Will start to typeset all math module solutions here as soon as I'm done with the requirements for the semester. Linear algebra solutions start at page 5. I have some problems regarding the problem at page 15. The Jacobian is clearly a zero matrix at the fixed point but we were tasked to prove that it is neutrally stable.

1 Dimensional Analysis

1. **Infinitely long wires**

Two infinitely long wires running parallel to the x-axis carry uniform charge densities. Find the potential at any point (x, y, z) using the origin as reference

Plugging these to Eq. (**??**), the potential function in each region is, then,

$$V(r,\theta) = \begin{cases} 0 & r \leq a \\ \left(\gamma_c E_0 r + \frac{\gamma_d E_0}{r^2}\right)(\cos(\theta)) & a < r < b \\ -E_0 r \cos\theta + (\gamma_b E_0/r^2)(\cos(\theta)) & r > b \end{cases} \tag{1}$$

To find the electric field inside the dielectric, we simply take the gradient of the potential. Observe that all of the coefficients are position-independent. Taking the gradient inside the dielectric,

$$\vec{E} = -\vec{\nabla}V = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}\right)\left(\gamma_c E_0 r + \frac{\gamma_d E_0}{r^2}\right)(\cos(\theta)) \tag{2}$$

$$= -\cos(\theta)\frac{\partial}{\partial r}\left(\gamma_c E_0 r + \frac{\gamma_d E_0}{r^2}\right)\hat{r} - \left(\gamma_c E_0 + \frac{\gamma_d E_0}{r^3}\right)\frac{\partial}{\partial \theta}(\cos(\theta))\hat{\theta} \tag{3}$$

$$= -\left(\gamma_c E_0 - 2\frac{\gamma_d E_0}{r^3}\right)\cos(\theta)\hat{r} + \left(\gamma_c E_0 + \frac{\gamma_d E_0}{r^3}\right)\sin(\theta)\hat{\theta} \tag{4}$$

Hence, inside the dielectric, the electric field is

$$\boxed{\vec{E} = E_0\left(\left(2\frac{\gamma_d}{r^3} - \gamma_c\right)\cos(\theta)\hat{r} + \left(\gamma_c + \frac{\gamma_d}{r^3}\right)\sin(\theta)\hat{\theta}\right) \qquad a < r < b} \tag{5}$$

To find the total charge induced at the conductor, we first find the charge density and integrate it over the surface of the spherical conductor because that is where they reside. Recall that at $r = a$, the boundary condition reads

$$0 - \varepsilon\frac{\partial V}{\partial r} = -\sigma_f \tag{6}$$

Differentiating the potential inside the dielectric at $r = a$, we get

$$\sigma_f(\theta) = \varepsilon\left(2\frac{\gamma_d E_0}{a^3} - \gamma_c E_0\right)\cos\theta \tag{7}$$

Integrating at polar domain $[0, \pi]$, we have

$$Q = 2\pi\int_0^\pi \sigma_f(\theta)\ \mathrm{d}\theta = 2\pi\varepsilon\left(2\frac{\gamma_d E_0}{a^3} - \gamma_c E_0\right)\int_0^\pi \cos(\theta)\ \mathrm{d}\theta \tag{8}$$

Of course, the integral is zero and hence,

$$\boxed{Q = 0} \tag{9}$$

2 Perturbation Theory

1. **Infinitely long wires**

   Two infinitely long wires running parallel to the x-axis carry uniform charge densities. Find the potential at any point (x, y, z) using the origin as reference

2. **Infinitely long wires**

   Two infinitely long wires running parallel to the x-axis carry uniform charge densities. Find the potential at any point (x, y, z) using the origin as reference

3 Linear Algebra

1. **Change of Basis**

On $\mathbb{R}_2[t]$, consider the bases $\mathbb{B} = \{1, t, t^2\}$ and $\mathbb{D} = \{1, t-1, t^2-2t-1\}$. Let $L\vec{p}(t) = \vec{p}(1-t)$. Compute $P_{\mathbb{BD}}$, $P_{\mathbb{DB}}$, $[L]_{\mathbb{B}}$, $[L]_{\mathbb{D}}$, and verify $[L]_{\mathbb{D}} = P_{\mathbb{DB}}[L]_{\mathbb{B}}P_{\mathbb{BD}}$.

Our first task is to find the transition (change-of-basis) matrices from basis set $\mathbb{B}$ to basis set $\mathbb{D}$ and vice versa. We will proceed to find the latter directly by inversion. Letting the index denote the order in the set, we have

$$P_{\mathbb{BD}} = \begin{bmatrix} \langle \mathbb{B}_1 | \mathbb{D}_1 \rangle & \langle \mathbb{B}_1 | \mathbb{D}_2 \rangle & \langle \mathbb{B}_1 | \mathbb{D}_3 \rangle \\ \langle \mathbb{B}_2 | \mathbb{D}_1 \rangle & \langle \mathbb{B}_2 | \mathbb{D}_2 \rangle & \langle \mathbb{B}_2 | \mathbb{D}_3 \rangle \\ \langle \mathbb{B}_3 | \mathbb{D}_1 \rangle & \langle \mathbb{B}_3 | \mathbb{D}_2 \rangle & \langle \mathbb{B}_3 | \mathbb{D}_3 \rangle \end{bmatrix} \tag{10}$$

We can express the basis set $\mathbb{D}$ in terms of the elements of $\mathbb{B}$ as

$$\mathbb{D} = \{\mathbb{B}_1, \mathbb{B}_2 - \mathbb{B}_1, \mathbb{B}_3 - 2\mathbb{B}_2 - \mathbb{B}_1\} \tag{11}$$

and hence

$$P_{\mathbb{BD}} = \begin{bmatrix} \langle \mathbb{B}_1 | \mathbb{B}_1 \rangle & \langle \mathbb{B}_1 | \mathbb{B}_2 - \mathbb{B}_1 \rangle & \langle \mathbb{B}_1 |, \mathbb{B}_3 - 2\mathbb{B}_2 - \mathbb{B}_1 \rangle \\ \langle \mathbb{B}_2 | \mathbb{B}_1 \rangle & \langle \mathbb{B}_2 | \mathbb{B}_2 - \mathbb{B}_1 \rangle & \langle \mathbb{B}_2 |, \mathbb{B}_3 - 2\mathbb{B}_2 - \mathbb{B}_1 \rangle \\ \langle \mathbb{B}_3 | \mathbb{B}_1 \rangle & \langle \mathbb{B}_3 | \mathbb{B}_2 - \mathbb{B}_1 \rangle & \langle \mathbb{B}_3 |, \mathbb{B}_3 - 2\mathbb{B}_2 - \mathbb{B}_1 \rangle \end{bmatrix} \tag{12}$$

Exploiting orthornomality of basis $\mathbb{B}$ and distributivity of inner products, we get

$$P_{\mathbb{BD}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

and inverting gives us

$$P_{\mathbb{DB}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

Finding the matrix representation of the linear transformation L in both bases, we get

$$[L]_{\mathbb{B}} = \begin{bmatrix} \langle \mathbb{B}_1 | L | \mathbb{B}_1 \rangle & \langle \mathbb{B}_1 | L | \mathbb{B}_2 \rangle & \langle \mathbb{B}_1 | L | \mathbb{B}_3 \rangle \\ \langle \mathbb{B}_2 | L | \mathbb{B}_1 \rangle & \langle \mathbb{B}_2 | L | \mathbb{B}_2 \rangle & \langle \mathbb{B}_2 | L | \mathbb{B}_3 \rangle \\ \langle \mathbb{B}_3 | L | \mathbb{B}_1 \rangle & \langle \mathbb{B}_3 | L | \mathbb{B}_2 \rangle & \langle \mathbb{B}_3 | L | \mathbb{B}_3 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

$$[L]_{\mathbb{D}} = \begin{bmatrix} \langle \mathbb{D}_1 | L | \mathbb{D}_1 \rangle & \langle \mathbb{D}_1 | L | \mathbb{D}_2 \rangle & \langle \mathbb{D}_1 | L | \mathbb{D}_3 \rangle \\ \langle \mathbb{D}_2 | L | \mathbb{D}_1 \rangle & \langle \mathbb{D}_2 | L | \mathbb{D}_2 \rangle & \langle \mathbb{D}_2 | L | \mathbb{D}_3 \rangle \\ \langle \mathbb{D}_3 | L | \mathbb{D}_1 \rangle & \langle \mathbb{D}_3 | L | \mathbb{D}_2 \rangle & \langle \mathbb{D}_3 | L | \mathbb{D}_3 \rangle \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

It is easy to see that [1]

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

Indeed, $[L]_{\mathbb{D}} = P_{\mathbb{DB}}[L]_{\mathbb{B}}P_{\mathbb{BD}}$ holds. $\square$

[1] See Mathematica file "basis_change.nb" for "proof"

2. **Kernel**

Let C^∞ denote the space of infinitely differentiable functions on the real line. Let $L = \mathrm{d}^2/\mathrm{d}t^2 + 3\mathrm{d}/\mathrm{d}t + 2$ be an operator on $C^\infty(\mathbb{R})$. Find a basis for $\mathrm{Ker}(L)$.

The kernel of a linear transformation is the set of all vectors in a given vector space that gets reduced to a zero vector. If $y \in C^\infty(\mathbb{R})$ is a vector, then $\mathrm{Ker}(L)$=$\{y \in C^\infty(\mathbb{R}) | Ly = 0\}$. That is,

$$Ly = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2y = 0 \tag{18}$$

Solving for such vector is tantamount to solving the given homogeneous differential equation. As always, such equation demands an exponential ansatz $\exp(\lambda t)$ that leads to its characteristic equation

$$\lambda^2 + 3\lambda + 2 = 0 \iff (\lambda + 2)(\lambda + 1) = 0 \tag{19}$$

Plugging the solutions to the ansatz gives us $y = C_1 \ \exp\ (-2t) + C_2 \ \exp\ (-t)$ for any arbitrary coefficient. Hence,

$$\mathrm{Ker}(L) = \{y \in C^\infty(\mathbb{R}) | C_1 \ \exp\ (-2t) + C_2 \ \exp\ (-t)\} \tag{20}$$

or simply

$$\mathrm{Ker}(L) = \mathrm{Span}\{(\exp(-2t)), (\exp(-t))\} \tag{21}$$

from where we can see that the two-dimensional kernel of the vector space has basis $\mathbb{B}$ expressed as

$$\boxed{\mathbb{B} = \{(\exp(-2t)), (\exp(-t))\}} \tag{22}$$

3. **Diagonalization**

> Prove that three diagonalizable operators on a finite-dimensional vector space are simultaneously diagonalizable if and only if each pair of operators commutes.

Let A, B, and C be three simultaneously diagonalizable operators on a finite-dimensional vector space and $\vec{b}$ is a common eigenvector where

$$A\vec{b} = \lambda_a \vec{b} \qquad B\vec{b} = \lambda_b \vec{b} \qquad C\vec{b} = \lambda_c \vec{b} \tag{23}$$

Then, consecutive application of operators $C \to B \to A$ on $\vec{b}$ gives us

$$A(B(C\vec{b})) = \gamma_a \gamma_b \gamma_c \vec{b} \tag{24}$$

while $A \to B \to C$ on $\vec{b}$ gives us

$$C(B(A\vec{b})) = \gamma_c \gamma_b \gamma_a \vec{b} \tag{25}$$

Observe how the order of operators correspond to the order of eigenvalues. One can think of tying a string to the operators and their corresponding eigenvalue. Now, the key here is that scalars do commute, and hence, any permutation of eigenvalues must be equal such that

$$\gamma_a \gamma_b \gamma_c \vec{b} = \gamma_a \gamma_c \gamma_b \vec{b} = \gamma_b \gamma_a \gamma_c \vec{b} = \gamma_b \gamma_c \gamma_a \vec{b} = \gamma_c \gamma_a \gamma_b \vec{b} = \gamma_c \gamma_b \gamma_a \vec{b} \tag{26}$$

Applying our "string" arguments, all pairwise permutation of the operators must also be equal. Hence, if three operators are simultaneously diagonalizable, they must pairwise commute. Let $\vec{b}_a$ be an eigenvector of A, $\vec{b}_b$ be an eigenvector of B, $\vec{b}_c$ be an eigenvector of C. For the inverse proof, we utilize the theorem which states that two commuting operators must be diagonalizable. Let $\vec{b}_1$, i=1,2,3, be simultaneous eigenvectors where

$$AB\vec{b}_1 = BA\vec{b}_1 \qquad BC\vec{b}_2 = CB\vec{b}_2 \qquad AC\vec{b}_3 = CA\vec{b}_3 \tag{27}$$

Applying the third missing operator to each equation, we have

$$CAB\vec{b}_1 = CBA\vec{b}_1 \qquad ABC\vec{b}_2 = ACB\vec{b}_2 \qquad BAC\vec{b}_3 = BCA\vec{b}_3 \tag{28}$$

If all three operators were to pairwise commute, we will have that

$$CAB\vec{b}_1 = ABC\vec{b}_1 = BAC\vec{b}_1; \;\; CAB\vec{b}_2 = ABC\vec{b}_2 = BAC\vec{b}_2; \;\; CAB\vec{b}_3 = ABC\vec{b}_3 = BAC\vec{b}_3 \tag{29}$$

For such equality to hold, this demands that there exists a vector $\vec{b}$ such that $\vec{b} = \vec{b}_1 = \vec{b}_2 = \vec{b}_3$. This tells us that if three operators pairwise commute, they must be simultaneously diagonalizable. Hence, three operators are simultaneously diagonalizable if and only if they pairwise commute. □

Diagonalization

Prove that for any number of diagonalizable operators on a finite-dimensional vector space are simultaneously diagonalizable if and only if each pair of operators commutes.

Now, we proceed to generalize the previous to arbitrary number of operators. Let A_n $n = 1, 2, 3, ..., m$ be m diagonalizable operators on a finite-dimensional vector space. Then, if they are diagonalizable, we have

$$\prod_{n=1}^{m} A_n \vec{b} = \prod_{n=1}^{m} \lambda_n \vec{b} \tag{30}$$

But λ_n pairwise commute for all $n = 1, 2, 3, ..., m$. Hence, from our previous string arguments, A_n must commute for all $n = 1, 2, 3, ..., m$. To prove the inverse statement, we claim that $\exists k$ such that for two commuting operators,

$$A_j A_i \vec{b}_k = A_i A_j \vec{b}_k \qquad i = 1, ..., m \quad j = 1, ..., m \qquad i \neq j \quad k = 1, ..., m!/(m-2)!2! \tag{31}$$

where each k denotes the index of a pair of commuting operators. Multiplying by the remaining $m - 2$ operators,

$$\prod_{n=1, n\neq i,j}^{m} A_n \left(A_j A_i \vec{b}_k = A_i A_j \vec{b}_k \right) \iff \prod_{n}^{m} A_n \vec{b}_k = \prod_{n}^{m} A_n \vec{b}_k \tag{32}$$

It follows that $\forall k = 1, 2, 3, ..., m!/(m-2)!2!$, $\exists \vec{b}$ such that

$$\vec{b}_1 = \vec{b}_2 = ... = \vec{b}_{m!/(m-2)!2!} := \vec{b} \tag{33}$$

thus proving that any m pairwise commuting operators must be simultaneously diagonalizable. Hence, any arbitrary number of diagonalizable operators on a finite-dimensional vector space are simultaneously diagonalizable if and only if they pairwise commute. □

4. **Dynamics**

Griffins and Dragons live in the Enchanted Forest. The number of dragons D and griffins G in the forest each year is determined by the populations of the previous year according to the formulas:

$$D(n) = 1.5D(n-1) + G(n-1) \tag{34}$$
$$G(n) = D(n-1) \tag{35}$$

If in year 0, there are 25 dragons and no griffins, find the population in year k, analyze long term stability, and find limiting ratio.

The governing equations are a coupled first-order two-dimensional system of difference equations. It can be expressed in matrix form as

$$\vec{x}(n) = A\vec{x}(n-1) \tag{36}$$

where

$$A = \begin{bmatrix} 1.5 & 1 \\ 1 & 0 \end{bmatrix} \tag{37}$$

To solve the equations, we diagonalize the matrix A. However, Sadun has already provided us with a closed form expression for solving the system by decoupling. The solutions expressed as

$$\begin{bmatrix} D(n) \\ G(n) \end{bmatrix} := \vec{x}(n) = \sum_{i=1}^{m} \lambda_i^n a_i(0)\vec{b}_i \tag{38}$$

Then, the problem is tantamount to finding the eigenvalues and eigenvectors of A and expressing the initial condition in the eigenbasis to find $a_i(0)$. We will first find the eigenvalues, the eigenvectors, and construct the transition matrix. The characteristc polynomial of A reads

$$\left(\lambda + \frac{1}{2}\right)(\lambda - 2) = 0 \tag{39}$$

Hence, $\lambda_1 = -1/2$ and $\lambda_2 = 2$. Finding the eigenvectors, we get

$$\vec{b}_1 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{40}$$

and we find that the transition matrix is

$$T = \begin{bmatrix} -1/2 & 2 \\ 1 & 1 \end{bmatrix} \tag{41}$$

Transforming the initial condition, we find that

$$\begin{bmatrix} -1/2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 25 \\ 0 \end{bmatrix} = \begin{bmatrix} -25/2 \\ 25 \end{bmatrix} \tag{42}$$

Using Eq. (38),

$$\vec{x}(n) = \left(-\frac{1}{2}\right)^n \left(\frac{25}{2}\right) \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + 2^n(25) \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{43}$$

Extracting the Griffins and Dragons, we get

$$D(k) = \left(-\frac{1}{2}\right)^k \left(-\frac{25}{4}\right) + 2^k(50) \tag{44}$$

$$G(k) = \left(-\frac{1}{2}\right)^k \left(\frac{25}{2}\right) + 2^k(25) \tag{45}$$

Observe that they both have unstable modes from λ_2. Hence, both population must explode. However, taking the ratio D/N , Mathematica says that their ratio approaches 2. Furthermore, the matrix A has the following properties:

$$\text{Det } A = 1 \qquad \text{Tr } A = 1.5 > 0 \qquad (\text{Tr } A)^2 - 4 \text{ Det } A = -\frac{7}{4} < 0 \tag{46}$$

These values signify that the origin is an unstable spiral.

Dynamics

Consider the three coupled blocks illustrated below, with each block having mass $m = 1$ and each spring having spring constant $k = 1$. Write down the equations of motion for this coupled system. Find the normal modes and oscillation frequencies.

Hooke's law gives us the following system for $m = 1$, $k = 1$

$$\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} = -2x_1 + x_2 + 0x_3 \tag{47}$$

$$\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} = 1x_1 - 2x_2 + x_3 \tag{48}$$

$$\frac{\mathrm{d}^2x_3}{\mathrm{d}t^2} = 0x_1 + x_2 - 2x_3 \tag{49}$$

The corresponding matrix of the second-order continuous system is

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \tag{50}$$

The eigenvalues are $\lambda_1 = -2 - \sqrt{2}$, $\lambda_2 = -2$, $\lambda_3 = -2 + \sqrt{2}$. The corresponding eigenvectors are

$$\vec{b}_1 = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} \qquad \vec{b}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{b}_3 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} \tag{51}$$

These are the normal modes of our system. Observe that $\lambda_i < 0$ for all i. Hence, we expect that all modes are neutrally stable. Recall that, by decoupling, the closed form analytic solution for such system is

$$\sum_{i=1}^{m} a_i(t)\vec{b}_i \tag{52}$$

where $a_i(t)$ depends on the nature of λ. Specifically, for $\lambda_i < 0$,

$$a_i(t) = \cos(\sqrt{-\lambda_i}t)a_i(0) + \frac{\dot{a}_i(0)}{\sqrt{-\lambda_i}}\sin(\sqrt{-\lambda_i}t) \tag{53}$$

Hence, if $\vec{x} := (x_1, x_2, x_3)^T$, we can first define $\omega_i = \sqrt{-\lambda_i}$ where

$$\omega_1 = \sqrt{2 + \sqrt{2}} \qquad \omega_2 = \sqrt{2} \qquad \omega_3 = \sqrt{2 - \sqrt{2}} \tag{54}$$

These are the oscillation frequencies of our system. Then, we now have the complete analytic solution for the equations of motion

$$\vec{x} = \begin{bmatrix} \cos(\omega_2 t)a_1(0) + \frac{\dot{a}_1(0)}{\omega_1 t}\sin(\omega_1 t) - \cos(\omega_2 t)a_2(0) + \frac{\dot{a}_2(0)}{\omega_2 t}\sin(\omega_2 t) + \cos(\omega_3 t)a_3(0) + \frac{\dot{a}_3(0)}{\omega_3 t}\sin(\omega_3 t) \\ -\sqrt{2}\cos(\omega_2 t)a_1(0) + \frac{\dot{a}_1(0)}{\omega_1 t}\sin(\omega_1 t) + \sqrt{2}\cos(\omega_3 t)a_3(0) + \frac{\dot{a}_3(0)}{\omega_3 t}\sin(\omega_3 t) \\ \cos(\omega_2 t)a_1(0) + \frac{\dot{a}_1(0)}{\omega_1 t}\sin(\omega_1 t) + \cos(\omega_2 t)a_2(0) + \frac{\dot{a}_2(0)}{\omega_2 t}\sin(\omega_2 t) + \cos(\omega_3 t)a_3(0) + \frac{\dot{a}_3(0)}{\omega_3 t}\sin(\omega_3 t) \end{bmatrix} \tag{55}$$

Hence,

$$x_1(t) = \cos(\omega_2 t)a_1(0) + \frac{\dot{a}_1(0)}{\omega_1 t}\sin(\omega_1 t) - \cos(\omega_2 t)a_2(0) + \frac{\dot{a}_2(0)}{\omega_2 t}\sin(\omega_2 t) + \cos(\omega_3 t)a_3(0) + \frac{\dot{a}_3(0)}{\omega_3 t}\sin(\omega_3 t) \tag{56}$$

$$x_2(t) = -\sqrt{2}\cos(\omega_2 t)a_1(0) + \frac{\dot{a}_1(0)}{\omega_1 t}\sin(\omega_1 t) + \sqrt{2}\cos(\omega_3 t)a_3(0) + \frac{\dot{a}_3(0)}{\omega_3 t}\sin(\omega_3 t) \tag{57}$$

$$x_3(t) = \cos(\omega_2 t)a_1(0) + \frac{\dot{a}_1(0)}{\omega_1 t}\sin(\omega_1 t) + \cos(\omega_2 t)a_2(0) + \frac{\dot{a}_2(0)}{\omega_2 t}\sin(\omega_2 t) + \cos(\omega_3 t)a_3(0) + \frac{\dot{a}_3(0)}{\omega_3 t}\sin(\omega_3 t) \tag{58}$$

5. **Exploration: Difference Equation**

Analyze stability of the following system

$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x} \tag{59}$$

Plugging the exponential ansatz to the difference equation gives us

$$\lambda^{n+2} = 5\lambda^{n+1} - 6\lambda^n \iff \lambda^2 = 5\lambda - 6 \tag{60}$$

giving us the solution

$$x(n) = c_1 2^n + c_2 3^n \tag{61}$$

Reducing the order by letting $y(n) = x(n+1)$ and converting to a matrix equation,

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} \tag{62}$$

with characteristic equation

$$(\lambda - 2)(\lambda - 3) = 0 \tag{63}$$

From the solutions of this, $\lambda = 2$ and $\lambda = 3$, we obtain the similar soluton at Eq. (61). Generalizing this with an equation of form $x(n) = ax(n-1) + bx(n-2)$ gives a characteristic equation of form $\lambda^2 - a\lambda - b = 0$ with solution

$$\lambda_\pm = \frac{a}{2} \pm \sqrt{a^2 + 4b} \tag{64}$$

It turns out the reduction of order gives out a matrix of form

$$\begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \tag{65}$$

with the exact same characteristic polynomial, and hence, solution. Now, we are tasked to find a matrix whose characteristic polynomial is $\lambda^2 + a_1\lambda + a_0$ which is the so-called companion matrix of the polynomial. This is also tantamount to finding a constant coefficient difference equation with similar characteristic equation when fed by an exponential ansatz. Defining an arbitrary 2x2 matrix

$$\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \tag{66}$$

with characteristic polynomial $\lambda^2 - \lambda(m_1 + m_4) + m_1 m_4 - m_2 m_3$. If this is to be equal to the specified form composed by a_0 and a_0, this forms two equations for four unknowns

$$a_1 = -m_1 - m_4 \tag{67}$$

$$a_0 = m_4 m_4 - m_3 m_2 \tag{68}$$

This implies that we have two free parameters. Doing this similarly for a 3x3 matrix with elements

$$\begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix} \tag{69}$$

In this case, we have three equations for nine unknowns. Hence, there are six arbitrary parameters that we can freely set for convenience. Setting them to ones and zeroes for both 2x2 and 3x3 matrices, we get

$$\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & a_0 \\ 1 & 0 & a_1 \\ 0 & 1 & a_2 \end{bmatrix} \tag{70}$$

See Mathematica file "exp_diffeq.nb" for derivation.

6. **Stability**

Analyze stability of the following system

$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x} \tag{71}$$

We can quickly infer the stability of the normal modes from their corresponding eigenvalues. The characteristic polynomial $-\lambda^3 - 2\lambda$ gives us the following eigenvalues

$$\lambda_1 = -\sqrt{2} \qquad \lambda_2 = \sqrt{2} \qquad \lambda_3 = 0 \tag{72}$$

This system has a neutrally stable mode $\vec{b}_1 = (1, -\sqrt{2}, 1)^T$ from λ_1, an unstable mode $\vec{b}_2 = (1, \sqrt{2}, 1)^T$ from λ_2, and a stable mode $\vec{b}_3 = (-1, 0, 1)^T$ from λ_3.

Let $\vec{x}(n) = A\vec{x}(n-1)$ with A a real 2x2 matrix whose determinant equals one. Show that the system is neutrally stable if -2<Tr A<2 but is unstable if |Tr(A)|>2.

Let A be a 2x2 real matrix with entries

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{73}$$

Then, the characteristic polynomial is given as

$$\lambda^2 - \lambda(\text{Tr } A) + \text{Det } A = 0 \tag{74}$$

Since matrix has unit determinant, using quadratic formula gives us

$$\lambda_\pm = \frac{\text{Tr}A}{2} \pm \frac{\sqrt{(\text{Tr } A)^2 - 4}}{2} \tag{75}$$

Observe that at the open interval (-2,2), λ is complex with imaginary part $i/2\sqrt{4 - (\text{Tr } A)^2}$. Hence, in this interval, the system is neutrally stable (oscillatory). Outside the interval (-2,2), $\lambda_+ > 0$. In the case of λ_-, we claim that $\lambda_- > 0$. Then,

$$\text{Tr } A\sqrt{(\text{Tr } A)^2 - 4} \iff (\text{Tr } A)2 > (\text{Tr })^2 - 4 \iff 0 > -4 \tag{76}$$

Hence, outside the open interval (-2,2), the system must be unstable for all modes.

Find a criterion involving p_1, p_2, k that will determine whether the deer population eventually grows, stabilizes, or shrinks to zero.

$$\vec{x}(n) = \begin{bmatrix} p_1 & k \\ p_1 & p_2 \end{bmatrix} \tag{77}$$

We are tasked to analyze the long term stability of the system and find relevant criteria. From Eq. (74), we see that

$$\lambda_{\pm} = \frac{\text{Tr}A}{2} \pm \frac{\sqrt{(\text{Tr } A)^2 - 4 \text{ Det } A}}{2} \tag{78}$$

Finding relevant quantities,

$$\text{Tr } A = p_1 + p_2 \qquad \text{Det } A = p_1(p_2 - k) \tag{79}$$

Then,

$$\lambda_{\pm} = \frac{p_1 + p_2}{2} \pm \frac{\sqrt{(p_1 + p_2)^2 - 4p_1(p_2 - k)}}{2} \tag{80}$$

From this expression, some observations can be made. The system has a positive trace. Hence, it can never achieve stability. A negative determinant requires that

$$k > p_2 \tag{81}$$

In this case, the origin is a saddle point. If the determinant is positive, it requires that

$$p_2 > k \tag{82}$$

and the system can either be an unstable node or an unstable spiral. For the fixed point to be an unstable node, the discriminant must be positive. That is,

$$(p_1 + p_2)^2 - 4p_1(p_2 - k) > 0 \tag{83}$$

For the fixed point to be an unstable spiral, the discriminant must be negative. That is,

$$(p_1 + p_2)^2 - 4p_1(p_2 - k) < 0 \tag{84}$$

One can plot the boundary region between spirals and nodes as a cone. These results demand that we restrict our domain to positive parameters. We can further generalize the criterion where we can include cannibalism or interspecies predation and the parameters can sometimes be negative. The following statements list stability criteria:

- a saddle point is assured if $p_1(p_2 - k) < 0$
- If $p_1(p_2 - k) > 0$, system will have unstable nodes if $p_1 + p_2 < 0$ and stable node if $p_1 + p_2 > 0$
- If $p_1(p_2 - k) < 0$, system will have unstable spiral if $p_1 + p_2 < 0$, stable spiral if $p_1 + p_2 > 0$, and a center if $p_1 = -p_2$

With these, we can conclude that population shrinks if $p_1 > p_2$, explodes if $p_1(p_2 - k) < 0$ or $p_1 < p_2$, and stabilizes (Liaponuv stable) if $p_1 = -p_2$.

Consider the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1^2 + x_2^2)\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \tag{85}$$

Show that the linearization at the fixed point $(0,0)^T$ is neutrally stable but some solutions to the nonlinear system actually grow with time.

The Jacobian of the system is

$$\begin{bmatrix} -2x_1x_2 & -x_1^2 - 3x_2^2 \\ 3x_1^2 + x_2^2 & 2x_1x_2 \end{bmatrix} \tag{86}$$

at the fixed point $(0,0)^T$, the Jacobian is a zero matrix.

Prove the Tennis Racket Theorem from the following Euler equations

$$\frac{d\omega_1}{dt} = \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \tag{87}$$
$$\frac{d\omega_2}{dt} = \frac{I_3 - I_1}{I_2}\omega_3\omega_1 \tag{88}$$
$$\frac{d\omega_3}{dt} = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 \tag{89}$$
$$\tag{90}$$

For convenience, we define the angular momenta factor as

$$\frac{d\omega_1}{dt} = A_1\omega_2\omega_3 \tag{91}$$
$$\frac{d\omega_2}{dt} = A_2\omega_3\omega_1 \tag{92}$$
$$\frac{d\omega_3}{dt} = A_3\omega_1\omega_2 \tag{93}$$
$$\tag{94}$$

The fixed points are

$$\begin{bmatrix} n_1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ n_2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ n_3 \end{bmatrix} \tag{95}$$

where $n_1 > 0$, $i = 1, 2, 3$ corresponding to constant ω_i. Physically, the fixed points represent that the system is rotating on a single axis with constant angular velocity n_i. At these fixed points, we only need to analyze the remaining two axes for stability. The corresponding Jacobians are

$$J_1 = \begin{bmatrix} 0 & A_2 n_1 \\ A_3 n_1 & 0 \end{bmatrix} \qquad J_2 = \begin{bmatrix} 0 & A_1 n_2 \\ A_3 n_2 & 0 \end{bmatrix} \qquad J_3 = \begin{bmatrix} 0 & A_1 n_3 \\ A_2 n_3 & 0 \end{bmatrix} \tag{96}$$

Observe that for all axis, all Jacobians are traceless. Hence, it would either be a stable center fixed point or an unstable saddle point. The governing criterion is the sign of its determinant. If Det J is positive, fixed point is a center (neutrally stable oscillation). If Det J is negative, fixed point is an unstable saddle. Finding the determinants

$$\text{Det} J_1 = -n_1^2 \frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3} \tag{97}$$

$$\text{Det} J_2 = -n_2^2 \frac{(I_2 - I_3)(I_2 - I_3)}{I_3 I_1} \tag{98}$$

$$\text{Det} J_3 = -n_3^2 \frac{(I_2 - I_3)(I_3 - I_1)}{I_2 I_1} \tag{99}$$

The condition $I_1 > I_2 > I_3 > 0$ implies that Det $J_1 > 0$, Det $J_2 < 0$, and Det $J_3 > 0$. Hence, at fixed points represent constant rotation about axis 1 and axis 3, the system is a center, and hence, neutrally stable. At axis 2, however, the system is an unstable saddle point. This proves the Tennis Racket theorem. □

7. **Exploration: Nonlinear ODEs**

Analyze the following system

$$\frac{\mathrm{d}}{\mathrm{d}t} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_1^2 - x_1 x_2 \\ 3x_2 - 2x_1 x_2 - x_2^2 \end{bmatrix} \tag{100}$$

The Jacobian of the system is

$$\begin{bmatrix} 2 - 2x_1 - x_2 & -x_1 \\ -2x_2 & 3 - 2x_1 - 2x_2 \end{bmatrix} \tag{101}$$

From the vector plot of the system, we get the following fixed points: $\vec{x}_1 = (0,0)^T$, $\vec{x}_2 = (0,3)^T$, $\vec{x}3 = (2,0)^T$, and $\vec{x}_4 = (1,1)^T$. The respective Jacobian matrices are

$$J_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad J_2 = \begin{bmatrix} 1 & 0 \\ -6 & -3 \end{bmatrix} \qquad J_3 = \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix} \qquad J_4 = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \tag{102}$$

At the first fixed point, Det $J_1 = 6$, Tr $J_1 = 5$, and (Tr J_1)^2-4 Det $J_1 = 1$. This is an unstable node. At the second fixed point, Det $J_2 = -3$, Tr $J_2 = -2$, and (Tr J_2)^2-4 Det $J_2 = 16$. This is a saddle point. At the third fixed point, Det $J_3 = 2$, Tr $J_3 = -3$, and (Tr J_3)^2-4 Det $J_3 = 1$. This is a stable node. At the fourth fixed point, Det $J_4 = -1$, Tr $J_4 = -2$, and (Tr J_4)^2-4 Det $J_4 = 8$. This is a saddle point. These can all be seen from the Mathematical file "exp_nonlinear.nb".

4 Complex Analysis

1. **Change of Basis**

   On $\mathbb{R}_2[t]$, consider the bases $\mathbb{B} = \{1, t, t^2\}$ and $\mathbb{D} = \{1, t-1, t^2 - 2t - 1\}$. Let $L\vec{p}(t) = \vec{p}(1-t)$. Compute $P_{\mathbb{BD}}$, $P_{\mathbb{DB}}$, $[L]_{\mathbb{B}}$, $[L]_{\mathbb{D}}$, and verify $[L]_{\mathbb{D}} = P_{\mathbb{DB}}[L]_{\mathbb{B}}P_{\mathbb{BD}}$.