1 Problem Set 8: Addition of Angular Momenta

1. **Tensor Notation**

Introduce the tensor product.

We start by motivating the problem set tasks by going through the relevant Griffiths section again. This is due to some personal discomfort with the notations used. For instance, the problem set requires us to find the matrix elements of a "dot product", $\vec{S}_1 \cdot \vec{S}_2$, which is seemingly a one-dimensional scalar quantity. Moreover, Griffiths presented some weird operations such as

$$|\alpha\rangle\,|\beta\rangle \tag{1}$$

which I have had a hard time figuring out what it really means. [1] With these, we will use tensor notation to remove as much ambiguity as possible. A two-particle composite state shall be redefined as a tensor product

$$|s_1 s_2 m_1 m_2\rangle = |s_1 m_1\rangle \otimes |s_2 m_2\rangle \tag{2}$$

where an operator $\hat{Q}$ acting on each state is defined as

$$\hat{Q}^{(1)} = \hat{Q} \otimes \hat{I} \tag{3}$$

$$\hat{Q}^{(2)} = \hat{I} \otimes \hat{Q} \tag{4}$$

With these, we have the following relations

$$\hat{S}^2 \otimes \hat{I}(|s_1 m_1\rangle \otimes |s_2 m_2\rangle) = s_1(s_1+1)\hbar^2(|s_1 m_1\rangle \otimes |s_2 m_2\rangle) \tag{5}$$

$$\hat{I} \otimes \hat{S}^2(|s_1 m_1\rangle \otimes |s_2 m_2\rangle) = s_2(s_2+1)\hbar^2(|s_1 m_1\rangle \otimes |s_2 m_2\rangle) \tag{6}$$

$$\hat{S}_z \otimes \hat{I}(|s_1 m_1\rangle \otimes |s_2 m_2\rangle) = m_1\hbar(|s_1 m_1\rangle \otimes |s_2 m_2\rangle) \tag{7}$$

$$\hat{I} \otimes \hat{S}_z(|s_1 m_1\rangle \otimes |s_2 m_2\rangle) = m_2\hbar(|s_1 m_1\rangle \otimes |s_2 m_2\rangle) \tag{8}$$

Executing the operator $\hat{S}_z \otimes \hat{I} + \hat{I} \otimes \hat{S}_z$ gives us $m = m_1 + m_2$. Now, we focus on spin-1/2 particles. In terms of Griffiths arrow notations, the four possible states are

$$|\uparrow\uparrow\rangle := |\uparrow\rangle \otimes |\uparrow\rangle = \left|\frac{1}{2}\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\frac{1}{2}\right\rangle \qquad m = 1 \tag{9}$$

$$|\uparrow\downarrow\rangle := |\uparrow\rangle \otimes |\downarrow\rangle = \left|\frac{1}{2}\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\left(-\frac{1}{2}\right)\right\rangle \qquad m = 0 \tag{10}$$

$$|\downarrow\uparrow\rangle := |\downarrow\rangle \otimes |\uparrow\rangle = \left|\frac{1}{2}\left(-\frac{1}{2}\right)\right\rangle \otimes \left|\frac{1}{2}\frac{1}{2}\right\rangle \qquad m = 0 \tag{11}$$

$$|\downarrow\downarrow\rangle := |\downarrow\rangle \otimes |\downarrow\rangle = \left|\frac{1}{2}\left(-\frac{1}{2}\right)\right\rangle \otimes \left|\frac{1}{2}\left(-\frac{1}{2}\right)\right\rangle \qquad m = -1 \tag{12}$$

Throughout the problem set, I will use the three versions interchangeably. Recall that the ladder operators have the following relations

[1] The rumours are true. Griffiths EM is vastly superior that Griffiths QM.

$$S_+ |\downarrow\rangle = \hbar |\uparrow\rangle \tag{13}$$
$$S_- |\uparrow\rangle = \hbar |\downarrow\rangle \tag{14}$$
$$S_+ |\uparrow\rangle = S_- |\downarrow\rangle = 0 \tag{15}$$

Applying the total lowering operator $\hat{S}_- := \hat{S}_- \otimes \hat{I} + \hat{I} \otimes \hat{S}_-$ on $|\uparrow\uparrow\rangle$ gives us

$$(\hat{S}_- \otimes \hat{I} + \hat{I} \otimes \hat{S}_-) |\uparrow\rangle \otimes |\uparrow\rangle = \hbar(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) \tag{16}$$

Hence, after normalization, we claim that the triplet combination is expressed as

$$|11\rangle = |\uparrow\uparrow\rangle \tag{17}$$
$$|10\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \tag{18}$$
$$|1(-1)\rangle = |\downarrow\downarrow\rangle \tag{19}$$

This leads us to our first task.

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2. **Confirmation of Triplet State**

Confirm that $\hat{S}_- |10\rangle = \sqrt{2}\hbar |1(-1)\rangle$

Applying the total lowering operator on $|10\rangle$ as tensor products,

$$\hat{S}_- \otimes \hat{I} + \hat{I} \otimes \hat{S}_- \left(\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \right) = \frac{1}{\sqrt{2}} \left(\hat{S}_- |\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes \hat{S}_- |\uparrow\rangle \right) \tag{20}$$

From Eq. (14),

$$\hat{S}_- |10\rangle = \frac{1}{\sqrt{2}} (\hbar |\downarrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes \hbar |\downarrow\rangle) = \frac{2\hbar}{\sqrt{2}} |\downarrow\rangle \otimes |\downarrow\rangle \tag{21}$$

Hence, since $|1(-1)\rangle = |\downarrow\downarrow\rangle$, we confirm that

$$\boxed{\hat{S}_- |10\rangle = \sqrt{2}\hbar |1(-1)\rangle} \square \tag{22}$$

Meanwhile, the singlet state is expressed as

$$|00\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \tag{23}$$

Since this is supposedly as singlet state, obviously, ladder operators will reduce it to zero. This leads us to our second task.

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3. **Confirmation of Singlet State**

Confirm that $\hat{S}_\pm |00\rangle = 0$

Applying the total lowering operator on $|00\rangle$ as tensor products,

$$\hat{S}_- \otimes \hat{I} + \hat{I} \otimes \hat{S}_- \left(\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right) = \frac{1}{\sqrt{2}} \left(\hat{S}_- |\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes \hat{S}_- |\uparrow\rangle \right) \tag{24}$$

From Eq. (14),

$$\hat{S}_- |10\rangle = \frac{1}{\sqrt{2}} \left(\hbar |\downarrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes \hbar |\downarrow\rangle \right) = \frac{\hbar}{\sqrt{2}} |\downarrow\downarrow\rangle - |\downarrow\downarrow\rangle = 0 \tag{25}$$

Hence,

$$\hat{S}_- |00\rangle = 0 \tag{26}$$

Applying the total raising operator on $|00\rangle$ as tensor products,

$$\hat{S}_+ \otimes \hat{I} + \hat{I} \otimes \hat{S}_+ \left(\frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \right) = \frac{1}{\sqrt{2}} \left(\hat{S}_+ |\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes \hat{S}_+ |\uparrow\rangle \right) \tag{27}$$

From Eq. (15), raising a spin-up state destroys it. Since $0 \otimes |\downarrow\rangle = |\downarrow\rangle \otimes 0 = 0$,

$$\hat{S}_+ |00\rangle = 0 \tag{28}$$

Therefore, we conclude that

$$\boxed{\hat{S}_\pm |00\rangle = 0} \; \square \tag{29}$$

Griffiths claimed that two spin-1/2 particles can carry a total spin of 1 or 0 if and only if all triplet states are eigenvectors of $\hat{S}^2$ with eigenvalue $2\hbar^2$ and the singlet state is an eigenvector of $\hat{S}^2$ with eigenvalue 0. Fixing Griffiths notations, total $\hat{S}_T^2$ should have been written as

$$\hat{S}_T^2 = (\hat{S}^2 \otimes \hat{I}) + (\hat{I} \otimes \hat{S}^2) + 2(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z) \tag{30}$$

We can now see that what the dot product really meant was

$$\vec{S}_1 \cdot \vec{S}_2 = \hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z \tag{31}$$

Indeed, it does make sense since we are later tasked to find a 4x4 matrix - fitting description of the tensor product described here. Anyway, Griffiths already proved the claim that

$$\hat{S}_T^2 |10\rangle = 2\hbar^2 |10\rangle \tag{32}$$

It is now our duty to prove the two remaining triplet states.

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4. **Eigenstates of $\hat{S}_T^2$**

Show that $\hat{S}_T^2 |11\rangle = 2\hbar^2 |11\rangle$ and $\hat{S}_T^2 |1(-1)\rangle = 2\hbar^2 |1(-1)\rangle$

Recall the the spin component operator relation can be expressed as

$$\hat{S}_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle \qquad \hat{S}_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle \tag{33}$$

$$\hat{S}_y |\uparrow\rangle = \frac{i\hbar}{2} |\downarrow\rangle \qquad \hat{S}_y |\downarrow\rangle = -\frac{i\hbar}{2} |\uparrow\rangle \tag{34}$$

$$\hat{S}_x |\uparrow\rangle = \frac{\hbar}{2} |\downarrow\rangle \qquad \hat{S}_x |\downarrow\rangle = \frac{\hbar}{2} |\uparrow\rangle \tag{35}$$

$$\hat{S}^2 |\uparrow\rangle = \frac{3\hbar^2}{4} |\uparrow\rangle \qquad \hat{S}^2 |\downarrow\rangle = \frac{3\hbar^2}{4} |\downarrow\rangle \tag{36}$$

The states can be represented in terms of tensor products

$$|11\rangle = |\uparrow\rangle \otimes |\uparrow\rangle \qquad |1(-1)\rangle = |\downarrow\rangle \otimes |\downarrow\rangle \tag{37}$$

Executing the operator $\hat{S}_T^2$ on $|11\rangle$,

$$\hat{S}_T^2 |11\rangle = (\hat{S}^2 \otimes \hat{I}) + (\hat{I} \otimes \hat{S}^2) + 2(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z)(|\uparrow\rangle \otimes |\uparrow\rangle) \tag{38}$$

$$\hat{S}_T^2 |1(-1)\rangle = (\hat{S}^2 \otimes \hat{I}) + (\hat{I} \otimes \hat{S}^2) + 2(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z)(|\downarrow\rangle \otimes |\downarrow\rangle) \tag{39}$$

Observe that using Eqs. (33) to (36),

$$(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z)(|\uparrow\rangle \otimes |\uparrow\rangle) = \left(\frac{\hbar}{2}|\downarrow\rangle \otimes \frac{\hbar}{2}|\downarrow\rangle\right) + \left(\frac{i\hbar}{2}|\downarrow\rangle \otimes \frac{i\hbar}{2}|\downarrow\rangle\right) + \left(\frac{\hbar}{2}|\uparrow\rangle \otimes \frac{\hbar}{2}|\uparrow\rangle\right) \tag{40}$$

$$= \frac{\hbar^2}{4}|\uparrow\uparrow\rangle \tag{41}$$

Similarly,

$$(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z)(|\downarrow\rangle \otimes |\downarrow\rangle) = \left(\frac{\hbar}{2}|\uparrow\rangle \otimes \frac{\hbar}{2}|\uparrow\rangle\right) + \left(-\frac{i\hbar}{2}|\uparrow\rangle \otimes -\frac{i\hbar}{2}|\uparrow\rangle\right) + \left(-\frac{\hbar}{2}|\downarrow\rangle \otimes -\frac{\hbar}{2}|\downarrow\rangle\right) \tag{42}$$

$$= \frac{\hbar^2}{4}|\downarrow\downarrow\rangle \tag{43}$$

Moreover,

$$((\hat{S}^2 \otimes \hat{I}) + (\hat{I} \otimes \hat{S}^2))(|\uparrow\rangle \otimes |\uparrow\rangle) = \frac{3\hbar^2}{4}|\uparrow\uparrow\rangle + \frac{3\hbar^2}{4}|\uparrow\uparrow\rangle = \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle \tag{44}$$

Similarly,

$$((\hat{S}^2 \otimes \hat{I}) + (\hat{I} \otimes \hat{S}^2))(|\downarrow\rangle \otimes |\downarrow\rangle) = \frac{3\hbar^2}{4}|\downarrow\downarrow\rangle + \frac{3\hbar^2}{4}|\downarrow\downarrow\rangle = \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle \tag{45}$$

Plugging these results into Eqs. (38) and (39), we see that, indeed, $|11\rangle$ and $|1(-1)\rangle$ are both eigenkets of $\hat{S}_T^2$ with eigenvalues $2\hbar^2$

$$\hat{S}_T^2 |11\rangle = \frac{3\hbar^2}{2}|\uparrow\uparrow\rangle + \frac{\hbar^2}{2}|\uparrow\uparrow\rangle = 2\hbar^2 |\uparrow\uparrow\rangle \tag{46}$$

$$\hat{S}_T^2 |1(-1)\rangle = \frac{3\hbar^2}{2}|\downarrow\downarrow\rangle + \frac{\hbar^2}{2}|\downarrow\downarrow\rangle = 2\hbar^2 |\downarrow\downarrow\rangle \tag{47}$$

we see that, indeed, $|11\rangle$ and $|1(-1)\rangle$ are both eigenkets of $\hat{S}_T^2$ with eigenvalues $2\hbar^2$

$$\boxed{\hat{S}_T^2 |11\rangle = 2\hbar^2 |11\rangle} \qquad \boxed{\hat{S}_T^2 |1(-1)\rangle = 2\hbar^2 |1(-1)\rangle} \ \square \tag{48}$$

With this exercise, we have derived some important relations. Together with the results from Griffiths, the eigenvalues and eigenfunctions of the operator $\hat{S} \otimes \hat{S} := \hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z$ can be deduced from the first four following relations of form $A|x\rangle = \lambda |x\rangle$, where λ is the eigenvalue corresponding to the eigenstate $|x\rangle$.

$$\boxed{(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z)(|\uparrow\rangle \otimes |\uparrow\rangle) = \frac{\hbar^2}{4}|\uparrow\uparrow\rangle} \tag{49}$$

$$\boxed{(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z)(|\downarrow\rangle \otimes |\downarrow\rangle) = \frac{\hbar^2}{4}|\downarrow\downarrow\rangle} \tag{50}$$

$$\boxed{(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z)\left(\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\right) = \frac{\hbar^2}{4}\left(\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\right)} \tag{51}$$

$$\boxed{(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z)\left(\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\right) = -\frac{3\hbar^2}{4}\left(\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\right)} \tag{52}$$

$$\boxed{(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z)(|\uparrow\rangle \otimes |\downarrow\rangle) = \frac{\hbar^2}{4}(2\,|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle)} \tag{53}$$

$$\boxed{(\hat{S}_x \otimes \hat{S}_x + \hat{S}_y \otimes \hat{S}_y + \hat{S}_z \otimes \hat{S}_z)(|\downarrow\rangle \otimes |\uparrow\rangle) = \frac{\hbar^2}{4}(2\,|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle} \tag{54}$$

Using these composite states as basis, we can construct the matrix representation of $\hat{S} \otimes \hat{S}$. Note that since $|\uparrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ are all eigenfunctions of $\hat{S}_T^2$, they must be mutually orthogonal. Constructing the matrix representation in ordered basis set $\beta = \{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$

$$[\![\hat{S} \otimes \hat{S}]\!]_\beta = \begin{bmatrix} \langle\uparrow\uparrow|\,\hat{S}\otimes\hat{S}\,|\uparrow\uparrow\rangle & \langle\uparrow\uparrow|\,\hat{S}\otimes\hat{S}\,|\downarrow\downarrow\rangle & \langle\uparrow\uparrow|\,\hat{S}\otimes\hat{S}\,|\uparrow\downarrow\rangle & \langle\uparrow\uparrow|\,\hat{S}\otimes\hat{S}\,|\downarrow\uparrow\rangle \\ \langle\downarrow\downarrow|\,\hat{S}\otimes\hat{S}\,|\uparrow\uparrow\rangle & \langle\downarrow\downarrow|\,\hat{S}\otimes\hat{S}\,|\downarrow\downarrow\rangle & \langle\downarrow\downarrow|\,\hat{S}\otimes\hat{S}\,|\uparrow\downarrow\rangle & \langle\downarrow\downarrow|\,\hat{S}\otimes\hat{S}\,|\downarrow\uparrow\rangle \\ \langle\uparrow\downarrow|\,\hat{S}\otimes\hat{S}\,|\uparrow\uparrow\rangle & \langle\uparrow\downarrow|\,\hat{S}\otimes\hat{S}\,|\downarrow\downarrow\rangle & \langle\uparrow\downarrow|\,\hat{S}\otimes\hat{S}\,|\uparrow\downarrow\rangle & \langle\uparrow\downarrow|\,\hat{S}\otimes\hat{S}\,|\downarrow\uparrow\rangle \\ \langle\downarrow\uparrow|\,\hat{S}\otimes\hat{S}\,|\uparrow\uparrow\rangle & \langle\downarrow\uparrow|\,\hat{S}\otimes\hat{S}\,|\downarrow\downarrow\rangle & \langle\downarrow\uparrow|\,\hat{S}\otimes\hat{S}\,|\uparrow\downarrow\rangle & \langle\downarrow\uparrow|\,\hat{S}\otimes\hat{S}\,|\downarrow\uparrow\rangle \end{bmatrix} \tag{55}$$

Eliminating all zero entries due to orthogonality,

$$[\![\hat{S} \otimes \hat{S}]\!]_\beta = \begin{bmatrix} \langle\uparrow\uparrow|\,\hat{S}\otimes\hat{S}\,|\uparrow\uparrow\rangle & 0 & 0 & 0 \\ 0 & \langle\downarrow\downarrow|\,\hat{S}\otimes\hat{S}\,|\downarrow\downarrow\rangle & 0 & 0 \\ 0 & 0 & \langle\uparrow\downarrow|\,\hat{S}\otimes\hat{S}\,|\uparrow\downarrow\rangle & \langle\uparrow\downarrow|\,\hat{S}\otimes\hat{S}\,|\downarrow\uparrow\rangle \\ 0 & 0 & \langle\downarrow\uparrow|\,\hat{S}\otimes\hat{S}\,|\uparrow\downarrow\rangle & \langle\downarrow\uparrow|\,\hat{S}\otimes\hat{S}\,|\downarrow\uparrow\rangle \end{bmatrix} \tag{56}$$

Plugging in the relations from Eqs. (49),(50),(53), and (54),

$$[\![\hat{S} \otimes \hat{S}]\!]_\beta = \begin{bmatrix} \hbar^2/4 & 0 & 0 & 0 \\ 0 & \hbar^2/4 & 0 & 0 \\ 0 & 0 & \hbar^2/2 & -\hbar^2/4 \\ 0 & 0 & -\hbar^2/4 & \hbar^2/2 \end{bmatrix} \tag{57}$$

Hence, the matrix representation is expressed as

$$\boxed{[\![\hat{S} \otimes \hat{S}]\!]_\beta = \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}} \tag{58}$$