# 1 Problem Set 1: Vector calculus

1. **Preliminaries**

Before proceeding to provide the solution of for the problem set, we shall first lay out some identities and equations. For the first part of the problem set, we should recall some vector expressions in index notation. First, we will attempt to express basic vector operations in in terms of index notation. Recall that dot products can be expressed as

$$\vec{A} \cdot \vec{B} = A_i B_i \tag{1}$$

Since divergences can be expressed in terms of the dot product, it is expressed as

$$\vec{\nabla} \cdot \vec{A} = \partial_i A_i \tag{2}$$

Cross products can be expressed as

$$\vec{A} \times \vec{B} = \hat{e}_i \varepsilon_{ijk} A_j B_k \tag{3}$$

Similarly, curls can be expressed as

$$\vec{\nabla} \times \vec{A} = \hat{e}_i \varepsilon_{ijk} \partial_j A_k \tag{4}$$

Lastly, gradients of a scalar field $\phi$ can be expressed as

$$\vec{\nabla} \phi = \partial_i \phi \tag{5}$$

Lastly, we will be using the Kronecker-Delta-Levi-Civita identity

$$\varepsilon_{kij} \varepsilon_{klm} = \delta_{ij} \delta_{lm} - \delta_{im} \delta_{jl} \tag{6}$$

Since some integrals keep appearing on the solutions. we may as well evaluate them first and simply reference them later when needed. Obviously, sine and cosine functions are zero when integrated over a complete cycle.

$$\int_0^{2\pi} \sin(x)\ \mathrm{d}x = -\cos(x)_0^{2\pi} = -(1-1) = 0 \tag{7}$$

$$\int_0^{2\pi} \cos(x)\ \mathrm{d}x = \sin(x)_0^{2\pi} = (0-0) = 0 \tag{8}$$

When, encountering these later, one can avoid doing integration when having these integrals as factors. Another integral is

$$\int_0^{\pi/2} \sin(x) \cos(x)\ \mathrm{d}x \tag{9}$$

Letting $u = \sin(x)$, $\mathrm{d}u = \cos(x)\mathrm{d}x$, the integral becomes

$$\int_0^1 u\ \mathrm{d}u = \frac{u^2}{2}|_0^1 = \frac{1}{2} \tag{10}$$

Lastly we evaluate the integral

$$\int_0^\infty \frac{x^2}{(x^2+\varepsilon^2)^{\frac{5}{2}}}\ \mathrm{d}x \tag{11}$$

for any constant $\varepsilon$. Carrying out a trigonometric substitution, $x = \varepsilon \tan\theta$, $\mathrm{d}x = \varepsilon \sec^2\theta\ \mathrm{d}\theta$ and noting that $r^2 + \varepsilon^2 = \varepsilon^2(1+\tan^2\theta) = (\varepsilon \sec\theta)^2$, the integral becomes

$$\int_0^{\pi/2} \frac{\varepsilon^2\ \tan^2\theta}{\varepsilon^5\ \sec^5\theta}\varepsilon\ \sec^2\theta\ \mathrm{d}\theta = \frac{1}{\varepsilon^2}\int_0^{\pi/2} \sin^2(\theta)\cos(\theta)\mathrm{d}\theta \tag{12}$$

Using a substitution $u = \sin(\theta)$, $\mathrm{d}u = \cos\theta\ \mathrm{d}\theta$, integral becomes

$$\frac{1}{\varepsilon^2}\int_0^1 u^2\mathrm{d}u = \frac{u^3}{3\varepsilon^2}|_0^1 = \frac{1}{3\varepsilon^2} \tag{13}$$

2. **Vector Identities** Prove the following vector identities:

$$\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} \tag{14}$$

We are tasked to prove three vector product rule identities. To do so, we have three options. Looking at the expressions, it screams to be rewritten into other expressions such as using the BAC-CAB rule. Another one is to use a more brute-force approach of picking a coordinate system and expressing the vectors into its component form. While it is straightforward, it might take us a while to prove. Another method is to use a combination of the first and the second approach: expressing into a convenient form first and converting into the component form. However, the first method is somehow problematic as it assumes that the identities used are already true. It feels uncomfortable proving some vector identities if the techniques used are also unproven vector identities (in this problem set).

A more direct first-principles approach is to use index notation using Einstein summation convention (ESC) which will be used here to prove the three identities. We shall begin. In index notation, the equation reads

$$\hat{e}_i\partial_i A_j B_j = \hat{e}_i\varepsilon_{ijk}A_j\varepsilon_{klm}\partial_l B_m + \hat{e}_i\varepsilon_{ijk}B_j\varepsilon_{klm}\partial_l A_m + A_j\partial_j\hat{e}_i B_i + B_j\partial_j\hat{e}_i A_i \tag{15}$$

The first two terms contain two $\varepsilon$s and it would be convenience to turn them into $\delta$s. To do so, we need to note that due to the cyclic permutation of the indices,

$$\varepsilon_{ijk} = \varepsilon_{kij} \tag{16}$$

and hence

$$\hat{e}_i\partial_i A_j B_j = \hat{e}_i\varepsilon_{kij}A_j\varepsilon_{klm}\partial_l B_m + \hat{e}_i\varepsilon_{kij}B_j\varepsilon_{klm}\partial_l A_m + A_j\partial_j\hat{e}_i B_i + B_j\partial_j\hat{e}_i A_i \tag{17}$$

Although the main advantage of manipulating expression in index notation is that most terms commute, we need to be wary with the differential operator as they do not commute. With these, the two $\varepsilon$s are safe and the equation reads

$$\hat{e}_i\partial_i A_j B_j = \hat{e}_i\varepsilon_{kij}\varepsilon_{klm}A_j\partial_l B_m + \hat{e}_i\varepsilon_{kij}\varepsilon_{klm}B_j\partial_l A_m + A_j\partial_j\hat{e}_i B_i + B_j\partial_j\hat{e}_i A_i \tag{18}$$

such that we can now carry out the identity using Eq. (6) giving us

$$\hat{e}_i\partial_i A_j B_j = \hat{e}_i(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_j\partial_l B_m + \hat{e}_i(\hat{e}_i\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})B_j\partial_l A_m + A_j\partial_j\hat{e}_i B_i + B_j\partial_j\hat{e}_i A_i \tag{19}$$

Due to the delta notation serving as *switches*, for the first term and third term, indices l turns to i and indices m turns to j. For the second term and fourth term, indices m turns to i and indices l turns to j. This simplifies the equation into

$$\hat{e}_i\partial_i A_j B_j = \hat{e}_i A_j\partial_i B_j - \hat{e}_i A_j\partial_j B_i + \hat{e}_i B_j\partial_i A_j - \hat{e}_i B_j\partial_j A_i + \hat{e}_i A_j\partial_j B_i + \hat{e}_i B_j\partial_j A_i \tag{20}$$

Second term cancels with fifth term. Fourth term cancels with sixth term. Equation now reads

$$\hat{e}_i\partial_i A_j B_j = \hat{e}_i A_j\partial_i B_j + \hat{e}_i B_j\partial_i A_j \tag{21}$$

This is precisely why the differential operator must not commute since product rule for differentiation states that

$$\partial_i A_j B_j = A_j\partial_i B_j + B_j\partial_i A_j \tag{22}$$

and hence Eq. (21) must be true proving the identity. ■

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \tag{23}$$

The next identity is also proven similarly. Converting into index notation, the identity reads

$$\partial_i \varepsilon_{ijk} A_j B_k = B_i \varepsilon_{ijk} \partial_j A_k - A_i \varepsilon_{ijk} \partial_j B_k \tag{24}$$

By linearity, we can move the constant terms of the left-hand side

$$\varepsilon_{ijk} \partial_i A_j B_k = \varepsilon_{ijk} B_i \partial_j A_k - \varepsilon_{ijk} A_i \partial_j B_k \tag{25}$$

To prove this equation, we use product rule to see that

$$\varepsilon_{ijk} \partial_i A_j B_k = \varepsilon_{ijk} B_k \partial_i A_j + \varepsilon_{ijk} A_j \partial_i B_k \tag{26}$$

Now, notice that Eq. (25), the left-hand side differential operators have j as indices. To convert the RHS of Eq. (26), into the RHS of Eq. (25), the trick is to rename the dummy indices as follows

$$\varepsilon_{ijk} B_k \partial_i A_j = \varepsilon_{jki} B_i \partial_j A_k = \varepsilon_{ijk} B_i \partial_j A_k \qquad \text{(cyclic permutation)} \tag{27}$$

$$\varepsilon_{ijk} A_j \partial_i B_k = \varepsilon_{jik} A_i \partial_j B_k = -\varepsilon_{ijk} A_i \partial_j B_k \qquad \text{(anti-cyclic permutation)} \tag{28}$$

Plugging these into Eq. (26), we see that

$$\varepsilon_{ijk} \partial_i A_j B_k = \varepsilon_{ijk} B_i \partial_j A_k - \varepsilon_{ijk} A_i \partial_j B_k \tag{29}$$

which is exactly what Eq. (25) states thus proving the identity. ■

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A}) \tag{30}$$

Similarly, we first convert the identity into an index notation

$$\begin{aligned}
\hat{e}_i \varepsilon_{ijk} \partial_j \varepsilon_{klm} A_l B_m =& B_j \partial_j \hat{e}_i A_i - A_j \partial_j \hat{e}_i B_i + \hat{e}_i A_i \partial_j B_j - \hat{e}_i B_i \partial_j A_j && (31)\\
&= \hat{e}_i B_j \partial_j A_i - \hat{e}_i A_j \partial_j B_i + \hat{e}_i A_i \partial_j B_j - \hat{e}_i B_i \partial_j A_j && (32)\\
&= \hat{e}_i A_i \partial_j B_j + \hat{e}_i B_j \partial_j A_i - \hat{e}_i B_i \partial_j A_j - \hat{e}_i A_j \partial_j B_i && (33)
\end{aligned}$$

To prove this, similar to our previous approach for double $\varepsilon$s, we use the $\varepsilon - \delta$ identity.

$$\begin{aligned}
\hat{e}_i \varepsilon_{ijk} \partial_j \varepsilon_{klm} A_l B_m &= \hat{e}_i \varepsilon_{kij} \varepsilon_{klm} \partial_j A_l B_m = \hat{e}_i (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_j A_l B_m && (34)\\
&= \hat{e}_i \partial_j A_i B_j - \hat{e}_i \partial_j A_j B_i && (35)\\
&= \hat{e}_i A_i \partial_j B_j + \hat{e}_i B_j \partial_j A_i - \hat{e}_i B_i \partial_j A_j - \hat{e}_i A_j \partial_j B_i && (36)
\end{aligned}$$

which is exactly what Eq. (33) states thus proving the identity.

3. **Spiked Density**

Prove that

$$\nabla^2 \frac{1}{r} = -4\pi\delta^3(\vec{r}) \tag{37}$$

For clarity, prove that

$$\lim_{\varepsilon\to 0} D(r,\varepsilon) \equiv \lim_{\varepsilon\to 0} -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\varepsilon^2}} = \delta^3(\vec{r}) \tag{38}$$

Proving the first identity is straightforward. The RHS demands the equation to be integrated. It turns out that the radius is irrelevant to the proof as long as it contains the origin. To see, integrating across a sphere centered at the origin, RHS gives us

$$\iiint -4\pi\delta^3(\vec{r})\ \mathrm{d}V = -4\pi \tag{39}$$

while LHS gives us

$$\iiint \nabla^2 \frac{1}{r}\ \mathrm{d}V = \iiint \vec{\nabla}\cdot\vec{\nabla}\frac{1}{r}\ \mathrm{d}V \tag{40}$$

By divergence theorem, it reduces into a closed surface integral

$$\iiint \vec{\nabla}\cdot\vec{\nabla}\frac{1}{r}\ \mathrm{d}V = \oiint \vec{\nabla}\frac{1}{r}\cdot\hat{n}\ \mathrm{d}A \tag{41}$$

where $\hat{n}$ is normal to the surface we are integrating in. We chose a spherical volume integration for a reason: normal vector is entirely radial such that $\hat{n} = \hat{r}$. This is so that, even though the gradient in spherical coordinates has messy terms in the angular components, we only need the radial one resulting to a non-zero dot product. That is

$$\vec{\nabla}\frac{1}{r} = -\frac{1}{r^2}\hat{r} + \ldots\ \hat{\theta} + \ldots\ \hat{\phi} \iff \vec{\nabla}\frac{1}{r}\cdot\hat{r} = -\frac{1}{r^2} \tag{42}$$

Moreover, the radial dependence cancels out since the spherical differential area is $\mathrm{d}A = r^2\ \sin(\theta)\ \ \mathrm{d}\theta\ \mathrm{d}\phi$. That is

$$\vec{\nabla}\frac{1}{r}\cdot\hat{n}\ \mathrm{d}A = -\frac{1}{r^2}r^2\ \sin(\theta)\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi = -\sin(\theta)\ \ \mathrm{d}\theta\ \mathrm{d}\phi \tag{43}$$

Evaluating the closed surface integral at Eq. (41),

$$\oiint \vec{\nabla}\frac{1}{r}\cdot\hat{n}\ \mathrm{d}A = -\oiint \sin(\theta)\ \ \mathrm{d}\theta\ \mathrm{d}\phi = -\int_0^{\pi} \sin(\theta)\ \mathrm{d}\theta \int_0^{2\pi}\ \mathrm{d}\phi = \cos(\theta)|_0^{\pi}\phi|_0^{2\pi} = -4\pi \tag{44}$$

Observe that $\iiint -4\pi\delta^3(\vec{r})\ \mathrm{d}V = -4\pi$ and $\iiint \nabla^2\frac{1}{r}\ \mathrm{d}V = -4\pi$. It is important to note that the integrands are only equal if the boundaries themselves do not alter the equality (i.e. integrals are equal for any choice of integration limits). This is justified from the radial independence discussed earlier. Another way to prove the equation is to rearrange the terms isolating the Delta function and to prove that the terms equating to the Delta function must behave like one. That is, we prove that

$$\lim_{\varepsilon\to 0} D(r,\varepsilon) \equiv \lim_{\varepsilon\to 0} -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\varepsilon^2}} = \delta^3(\vec{r}) \tag{45}$$

First, we should convert $D(r,\varepsilon)$ into a more elementary form (without the derivative). As usual, we should always check for symmetry so that we don't waste time. The function is angle-independent. Hence, its Laplacian in spherical coordinate must only contain the radial derivative term turning it into ordinary differentiation. That is,

$$D(r,\varepsilon) = -\frac{1}{4\pi r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\left(\frac{-r}{\sqrt{r^2+\varepsilon^2}}\right)\right) = \frac{1}{4\pi r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^3}{\sqrt{r^2+\varepsilon^2}}\right) \tag{46}$$

We can use the product rule to evaluate the derivative because why not.

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^3}{\sqrt{r^2+\varepsilon^2}}\right) = \frac{\mathrm{d}}{\mathrm{d}r}(r^3(r^2+\varepsilon^2)^{-\frac{3}{2}}) = r^3\left(-\frac{3}{2}(r^2+\varepsilon^2)^{-\frac{5}{2}} + 3r^2(r^2+\varepsilon^2)^{-\frac{3}{2}}\right) \tag{47}$$

$$= -3r^4(r^2+\varepsilon^2)^{-\frac{5}{2}} + 3r^2(r^2+\varepsilon^2)^{-\frac{3}{2}} \tag{48}$$

$$= \frac{3r^2}{(r^2+\varepsilon^2)^{\frac{5}{2}}}(-r^2+r^2+\varepsilon^2) = \frac{3\varepsilon^2 r^2}{(r^2+\varepsilon^2)^{\frac{5}{2}}} \tag{49}$$

Hence,

$$D(r,\varepsilon) = \frac{1}{4\pi}\frac{3\varepsilon^2}{(r^2+\varepsilon^2)^{\frac{5}{2}}} \tag{50}$$

With this resulting expression, we can proceed to analyze the behavior at the specified points if it indeed corresponds with a Dirac delta function. First, it must have an infinite spike at $r = 0$. Checking limiting behavior,

$$\lim_{\varepsilon\to 0} D(0,\varepsilon) = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\frac{3\varepsilon^2}{(\varepsilon^2)^{\frac{5}{2}}} = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\frac{3\varepsilon^2}{(\varepsilon^2)^{\frac{5}{2}}} = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\frac{3}{\varepsilon^3} = +\infty \tag{51}$$

As expected, there is a spike at $r = 0$. Next, we prove that it vanishes everywere when $r \neq 0$. Letting $r = c$ where $c$ is an arbitrary constant,

$$\lim_{\varepsilon\to 0} D(c,\varepsilon) = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\frac{3\varepsilon^2}{(c^2+\varepsilon^2)^{\frac{5}{2}}} = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\frac{3\varepsilon^2}{(c^2+\varepsilon^2)^{\frac{5}{2}}} = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\frac{3}{\varepsilon^3} = \frac{1}{4\pi}\frac{0}{c^2} = 0 \tag{52}$$

It, indeed, vanished. Lastly, it needs to have an area of unity if integrated over all of space. We will use spherical coordinates so imagine integrating an infinitely-sized sphere.

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\frac{3\varepsilon^2}{4\pi(r^2+\varepsilon^2)^{\frac{5}{2}}}r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{3\varepsilon^2}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\frac{r^2}{(r^2+\varepsilon^2)^{\frac{5}{2}}}\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{53}$$

Using Fubini's theorem to divide into iterated integrals,

$$\frac{3\varepsilon^2}{4\pi}\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty}\frac{r^2}{(r^2+\varepsilon^2)^{\frac{5}{2}}}\mathrm{d}r \tag{54}$$

$$= \frac{3\varepsilon^2}{4\pi}(2)(2\pi)\int_0^{\infty}\frac{r^2}{(r^2+\varepsilon^2)^{\frac{5}{2}}}\,\mathrm{d}r \tag{55}$$

$$= 3\varepsilon^2\int_0^{\infty}\frac{r^2}{(r^2+\varepsilon^2)^{\frac{5}{2}}}\,\mathrm{d}r \tag{56}$$

Using Eq. (13), this integral evaluates to the exact multiplicative inverse of its coefficient. Hence[1],

$$\iiint \lim_{\varepsilon\to 0} D(r,\varepsilon)\,\mathrm{d}r = \lim_{\varepsilon\to 0}\iiint D(r,\varepsilon)\,\mathrm{d}r = \lim_{\varepsilon\to 0} 1 = 1 \tag{57}$$

Clearly, $\lim_{\varepsilon\to 0} D(r,\varepsilon)$ has the properties of a Dirac delta function and the equation must hold. ■

[1] Here, I assumed the interchange of limit and integral to be justified and I assume that such rigorous proof is out of the coverage of the course.

4. **Vector Area**

Find $\vec{a} = \int_S \mathrm{d}\vec{a}$ if $S$ is a sphere.

Show that a closed surface must have a zero vector area and that vector areas only depends on the perimeter.

Specifically, show that it conforms to the equation

$$\vec{a} = \frac{1}{2}\oint \vec{r} \times \mathrm{d}\vec{l} \tag{58}$$

Show that the following identity holds

$$\oint (\vec{c} \cdot \vec{r})\, \mathrm{d}\vec{l} = \vec{a} \times \vec{c} \tag{59}$$

To find the area vector $\vec{a}$ of a sphere, we evaluate the integral $\int_S \mathrm{d}\vec{a}$. The differential vector area in spherical coordinate is

$$\mathrm{d}\vec{a} = \hat{r} r^2 \sin\theta \,\mathrm{d}\theta\, \mathrm{d}\phi \tag{60}$$

One possible pitfall is to assume that $\hat{r}$ is angle independent since $r$ is. Physical intuition, however, tells us that the direction of $\hat{r}$ changes as the angle changes. To express this dependence explicitly, we express $\hat{r}$ in cartesian coordinates.

$$\hat{r} = \sin\theta \cos\phi\, \hat{x} + \sin\theta \sin\phi\, \hat{y} + \cos\theta\, \hat{z} \tag{61}$$

Integrating a sphere and noting that r remains constant $r = R$,

$$\vec{a} = \int_0^{2\pi} \int_0^{\pi/2} r^2 \sin\theta\, (\sin\theta \cos\phi\, \hat{x} + \sin\theta \sin\phi\, \hat{y} + \cos\theta\, \hat{z}) \mathrm{d}\theta\, \mathrm{d}\phi \tag{62}$$

Thanks to our results in the preliminary section, we can first see that there are sin$\phi$ and cos$\phi$ integrand terms and that such terms must be zero. Hence, we only need to evaluate one integral!

$$\vec{a} = R^2 \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi/2} \sin\theta \cos\theta\, \mathrm{d}\theta \hat{z} \tag{63}$$

From Eq. (10),

$$\vec{a} = R^2 (2\pi) \left(\frac{1}{2}\right) \hat{z} = \pi R^2 \hat{z} \tag{64}$$

Note that we are after a hemispherical **bowl**. What we have calculated is actually an inverted bowl. Hence, by symmetry, we simply reverse the direction giving us

$$\vec{a} = -\pi R^2 \hat{z} \tag{65}$$

for a proper hemispherical bowl (the one used for storing soup). We proceed to prove that a closed surface must necessarily have a zero vector area. The key to the proof is that, for a closed surface, we can utilize the divergence theorem. Consider a constant vector field $\vec{c}$. Since it is constant, it must be divergenceless

$$\vec{\nabla} \cdot \vec{c} = 0 \tag{66}$$

Consider the closed surface integral of a constant vector field $\vec{c}$. Using divergence theorem,

$$\oiint \vec{c} \cdot \mathrm{d}\vec{A} \iff \iiint \vec{\nabla} \cdot \vec{c}\, \mathrm{d}V \tag{67}$$

But since $\vec{c}$ is divergenceless, LHS must be zero. It is important to note that the constant vector $\vec{c}$ can be thrown out of the integral. Since it is constant, it does not contribute to the integration of each component of the vector area (aside from scaling). Hence, we can recover the entire $\vec{c}$ after integration and so

$$\oiint \vec{c} \cdot \mathrm{d}\vec{a} = 0 \iff \vec{c} \cdot \oiint \mathrm{d}\vec{a} = 0 \iff \oiint \mathrm{d}\vec{a} = 0 \qquad \forall\, \vec{c} \tag{68}$$

Therefore, for any closed surface (encoded in the usage of divergence theorem), $\vec{a} = 0$ ■

Using this result, we can prove the sole dependence of vector area to the perimeter of the surface boundary. Observe two surfaces $S_1$ and $S_2$ sharing the same perimeter shown below.

insert figure

Adding or subtracting these surface will inevitable form a closed surface ($S = S_1 + S_2$ is a closed surface). We can, then, proceed to calculate the vector area of this closed surface as a sum of the two vector areas and noting that the vector area of a closed surface is zero.

$$\oint_S \mathrm{d}\vec{a} = \int_{S_1} \mathrm{d}\vec{a} - \int_{S_2} \mathrm{d}\vec{a} = \vec{0} \tag{69}$$

Hence, equation must hold if and only if

$$\int_{S_1} \mathrm{d}\vec{a} = \int_{S_2} \mathrm{d}\vec{a} \tag{70}$$

That is, the two surfaces must have the same vector area thus proving that vector areas must only depend on the perimeter of their boundaries. ■

Since we have already shown this dependence to the perimeter, we can go further and find a closed form expression of this dependence. The key to this derivation is to divide the surface into flat infinitesimal triangles with sides $\mathrm{d}\vec{l}$ and $\vec{r}$ shown below.

insert figure

Recall that the normal vector to the area spanned by $\mathrm{d}\vec{l}$ and $\vec{r}$ is given by a cross product. This is given by the vector whose magnitude is equal to the area of the parallelogram spanned by the two vectors.

The vector area of that parallelogram is given as

$$\mathrm{d}\vec{a}' = \vec{r} \times \mathrm{d}\vec{l} \tag{71}$$

Since we are interested on the triangular area, we simply divide $\vec{a}'$ by 2.

$$\mathrm{d}\vec{a} = \frac{1}{2}\vec{r} \times \mathrm{d}\vec{l} \tag{72}$$

To find the vector area of the entire surface, we simply evaluate a closed loop line integral about its perimeter using Eq. (72) and we get

$$\vec{a} = \oint \frac{1}{2}\vec{r} \times \mathrm{d}\vec{l} \tag{73}$$

and thus proving the relation of the vector area with its perimeter. ■

The last task is to prove the relation given by Eq. (59). Observing both sides of the equation, LHS is a line integral while RHS contains a cross product. It is tempting to use Stokes' theorem to prove the relation. However, upon further inspection, the equation is a vector equation while Stokes' theorem deals with a scalar equation. Instead, a variation of Stoke's theorem can be used. If $m$ is a scalar field, then

$$\oint m \,\mathrm{d}\vec{l} = \int \mathrm{d}\vec{a} \times \vec{\nabla} m = -\int \vec{\nabla} m \times \mathrm{d}\vec{a} \tag{74}$$

Letting $m = \vec{c} \cdot \vec{r}$, we find that

$$\oint \vec{c} \cdot \vec{r} \,\mathrm{d}\vec{l} = -\int \vec{\nabla}(\vec{c} \cdot \vec{r}) \times \mathrm{d}\vec{a} \tag{75}$$

Using the product rule for gradients

$$\vec{\nabla}(\vec{c}\cdot\vec{r}) = \vec{c}\times(\vec{\nabla}\times\vec{r}) + \vec{r}\times(\vec{\nabla}\times\vec{c}) + (\vec{c}\cdot\vec{\nabla})\vec{r} + (\vec{r}\cdot\vec{\nabla})\vec{c} \tag{76}$$

To find the first term, we first find the curl of the position vector. The curl in spherical coordinate is hideous. However, expressing this in cartesian coordinates, one can clearly see that since all components of the position vector is independent (no partial dependence between components), its curl must be zero. To find the second term, simply note that the curl of a constant field must be zero. To find the fourth term, simply note that this is a differentiated components of a constant field scaled with the corresponding component of $\vec{r}$ and, thus, must also be zero. Only the third term survives giving us

$$\vec{\nabla}(\vec{c}\cdot\vec{r}) = (\vec{c}\cdot\vec{\nabla})\vec{r} = c_x\frac{\mathrm{d}x}{\mathrm{d}x}\hat{x} + c_y\frac{\mathrm{d}y}{\mathrm{d}y}\hat{y} + c_z\frac{\mathrm{d}z}{\mathrm{d}z}\hat{z} = \vec{c} \tag{77}$$

The Eq. (75) now reads

$$\oint \vec{c}\cdot\vec{r}\,\mathrm{d}\vec{l} = -\int \vec{c}\times\mathrm{d}\vec{a} \tag{78}$$

The constant can be thrown out of the integral

$$\oint \vec{c}\cdot\vec{r}\,\mathrm{d}\vec{l} = -\vec{c}\times\int \mathrm{d}\vec{a} = -\vec{c}\times\vec{a} \tag{79}$$

By anti-commutativity of the cross product, we then prove that the identity

$$\oint \vec{c}\cdot\vec{r}\,\mathrm{d}\vec{l} = \vec{a}\times\vec{c} \tag{80}$$

holds true. ■