\varepsilon_0}} \tag{28}$$

Again, without including s-dependence, I can't seem to think of any way to find out a closed form expression of $\vec{E}$ or V.

2. **A Non-Uniform Charge Distribution on a Surface**

Given a surface density

$$\sigma(z=0,\rho) = \frac{-qd}{2\pi(\rho^2+s^2)^{3/2}} \tag{29}$$

a) Find total charge Q on the plane
b) Show that the potential of the charge configuration on the z-axis is identical to a point charge Q potential on the axis at $z=-s$.

First, we are tasked to find the total charge on the $z=0$ plane. We integrate using cylindrical coordinate system.

$$Q = \int \sigma \, \mathrm{d}a \tag{30}$$

Using a cylindrical Jacobian,

$$Q = \int \sigma\rho \, \mathrm{d}\rho \, \mathrm{d}\phi = \int \frac{-qd}{2\pi(\rho^2+s^2)^{3/2}}\rho \, \mathrm{d}\rho \, \mathrm{d}\phi \tag{31}$$

$$= \frac{-qd}{2\pi}\int_0^\infty \frac{\rho}{(\rho^2+s^2)^{3/2}} \, \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \tag{32}$$

To evaluate the integral, simply execute a substitution $u=\rho^2+s^2$, $\mathrm{d}u = 2\rho \, \mathrm{d}\rho$. The limits transform as $u=s^2$ to $u=\infty$. Executing the substitution and evaluating the azimuthal integral,

$$Q = \frac{-qd}{2}\int_{s^2}^\infty \frac{1}{u^{3/2}} \, \mathrm{d}u = \frac{qd}{2}\frac{2}{u^{1/2}}\Big|_{s^2}^\infty = -\frac{qd}{s} \tag{33}$$

Hence, the total charge in the plane is

$$\boxed{Q = -\frac{qd}{s}} \tag{34}$$

Finding the potential of the charge configuration, we can use the expression at Eq. (5) provided that we throw the density back into the integral. That is,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma}{(r^2+r'^2-2rr'\cos\gamma)^{1/2}}\rho' \, \mathrm{d}\rho' \, \mathrm{d}\phi' \tag{35}$$

In this configuration, we only wish to evaluate the potential at the z-axis. Since the source is always perpendicular to the field point, $\cos\gamma = 0$. Moreover, in the cylindrical basis, the position vectors are given as $\vec{r} = (0,0,z)$ and $\vec{r'} = \rho', 0, 0$ giving us the norms $r=z$ and $r'=\rho$. The integral reduces to

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma}{(z^2+\rho'^2)^{1/2}}\rho' \, \mathrm{d}\rho' \, \mathrm{d}\phi' \tag{36}$$

Plugging in the surface density,

$$V(\vec{r}) = \frac{-qd}{8\pi^2\varepsilon_0}\int_0^\infty \frac{1}{(s^2+\rho'^2)^{3/2}}\frac{1}{(z^2+\rho'^2)^{1/2}}\rho' \, \mathrm{d}\rho' \int_0^{2\pi} \mathrm{d}\phi' \tag{37}$$

The task is to find an appropriate substitution. A substitution would be $u=(z^2+\rho'^2)^{1/2}$, $\mathrm{d}u = \rho'/(\rho'^2+z^2)$. Now, the limits transforms as $u=z$ to $u=\infty$. Executing the substitution and evaluating the azimuthal integral,

$$V(\vec{r}) = \frac{-qd}{4\pi\varepsilon_0}\int_z^\infty \frac{1}{(u^2-z^2+s^2)^{3/2}}\mathrm{d}u \tag{38}$$

Using a trigonometric substitution $u = (s^2 - z^2)\ \tan\gamma$, $\mathrm{d}u = (s^2 - z^2)\ \sec\gamma\ \mathrm{d}\gamma$, then $(u^2 - z^2 + s^2)^{3/2} = (s^2 - z^2)^3\ \sec^3\gamma$, shifting the limits as $\gamma = \tan^{-1}(z/(s^2 - z^2))$ to $\gamma = \pi/2$. Then

$$V(\vec{r}) = \frac{-qd}{4\pi\varepsilon_0}\frac{1}{(s^2 - z^2)}\int_{\tan^{-1}(z/(s^2-z^2))}^{\pi/2} \cos\gamma\ \mathrm{d}\gamma \tag{39}$$

Luckily, the integral is elementary. Integrating,

$$\begin{aligned} V(\vec{r}) &= \frac{-qd}{4\pi\varepsilon_0}\frac{1}{(s^2 - z^2)}\left(\sin\gamma\right)\Big|_{\tan^{-1}(z/(s^2-z^2))}^{\pi/2} && (40)\\ &= \frac{-qd}{4\pi\varepsilon_0}\frac{1}{(s^2 - z^2)}\left(1 - \sin(\tan^{-1}(z/(s^2 - z^2)))\right) && (41) \end{aligned}$$

By appropriate trigonometry identity, $(1 - \sin(\tan^{-1}(z/(s^2 - z^2))) = (s - z)/s$ such that the potential of the given configuration at the z-axis is

$$\boxed{V(\vec{r}) = -\frac{qd}{4\pi\varepsilon_0}\frac{1}{s(s+z)}} \tag{42}$$

Now, we find the potential at $\vec{r} = (0, 0, z)$ of a point charge with charge $Q = -q(d/s)$ placed at $\vec{r}' = (0, 0, -s)$. Plugging this into the potential expression of a point charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r - r'} = -\frac{1}{4\pi\varepsilon_0}\frac{q}{z+s}\frac{d}{s} \tag{43}$$

Hence, the potential of a point charge with charge Q at $\vec{r}' = (0, 0, -s)$ is

$$\boxed{V(\vec{r}) = -\frac{qd}{4\pi\varepsilon_0}\frac{1}{s(s+z)}} \tag{44}$$

3. **Coaxial Cable**

Find the electric field of a coaxial cable composed of a solid cylinder with radius a, and a cylindrical shell with radius b both having uniform charge density separated by some distance. The entire structure is electrically neutral.

Cylindrical symmetry demands utility of Gaussian surfaces. To execute the task, imagine that we are superimposing a growing coaxial cylindrical Gaussian surface starting from zero radius and extending towards infinity seen at Figure 4. For each regions (inside the solid cylinder, in the separation area of the two cables, and outside the cylindrical shell), we shall find the field $\vec{E}$.

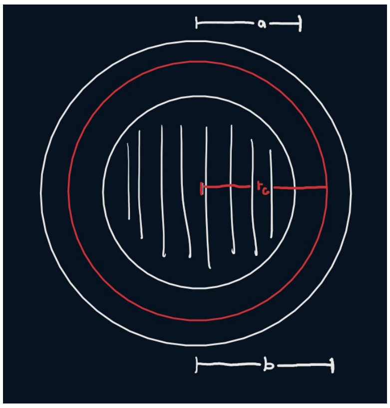

Figure 4: Red circle is a cross section of Gaussian cylinder, white circles are coaxial cables cross section.

Gauss's law states that

$$\oint \vec{E} \cdot \, \mathrm{d}\vec{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \tag{45}$$

The assumption is that the fields at a constant radius have a constant magnitude with direction radially pointing outward which we can tack in later as $\hat{n}$ as to align with the differential vector areas $\mathrm{d}\vec{a}$. These two assumptions permit us to write Gauss's law as

$$|\vec{E}| = \left(\frac{Q_{\text{enc}}/\varepsilon_0}{\oint \, \mathrm{da}}\right) \iff \vec{E} = \left(\frac{Q_{\text{enc}}/\varepsilon_0}{\oint \, \mathrm{da}}\right)\hat{n} = \left(\frac{Q_{\text{enc}}/\varepsilon_0}{A_G}\right)\hat{n} \tag{46}$$

where A_G denotes the area of the Gaussian surface. Note that we are only concerned with the walls of the cylinder and not the lids as the latter does not contribute to the integral by being perpendicular with the field. Before diving into the calculations, we should observe the Q_{enc} grows as the Gaussian surface grows from radius $s = 0$ until $s = a$, then stays constant equal to the total charge $Q = \rho V_C$, where V_C is the volume of the solid cylinder with given a height h_G and dips to zero outside the two coaxial cable $s = b$. Notice that we have coincided the radius of the Gaussian surface with the radial distance s. Inside the cylinder, $Q_{\text{enc}} = \rho V_G$ where V_G denoted the volume of the Gaussian surface. The electric field is, then,

$$\vec{E} = \left(\frac{\rho V_G}{\varepsilon_0 A_G}\right)\hat{s} = \left(\frac{\rho\pi s^2 h_G}{\varepsilon_0 2\pi s h_G}\right)\hat{s} = \left(\frac{\rho}{2\varepsilon_0}s\right)\hat{s} \tag{47}$$

In the intermediate region,

$$\vec{E} = \left(\frac{\rho V_C}{\varepsilon_0 A_G}\right)\hat{s} = \left(\frac{\rho\pi a^2 h_G}{\varepsilon_0 2\pi s h_G}\right)\hat{s} = \left(\frac{\rho a^2}{2\varepsilon_0}\frac{1}{s}\right)\hat{s} \tag{48}$$

Outside, the field is trivially zero since the cable is electrically neutral overall

$$\vec{E} = \vec{0} \tag{49}$$

Hence, we found the field by simply probing the system with a symmetric Gaussian surface. The field can be expressed as

$$\vec{E} = \begin{cases} \left(\frac{\rho}{2\varepsilon_0} s\right) \hat{s} & s \leq a \\ \left(\frac{\rho a^2}{2\varepsilon_0} \frac{1}{s}\right) \hat{s} & a < s \leq b \\ \vec{0} & s > b \end{cases} \iff |\vec{E}| = \begin{cases} \frac{\rho}{2\varepsilon_0} s & s \leq a \\ \frac{\rho a^2}{2\varepsilon_0} \frac{1}{s} & a < s \leq b \\ 0 & s > b \end{cases} \tag{50}$$

The trend of the electric field goes like this. As the Gaussian surface is increasing, the field grows linearly until it reaches $s = a$, then starts to decrease with $1/s$ and plummets zero upon reaching $s = b$. We can plot the magnitude of the field as

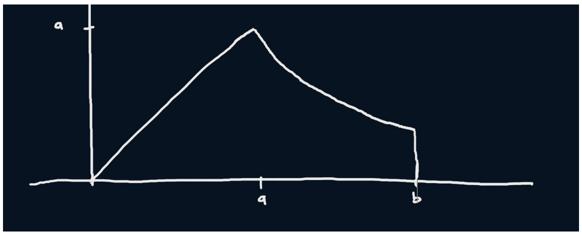

Figure 5: Although continuous, point a is not smooth and point b has a step discontinuity.

where we have set the units $\rho/2\varepsilon_0 = 1$. Observe that at the boundary $s = a$, the function is smooth while there exists a step discontinuity at $s = b$.

4. **Capacitance**

Find the capacitance per unit length of two coaxial metal cylindrical tubes of radii a and b.

The problem is somehow an extension of the previous one. Recall that capacitance is the proportionality constant of charge and potential. That is,

$$C = \frac{Q}{V} \tag{51}$$

where V is the potential difference and Q is the charge of each conductor. To find the potential difference, we can find the field $\vec{E}$ first and then use the definition of a potential such that the potential difference $V(b) - V(a)$ is given as

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot \, \mathrm{d}\vec{l} \tag{52}$$

We set that the inner cylinder has a higher potential such that the defined potential difference is positive. To execute the integral, we need to find the field between the two cylindrical tubes. The usual route is to exploit symmetry and use Gauss's law. Luckily, we have already derived an expression for the field in the previous problem at Eq. (50). That is,

$$\vec{E} = \frac{\rho a^2}{2\varepsilon_0}\frac{1}{s}\hat{s} \tag{53}$$

Since we are tasked to find the $Q : V$ proportionality constant, we need to reparametrize the expression in terms of Q:

$$Q = \rho V = \rho \pi a^2 L \iff \rho a^2 = \frac{Q}{\pi L} \tag{54}$$

Hence, the electric field can now be expressed as

$$\vec{E} = \frac{Q}{2\varepsilon_0 \pi L}\frac{1}{s}\hat{s} \tag{55}$$

We can now find the potential difference using this expression.

$$V(b) - V(a) = -\int_a^b \frac{Q}{2\varepsilon_0 \pi L}\frac{1}{s}\hat{s} \cdot \, \mathrm{d}\vec{l} \tag{56}$$

Of course, we set the integration path parallel with the field

$$V(b) - V(a) = -\int_a^b \frac{Q}{2\varepsilon_0 \pi L}\frac{1}{s}\hat{s} \cdot \hat{s} \, \mathrm{d}s = -\int_a^b \frac{Q}{2\varepsilon_0 \pi L}\frac{1}{s} \, \mathrm{d}s \tag{57}$$

The integral is elementary and integration is trivial.

$$V(b) - V(a) = -\frac{Q}{2\varepsilon_0 \pi L} \ln s|_a^b = \frac{Q}{2\varepsilon_0 \pi L} \ln s|_b^a = \frac{Q}{2\varepsilon_0 \pi L} \ln\left(\frac{a}{b}\right) \tag{58}$$

The ratio per unit length $(Q/V)/L$ is

$$\frac{Q}{VL} = \frac{Q}{\frac{Q}{2\varepsilon_0 \pi L}\ln\left(\frac{a}{b}\right) L} = \frac{2\pi\varepsilon_0}{\ln(a/b)} \tag{59}$$

which finally gives us the capacitance of the system

$$\boxed{C = \frac{2\pi\varepsilon_0}{\ln(a/b)}}$$

Observe that, as always discussed, the capacitance is a purely geometric property.

5. **Potential and the Hemispherical Bowl**

Find the potential difference between the north pole and the center of a uniformly charged inverted hemispherical bowl of radius R.

We are tasked to analyze the potentials of a hemisphere shown below:

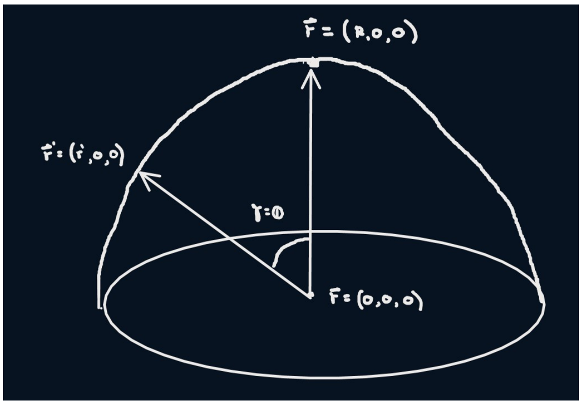

Figure 6: Spherical coordinate system was used for obvious reasons with the γ coinciding with θ

To find the potential difference between the north pole and the center, we have two ways. First is to find the field inside the hemisphere and to extract the potential. However, I think it would take us less time and effort if we calculate the potential directly from the charge density. Recall that the potential is given as

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma(\vec{r})}{|\vec{r} - \vec{r}'|} \, \mathrm{d}a \tag{60}$$

where using the cosine law gives us

$$V(\vec{r}) = \frac{\sigma}{4\pi\varepsilon_0} \int \frac{1}{(r^2 + r'^2 - 2rr'\cos\gamma)^{1/2}} \, \mathrm{d}a \tag{61}$$

This problem showcases the significance of the γ term which is repeatedly cancelled out in the previous problems. Obviously, we use a spherical coordinate system to analyze the problem. In the spherical basis, the source position vector is $\vec{r}' = (r', 0, 0)$ while the field position vector is $\vec{r} = (r, 0, 0)$. The north pole is given by $\vec{r} = (R, 0, 0)$ while the center is simply the origin $\vec{r} = (0, 0, 0)$. Evaluating the integral generally would be cumbersome. What we shall do is evaluate the integrals separately at each field point and then do the subtraction in the end. At the center, using a spherical Jacobian, the integral reduces to

$$V(0) = \frac{\sigma}{4\pi\varepsilon_0} \int \frac{1}{r'} r'^2 \sin\theta' \, \mathrm{d}\theta' \, \mathrm{d}\phi' \tag{62}$$

Observe that throughout the integration, we are fixing the source position magnitude at $r' = R$. Integrating,

$$V(0) = \frac{\sigma R}{4\pi\varepsilon_0} \int_0^{\pi/2} \sin\theta' \, \mathrm{d}\theta' \int_0^{2\pi} \mathrm{d}\phi' = \frac{\sigma R}{4\pi\varepsilon_0} (-\cos|_0^{\pi/2})(2\pi) = \frac{\sigma R}{2\varepsilon_0} \tag{63}$$

We use a different simplification scheme for the north pole case. At the north pole, note that $r' = r = R$. Hence, the denominator of the integrand at Eq. (61) reduces to

$$r^2 + r'^2 - 2rr'\cos\gamma = 2R^2 - 2R^2\cos\gamma = 2R^2(1 - \cos\gamma) \tag{64}$$

Moreover, the angle between the two position vectors γ is precisely the polar angle θ. The potential integral is, then,

$$V(R) = \frac{\sigma}{4\pi\varepsilon_0} \int \frac{1}{(2R^2(1 - \cos\theta'))^{1/2}} R^2 \sin\theta' \, \mathrm{d}\theta' \, \mathrm{d}\phi' \tag{65}$$

Simplifying and integrating,

$$V(R) = \frac{\sigma R}{4\sqrt{2}\pi\varepsilon_0} \int_0^{\pi/2} \frac{\sin\theta'}{(1-\cos\theta')^{1/2}}\ \mathrm{d}\theta' \int_0^{2\pi}\ \mathrm{d}\phi' \tag{66}$$

A substitution would be $u = 1 - \cos\theta'$, $\mathrm{d}u = -\sin\theta'\mathrm{d}\theta'$. The limits transform as $u = 0$ to $u = 1$

$$V(R) = \frac{\sigma R}{2\sqrt{2}\varepsilon_0} \int_0^1 \frac{1}{(u)^{1/2}}\ \mathrm{d}u = \frac{\sigma R}{2\sqrt{2}\varepsilon_0}\frac{u^{1/2}}{1/2}|_0^1 = \frac{\sigma R}{\sqrt{2}\varepsilon_0} \tag{67}$$

Using the results from Eqs. (63) and (67), we have the potential difference between the north pole and center expressed as

$$V(R) - V(0) = \frac{\sigma R}{2\varepsilon_0} - \frac{\sqrt{2}\sigma R}{2\varepsilon_0} \tag{68}$$

Hence,

$$\boxed{V(R) - V(0) = \frac{\sigma R}{2\varepsilon_0}(\sqrt{2} - 1)}$$

6. **Alternative Coulomb's Law**

Reformulate electrostatics if superposition still holds and Coulomb's law is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\vec{r}-\vec{r}'|^2}\left(1+\frac{|\vec{r}-\vec{r}'|}{\lambda}\right)\exp(-|\vec{r}-\vec{r}'|/\lambda)\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} \tag{69}$$

a) Derive expression for $\vec{E}$
b) Is this conservative?
c) Derive expression for V
d) Derive Gauss's law: start from a point charge.
e) Generalize for arbitrary system.
f) Draw the electrostatic triangle diagram.
g) Show that charges do reside on the volume and not only on the surface.

We can simply declare the phenomenon an existence of *dark charge* to account for the additional electrostatic force in the data. QED

Just kidding. Recall that the entire electrostatics can be derived from two principles: Coulomb's law and superposition. We are lucky that superposition principle still holds and everything adds up linearly. We start from Coulomb's law. The force felt by charge Q due to charge q_i is [3]

$$\vec{F}_i = \frac{1}{4\pi\varepsilon_0}\frac{Qq_i}{|\vec{r}-\vec{r}'|_i^2}\left(1+\frac{|\vec{r}-\vec{r}'|_i}{\lambda}\right)\exp(-|\vec{r}-\vec{r}'|_i/\lambda)\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} \tag{70}$$

By superposition, the total force due to total charge $\sum_i^n q_i$ is

$$\begin{aligned}
\vec{F} &= \sum_{i=0}^{n}\frac{1}{4\pi\varepsilon_0}\frac{Qq_i}{|\vec{r}-\vec{r}'|_i^2}\left(1+\frac{|\vec{r}-\vec{r}'|_i}{\lambda}\right)\exp(-|\vec{r}-\vec{r}'|_i/\lambda)\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} && (71)\\
&= Q\sum_{i=0}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_i}{|\vec{r}-\vec{r}'|_i^2}\left(1+\frac{|\vec{r}-\vec{r}'|_i}{\lambda}\right)\exp(-|\vec{r}-\vec{r}'|_i/\lambda)\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} && (72)\\
&:= Q\vec{E} && (73)
\end{aligned}$$

Hence, the electric field $\vec{E}$ is expressed as

$$\boxed{\vec{E} = \sum_{i=0}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_i}{|\vec{r}-\vec{r}'|_i^2}\left(1+\frac{|\vec{r}-\vec{r}'|_i}{\lambda}\right)\exp(-|\vec{r}-\vec{r}'|_i/\lambda)\frac{\vec{r}-\vec{r}'_i}{|\vec{r}-\vec{r}'|_i}} \tag{74}$$

Does this admit a scalar potential? We start our handwavy argument by placing a point charge at the origin. How does the field behave? The field is expressed as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r^2}\left(1+\frac{r}{\lambda}\right)\exp\left(-\frac{r}{\lambda}\right)\hat{r} \tag{75}$$

Note that the expression is only described solely by the radial parameter. It has radial symmetry which implies a zero curl. That is,

$$\boxed{\vec{\nabla}\times\vec{E} = \vec{0}} \tag{76}$$

So yes, it does admit a scalar potential. This implies a conservative vector field. If you are not convinced enough, we will derive the potential from this field. Due to conservativeness, we can define a path-independent potential as

$$\boxed{V := -\int_{\infty}^{r}\vec{E}\cdot \mathrm{d}\vec{l}} \tag{77}$$

[3] I apologize for the cluttered notation. I don't know how to write a script-r and don't want to confuse with some other quantity. I wrote them explicitly to remove any ambiguity

At the origin, Eq. (74) reduces to

$$\vec{E} = \sum_{i=0}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i'^2} \left(1 + \frac{r_i'}{\lambda}\right) \exp(-r_i'/\lambda)\hat{r}_i \tag{78}$$

where $\hat{r}_i$ refers to a unit vector pointing from charge q_i towards the origin. The field of a single point charge would then be

$$\vec{E}_i = \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i'^2} \left(1 + \frac{r_i'}{\lambda}\right) \exp(-r_i'/\lambda)\hat{r}_i \tag{79}$$

Finding the potential using Eq. (77),

$$V_i = -\frac{1}{4\pi\varepsilon_0} \int_{\infty}^{r} \frac{q_i}{r_i'^2} \left(1 + \frac{r_i'}{\lambda}\right) \exp(-r_i'/\lambda)\hat{r}_i \cdot \hat{r}_i \mathrm{d}r_i' \tag{80}$$

where we take the integration path to be parallel with $\hat{r}_i$. Simplifying the integrand,

$$V_i = -\frac{q_i}{4\pi\varepsilon_0} \left(\int_{\infty}^{r} \frac{1}{r_i'^2} \exp(-r_i'/\lambda)\mathrm{d}r_i' + \frac{1}{\lambda} \int_{\infty}^{r} \frac{1}{r_i'} \exp(-r_i'/\lambda)\mathrm{d}r_i' \right) \tag{81}$$

Integrating by parts by integrating $1/r_i'^2$ and differentiating $\exp(-r_i'/\lambda)$, the first term is

$$\int_{\infty}^{r} \frac{1}{r_i'^2} \exp(-r_i'/\lambda)\mathrm{d}r_i' = -\frac{1}{r_i'} \exp(-r_i'/\lambda)|_{\infty}^{r} - \frac{1}{\lambda} \int_{\infty}^{r} \frac{1}{r_i'} \exp(-r_i'/\lambda)\mathrm{d}r_i' \tag{82}$$

Evaluating and plugging this into Eq. (81),

$$V_i = -\frac{q_i}{4\pi\varepsilon_0} \left(-\frac{1}{r} \exp(-r/\lambda) - \frac{1}{\lambda} \int_{\infty}^{r} \frac{1}{r_i'} \exp(-r_i'/\lambda)\mathrm{d}r_i' + \frac{1}{\lambda} \int_{\infty}^{r} \frac{1}{r_i'} \exp(-r_i'/\lambda)\mathrm{d}r_i' \right) \tag{83}$$

Hence, we see that second and third term cancel and we have the potential of a point charge at the origin expressed as

$$V_i = \frac{q_i}{4\pi\varepsilon_0} \frac{\exp(-r/\lambda)}{r} \tag{84}$$

Generalizing to arbitrary field point, we have

$$\boxed{V_i = \frac{q_i}{4\pi\varepsilon_0} \frac{\exp(-(r - r_i')/\lambda)}{r - r_i'}} \tag{85}$$

Now, we proceed to derive Gauss's law starting from a point charge. Using Eq. (74), the electric field of a point charge placed at the origin in

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r^2} \left(1 + \frac{r}{\lambda}\right) \exp(-r/\lambda)\hat{r} \tag{86}$$

Enclosing with a symmetric spherical Gaussian surface centered at the origin with radius r, the flux is

$$\begin{aligned}
\oint \vec{E} \cdot \mathrm{d}\vec{a} &= \int \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r^2} \left(1 + \frac{r}{\lambda}\right) \exp(-r/\lambda)\hat{r} \cdot \hat{r} r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi & (87)\\
&= \frac{q_i}{4\pi\varepsilon_0} \left(1 + \frac{r}{\lambda}\right) \exp(-r/\lambda) \int_0^{\pi} \sin\theta \, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi & (88)\\
&= \frac{q_i}{\varepsilon_0} \left(1 + \frac{r}{\lambda}\right) \exp(-r/\lambda) & (89)
\end{aligned}$$

Our task is to prove that the above expression must conform to the relation

$$\frac{q_i}{\varepsilon_0} \left(1 + \frac{r}{\lambda}\right) \exp(-r/\lambda) = \frac{q_i}{\varepsilon_0} - \frac{1}{\lambda^2} \oint V \mathrm{d}\tau \tag{90}$$

where the second term is a closed volume integral of a sphere with radius r. From Eq. (84),

$$\oint V \mathrm{d}\tau = \frac{q_i}{4\pi\varepsilon_0}\oint \frac{\exp(-r/\lambda)}{r}\mathrm{d}\tau \tag{91}$$

Integrating via a spherical Jacobian and priming r for clarity, the RHS integral factor is

$$I = \int_0^r r'\exp(-r'/\lambda)\mathrm{d}r' \int_0^\pi \sin\theta \ \mathrm{d}\theta \int_0^{2\pi} \ \mathrm{d}\phi \tag{92}$$

Application of integration by parts twice on the first integral factor and evaluating the angular integrals gives us

$$\begin{aligned} I &= -4\pi(r'\lambda\exp(-r'/\lambda) + \lambda^2\exp(-r'/\lambda))|_0^r & (93)\\ &= 4\pi(r\lambda\exp(-r/\lambda) + \lambda^2\exp(-r/\lambda) - \lambda^2) & (94)\\ & & (95) \end{aligned}$$

From Eq. (91), we then get

$$\begin{aligned} \oint V\mathrm{d}\tau &= -\frac{q_i}{\varepsilon_0}(r\lambda\exp(-r/\lambda) + \lambda^2\exp(-r/\lambda) - \lambda^2) & (96)\\ &= -\frac{q_i}{\varepsilon_0}\lambda^2\left(\frac{r}{\lambda}\exp(-r/\lambda) + \exp(-r/\lambda) - 1\right) & (97)\\ &= \frac{q_i}{\varepsilon_0}\lambda^2 - \frac{q_i}{\varepsilon_0}\lambda^2\left(\frac{r}{\lambda}\exp(-r/\lambda) + \exp(-r/\lambda)\right) & (98) \end{aligned}$$

Plugging this into the RHS of Eq. (90),

$$\begin{aligned} \frac{q_i}{\varepsilon_0} - \frac{1}{\lambda^2}\oint V\mathrm{d}\tau &= \frac{q_i}{\varepsilon_0} - \frac{1}{\lambda^2}\left(\frac{q_i}{\varepsilon_0}\lambda^2 - \frac{q_i}{\varepsilon_0}\lambda^2\left(\frac{r}{\lambda}\exp(-r/\lambda) + \exp(-r/\lambda)\right)\right) & (99)\\ &= \frac{q_i}{\varepsilon_0} - \frac{q_i}{\varepsilon_0} + \frac{q_i}{\varepsilon_0}\left(\frac{r}{\lambda}\exp(-r/\lambda) + \exp(-r/\lambda)\right) & (100)\\ &= \frac{q_i}{\varepsilon_0}\left(1 + \frac{r}{\lambda}\right)\exp(-r/\lambda) & (101) \end{aligned}$$

which is precisely Eq. (90) thus proving the identity

$$\boxed{\oint_S \vec{E}_i \cdot \mathrm{d}\vec{a} + \frac{1}{\lambda^2}\int_{\mathrm{vol}} V_i\mathrm{d}\tau = \frac{q_i}{\varepsilon_0}} \tag{102}$$

for any spherical gaussian surfaces and volumes. Generalizing this to any arbitrary configuration, we invoke the same arguments as in the "classical"[4] case. Recall that deriving Gauss's law rests on the assumption of a superposition-obeying linear fields $\vec{E}$. Luckily for us, potentials also obey linear superposition such that $V = \sum_i V_i$. This way, we can simply take the sum of each term as follows

$$\begin{aligned} \sum_i \left(\oint_S \vec{E}_i \cdot \mathrm{d}\vec{a} + \frac{1}{\lambda^2}\int_{\mathrm{vol}} V_i\mathrm{d}\tau\right) &= \sum_i \left(\frac{q_i}{\varepsilon_0}\right) & (103)\\ \iff \sum_i \left(\oint_S \vec{E}_i \cdot \mathrm{d}\vec{a}\right) + \sum_i \left(\frac{1}{\lambda^2}\int_{\mathrm{vol}} V_i\mathrm{d}\tau\right) &= \sum_i \left(\frac{q_i}{\varepsilon_0}\right) & (104) \end{aligned}$$

giving us the non-classical Gauss's law

$$\boxed{\oint_S \vec{E} \cdot \mathrm{d}\vec{a} + \frac{1}{\lambda^2}\int_{\mathrm{vol}} V\mathrm{d}\tau = \frac{Q_{\mathrm{enc}}}{\varepsilon_0}} \tag{105}$$

[4] I'll refer to this formulation of Coulomb's law as non-classical.

We can readily convert this into its differential form by similarly invoke divergence theorem on the first term and expanding Q_{enc} in terms of density

$$\int_{\text{vol}} \vec{\nabla} \cdot \vec{E} \mathrm{d}\vec{\tau} + \frac{1}{\lambda^2} \int_{\text{vol}} V \mathrm{d}\tau = \int_{\text{vol}} \frac{\rho}{\varepsilon_0} \mathrm{d}\tau \tag{106}$$

which gives us, by equal integral - equal integrand argument, Gauss's law in differential form

$$\boxed{\vec{\nabla} \cdot \vec{E} + \frac{1}{\lambda^2} V = \frac{\rho}{\varepsilon_0}} \tag{107}$$

Finally, since the electric field can be expressed as a gradient of the potential, we can derive the non-classical Poisson's equation as

$$\vec{\nabla} \cdot \vec{E} + \frac{1}{\lambda^2} V = \frac{\rho}{\varepsilon_0} \iff \vec{\nabla} \cdot (-\vec{\nabla} V) + \frac{1}{\lambda^2} V = \frac{\rho}{\varepsilon_0} \tag{108}$$

Hence, the non-classical Poisson's equation is

$$\boxed{-\nabla^2 V + \frac{1}{\lambda^2} V = \frac{\rho}{\varepsilon_0}} \tag{109}$$

With these results, we have derived electrostatics with the non-classical Coulomb's law. It is important to note that the potential was derived similarly as the classical one. Hence, the relationship between $\vec{E}$ and V remains invariant with this modification. Visually, we can summarize these results with an electrostatic triangle shown below.

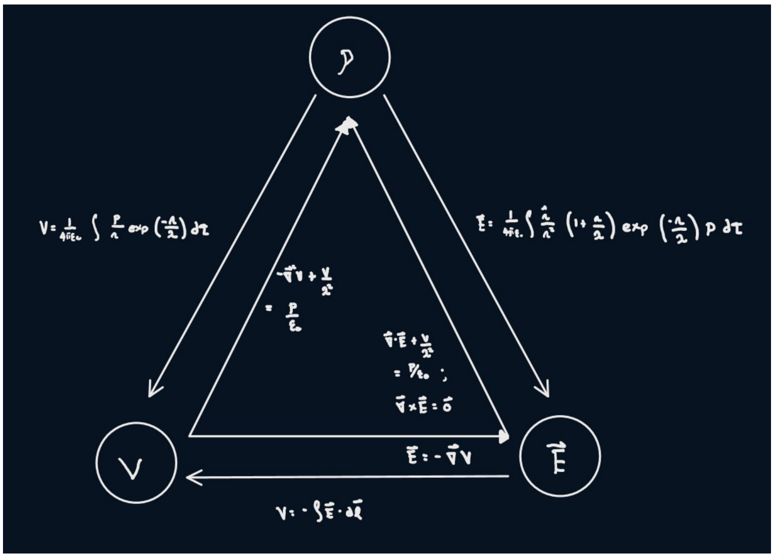

Figure 7: Extra correction terms were included except the E-V relations in the electrostatic triangle

Lastly, unlike in the classical case of conductors, some charges distribute themselves within the volume interior. With the similar $\vec{E} = \vec{0}$ assumption, inside the conductor, Gauss's law states that

$$0 + \frac{1}{\lambda^2} V = \frac{\rho}{\varepsilon_0} \tag{110}$$

Observe that we have a non-zero charge density inside. Moreover, it has a uniform value of

$$\rho = \frac{\varepsilon_0}{\lambda^2} V \tag{111}$$

The remainder of the charges would reside on the surface because where else could they be. Although it would be negligible in our scale, this might be relevant when we construct our first galactic conductor mega-structure with enormous potentials.

7. **Reflections**

Instead of having reflections individually, I compiled them all here in a single page. Some tasks were challenging but the overall problem set is very rewarding. The problems cover the entire scope of electrostatics with some focus on the significance of using potentials and exploiting symmetries. Also, given the nature of how I write my problem sets in a narrative tone, I think a prior layout of the solution before starting the problem would be redundant.

- **Problem 1** The first problem is the hardest problem, personally. The first two tasks were relatively straightforward but the last two really are challenging for me. The task was to find the electric field pattern in the plane of the disk (z=0) where I was forced to evaluate a nasty integral . Unlike the rim case where the integral can be simplified, analyzing the behavior to be able to plot it means solving a general integral potential and then taking its gradient. I can't seem to find a general expression for the potential or the field of the configuration. The radial dependence derivation can be seen in this paper.
- **Problem 2** The second problem is somehow easy to execute but I find myself having a hard time interpreting the parameters. What do d and q mean, physically? I know that s was used as some parameter to find some equivalence between the given charge configuration and a point charge as a function of s. That being said, the entire method of solving the problem is very straightforward. Integrals were not that difficult to execute.
- **Problem 3** The third problem is the easiest of the entire problem set. We basically abandoned the notion of direction and just probed with a symmetric cylindrical Gaussian surface. Of course, this simplification costs multiple assumptions to make but it really is not a giant leap. One simply needs to assume a cylindrical field symmetry.
- **Problem 4** I find that this fourth problem is somehow still part or an extension of the third problem. The task is tantamount to simply finding the potential difference and cleaning them up in the end to find the proportionality constant (capacitance). So I take my words back, this is actually the easiest of the entire problem set.
- **Problem 5** The fifth problem is quite interesting, although there is a very apparent spherical symmetry to exploit. Finding the potential difference only means taking the difference of two integrals. This highlights the significance and comfort (on my part) of expanding the separation vector norm using the cosine law from the very beginning. The γ (separation angle) coincided with the θ polar angle which leads to a smooth integration. This is facilitated by the fact that the points of interest are very convenient which reduces the complexity of the integral significantly.
- **Problem 6** This is, so far, my favorite electrostatic problem to answer. The task was to rederive all electrostatic principles by modifying Coulomb's law. This is very reminiscent of some tasks I have to make a while back when accounting for the excess mass distribution due to some anomaly in the galactic rotation, supposedly caused by dark matter. I have read that a solution was to modify Newtonian dynamics itself to account for the anomaly. This problem nailed my understanding of the basics - how to derive each relevant rules (Gauss's, Poissons's, etc.). Although the task was somehow longer than the others (and somewhat challenging to typeset), this is the task that I've enjoyed the most. Accounting for the modification in Coulomb's law and see this as some cute little additional terms to Gauss's law or Poisson's equation is really satisfying. Overall, the task was straightforward and I didn't find any major roadblock although I was terrified at first when I thought they would modify superposition too.

For the most part, aside from the first problem, the electrostatic problem set posed little to no conceptual roadblocks and not really that tedious (although typesetting is a different story). I find myself reviewing Griffith's textbook as a reference and some help from my Math 2X notes for some neat identities from time to time. Given the first problem, though, I still think I lack the mathematical capabilities to evaluate nasty integrals or find ways to go around it. I'll rate the material as a YELLOW-difficulty one with this said barriers. That being said, I'm looking forward to earning more tools to be comfortable with my EM class.

With this problem set, I became comfortable in expanding and contracting separation vectors at will (which is what I'm really confused about in Physics 107), and finding relevant symmetries for the problem.