GRAVITY GROUP APPLICATION

Stellar toy models: a review

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Abstract. We review a paper that provides a model approximating a neutron star as a thin spherical shell for a gravitational collapse using a purely Newtonian approach that greatly simplifies the mathematical requisite. The model justifies the collapse by the equality of points of maximum mass and instability. The effective polytropic exponent γ that denotes the point of collapse also can be directly related to the phenomenon that the model describes having found that $\gamma = 3/2$ under Newtonian gravity could accurately describe a neutron star collapse. We provide a way of extending the range of applicability of the simplification by deriving a way to customize γ value by using a different mass distribution. One could construct different mass distributions to generate a specific γ value and use the Lane-Emden equation to find the corresponding application. Since γ is designed to be EoS-independent, one could use EoS as a fine-tuning parameter to improve modelling accuracy.

1 Introduction

A black hole is as mysterious as it seems. Some mystery lies in the fact that it literally separates what can be observed from its interior since nothing, not even light, can escape once they got in. However, a more apparent mystery is the fact that it would need advanced mathematics to be able to truly grasp, let alone mathematically describe, its dynamics. One needs to take a course in general relativity using tensors and differential geometry to have a rigorous understanding of a gravitational collapse. However, in their paper about Equilibrium and stability of thin spherical shells in Newtonian and relativistic gravity, LeMaitre and Poisson provided a way to go around the hard road[1]. One only needs to make an approximation.

As the neutron star continues to accrete matter from its companion, it increases its mass up to a point where it achieves its maximum. The main point of LeMaitre and Possion's paper is to bypass these internal dynamics by simplifying the problem into a version where it could be derived and understood without advanced mathematical requirements such as the intricacies of general relativity. In the process of doing so, the main insights of the collapse can be made more accessible. The toy-model is done by considering the star as a thin spherical shell subject to outwards radial pressure and inwards radial gravitational pull. The complex fluid dynamics of the stellar interior are all contained in an equation of state (EoS) . Appropriate EoS is chosen to match the model into realistic ones. This way, the problem of exploring the point of collapse subject to the different complex dynamics of the internal stellar process can be reduced to a problem of statics and dynamics discussed in freshmen physics courses. By reducing hundreds of pages of derivation into simplified equations[1], this claim is certainly profound as it can be extended to other astronomical phenomena. As such, including a rudimentary discussion, although erring on the qualitative side, can be made significantly early to people trying to understand the said material.

The paper started with deriving an equation of the shell's motion using the Newtonian approach $\Sigma F = m\ddot{R}$ subject to both internal outwards pressure and inwards gravitational force. The analysis is, then, akin to a static-body problem where the acceleration term is set to zero. Using the static equation, relations for mass and radius as functions of surface density and pressure were derived. These serves as the main functions subject to the stability analysis of the thin shell model. Using a specific EoS, turning points for the mass (and radius) functions were found corresponding to the point of gravitational collapse onset. The crux of the paper is that, by exploiting the barotropic form $p = p(\sigma)$ of the EoS (i.e., pressure is only dependent of density), a quantity Γ that signifies the boundary between a real and imaginary frequency in the perturbation equation $\delta\ddot{R} = -\frac{Gm}{2R_o^3}(2\Gamma - 3)\delta R$ is equal to the quantity γ where maximum mass occurs with respect to density. This implies the direct relationship between maximum mass and point of instability where a neutron star collapses into a black hole. Due to it fundamentally being a relativistic entity, improvements upon the approximation can be made by allowing relativistic corrections. Present on the other half of the paper is a redefinition of the concept of relativistic time and relativistic mass on behalf of the earlier-used Newtonian quantities. Needless to say, the entire paper is a massive approximation to the entire process of a black hole collapse that one may say it leans to a qualitative form of analysis. Even then, the paper claimed that whatever realism lost by the approximation is gained through the simplicity it provides "by a wide margin".

As we shall see later, γ will directly relate to what phenomenon one is trying to describe. The first part of the paper proved that $\gamma = 3/2$ for a Newtonian gravity while relativistic corrections has an EoS-dependent γ reducing to 3/2 in the Newtonian limit. For instance, in terms of the index n (discussed further in Section (4.2)), $n = 0$ is fit to model a solid, incompressible sphere as will be shown in Section (4.1), $n = 1.5$ is a good model for a convective star [2][3], $n = 3$ is a good model for white dwarfs[4], or main sequence star[5] depending on the equation of state, $n = 5$ corresponds to a simplest plausible model of a self-consistent stellar system , or $n = \infty$ can model an isothermal self-gravitating sphere with similar structure to a collisionless system of stars like a globular cluster.

The goal of this review is to go through the step-by-step derivation of the paper and to apply similar methods in finding out how to construct a variation of the thin shell model differing in geometry resulting to a different γ. The hope is that we could develop a general algorithm, system, or even a function that could facilitate in computing various γ given a specific mass distribution. This could provide us a way to extend the simplification mechanism of the paper outside the realm of a neutron star collapse and open new opportunity to various applications to different phenomena and fields.

2 Equation of motion

2.1 Thin shell model

2.1.1 Thin shell equation of motion

We start by approximating the entire stellar structure, together with its thermodynamics, fluid dynamics, relativistic properties, and other complex properties that hinders us from finding out an elementary analysis, as a hollowed out shell. Instead, we summarize all those properties and compress them into two forces of interaction: pressure and gravitation. These two forces are the only interaction that we will consider in the entire analysis. One can directly use Newton's second law to find for an equation of motion. This would provide us a way to derive the equilibrium configuration relations that we need to analyze. To be able to analyze equilibrium conditions, the equation of motion was first derived using Newton's second law. As discussed in the paper, exploiting spherical symmetry allows us to avoid the necessity to use vectorial analysis. Letting m be the shell's mass, R be the shell's radius, F_g be the gravitational force, and F_p be the force from the internal pressure of the shell,

$$F_g + F_p = m\ddot{R} \tag{1}$$

From here, we find expressions for both F_g and F_p in terms of the shell's radius. It is useful to exploit spherical symmetry by considering a point mass on the surface of the thin shell with distance R from the center and mass m (equal to shell mass) subject to the same forces F_g and F_p. This point mass represents the entire shell radial motion in one dimension and finding the equation of motion of this point mass is equivalent to finding the equation of motion of the shell. Now, we proceed with finding appropriate expression for F_p. Directly following from the derivation of the paper, the following relations were utilized

$$dW = pdA \iff dW = F_p dR \tag{2}$$

from where one can directly extract an expression for F_p by differentiating $4\pi R^2$, the surface area of the spherical shell

$$\frac{\mathrm{d}A}{\mathrm{d}R} = 8\pi R \iff \mathrm{d}A = 8\pi R\mathrm{d}R \tag{3}$$

Upon close inspection however, Eq. (2) has some unit inconsistencies. Explicitly writing the units, we see that

$$dW = p\mathrm{d}A \iff [J] \stackrel{?}{=} [\frac{N}{m^2} * m^2] \iff [N * m] \neq [N]$$

This apparent unit inconsistency arises from the differential approximation

$$\mathrm{d}A \approx \mathrm{d}V$$

Obviously, quantities of area could not be directly equated to quantities of volume although a common occurrence throughout the paper is to use a different set of units to make the calculations easier. To see why the approximation $\mathrm{d}A \approx \mathrm{d}V$ holds, consider the volume of the spherical shell with thickness ϵ as the difference of the volume of two spheres with radii $r_1 = R + \epsilon$ and $r_0 = R$. Hence, the shell's volume is given as

$$V = \frac{4}{3}\pi(R+\epsilon)^3 - \frac{4}{3}\pi R^3 \tag{4}$$

Since the shell's thickness is given to be small, we can use binomial approximation to simplify Eq. (4) such that

$$V = (4\pi\epsilon)R^2 \tag{5}$$

Assuming that the expansion (or contraction) of the shell is made such that the thickness remains constant, we can linearize the volume expression about some radius R_0 as follows.

$$V = V(R_0) + \frac{\mathrm{d}V}{\mathrm{d}R}|_{R=R_0}(R - R_0) \tag{6}$$

Linearizing Eq. (5) and differentiating, we see that the differential volume element for small values of ϵ is expressed as

$$\mathrm{d}V = 8\pi R_0(\epsilon \mathrm{d}R) \approx 8\pi R_0 \mathrm{d}R \tag{7}$$

which is the expression of the differential area $\mathrm{d}A$ and, hence, $\mathrm{d}V \approx \mathrm{d}A$. On the other hand, if we allow a type of expansion that fixes the inner radius while varying the thickness instead (i.e. volume depends on thickness ϵ as a variable), we will have the differential volume element to be expressed as

$$\mathrm{d}V = 4\pi R_0^2 \mathrm{d}\epsilon \tag{8}$$

where the approximation would be rendered inapplicable as the thickness increases. The model, hence, rests on an assumption of a non-varying, infinitesimal but finite, shell thickness. Moving on, plugging Eq. (3) to Eq. (2) gives us

$$dW = 8\pi pRdR \tag{9}$$

From which the force from internal pressure is extracted

$$F_p = 8\pi pR \tag{10}$$

We proceed with finding the next piece to our puzzle, F_g, to complete our equation of motion. We find that

$$\frac{F_g}{A} = \frac{-G\sigma m}{2R^2} \iff F_g = -\frac{Gm^2}{2R^2} \tag{11}$$

Finally, plugging in Eq. (11) and Eq. (10) into Eq. (1), we find that

$$m\ddot{R} = -\frac{Gm^2}{2R^2} + 8\pi pR$$

Dividing by area, we get

$$\sigma\ddot{R} = -\frac{Gm\sigma}{2R^2} + \frac{8\pi pR}{A}$$

or

$$\ddot{R} = -\frac{Gm}{2R^2} + \frac{8\pi pR}{m} \tag{12}$$

This is the equation of motion of a thin spherical shell used in analyzing equilibrium configurations in the later parts of the paper. Though it might be tempting to solve the differential equation, or at least to check whether a closed analytic form of the solution exists, we can skip that as we would only need to find equilibrium conditions. We can also extend the analysis to thin shell of arbitrary shape. However, as we shall see later, doing so might be irrelevant as astrophysical bodies tend to compress into a spherical shape at equilibrium. Then again, the thin spherical shell only serves as an approximation to give an elementary model for a neutron star collapse. Stars are far from resembling a hollowed out shell but other stuffs do. If, for some reason, an astronomical self-gravitating shell are constructed, we can directly use Eq. (12) to provide a quick way to analyze these things. One can directly carry similar calculations over to actual physical thin shells. Before proceeding with the rest of the paper, we can explore an interesting case of constructing a huge thin spherical megastructure.

2.1.2 Thin Spherical Megastructures

To give us some rudimentary analysis of the stability of a megastructure with a thin spherical shell structure, the equation of motion previously derived can be directly used. At equilibrium, using $\ddot{R} = 0$, Eq. (12) reduces to

$$\frac{Gm}{2R^2} = \frac{8\pi pR}{m}$$

Rearranging,

$$m^2 = \frac{16\pi p}{G} R^3 \tag{13}$$

Due to some earlier approximation, the unit inconsistency has propagated into different parts of the paper. To salvage it, we can define a pressure coefficient α such that α is equal in magnitude with p but with new units to conform with the equation. The coefficient α also signifies internal pressure from the interior of the structure. Using α,

$$m^2 = \alpha \frac{16\pi}{G} R^3 \tag{14}$$

Letting $a = \frac{16\pi}{G} \approx 7.53 \times 10^{11} \frac{[kg][s^2]}{[m^3]}$, α, then, have units $\frac{[kg]}{[s^2]}$. Taking the positive root, we can find an equation for mass as a function of the structure's radius $m = m(R)$.

$$m = \sqrt{a\alpha} R^{3/2} \iff R = \left(\frac{m^2}{a\alpha}\right)^{1/3} \tag{15}$$

Here, α can represent any internal force that supports the structure: be it pressure or mechanical support for the sphere the prevents it from gravitational collapse. Building such structure given a specific radius, one must find a balance between adjusting the mass and providing the internal pressure. It is insightful to note that m contributes more to the radius than α which might be useful in optimizing the trade-off between both quantities. Needless to say, this is a primitive way of finding out the quantities at equilibrium of such structure. Later in the review, we extend this analysis to a specific type of megastructure, the Dyson Sphere, where a massive body is encapsuled inside the structure. For all of these structures, we only consider that the object is stationary such that we can ignore other interactions such as inertial and rotational forces.

2.2 Solid body model

2.2.1 Solid body equation of motion

It is common knowledge that the star is not precisely a hollowed-out shell. The earlier model used thin spherical shell as an approximation yet nothing prevents us to consider a solid version of the approximation using algebra and basic calculus as we did. Moreover, a neutron star resembles a solid sphere more than a thin spherical shell providing us a motivation to test this model out. Similarly, We can also treat this solid sphere as a black box where we consider specific EoS that may fit our application. It is of similar nature of methods to arrive with an equation of motion using the same derivation. Here, we start off by using a correct form of the differential PV work.

$$dW = pdV \tag{16}$$

Since $V_{sphere} = \frac{4}{3}\pi R^3$,

$$\frac{dV}{dR} = 4\pi R^2 \iff dV = 4\pi R^2 dR \tag{17}$$

Plugging in Eq. (16) into Eq. (17), we can directly extract the force from internal pressure.

$$F_p = 4\pi p R^2 \tag{18}$$

We provide a way of simplifying this solid body approximation by exploiting the fact that we already have derived a thin shell approximation. In the case of a solid sphere, we can divide the body into two parts: the interior and the exterior. The exterior is comprised of the thin shell outer layer of the sphere while the interior is what remains. We have already derived an expression for the self-gravitation of the shell. Now, we are tasked to find the interaction between the interior and the exterior parts. This can be done by invoking the Shell theorem to further simplify the problem via its spherical symmetry achieved by considering the following model. Basically, the Shell theorem allows us to consider any spherically symmetric body possessing a gravitational field into a single points located at its center.

Let M be the mass of the entire solid sphere. Consider the outermost shell of the solid sphere having a mass m with distance R from the center. We can collapse the entire solid sphere into a single point with mass $\mu = M - m$ leaving us with the thin shell from which we can find the forces and, hence, the equation of motion. Let F_{g1} be the interior-exterior interaction while F_{g2} be the exterior-exterior interaction that we have already derived earlier. To find F_{g2}, using the Shell theorem and applying Newton's law of gravitation,

$$F_{g1} = \frac{-G\mu m}{R^2} \iff F_{g1} = \frac{-G(M-m)m}{R^2} \iff F_{g1} = \frac{-GMm}{R^2} + \frac{Gm^2}{R^2}$$

Combining this with F_{g2} using the expression from Eq. (11), we find that

$$\begin{aligned} F_g &= F_{g1} + F_{g2} \\ &= \frac{-GMm}{R^2} + \frac{Gm^2}{R^2} - \frac{Gm^2}{2R^2} \\ &= \frac{G}{R^2}\left(\frac{m^2}{2} - Mm\right) \end{aligned} \tag{19}$$

From equations (18) and (19), the net force is then,

$$F = 4\pi p R^2 - \frac{G}{R^2}\left(\frac{m^2}{2} - Mm\right) \tag{20}$$

Using Newton's second law and dividing by mass,

$$m\ddot{R} = 4\pi p R^2 - \frac{G}{R^2}\left(\frac{m^2}{2} - Mm\right) \iff \ddot{R} = -\frac{G}{R^2}\left(\frac{m}{2} - M\right) + \frac{4\pi p R^2}{m} \tag{21}$$

This gives us the equation of motion for a solid-sphere approximation. As far as our model goes, R also represents the position of a single point located on the sphere's surface R distance from the center. Observing the resulting equation of motion, we now have an additional parameter: the entire solid body mass M that we need to take into account. Nothing prevents us from finding a general expression of the equation of motion from any arbitrary spherically-symmetric solid body approximation as we shall attempt shortly. Retracing our steps, we can find the force from the internal pressure as follows

$$dW = pdV$$

To be able to extract the force, we need to find an expression of the form

$$dW = F_p dR \tag{22}$$

Using the chain rule, we express the differential $dV = \frac{\mathrm{d}V}{\mathrm{d}R}\mathrm{d}R$ such that

$$dW = p\frac{dV}{dR}dR \tag{23}$$

From Eq. (22) and Eq. (23), we find that the force from the internal pressure is

$$F_p = p\frac{dV}{dR} \tag{24}$$

We can express the volume V of a spherically-symmetric body as a triple integral

$$V = \int_0^{2\pi} \int_0^{\pi} \sin\theta \int_0^R r^2 dr d\theta d\phi \tag{25}$$

Hence, from Eq. (26),

$$F_p = p\frac{d}{dR}\left(\int_0^{2\pi} \int_0^{\pi} \sin\theta \int_0^R r^2 dr d\theta d\phi\right) = p\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \sin\theta d\theta \frac{d}{dR}\int_0^R r^2 dr$$

By evaluating the iterated integrals,

$$\begin{aligned} F_p &= p\frac{d}{dR}\left(\frac{4}{3}\pi R^3\right) \\ &= 4\pi p R^2 \end{aligned}$$

Hence, we now have our expression for the force on the surface of a general spherically-symmetric body at distance R from the center. It can be observed that the pressure force varies with varying distance from the center. It is only the case with a spherical surface that the pressure force is uniform. Now that we have our general expression for the pressure force, we proceed with deriving a general expression for the gravitational force. To recap the method for the solid sphere approximation in finding the gravitational force, we divided the solid body into the interior and exterior parts. The total gravitational force on the body's surface, from which we find the equation of motion, is composed of the interior-exterior interaction and exterior-exterior interaction. The key-point to note in the method is the utilization of the Shell theorem to contract the entire interior part of any spherically symmetric body into a single point located at its center. Hence, one can generalize this solid-body problem to a spherically symmetric body by decomposing the gravitational force into two parts, the interior-exterior interaction F_{gi} and the exterior-exterior interaction F_{ge}, such that $F_g = F_{gi} + F_{ge}$.

Contracting the interior part, the expression of F_{gi} is the same for all spherically symmetric bodies. As we have derived earlier,

$$F_{gi} = \frac{-GMm}{R^2} + \frac{Gm^2}{R^2}$$

What characterizes differently-shaped spherically symmetric bodies, however, is the exterior-exterior interaction that arises from a different geometry. Putting all forces together,

$$F_i = 4\pi p R_i^2 - \frac{GMm}{R_i^2} + \frac{Gm^2}{R_i^2} + F_{ge} \tag{26}$$

Eq. (26) refers to the force exerted on the surface of the body located at distance $R = R_i$. One can observe that F_p, F_{gi} and F_{ge} are all functions of R. Hence, if the exterior shell is not equidistant from the center, different parts would be subject to different magnitudes of gravitational force. If the system is non-rigid (i.e. the exterior shell can change shape), the body would eventually achieve an equilibrium state where all forces would balance out. In this case, the two opposing forces are gravity and internal pressure must be equal. Since both forces depend on distance R from the center, zero net force must mean that the surface must be equidistant at equilibrium. This would only hold if and only if the body is a sphere at equilibrium [1] As the system achieves equilibrium, the surface turns into a sphere, $R_i = R$ for all i. That is, all points of the surface are now equidistant from the center. Eq. (26) reduces to Eq. (20) giving us an equation of motion equal to Eq. (21), that of a solid sphere. Since any arbitrarily shaped body will compress back into a sphere, we conclude that the equilibrium condition arising from the equation of motion are all similar regardless of the initial shape. Hence, F_{ge} will always assume its spherical form

$$F_{ge} = -\frac{Gm^2}{2R^2}$$

Then, for any arbitrary solid body, the force at any of its surface at equilibrium is given as

$$F = 4\pi p R^2 - \frac{GMm}{R^2} + \frac{Gm^2}{R^2} - \frac{Gm^2}{2R^2} \tag{27}$$

$$= 4\pi p R^2 - \frac{GMm}{R^2} - \frac{Gm^2}{2R^2} \tag{28}$$

Finding the equation of motion of the solid sphere's surface,

$$m\ddot{R} = 4\pi p R^2 - \frac{GMm}{R^2} - \frac{Gm^2}{2R^2}$$

Dividing by m,

$$\ddot{R} = \frac{4\pi p R^2}{m} - \frac{GM}{R^2} - \frac{Gm}{2R^2} \tag{29}$$

Comparing this with our previous equation of motion for the thin shell approximation, we now have the total mass M of the entire solid as a parameter. Due to our Shell theorem interior contraction, we can still carry out similar analysis of equations of state using p and σ as before. Letting $m = \sigma A = 4\pi R^2 \sigma$,

$$\ddot{R} = \frac{p}{\sigma} - \frac{GM}{R^2} - 2G\pi\sigma \tag{30}$$

This gives us the EoS for stellar body using the solid body approximation. Note that even though we are considering a solid body, we stick with using surface density as a parameter as to execute the same form of analysis with the paper. We then compare the results of this form of approximation.

2.2.2 Megastructure revisited: The Dyson Sphere

In this case, the spherical shell envelopes a massive stellar body. We can consider a specific case of enclosing the entire sun with a material strong enough to withstand the forces of interaction. A specific example is the Dyson sphere: a hypothetical megastructure that is essentially a spherical solar panel enclosing the sun. Using the Shell theorem, we can consider the sun as a single point located at the center of the sphere. In this scenario, we need to consider the gravitational pull of the host star that we haven't in our earlier discussion with thin shell megastructures. We can directly use the result of Eq. (30) for our stability analysis. Setting $\ddot{R} = 0$,

[1] A direct counterargument to flat-earthers.

$$\begin{aligned} p &= \frac{GMm}{4\pi R^4} + 2G\pi\frac{m^2}{16\pi^2 R^4} \\ &= \frac{G}{4\pi R^4}\left(Mm + \frac{m^2}{2}\right) \end{aligned}$$

Our goal is to find an expression for the sphere's radius. Rearranging the above equation,

$$R(m,p) = \left(\frac{G}{4\pi p}\left(Mm + \frac{m^2}{2}\right)\right)^{1/4} \tag{31}$$

It can be observed that in order to build the struc ture, their mass and radius must be directly related. In the specific case of the sun as our host star, we can find a mass-radius relation by plugging in specific value for M and p. Finding out these two parameters, we can directly use the value for solar mass $M = 1.989 \times 10^{30}$ kg. For p, we can use an expression in terms of the solar constant G_{sc}. Assuming a perfectly absorbing, spherical surface,

$$p = \frac{1}{(1.495 \times 10^{11} m)^2}\frac{G_{sc}}{cR^2}$$

Plugging this in to Eq. (31),

$$R(m) = \left(\frac{(1.495 \times 10^{11} m)^2 Gc}{4\pi G_{sc}}\left(Mm + \frac{m^2}{2}\right)\right)^{1/2} \tag{32}$$

We can likewise redefine the leading coefficient to have consistent units by letting $\beta = |\frac{(1.495\times 10^{11} m)^2 Gc}{4\pi G_{sc}}|$ with units $\frac{[m^2]}{[kg]}$ such that

$$R(m) = \beta\left(Mm + \frac{m^2}{2}\right)^{1/2} \tag{33}$$

Calculating the value of β,

$$\beta = |\frac{(1.495 \times 10^{11})^2 Gc}{4\pi G_{sc}}| = \frac{(1.495 \times 10^{11} m)^2 * 6.67 \times 10^-11 * 299792458}{4\pi * 1.361 \times 10^3} = 2.61 \times 10^{16}$$

Plugging in the solar mass,

$$R(m) = \beta\left(1.989 \times 10^{30}\text{ kg } * m + \frac{m^2}{2}\right)^{1/2} \tag{34}$$

Observe that the second term is small and can be neglected. This gives a a compact form for estimating how big such structure should be. One should also observe that the structures mass is significant as it serves as a multiplier to the large solar mass. Compactifying, we can generalize it to encapsule any celestial body:

$$R(m) = \beta\,(Mm)^{1/2} \tag{35}$$

This, of course, only accounts for a static structure that is gravitationally stable while only using stellar radiation as a counterforce. A numerical calculation shows that to build a structure having a 1 kg mass, it needs to have a radius of 3.68×10^{31} m. That is nowhere near even when compared to the radius of the entire solar system at 4.55×10^{12} m. Such a system would be highly unstable if constructed with a practical radius. One could use the parameter m to reduce the required radius. To construct a structure at around twice the solar radius, it would require the structure to have a mass of about 10^{-45} kg. This poses absurd engineering problem. It would be more stable to build a swarm of orbiting reflectors instead.

3 Stability about the equilibrium

3.1 Thin shell approximation

Now that we have inspected various forms of toy model approximations and derived equations of motion accordingly, our next task is to find out the equilibrium relations. As we have mentioned earlier, these are the mass and pressure functions. We can directly use the respective equations of motion to find the static equation by setting $\ddot{R} = 0$. The goal is to find expressions for mass m and radius R in terms of pressure p and density σ such that

$$m = m(p, \sigma) \qquad\qquad R = R(p, \sigma)$$

Using an appropriate EoS of the barotropic form $p = p(\sigma)$, the following quantities can be expressed as

$$m = m(\sigma) \qquad\qquad R = R(\sigma)$$

In these forms, we can analyze the stationary point $\frac{dm}{d\sigma}$, the point at which the toy model of the neutron star achieves a maximum mass. Using the thin shell equation of motion from Eq. (12), and setting $\ddot{R} = 0$,

$$\begin{aligned} 0 &= -\frac{Gm}{2R^2} + \frac{8\pi pR}{m} \\ \frac{Gm}{2R^2} &= \frac{8\pi pR}{m} \\ p &= \frac{Gm^2}{16\pi R^3} \end{aligned} \tag{36}$$

The surface area is given as

$$\sigma = \frac{m}{4\pi R^2} \tag{37}$$

Simultaneously solving for equations (36) and (37), we have the following relation at equilibrium

$$m = \frac{4}{\pi G^2}\frac{p^2}{\sigma^3} \qquad\qquad R = \frac{1}{\pi G}\frac{p}{\sigma^2} \tag{38}$$

At this point, we are free to choose whatever form of EoS we desire. The key to remember is that it must be barotropic. It means that pressure must only be a function of density. Hence, we need to consider that in this toy model, aside from the obvious geometric approximation is the fact that the entire fluid dynamics of the star is restricted to a barotropic form of EoS. The paper rationalized that while main sequence stars are far from being accurately modelled by this form, neutron stars are a good candidate on the other hand. This is a restriction that needs to be noted when applying the same analysis to different kinds of star. This specific form is a necessity that would be apparent later. Without this assumption, we could not directly relate the onset of the maximum mass and instability and our analysis would be rendered moot.
The first form of EoS that we will tackle is the polytropic form

$$p = K\sigma^\gamma \tag{39}$$

It is a good exercise in algebra for the reader to show that for the given polytropic form,

$$m = \frac{4K^2}{\pi G^2}\sigma^{2\gamma-3} \qquad\qquad R = \frac{K}{\pi G}\sigma^{\gamma-2} \tag{40}$$

Observing the mass equation, $\gamma = \frac{3}{2}$ acts like a turning point where all γ values below result in an inverse mass-density relation while all γ values above result in a direct mass-density relation. However, note that γ is a constant. It does not signify an extremum point for mass since at all values, mass remains a monotonic function of density. In a polytropic EoS, differentiating Eq. (40) gives us an expression for γ:

$$\gamma = \frac{dp}{d\sigma}\frac{\sigma}{p} \tag{41}$$

We define the effective polytropic exponent in Eq. (41) to be true for all EoS that is, in general, a function of density. It is important to note that we can choose any EoS, so long that it must be a non-monotonic function (i.e. must have a turning point) of density and we can have that turning point to occur at $\gamma = \frac{3}{2}$. This 3/2 point is a general result of the following. This is a direct consequence of how the mass function was expressed. From Eq. (38), since the turning point of mass must occur at $\frac{\mathrm{d}m}{\mathrm{d}\sigma} = 0$, by chain rule,

$$\frac{\mathrm{d}m}{\mathrm{d}\sigma} = 0 \iff \frac{\partial m}{\partial p}\frac{\partial p}{\partial \sigma} + \frac{\partial m}{\partial \sigma} = 0$$

Differentiating the mass function and rearranging,

$$\frac{\sigma}{p}\frac{\mathrm{d}p}{\mathrm{d}\sigma} = \frac{3}{2} \tag{42}$$

Using Eq. (41), it turns out that at the turning point, $\gamma = \frac{3}{2}$ regardless of what form the EoS takes. One can explicitly express the turning point of mass in terms of the central density σ. From Eq. (42),

$$\sigma = \frac{3}{2}\frac{p}{\mathrm{d}p/\mathrm{d}\sigma} \tag{43}$$

The utilized EoS that the paper claims to have physical realism has a form $p = p_0[\sin\left(\frac{\sigma}{\sigma_0}\right)]^{5/3}$. Using Eq. (43), the σ_m at which mass reaches maximum is

$$\sigma_m = \frac{5}{3}P_o^{5/3}\cos\frac{\sigma_m}{\sigma_o}$$

A first-order approximation gives us

$$\sigma_m = \frac{\sqrt{P_o^{2/3}\sigma_o^2(200P_o^{10/3} + 9n^2)} - 3P_o^{1/3}\sigma_o^2}{10P_o^2}$$

Needless to say, a constant gamma with an arbitrary EoS results to an arbitrary turning point for maximum mass. For physical realism, this toy model needs to engineer the EoS to accommodate modelling accuracy. The key to note here is that γ represents quantities of pressure and density at equilibrium. It could be described more accurately by using the following notation

$$\gamma = \frac{\mathrm{d}p}{\mathrm{d}\sigma}\frac{\sigma_0}{p_0} \tag{44}$$

This would be important to remember when we compare γ with Γ on the next chapter.

3.1.1 Perturbation equation

The paper, then, perturbed the system and analyzed the resulting perturbation equation. This is done by a first-order perturbation expressions of the quantities $R = R_0 + \delta R$, $p = p_0 + \delta p$, $\sigma = \sigma_0 + \delta\sigma$ with time-dependent perturbations. Plugging these quantities into the equation of motion at Eq. (12),

$$\delta\ddot{R} = -\frac{Gm}{2(R_0 + \delta R)^2} + \frac{8\pi p(R_0 + \delta R)}{m}$$

Letting the right hand side be $f(\delta R)$, expanding in a first-order Taylor series centered at 0,

$$f(\delta R) = f(0) + \frac{\mathrm{d}f}{\mathrm{d}\delta R}|_{\delta R=0}(\delta R)$$

We can find the right hand side from the following substitutions.

$$p = p_0 + \delta p$$

Defining $\Gamma = \frac{\delta p}{\delta \sigma}\frac{\sigma_0}{p_0}$,

$$p = p_0(1 + \frac{\Gamma}{\sigma_0}\delta\sigma)$$

Lastly, since $\delta\sigma = -\frac{m}{2\pi R_0^3}\delta R$,

$$p = p_0\left(1 - \frac{\Gamma m}{2\sigma_0 \pi R_0^3}\delta R\right)$$

Using $p_0 = \frac{Gm^2}{16\pi R_0^3}$ and $\sigma_0 = \frac{m}{4\pi R_0^2}$, we now find the terms for the Taylor expansion. Evaluation of the first term gives us $f(0) = 0$. To find the second term, differentiation and evaluation gives us

$$\frac{\mathrm{d}f}{\mathrm{d}\delta R}|_{\delta R=0}(\delta R) = -\frac{Gm}{2R_0^3}(2\Gamma - 3)\delta R$$

which gives us the perturbation equation

$$\delta\ddot{R} = -\frac{Gm}{2R_0^3}(2\Gamma - 3)\delta R \tag{45}$$

To give us insights about the stability about the equilibrium point R_0, we can directly solve Eq. (45). Let $\delta R = e^{\lambda t}$ and $z = -\frac{Gm}{2R_0^3}(2\Gamma - 3)$. Then,

$$\lambda^2 = z \tag{46}$$

The solution depends on the nature of the roots of λ. In general,

$$\delta R = e^{\mathrm{Re}(\lambda)t}(C_1 \cos(\mathrm{Im}(\lambda)t) + C_2 \sin(\mathrm{Im}(\lambda)t)) \tag{47}$$

We can take only the positive root of λ. From Eq. (46), $\lambda = \sqrt{z}$. Observing the expression on Eq. (47), $\mathrm{Re}(\lambda) = 0$ when $\Gamma < \frac{3}{2}$ while $\mathrm{Im}(\lambda) = 0$ when $\Gamma > \frac{3}{2}$. That is

$$\delta R = \begin{cases} C_1 e^{(\mathrm{Re}(\lambda))}, & \text{if } \Gamma < \frac{3}{2} \\ 0, & \text{if } \Gamma = \frac{3}{2} \\ C_1 \cos(\mathrm{Im}(\lambda)t) + C_2 \sin(\mathrm{Im}(\lambda)t), & \text{if } \Gamma > \frac{3}{2} \end{cases}$$

The transition from an exponentially growing perturbation into a constant frequency oscillation occurs at $\Gamma = \frac{3}{2}$. The crux of the paper lies here. It is evident that the value of polytropic exponent γ discussed on the previous section to be the point where maximum mass occurs is equal to the value of Γ where the transition from stability of the perturbation occurs. This lies on how Γ was defined and our barotropic EoS constraint. Observe that

$$\Gamma = \frac{\delta p}{\delta \sigma}\frac{\sigma_0}{p_0}$$

which possesses a striking similarity with Eq. (41) differing only on the first fractions $\frac{\mathrm{d}p}{\mathrm{d}\sigma}$ and $\frac{\delta p}{\delta \sigma}$. Remember that $\frac{\mathrm{d}p}{\mathrm{d}\sigma}$ refers to the derivative of pressure (at equilibrium) with respect to density (at equilibrium) while $\frac{\delta p}{\delta \sigma}$ refers to the ratio of the perturbation of pressure to density as δR deviates away from R_0 - a ratio of dynamically changing perturbations. When the perturbations are small, it can be argued from limit definitions that $\frac{\mathrm{d}p}{\mathrm{d}\sigma} = \frac{\delta p}{\delta \sigma}$. The difference can be clearly seen when we otherwise assume a non-barotropic form with the pressure depending on another parameter - specific entropy s with form $p = p(\sigma, s)$. The derivative would, then, be expressed as

$$\frac{\mathrm{d}p}{\mathrm{d}\sigma} = \frac{\partial p}{\partial \sigma} + \frac{\partial p}{\partial s}\frac{\partial s}{\partial \sigma} \neq \frac{\delta p}{\delta \sigma}$$

However, when p does not depend on other variables except σ, the equality holds and we see that Γ=γ. This equality allows us to conclude that the point at where maximum mass occurs (contained by γ) is the same as the point where dynamic instability begins (contained by Γ) when the EoS is barotropic.

3.1.2 Effective potential

One can also include energy considerations to further analyze the system. To do so, we can find an effective potential energy for both gravitation and pressure forces. It is apparent that the work done by the two forces only depend on the shell's radius R. That is, when the shell expands from a given radius and contracts back to that given radius, the total work is zero. Hence, both must be a conservative force and can be derivable from the potentials U_p and U_g, pressure potential energy and gravitational potential energy, respectively. Finding U_p,

$$\frac{\partial U_p}{\partial R} = -8\pi p R \iff U_p = -4\pi p R^2 \tag{48}$$

Finding U_g,

$$\frac{\partial U_g}{\partial R} = \frac{Gm^2}{2R^2} \iff U_g = -\frac{Gm^2}{2R} \tag{49}$$

Then, we can define the effective potential as

$$U \equiv -\left(4\pi p R^2 + \frac{Gm^2}{2R}\right) \tag{50}$$

where differentiating/finding the gradient (inwards) gives us the net force.

$$F = -\frac{\partial U}{\partial R} = 8\pi p R - \frac{Gm^2}{2R^2} \tag{51}$$

To analyze stability, we can differentiate further and invoke the second derivative test

$$\frac{\partial^2 U}{\partial R^2} = -8\pi p + \frac{Gm^2}{R} \tag{52}$$

From Eq. (52), the second derivative test gives us a constraint such that a stable equilibrium exists. For a local minima to exist, Eq. (52) must be greater than zero. That is

$$R < \frac{Gm^2}{8\pi p} \tag{53}$$

We can express mass as a function of pressure and density using Eq. (38) and arrive at the constraint inequality

$$R < \frac{2}{(\pi G)^3}\frac{p^2}{\sigma^6} \tag{54}$$

An EoS $p = p(\sigma)$, R can be directly expressed as a function of density. Above this radius, no local minima for our effective potential can exist. Meanwhile, solving for mass using mass-radius relation at Eq. (37) gives us

$$m > \left(\frac{16\pi}{G^2}\frac{p^2}{\sigma}\right)^{1/3} \tag{55}$$

Note that equations (54) and (55) are not the same as our equilibrium quantities at Eq. (38). These are, instead, additional constraints for stability - a given range required such that the potential function concaves upwards. It is apparent that Eq. (54) is a minimum condition rather than a maximum one. Below the given mass threshold, no stable equilibrium, that is a local minimum, can occur. Lastly, for a Lagrangian approach[2] of analysis, one can define the Lagrangian of the system as

$$\mathcal{L}(R, \dot{R}) = \frac{1}{2}m\dot{R}^2 + \left(4\pi p R^2 + \frac{Gm^2}{2R}\right) \tag{56}$$

[2]This would give the same results as in the Newtonian case.

3.2 Solid body toy model

Since at equilibrium, any celestial body of arbitrary shape with sufficient mass will eventually revert back to a sphere, we will stick with a solid sphere model. Using the derived equation of motion from Eq. (30), we similarly set $\ddot{R} = 0$. That is,

$$0 = \frac{p}{\sigma} - \frac{GM}{R^2} - 2G\pi\sigma \tag{57}$$

Our task is to solve for $m = m(p, \sigma)$ and $R = R(p, \sigma)$. Solving for both p and σ from Eq. (58),

$$p = \frac{G}{4\pi R^4}\left(Mm + \frac{m^2}{2}\right) \tag{58}$$

One notable difference from our previous pressure equation, aside from M dependence is the R^4 dependence (as opposed to the previous R^3 dependence) Still using $\sigma = \frac{m}{4\pi R^2}$, we can first simplify Eq. (58). Eliminating R^4,

$$p = \frac{G\sigma}{64\pi^3}\left(\frac{M}{m} + \frac{1}{2}\right)$$

Giving us an equation for mass and radius

$$m = \frac{2GM\sigma}{128\pi^3 p - \sigma G} \iff R = \sqrt{\frac{GM}{256\pi^3 p - 2\sigma G}} \tag{59}$$

The resulting expressions are not as simple compared to the thin shell mass (and radius) relation to density and pressure. At this point, the paper tested different forms of EoS for the thin shell such as the polytropic form $p = K\sigma^\gamma$ and the sinusoidal form $p = p_0[\sin(\sigma/\sigma_0)]^{5/3}$, and found out that the turning point occurs at $\gamma = 3/2$ for both cases. Here, we could skip testing out specific cases and proceed with the analysis using a general form $p = p(\sigma)$. From Eq. (59),

$$m = \frac{2GM\sigma}{128\pi^3 p(\sigma) - \sigma G}$$

Finding the stationary points,

$$\frac{dm}{d\sigma} = \frac{256\pi^3 M}{(128\pi^3 p(\sigma) - \sigma G)^2}(p(\sigma) - \sigma p'(\sigma))$$

Defining an effective polytropic exponent as $\gamma = \frac{dp}{d\sigma}\frac{\sigma}{p}$

$$\begin{aligned}\frac{dm}{d\sigma} &= \frac{256\pi^3 M}{(128\pi^3 p(\sigma) - \sigma G)^2}(p(\sigma) - \gamma p(\sigma)) \\ &= \frac{256\pi^3 M p(\sigma)}{(128\pi^3 p(\sigma) - \sigma G)^2}(1 - \gamma)\end{aligned} \tag{60}$$

Apparently, the maximum mass occurs at $\gamma = 1$. Comparing this with the realistic case $<\gamma>=4/3$, our approximation is actually worse[3] than the case of the thin shell where $\gamma = 3/2$. The same process can be executed to find the stationary point for the radius. Since we have already shown on the previous section that $\Gamma = \gamma$, it would be futile to solve for the perturbation equation if our only goal is to find the point of instability offset, and, hence we could stop here. While $\gamma = 1$ polytropic exponent may not be suited for neutron stars (or stars in general), it may be used to model a rocky planet instead (like earth) as we will discuss in the next section.

[3]This is in the context of applicability to neutron stars.

4 Customizing a toy model

4.1 Lane-Emden equation

The paper presented a polytropic EoS of form $p = K\sigma^{\gamma}$, where γ is termed as the polytropic exponent. However, this is commonly known as the adiabatic index[4]. Another expression for a polytrope is of the form $p = K\sigma^{(n+1)/n}$, where n is termed as the polytropic index (as opposed to γ). We can observe that

$$\gamma = (n+1)/n \tag{61}$$

The density of a star as a function of its radius is described by the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n \tag{62}$$

where variable substitutions were made such that $\sigma(r) = \sigma_c\theta^n(\xi)$. From Eq. (61), the solid body model has a polytropic index $n = 0$ since $\gamma = 1$. In order to solve the Lane-Emden equation, numerical methods are generally needed to find θ_n. However, there are specific values for n where analytic solutions can be solved as we shall tackle later. For a given solution θ_n, the density function is given as

$$\sigma = \sigma_c\theta_n^n(\xi) \tag{63}$$

Clearly, a polytropic index $n = 0$ gives us a constant density $\sigma = \sigma_c$. A constant density solution with respect to a radius implies an incompressible sphere. Apparently, the solid body model that we have constructed fails to be applicable to neutron stars, or stars in general. It could be made useful, on the other hand, in the analysis of rocky planets and other incompressible spherical bodies. Solving the Lane-Emden equation for $n = 0$ with boundary conditions $\theta(0) = 1$ and $\theta'(0) = 0$ gives us an analytic solution

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 \tag{64}$$

The thin shell model presents an adiabatic index $\gamma = \frac{3}{2}$ which corresponds to a polytropic index $n = 2$. Fortunately, this is also one of the instances where analytic solutions can be solved. Assuming a power series solution ansatz, similarly given the boundary conditions $\theta(0) = 1$ and $\theta'(0) = 0$, the solution is

$$\theta(\xi) = \frac{sin(\xi)}{\xi} \tag{65}$$

Plugging this into Eq. (63),

$$\sigma = \sigma_c\frac{sin^2(\xi)}{\xi^2} \tag{66}$$

Since ξ is a constant multiple of R which is a function of density, making the substitutions from ξ to R, and plugging it into the polytropic EoS, we can recognize that the function $p = p(\sigma)$ takes the form akin to the one presented by the paper - a decaying sinusoid. Could the solid body be applied likewise to stellar bodies, a polynomial EoS akin to Eq. (64) may have been fit. One could make extensive use of Eq. (62) to match up γ into whatever model one is trying to simplify. Analytical solutions can be derived for limited values of n; generally, a numerical solution is needed.

[4]Unfortunately, the paper also termed Γ as adiabatic index

4.2 Mass function condition

So far, we have tried different forms of toy models: one that assumes a thin spherical shell and another that assumes a solid sphere. We can continue to build other models by executing the same process. Newton's second law gives us a way to have a static equation that relates both pressure and gravitational force from any arbitrary mass distributions.

$$F_p + F_g = 0 \iff p\frac{\mathrm{d}V}{\mathrm{d}R} + F_g(m, M, R, \sigma) = 0$$

from which we can derive a equilibrium mass function expressed as

$$m = m(M, \sigma, p)$$

Executing the chain rule,

$$\frac{\mathrm{d}m}{\mathrm{d}\sigma} = \frac{\partial m}{\partial p}\frac{\partial p}{\partial \sigma} + \frac{\partial m}{\partial \sigma} = 0 \tag{67}$$

Assuming a barotropic EoS and using the definition of γ at Eq. (41), this gives us a partial differential equation.

$$\gamma\frac{\partial m}{\partial p}p + \frac{\partial m}{\partial \sigma}\sigma = 0 \tag{68}$$

We can directly solve for γ.

$$\gamma = -\frac{\partial m/\partial \sigma}{\partial m/\partial p}\frac{p}{\sigma} \tag{69}$$

Note that since the equilibrium surface mass function m, total mass function M, and surface density σ are ultimately functions of the model's mass distribution, γ must also be a function of the mass distribution. Each mass distribution will correspond to a polytropic exponent albeit not necessarily unique. This way, we can "customize"γ to fit whatever model we are trying to make proxies of. Observe that, in general, γ is not constant. It depends on how the mass function was expressed. The thin shell approximation is considered to be a good simplifying model since the mass function was expressed in such a way that γ has a constant value (i.e. the function dm/dt has at least one root). This could reduce our set of possible mass distributions that is able to model a function with a turning point. It has to be noted that γ here refers to the set of polytropic exponents relative to a given mass function expression. This is as opposed to the paper's variable γ with respect to σ expressed relative to a given mass function expression of the thin shell. To screen the expressions and derive a constraining condition, the form of the mass expression could be derived by solving Eq. (68) . By assuming a separable mass function ansatz of form $m(p, \sigma) = P(p)\Sigma(\sigma)$, separation of variables gives us two building block solutions of form $P(p)$ and $\Sigma(\sigma)$. This gives us the mass function solution parametrized by two constants C_1 and C_2.

$$m(p, \sigma) = C_1 p^{C_2/\gamma}\sigma^{-C_2} \tag{70}$$

Given that the mass function can be expressed in the form given by Eq. (70), a constant γ at the turning point can be assured. As an example, an approximate mass expression of the solid body is given as

$$m = \frac{GM\sigma}{64\pi^3 p}$$

implying that $\gamma = 1$ as we have previously derived. Using Eqs. (69) and (70), one can tune γ to a desired value by picking the appropriate mass distribution. A constant γ gives us a way to express the turning point of the model's mass while the choice of the EoS gives us the knob to make the model more physically realistic. A mass expression that results to a constant gamma gives us a way to customize using the appropriate index via the Lane-Emden equation.

5 Relativistic correction

The second part of the paper introduces relativistic corrections to the previous analysis. This is done so by redefining the concepts of mass and time from Newtonian to relativistic. Although the corrections introduced arbitrary complexities, the mathematical procedure were similar to the first part. The equation of motion was first derived to find expressions for quantities at equilibrium. Then, the system was perturbed to find the offset of instability. As in the previous case, the paper presented again that $\gamma = \Gamma$ showing that the maximum mass occurs at the offset of instability and, hence, providing a relativistic shell proxy of the neutron star. Using relativistic expression for mass and relativistic time derivatives relative to proper times, the relativistic equation of motion is derived as

$$\ddot{R} = -\sqrt{1+\dot{R}^2}\left(\frac{m}{2R^2} - \frac{8\pi pR}{m}\left(\sqrt{1+\dot{R}^2} - m/R\right)\right) \tag{71}$$

By executing similar process and introducing necessary parametrization, the polytropic exponent was derived as

$$\gamma = \frac{6-8C+3C^2}{(1-C)(4-3C)} \tag{72}$$

Meanwhile, perturbing the system gives a perturbation equation

$$\delta\ddot{R} = -\frac{1}{4R_0^2}\frac{C}{(1-C)^2}(2(2\Gamma-3) - (7\Gamma-8)C + 3(\Gamma-1)C^2)\delta R \tag{73}$$

The offset of instability occurs at

$$\Gamma = \frac{6-8C+3C^2}{(1-C)(4-3C)}$$

It can be observed that the relativistic case offers a polytropic exponent that depends on density. The compactness parameter C depends on both p and σ expressed as $C = (4p/\sigma)/(1+4p/\sigma)$. This results to a turning point that is both a function of p and σ (i.e. $\gamma = \gamma(p, \sigma)$) implying that the mass function could not be expressed in the form written at Eq. (70). Moreover, a variable γ ultimately implies a dependence on the chosen EoS. That is, unlike the previous Newtonian model, each EoS has its own polytropic exponent where the mass function is stationary. The paper utilized two concepts of mass: material mass m and gravitational mass M derived as

$$m = \frac{4p^2}{\pi\sigma^3(1+4p/\sigma)^2}, \qquad M = \frac{4p^2(1+2p/\sigma)}{\pi\sigma^3(1+4p/\sigma)^3}$$

While finding out the stationary point of the material mass m has no particular significance as was stated by the paper, it was by finding out the stationary point of the gravitational mass M that we have derived the inequality condition that gave us γ. Observing how M was expressed, it can be noted again that it does not comply with Eq. (70). Another difference of the relativistic model from the Newtonian one is that the polytropic EoS produced a turning point.

The final part of the paper is deriving a complete non-approximated perturbation equation to find out the behavior of a relativistic shell from the initial onset of instability onwards without assuming a small δR. Numerical integration modelled how the shell's radius evolves with respect to the proper time given different γ. Essentially, it found out that the evolution becomes unstable growing without bounds either outwards or inwards upon crossing Γ. Below this threshold, the perturbation δR remains small and the evolution is stable. The newtonian shell of the thin shell can be recovered in the limiting case of the relativistic as C is diminished.

While the relativistic version of the model may be considered as more accurate, the simplicity of the Newtonian approach fits our goal more.

6 Conclusion

The reviewed paper made thin shells as proxies for the dynamics of neutrons stars to qualitatively describe its collapse into a black hole. By considering two forces of interaction - internal pressure and gravitation, an equation of motion and a first-order perturbation equation were derived. Using the equation of motion, a mass expression was also derived in terms of pressure and density and could ultimately be expressed solely in terms of density by the usage of an EoS. The derivation of an effective potential presented in Section (3.1.2) hints a possible extension toy-model kind of analysis using an energy approach. This approach provided us an additional constraint (Eq. (55)) which ensures a point of maximum mass to exist.

The main point of the paper is to show that the maximum mass occurs at the same point as the onset of perturbation instability denoted by the $\gamma = \Gamma$ equality. It was presented that thin shell subject to Newtonian gravity has a constant $\gamma = \Gamma = 3/2$ and is independent of the choice of EoS as was justified in Section (3.1.1). We have explicitly expressed that the point of collapse occurs at $\sigma_m = (\sqrt{P_o^{2/3}\sigma_o^2(200P_o^{10/3} + 9n^2)} - 3P_o^{1/3}\sigma_o^2)/10P_o^2$ in the given decaying sinusoidal EoS. As was justified in Section (4.1), the decaying sinusoidal EoS was deemed fit as a direct result of the specific solution of the Lane-Emden equation when $\gamma = 3/2$. The paper then proceeded with applying relativistic corrections to the derivation giving us an EoS-dependent γ. This form of simplification is useful in making such analytical description (albeit leaning into a qualitative nature) of an otherwise complicated system, such as a black hole collapse, accessible to a broader audience.

It can be argued that for such simple system, it would only require knowledge of algebra and basic calculus to see why a neutron star collapses after its mass reaches a specific point. If such phenomenon can be simplified this way, this begs the question of what else can be. There are various phenomena that are obscured by advanced and rigorous maths and this simplification may be the way to make them accessible to readers that only hopes to grasp the qualitative reasoning behind the phenomena. This brings us to the main point of the paper review: a step-by-step re-derivation and extension to possibly generalize this simplification concept motivated by the thin shell approximation. As was discussed in Section (4.1), the Lane-Emden equation gives us a way to parametrize possible ranges of applications via γ as an index. Since it would be useful for this application to have a consistent γ for each model, the mass function can be chosen specifically for this purpose. This is achieved when the function has a form similar to our derived expression in Eq. (70), a mass *law* of form $m = C_1 p^{C_2/\gamma}\sigma^{-C_2}$. C_1 denotes an arbitrary coefficient while C_2 denotes an arbitrary constant that links the exponent of p and σ. We have shown that γ can be directly extracted using Eq. (69) or it may be implied from the mass law. As a concrete example of this application, we have presented in Section (3.2) a solid body model by generalizing the pressure expression and using the Shell theorem to find an EoS for a spherical solid body. γ was derived both ways and we found that $\gamma = 1$ in both cases. A generalization of finding a custom model can be found in Section (4.2) where both pressure and gravitation forces were expressed generally.

To complete the toy model customization, one could introduce a general Newtonian gravitational expression to any arbitrary shape, compactify the expression if possible using approximations, and then parametrize the whole expression with respect to the model's geometry. The hope is that one can find a function that takes in a geometric parameter describing the mass distribution spitting out the γ corresponding to that distribution. The resulting mass expression can, then, be tabulated with respect to γ and the generalization of the approximating toy model can be considered complete. To improve modelling accuracy, the choice of the EoS can serve as a fine-tuning mechanism. Again, the ultimate goal of all of these is to be able to use the simplified models in making more scientific phenomena accessible to a broader audience.

7 Acknowledgments

I would like to thank Sir Ian Vega for extending the deadline of submission. Due to a recent untimely health compromise, the review may have not been completed if not for the extension.

8 Appendix

One objective of the simplification is the integration of more scientific phenomena on textbooks. The next pages are sample outlines of how an introductory textbook chapter would look like using the simplified models. The rest of this page was intentionally left blank.

$\mathcal{N}$.NEUTRON STAR COLLAPSE

(*A brief introduction of the binary star system, one being a neutron star, is discussed here and its collapse into a black hole as mass increases. This section aims to provide and overview of the thin shell model approximation.*) Lorem ipsum dolor sit amet, consectetur adipiscing elit. Nullam vitae scelerisque purus. Etiam sit amet nisi id massa vestibulum tempus. Nam vel nulla nulla. Etiam consequat dolor eu bibendum lobortis. Phasellus a efficitur enim. Aliquam nec facilisis nunc. Fusce scelerisque interdum scelerisque. Nulla nec dolor eget tortor suscipit varius eget non justo. Duis in magna vel justo dictum egestas non at quam. Phasellus ac quam non justo aliquam rutrum at vel augue. Suspendisse potenti. Praesent nec tincidunt mi. Mauris condimentum sollicitudin urna.

Learning objectives

In this chapter, you will learn how...

- to find the equation of motion of a thin spherical shell
- to derive the mass expression at equilibrium
- to find out when the mass approaches a maximum
- to derive the perturbation equation
- to identify when the system becomes unstable
- to conceptually understand why a neutron star collapses

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N.1. Equation of motion

(*When presented in an introductory physics textbook, one may reference the chapter of using Newton's laws to derive the equation of motion.*) Lorem ipsum dolor sit amet, consectetur adipiscing elit. Nullam vitae scelerisque purus. Etiam sit amet nisi id massa vestibulum tempus. Nam vel nulla nulla. Etiam consequat dolor eu bibendum lobortis. Phasellus a efficitur enim. Aliquam nec facilisis nunc. Fusce scelerisque interdum scelerisque. Nulla nec dolor eget tortor suscipit varius eget non justo. Duis in magna vel justo dictum egestas non at quam. Phasellus ac quam non justo aliquam rutrum at vel augue. Suspendisse potenti. Praesent nec tincidunt mi. Mauris condimentum sollicitudin urna.

$$\ddot{R} = -\frac{Gm}{2R^2} + \frac{8\pi pR}{m}$$

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N.2. Quantities at equilibrium

(*From the equation of motion, the equilibrium mass expression can be derived.*) Lorem ipsum dolor sit amet, consectetur adipiscing elit. Nullam vitae scelerisque purus. Etiam sit amet nisi id massa vestibulum tempus. Nam vel nulla nulla. Etiam consequat dolor eu bibendum lobortis. Phasellus a efficitur enim. Aliquam nec facilisis nunc. Fusce scelerisque interdum scelerisque. Nulla nec dolor eget tortor suscipit varius eget non justo. Duis in magna vel justo dictum egestas non at quam. Phasellus ac quam non justo aliquam rutrum at vel augue. Suspendisse potenti. Praesent nec tincidunt mi. Mauris condimentum sollicitudin urna.

$$m = \frac{4}{\pi G^2}\frac{p^2}{\sigma^3} \qquad R = \frac{1}{\pi G}\frac{p}{\sigma^2}$$

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(*The equation of state will be discussed here - what it means physically and what it implies in our model. Discussion of the polytropic exponent follows and how it relates to the maximum mass.*) Lorem ipsum dolor sit amet, consectetur adipiscing elit. Nullam vitae scelerisque purus. Etiam sit amet nisi id massa vestibulum tempus. Nam vel nulla nulla. Etiam consequat dolor eu bibendum lobortis. Phasellus a efficitur enim. Aliquam nec facilisis nunc. Fusce scelerisque interdum scelerisque. Nulla nec dolor eget tortor suscipit varius eget non justo. Duis in magna vel justo dictum egestas non at quam. Phasellus ac quam non justo aliquam rutrum at vel augue. Suspendisse potenti. Praesent nec tincidunt mi. Mauris condimentum sollicitudin urna.

$$\gamma = \frac{dp}{d\sigma}\frac{\sigma}{p}$$

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■ **Example N.1:** *Calculate the polytropic exponent*

(*Here, the reader must find that $\gamma = \frac{3}{2}$ regardless of the equation of state.*) Curabitur urna sem, laoreet non neque non, euismod venenatis dolor. Suspendisse dictum, massa vitae rutrum tincidunt, sem velit tempus ipsum, in dignissim est metus eu diam. Sed facilisis mauris vitae ipsum laoreet maximus. Nullam tristique id urna id sodales. Nulla eget placerat odio, a gravida libero. Mauris vel tellus enim. Aenean massa eros, lobortis eget sollicitudin gravida, scelerisque ac mi. In aliquam sit amet lorem id placerat. Nunc dui odio, dapibus ullamcorper faucibus id, accumsan nec

N.3. Perturbation equation

(*The perturbation equation will be derived here. This section must reference to the simple harmonic chapter of the textbook to facilitate understanding of the frequency and the tipping point of stability.*) Lorem ipsum dolor sit amet, consectetur adipiscing elit. Nullam vitae scelerisque purus. Etiam sit amet nisi id massa vestibulum tempus. Nam vel nulla nulla. Etiam consequat dolor eu bibendum lobortis. Phasellus a efficitur enim. Aliquam nec facilisis nunc. Fusce scelerisque interdum scelerisque. Nulla nec dolor eget tortor suscipit varius eget non justo. Duis in magna vel justo dictum egestas non at quam. Phasellus ac quam non justo aliquam rutrum at vel augue. Suspendisse potenti. Praesent nec tincidunt mi. Mauris condimentum sollicitudin urna.

$$\delta\ddot{R} = -\frac{Gm}{2R_0^3}(2\Gamma - 3)\delta R$$

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■ **Example N.2:** *Find the turning point frequency*

(*The reader must find out that the tipping point frequency occurs when* $\Gamma = \frac{3}{2}$.) Curabitur urna sem, laoreet non neque non, euismod venenatis dolor. Suspendisse dictum, massa vitae rutrum tincidunt, sem velit tempus ipsum, in dignissim est metus eu diam. Sed facilisis mauris vitae ipsum laoreet maximus. Nullam tristique id urna id sodales. Nulla eget placerat odio, a gravida libero. Mauris vel tellus enim. Aenean massa eros, lobortis eget sollicitudin gravida, scelerisque ac mi. In aliquam sit amet lorem id placerat. Nunc dui odio, dapibus ullamcorper faucibus id, accumsan nec

N.4. Collapse of a neutron star

This section is a concluding part of the entire chapter in light of $\gamma = \Gamma$ equality. The hope is that the reader should qualitatively know why a neutron star collapses using only Newtonian principles. A debriefing discussion must follow stating the extent of the approximation and a brief overview of the realistic case in the realm of general relativity. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Nullam vitae scelerisque purus. Etiam sit amet nisi id massa vestibulum tempus. Nam vel nulla nulla. Etiam consequat dolor eu bibendum lobortis. Phasellus a efficitur enim. Aliquam nec facilisis nunc. Fusce scelerisque interdum scelerisque. Nulla nec dolor eget tortor suscipit varius eget non justo. Duis in magna vel justo dictum egestas non at quam. Phasellus ac quam non justo aliquam rutrum at vel augue. Suspendisse potenti. Praesent nec tincidunt mi. Mauris condimentum sollicitudin urna.

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Notice how brief and simple the discussions were. This form of discussions can be extended to other applications as necessary - as a problem question or auxiliary discussions. If one can derive the appropriate model for other γ values, this can be extended to other bodies aside from neutron star.

REFERENCES

[1] P. LeMaitre, E. Poisson (2019), Equilibrium and stability of thin spherical shells in Newtonian and relativistic gravity, Astronomical Institute Utrecht, September 2019, pp. 961-970

[2] S. Chandrasekhar [1939] (1958). An Introduction to the Study of Stellar Structure, New York: Dover. ISBN 0-486-60413-6

[3] C. J. Hansen, S. D. Kawaler, V. Trimble (2004). Stellar Interiors – Physical Principles, Structure, and Evolution, New York: Springer. ISBN 0-387-20089-4

[4] Sagert, I., Hempel, M., Greiner, C., Schaffner-Bielich, J. (2006). Compact stars for undergraduates. European journal of physics, 27(3), 577.

[5] O. R. Pols (2011), Stellar Structure and Evolution, Astronomical Institute Utrecht, September 2011, pp. 64-68

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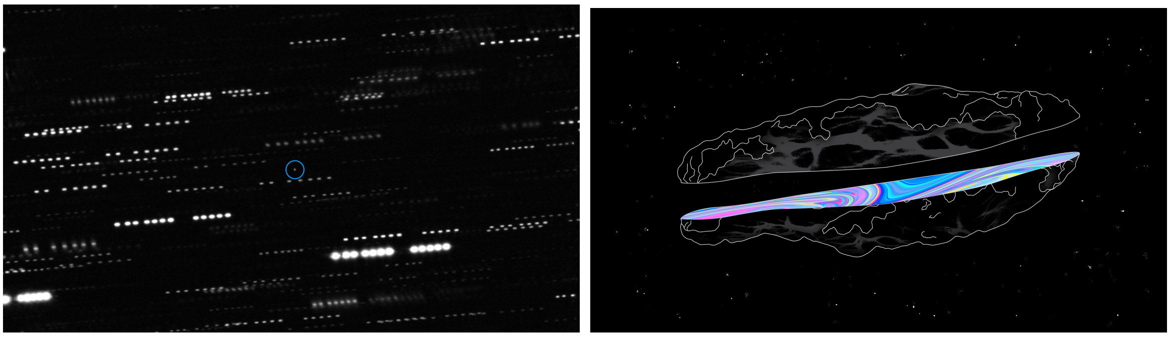

Figure shows 1I/2017 U1 ‘Oumuamua, the first known interstellar object to visit our solar system. Theoretical physicist Avi Loeb and postdoc Shmuel Bialy have written a paper that explores its possibility of being an radiation-accelerated, artificial solar sail.