1 Problem Set 4: Electric Fields in Matter

1. **Dipole near a grounded conducting plane**

> A perfect dipole $\vec{p}$ is situated a distance z above an infinite grounded conducting plane. The dipole makes an angle θ with the perpendicular to the plane. Find the torque and stable orientation of the dipole.

We are tasked to find the torque and stable configuration of an ideal dipole near the vicinity of a grounded conducting plane. The system is given by the following figure.

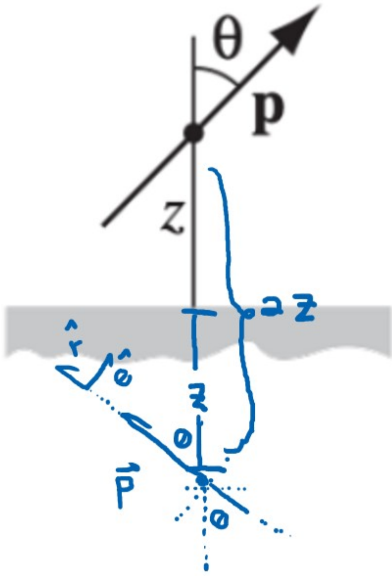

Figure 1: Due to symmetry of cosine, the image is not a total additive inverse of the original dipole vector for convenience of calculations.

Recall that we are already given the dipole moment and, hence, the potential configuration is simply

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^2} \tag{1}$$

But wait, there is an infinite conducting plane nearby. The dipole will induce some field in the nearby plane and will alter the net potential and field configuration. Recall that we have already tackled this same problem before - the classic image problem. Finding the potential is tantamount to solving a boundary value problem using Laplace's equation and the way to go around this infinite plane scenario is conjuring an equivalent image of the system. One simply replaces the entire plane with a charge (in this case, a dipole) of opposite sign equidistant with the symmetry axis. Since the dipole moment $\vec{p}$ was defined as

$$\vec{p} := \int \vec{r}\rho(\vec{r})\ \mathrm{d}\tau' \tag{2}$$

Flipping the sign of the entire charge distribution will flip the sign of the dipole moment vector

$$\int \vec{r}(-\rho(\vec{r}))\ \mathrm{d}\tau' = -\vec{p} \tag{3}$$

Hence, we have solved the auxillary problem by placing a dipole at -z with dipole moment $\vec{p}$. Now, the problem is reduced to the following statement: What is the torque on the dipole caused by the image dipole? The torque on a dipole, such as the alignment of a polar molecule, caused by a field is expressed as

$$\vec{N} = \vec{p} \times \vec{E} \tag{4}$$

But $\vec{E}$ is simply the field of the image dipole $\vec{E}_{\text{im}}$. Luckily, we have already derived the field of a dipole in the previous chapter. It is expressed (in spherical coordinates) as

$$\vec{E}(\vec{r}) = \frac{p}{4\pi\varepsilon_0 r^3}(2\ \cos\theta\hat{r} +\ \sin\theta\hat{\theta}) \tag{5}$$

From the figure above, the distance of the two dipoles is $r = 2z$. Hence, at the location of the main dipole, the field of the image dipole is

$$\vec{E}_{\text{im}} = \frac{p}{32\pi\varepsilon_0 z^3}(2\ \cos\theta\hat{r} +\ \sin\theta\hat{\theta}) \tag{6}$$

Hence, the torque on the main dipole is

$$\vec{N} = \vec{p} \times \left(\frac{p}{32\pi\varepsilon_0 z^3}(2\ \cos\theta\hat{r} +\ \sin\theta\hat{\theta})\right) \tag{7}$$

Of course, in the process, we have moved the origin into the image dipole. One might argue "Wouldn't this change the representation of dipole moment?". Well, yes but it doesn't matter. The symmetry of the the situation assures that whatever $\vec{p}$ is, it must follow that the image must have an additive inverse of $\vec{p}$. Expanding $\vec{p}$ in spherical coordinates (by reorienting the coordinate system such that $\vec{p}_{\text{im}}$ point towards the +z axis,

$$\vec{p} = p\ \cos\ \theta\hat{r} + p\ \sin\ \theta\hat{\theta} = p(\cos\ \theta\hat{r} + \sin\ \theta\hat{\theta}) \tag{8}$$

Updating the torque expression,

$$\begin{aligned}
\vec{N} &= p\left(\cos\ \theta\hat{r} + \sin\ \theta\hat{\theta}\right) \times \left(\frac{p}{24\pi\varepsilon_0 z^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})\right) && (9)\\
&= \frac{p^2}{24\pi\varepsilon_0 z^3}((\cos\ \theta\hat{r} + \sin\ \theta\hat{\theta}) \times (2\cos\theta\hat{r} + \sin\theta\hat{\theta}))) && (10)\\
&= -\frac{p^2}{32\pi\varepsilon_0 z^3}(\sin\theta\cos\theta\hat{\phi}) && (11)
\end{aligned}$$

where the cross product can be executed quickly from the determinant definition. Hence, the torque on the dipole is

$$\vec{N} = -\frac{p^2}{32\pi\varepsilon_0 z^3}(\sin\theta\ \cos\theta)\hat{\phi} \tag{12}$$

By right-hand rule, this points out of the plane. This expression is a product of out-of-phase periodic functions. It would be more insightful to express them as a single function. Recall that $\sin\alpha\ \cos\beta = 1/2(\sin(\alpha+\beta) + \sin(\alpha-\beta))$. Hence,

$$\sin\theta\ \cos\theta = \frac{1}{2}\ \sin 2\theta \tag{13}$$

Then,

$$\boxed{\vec{N} = -\frac{p^2\ \sin 2\theta}{64\pi\varepsilon_0 z^3}\hat{\phi}} \tag{14}$$

We have two minima: at $\theta = 0$ and $\theta = \pi$ corresponding to zero torque, and hence, two stable points. To see this, note that $N \propto \sin 2\theta$. It follows N points out of the page for quadrant I and quadrant III while N points into the page for quadrant II and quadrant IV. These changes in sign occur at $\theta = 0$ and $\theta = \pi$.

$$\boxed{\text{If initial angle is at domain } [0, \pi/2)\text{, dipole generally stabilizes at } \theta = 0} \tag{15}$$

$$\boxed{\text{If initial angle is at } [\pi, 3\pi/2)\text{, it generally stabilizes at } \theta = \pi} \tag{16}$$

Observe that we've excluded points $\theta = \pi/2$ and $\theta = 3\pi/2$ because they are unstable fixed points and where they will eventually rest depends on the perturbation direction.

2. **A cavity enclosed by a dielectric**

The figure below shows a spherical cavity scooped out of an infinite medium with dielectric constant ε_r. If the whole system is immersed in uniform external field $\vec{E_0} = E_0\hat{z}$, find the potential and field everywhere.

We have already tackled this kind of problem in the previous chapter - solving Laplace's equation given the following boundary conditions. For instance, we have already shown that $V = -E_0 r\cos\theta$ for such uniform external field configuration (see problem set 3 item 3).

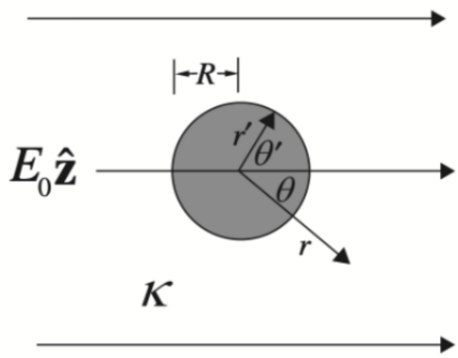

Figure 2: A cavity in an infinite dielectric medium with constant electric field

However, the boundary conditions with linear dielectrics pose slight variations. The potential is continuous at the boundaries but their derivatives must conform to the following relation

$$\varepsilon_{\text{in}}\frac{\partial V_{\text{in}}}{\partial n} - \varepsilon_{\text{out}}\frac{\partial V_{\text{out}}}{\partial n} = -\sigma_f \tag{17}$$

where σ_f is the charge density of free charge. In this problem, we have a spherical cavity. Hence, $\sigma_f = 0$. Moreover, inside, $\varepsilon_{\text{in}} = \varepsilon_0$ and, outside, we let $\varepsilon_{\text{out}} = \varepsilon$. Recalling the solution for Laplace's equation in spherical coordinates, inside the sphere,

$$V_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta)) \tag{18}$$

and outside the sphere,

$$V_{\text{out}}(r,\theta) = \sum_{l=0}^{\infty}\left(C_l r^l + \frac{D_l}{r^{l+1}}\right)P_l(\cos(\theta)) \tag{19}$$

Subject to the following boundary conditions

$$\begin{cases} V_{\text{in}} = V_{\text{out}} & r = R \\ \varepsilon_0(\partial V_{\text{in}}/\partial n) = \varepsilon(\partial V_{\text{out}}/\partial n) & r = R \\ V_{\text{out}} \to -E_0 r\cos\theta & r >> R \end{cases} \tag{20}$$

We start by cleaning up the expressions from the following observations. Outside, V_{out} can't explode. Hence, $C_l = 0$. Inside, V_{in} can't explode. Hence, $B_l = 0$. Moreover, it must approach a certain limiting case expressed in the third condition. Hence, updating our potential expression,

$$V(r,\theta) = \begin{cases} V_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta)) & r < R \\ V_{\text{out}} = -E_0 r\cos\theta + \sum_{l=0}^{\infty}(D_l/(r^{l+1}))P_l(\cos(\theta)) & r > R \end{cases} \tag{21}$$

Continuity condiition, $V(R,\theta) = V_{\text{in}}|_R = V_{\text{out}}|_R$ gives us

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos(\theta)) = -E_0 R\cos\theta + \sum_{l=0}^{\infty} \frac{D_l}{R^{l+1}} P_l(\cos(\theta)) \tag{22}$$

Eyeballing, due to orthonormality of the Legendre polynomials, equation only holds if summation only "turns on" at $l = 1$ and "turns off" otherwise. To see this clearly, we can rearrange the equation as

$$\left(A_1 R - \frac{D_1}{R^2}\right) \cos\theta + \sum_{l=0, l\neq 1}^{\infty} \left(A_l R^l P_l - \frac{D_l}{R^{l+1}}\right) P_l(\cos(\theta)) = -E_0 R\cos\theta \tag{23}$$

The summation must be disposed and the remaining terms must be equal. This translates to the following conditions

$$\begin{cases} A_l R^l = D_l / R^{l+1} & l \neq 1 \\ A_1 R = D_1/R^2 - E_0 R \end{cases} \tag{24}$$

Next, we invoke the derivative condition at $r = R$ by differentiating Eq. (21). This gives us

$$\varepsilon_0 \left(\sum_{l=0}^{\infty} l A_l R^{l-1} P_l \cos\theta\right) = \varepsilon \left(-E_0 \cos\theta - \sum_{l=0}^{\infty} \frac{(l+1)D_l}{R^{l+2}} P_l \cos\theta\right) \tag{25}$$

With similar arguments to the continuity equation, we rewrite the expression as

$$\sum_{l=0}^{\infty} \left(\varepsilon_0 l A_l R^{l-1} + \varepsilon \frac{(l+1)D_l}{R^{l+2}}\right) P_l \cos\theta = -E_0 \varepsilon \cos\theta \tag{26}$$

which gives us the following condition

$$\begin{cases} \varepsilon_0 l A_l R^{l-1} = -\varepsilon(l+1) D_l / R^{l+2} & l \neq 1 \\ \varepsilon_0 A_1 = -\varepsilon(E_0 + 2D_1/R^3) \end{cases} \tag{27}$$

Now, observe the first conditions at both Eqs. (24) and (27). The two R terms are linearly independent $\forall\, l$. Hence, $A_l = B_l = 0 \;\forall\, l \neq 1$. The remaining surviving coefficients can be easily found from the second conditions of Eqs. (24) and (27). This gives us two equations for two unknowns:

$$\begin{cases} A_1 R = D_1/R^2 - E_0 R \\ \varepsilon_0 A_1 = -\varepsilon(E_0 + 2D_1/R^3) \end{cases} \tag{28}$$

Solving the linear system

$$\begin{bmatrix} R & -\frac{1}{R^2} \\ \varepsilon_0 & \frac{2\varepsilon}{R^3} \end{bmatrix} \begin{bmatrix} A_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} -E_0 R \\ -E_0 \varepsilon \end{bmatrix} \tag{29}$$

Inversion gives us

$$\begin{bmatrix} A_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} -\frac{3\varepsilon E_0}{\varepsilon_0 + 2\varepsilon} \\ \frac{(\varepsilon_0 - \varepsilon) R^3 E_0}{\varepsilon_0 + 2\varepsilon} \end{bmatrix} \tag{30}$$

Plugging this into Eq. (21), we have

$$V(r, \theta) = \begin{cases} \alpha_{\text{in}} E_0 r\cos(\theta) & r < R \\ \alpha_{\text{out}} E_0 r\cos(\theta) & r > R \end{cases} \tag{31}$$

Converting to relative permittivity [1], since $\varepsilon = \varepsilon_0 \varepsilon_r$, we have

[1] I already have alot of opportunities to do this earlier. For some reason, I didn't. It is not that simplified but is consistent nonetheless.

$$\alpha_{\text{in}} := -\frac{3\varepsilon_0\varepsilon_r}{\varepsilon_0 + 2\varepsilon_0\varepsilon_r} \tag{32}$$

$$\alpha_{\text{out}} := -\left(\frac{R^3}{r^3}\right)\frac{3\varepsilon_0\varepsilon_r}{\varepsilon_0 + 2\varepsilon_0\varepsilon_r} \iff \frac{\mathrm{d}\alpha_{\text{out}}}{\mathrm{d}r} = 3\left(\frac{R^3}{r^4}\right)\frac{3\varepsilon_0\varepsilon_r}{\varepsilon_0 + 2\varepsilon_0\varepsilon_r} = -3\frac{\alpha_{\text{out}}}{r} \tag{33}$$

where we've differentiated the position dependent α in anticipation to finding the electric field. Taking the gradient of the potential to obtain the field,

$$\vec{E} = -\vec{\nabla}V = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{\partial}{\partial\theta}\hat{\theta}\right)V \tag{34}$$

Taking the r derivatives,

$$\frac{\partial}{\partial r}\left(\alpha_{\text{in}}E_0 r\cos(\theta)\right) = \alpha_{\text{in}}E_0\cos(\theta) \tag{35}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(\alpha_{\text{out}}E_0 r\cos(\theta)\right) &= \alpha_{\text{out}}E_0\cos(\theta) + \frac{\mathrm{d}\alpha_{\text{out}}}{\mathrm{d}r}E_0 r\cos(\theta) &(36)\\ &= -2\alpha_{\text{out}}E_0\cos(\theta) &(37)\end{aligned}$$

The θ derivatives are trivial where cosine simply transforms to negative sine. Furthermore, if we define $\alpha = \alpha_{\text{in}} = \gamma_r(r)\alpha_{\text{out}}$, where $\gamma_r(r) := R^3/r^3$, the cubic ratio of distance, then

$$\boxed{\vec{E}(r,\theta) = \begin{cases} -\alpha E_0(\cos\,\theta\hat{r} + \sin\,\hat{\theta}) & r < R \\ -\alpha\gamma(r)E_0(-2\cos\,\theta\hat{r} + \sin\,\hat{\theta}) & r > R \end{cases}} \tag{38}$$

where the potential is expressed as

$$\boxed{V(r,\theta) = \begin{cases} \alpha E_0 r\cos(\theta) & r < R \\ \alpha\gamma_r(r)E_0 r\cos(\theta) & r > R \end{cases}} \tag{39}$$

In these expressions, α is a pure constant and we can clearly see the dependence in position of the quantities. Also, in this form, except for E_0 and r, all other quantities are dimensionless. It is easy to see that all expressions of $\vec{E}$ have dimensions of E_0, which is an electric field quantity, and all expressions of V have dimensions of $E_0 r$, which has units of electrostatic potential.

3. **Capacitor with dielectric**

The space between the plates of a parallel-plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness a, so the the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2 and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate $-\sigma$.

a) Find the electric displacement in each slab.
b) Find the electric field in each slab.
c) Find the polarization in each slab.
d) Find the potential difference between the plates
e) Find the location and amount of all abound charges
f) Check consistency of item b and item e.

We are tasked to analyze various quantities of a parallel capacitor with dielectric "fillings".

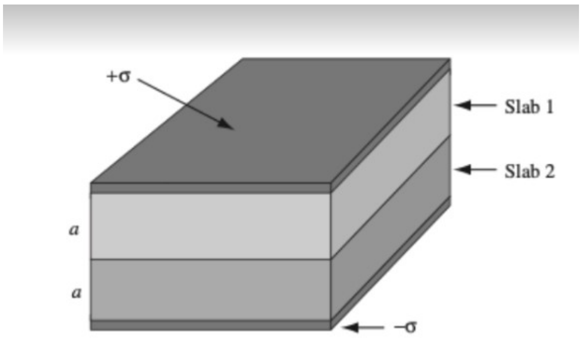

Figure 3: Inserting dielectric in a capacitor is a common way to increase capacitance

The first task is to find the electric displacement in each slab. Recall that we have derived an analogous version for Gauss's law, This time, it already includes all effects of polarization. It is expressed as

$$\oint \vec{D} \cdot \, \mathrm{d}\vec{a} = Q_{f_{\text{enc}}} \tag{40}$$

where we have made reference only to the free charges as it already contains the effects of bound charges. What we're given is plate free charge density, slab thickness, and slab dielectric constant. In order to use Gauss's law to find $\vec{D}$, the key observation is to note that dielectrics generally do not have free charges. Hence, we can probe a cylindrical Gaussian surface (as we have for earlier applications of Gauss's law) to straddle between the outside region and into the slab filling. This way, the only free charge contained in the Gaussian volume is of the capacitor plate. That is, $Q_{f_{\text{enc}}} = +\sigma A_{\text{lid}}$ and we have

$$\oint \vec{D} \cdot \, \mathrm{d}\vec{a} = +\sigma A_{\text{lid}} \tag{41}$$

In hindsight, we did use Gauss's law because we foresee a symmetry. From our previous experience with parallel-plate capacitors, such configuration induces no field outside but only perpendicular field inside. Hence, we only consider a single face (bottom lid) of our Gaussian surface this permits us the full Gaussian symmetry simplifcation

$$|\vec{D}|A_{\text{lid}} = +\sigma A_{\text{lid}} \tag{42}$$

Interestingly, we have

$$D = \sigma\hat{z} \tag{43}$$

where we have chosen $\hat{z}$ to point down. We can also do this at the bottom plate of the capacitor and get

$$|\vec{D}|A_{\text{lid}} = -\sigma A_{\text{lid}} \tag{44}$$

Here, the lid normal direction is taken upward. However, since the RHS is negative, $\vec{D}$ must point downwards. It is interesting to note that for both slabs and hence for all regions in the interior of our capacitor sandwich,

$$\boxed{\vec{D} = \sigma\hat{z}} \tag{45}$$

Now, we are tasked to find the field $\vec{E}$ for each slab. How could we even extract $\vec{E}$ from $\vec{D}$ if we have skipped the entire process of finding the polarization density $\vec{P}$ (think of finding tension but one used Lagrangian formalism). Luckily, the fillings are linear dielectrics. They conform to the neat expression

$$\vec{D} = \varepsilon\vec{E} \tag{46}$$

where ε is called the permittivity of the material and is defined as $\varepsilon := \varepsilon_0(1+\chi_e)$ and χ_e is the susceptibility inherent to the material. However, what we're given is actually the relative permittivity of the material ε_r (dielectric constant) given by $\varepsilon_r = \varepsilon/\varepsilon_0$. Hence

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0\varepsilon_r} = \frac{\sigma}{\varepsilon_0\varepsilon_r}\hat{z} \tag{47}$$

Since for slab 1: $\varepsilon_r = 2$ and for slab 2: $\varepsilon_r = 3/2$, plugging in gives us

$$\boxed{\text{Slab 1: } \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z}} \tag{48}$$

$$\boxed{\text{Slab 2: } \vec{E} = \frac{2\sigma}{3\varepsilon_0}\hat{z}} \tag{49}$$

Linear dielectrics also gives us a neat expression for polarization density and electric field

$$\vec{P} = \varepsilon_0\chi_e\vec{E} \tag{50}$$

Converting susceptibility into relative permittivity,

$$\varepsilon_r = 1 + \chi_e \iff \chi_e = \varepsilon_r - 1 \tag{51}$$

Hence,

$$\vec{P} = \varepsilon_0(\varepsilon_r - 1)\vec{E} \tag{52}$$

We merely have to plug in values of relative permittivity. For slab 1, $\varepsilon_r - 1 = 2 - 1 = 1$. For slab 2, $\varepsilon_r - 1 = 3/2 - 1 = 1/2$. Hence, the polarization densities for each slab can be directly taken

$$\boxed{\text{Slab 1: } \vec{P} = \frac{\sigma}{2}\hat{z}} \tag{53}$$

$$\boxed{\text{Slab 2: } \vec{P} = \frac{\sigma}{3}\hat{z}} \tag{54}$$

Another simplification offered by the parallel plate capacitor, aside from neat parallel directions of fields, is its uniformity in magnitude. With these, the entire line integral simplifies to

$$V(a) - V(b) = -\int_a^b \vec{E}\cdot \mathrm{d}\vec{l} = E\Delta z \tag{55}$$

Traversing the top plate towards the bottom plate, we merely have to cut the integral resulting into a summation of two terms

$$\Delta V = E_1 a + E_2 a = (E_1 + E_2)a \tag{56}$$

since the thickness is a for both slabs. Plugging in Eqs. (48) and (49), we get

$$\Delta V = \left(\frac{\sigma}{2\varepsilon_0} + \frac{2\sigma}{3\varepsilon_0} \right) a = \frac{7\sigma}{6\epsilon_0} a \tag{57}$$

Hence, the total potential difference across that parallel-plate capacitor is

$$\boxed{\Delta V = \frac{7\sigma a}{6\epsilon_0}} \tag{58}$$

Now, we are tasked to located the bound charges for each slab. Recall again that the effect of polarization to a dielectric is to "paint" its surface with bound charge expressed as

$$\sigma_b = \vec{P} \cdot \vec{n} \tag{59}$$

Since fillings are linear dielectrics, $\vec{P}$ must have the same direction as $\vec{E}$ (which we know to be pointing downwards). Hence, we should forget probing the sides of the fillings (which is perpendicular to the field) and focus on there top and bottom faces. Analyzing slab 1, the normal vector its top face is anti-parallel with the field and the normal vector of its bottom face is parallel with the field. That is,

$$\text{Slab 1 (top): } \sigma_b = \vec{P} \cdot \vec{n} = \frac{\sigma}{2} \hat{z} \cdot (-\hat{z}) = -\frac{\sigma}{2} \tag{60}$$

$$\text{Slab 1 (bottom): } \sigma_b = \vec{P} \cdot \vec{n} = \frac{\sigma}{2} \hat{z} \cdot (\hat{z}) = \frac{\sigma}{2} \tag{61}$$

Meanwhile, analyzing slab 2, the normal vector its top face is parallel with the field and the normal vector of its bottom face is anti-parallel with the field. That is,

$$\text{Slab 2 (top): } \sigma_b = \vec{P} \cdot \vec{n} = \frac{\sigma}{3} \hat{z} \cdot (-\hat{z}) = -\frac{\sigma}{3} \tag{62}$$

$$\text{Slab 2 (bottom): } \sigma_b = \vec{P} \cdot \vec{n} = \frac{\sigma}{3} \hat{z} \cdot (\hat{z}) = \frac{\sigma}{3} \tag{63}$$

Hence,

$$\boxed{\text{Slab 1 (top): } \sigma_b = -\frac{\sigma}{2}} \qquad \boxed{\text{Slab 1 (bottom): } \sigma_b = \frac{\sigma}{2}} \tag{64}$$

$$\boxed{\text{Slab 2 (top): } \sigma_b = -\frac{\sigma}{3}} \qquad \boxed{\text{Slab 2 (bottom): } \sigma_b = \frac{\sigma}{3}} \tag{65}$$

Do these findings make sense? Observe the capacitor system. It is composed of two regions of constant electric field. Hence, we can divide the capacitor sandwich into two sub-sandwiches and think of them as two separate parallel plate capacitors in the meantime having surface density σ_c on the plates with no dielectrics. We name the top capacitor containing original top plate and slab 1 Capacitor A and the bottom capacitor containing the original bottom plate and slab 2 Capacitor B. For all capacitors, field direction is downward. Recall that such capacitor must have a field of $\vec{E} = (\sigma_c/\varepsilon_0)\hat{z}$. For our bound charges to make sense, from Eqs. (48) and (49), $\sigma_c = \sigma/2$ for the top capacitor and $\sigma_c = 2\sigma/3$ for the bottom capacitor. The top plate and top surface of slab 1 (comprising the top plate of Capacitor A) has combined charge of $\sigma - \sigma/2 = \sigma/2$. The bottom surface of slab 1, top surface of slab 2, bottom surface of slab 2, and bottom plate (comprising the bottom plate of Capacitor A) has combined charge of $\sigma/2 - \sigma/3 + \sigma/3 - \sigma = -\sigma/2$. Hence, for Capacitor A, $\boxed{\sigma_c = \sigma/2}$ as it should be. Similar arguments for Capacitor B, with its top plate comprised by original top plate, top face of slab 1, bottom face of slab 1, top face of slab 2 with combined charge of $\sigma - \sigma/2 + \sigma/2 + \sigma/3 = 2\sigma/3$ and its bottom plate is comprised by bottom face of slab 2 and original bottom plate with combined charge of $\sigma/3 - \sigma = -2\sigma/3$. Hence, for Capacitor B, $\boxed{\sigma_c = 2\sigma/3}$ as it should be.

4. **A dipole in a dielectric sphere**

A point dipole $\vec{p}$ is embedded at the center of sphere of linear dielectric material with radius R and dieletric constant ε_r

a) Find the electric potential inside and outside the sphere.
b) Find the electric field outside the sphere.

This problem is reminiscent of item 2 - another boundary value problem whose conditions come from i) continuity condition ii) derivative condition iii) limiting condition. Looking back at item 2, the limiting condition was from the constant infinitely-ranged field telling us the value of potential at infinity. Although the first two conditions do apply here, we examine a different limiting condition present in this problem. That being said, it is our first task to find out the potential as we approach the center of the sphere. Assuming that we are given an ideal dipole [2], recall that we have already derived the expression of a dipole's potential. It is expressed as

$$V_{\text{dip}}(r,\theta) = \frac{p\ \cos\theta}{4\pi\varepsilon_0 r^2} \tag{66}$$

However, recall from arguments leading to Eq. 4.26 of Griffiths about the "modification" Coulomb's law that the effect of enclosing a field with a dielectric is to reduce it by a $1/\varepsilon_r$ constant. Hence, as we close in on the center, what we would expect to get is $V_{\text{dip}}(r,\theta) = p\ \cos\theta/(4\pi\varepsilon_0\varepsilon_r r^2)$. Modifying Eq. (20), we have the boundary conditions expressed as

$$\begin{cases} V_{\text{in}} = V_{\text{out}} & r = R \\ \partial V_{\text{in}}/\partial n - \partial V_{\text{out}}/\partial n = -\sigma_b/\varepsilon_0 & r = R \\ V_{\text{in}} \to p\cos\theta/4\pi\varepsilon_0\varepsilon_r r^2 & r << R \end{cases} \tag{67}$$

where the solution to Laplace's equation is

$$V(r,\theta) = \begin{cases} V_{\text{in}} = p\cos\theta/4\pi\varepsilon_0\varepsilon_r r^2 + \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta)) & r < R \\ V_{\text{out}} = \sum_{l=0}^{\infty}(B_l/(r^{l+1}))P_l(\cos(\theta)) & r > R \end{cases} \tag{68}$$

As usual, we will solve Laplace's equation in spherical coordinates using these boundary conditions. Observe that contrary to item 2, we have used a more native version of the boundary condition - the form that does not refer solely on the free charge. Observe that condition 1 is similar to item 2. Hence, we can borrow some results from there. Modifying Eq. (22) by replacing the limiting condition, we have

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos(\theta)) = \frac{p\ \cos\theta}{4\pi\varepsilon_0\varepsilon_r r^2} + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos(\theta)) \tag{69}$$

and carrying over similar arguments, we have

$$\begin{cases} A_l R^l = \frac{B_l}{R^{l+1}} & l \neq 1 \\ A_1 R = B_1/R^2 - (1/4\pi\varepsilon_0)(p/\varepsilon_r R^2) \end{cases} \tag{70}$$

Modified Eq. (27) also gives us

$$-\sum_{l=0}^{\infty}\left(lA_l R^{l-1} + \frac{(l+1)B_l}{R^{l+2}}\right)P_l\ \cos\theta + \frac{1}{4\pi\varepsilon_0}\frac{2p\ \cos\theta}{\varepsilon_r R^3} = -\frac{\sigma_b}{\varepsilon_0} \tag{71}$$

Now, how would we find the bound charge, σ_b residing at the surface? Recall from our very first introduction for bound charges (as they were first derived as mere mathematical objects) that $\sigma_b := \vec{P}\cdot\hat{n}$. In this case, $\hat{n}$ is entirely radial: $\hat{n} = \hat{r}$. Thus, $\sigma_b = \vec{P}\cdot\hat{r}$. Conveniently again, we are analyzing a linear dielectric. $\vec{P}$ is proportional to $\vec{E}$ related as

[2] I mean, that is the only logical interpretation of a point dipole.

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = -\varepsilon_0 \chi_e \frac{\partial V}{\partial r} \tag{72}$$

Noting that these charges reside on the surface of the sphere, we will use the derivative term inside. Hence,

$$-\frac{\sigma_b}{\varepsilon_0} = \chi_e \frac{\partial V_{\text{in}}}{\partial r} = \chi_e \left(-\frac{1}{4\pi\varepsilon_0} \frac{2p\ \cos\theta}{\varepsilon_r R^3} + \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\ \cos\theta) \right) \tag{73}$$

Then, the full derivative equation reads

$$-\sum_{l=0}^{\infty} \left(l A_l R^{l-1} + \frac{(l+1)B_l}{R^{l+2}} \right) P_l\ \cos\theta + \frac{1}{4\pi\varepsilon_0} \frac{2p\ \cos\theta}{\varepsilon_r R^3} = \chi_e \left(-\frac{1}{4\pi\varepsilon_0} \frac{2p\ \cos\theta}{\varepsilon_r R^3} + \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\ \cos\theta) \right) \tag{74}$$

Simplifying,

$$\sum_{l=0}^{\infty} \left(l A_l R^{l-1} (\chi_e + 1) + \frac{(l+1)B_l}{R^{l+2}} \right) P_l(\ \cos\theta) = \frac{1}{4\pi\varepsilon_0} \frac{2p\ \cos\theta}{\varepsilon_r R^3} (\chi_e + 1) \tag{75}$$

With great foresight that we have (from previous experience in item 2 and other similar electrostatic problems), the coefficients of A_l and B_l are both linearly independent for both conditions. Hence, for $l \neq 1$, the parenthesis term can only be zero if and only if $A_l = B_l = 0$. Again, for $l = 1$, the Legendre polynomial reduces to a cosine term. Hence, we will now only consider the $l = 1$ conditions. Comparing the components of the cosine basis for both sides evaluating at $l = 1$, and from Eq. (69), we now have two equations for two unknowns

$$\begin{cases} A_1 R = B_1/R^2 - (1/4\pi\varepsilon_0)(p/\varepsilon_r R^2) \\ A_1(\chi_e + 1) + 2B_1/R^3 = (1/4\pi\varepsilon_0)(2p/\varepsilon_r R^3)(\chi_e + 1) \end{cases} \tag{76}$$

These are two nonhomogeneous linear equations and, again, we utilize linear algebra exactly for these moments. In matrix form,

$$\begin{bmatrix} R & -1/R^2 \\ \varepsilon_r & 2/R^3 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} -p/(4\varepsilon_0 \varepsilon_r \pi R^2) \\ p/(2\varepsilon_0 \pi R^3) \end{bmatrix} \tag{77}$$

where we have plugged in $\varepsilon_r = 1 + \chi_e$. Inverting the matrix and simplifying, we get that

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} p(\varepsilon_r - 1)/((2\varepsilon_0 \varepsilon_r (2 + \varepsilon_r) \pi R^3)) \\ 3p/(8\varepsilon_0 \pi + 4\varepsilon_0 \varepsilon_r \pi) \end{bmatrix} \tag{78}$$

Plugging these coefficients into Eq. (68),

$$V(r, \theta) = \begin{cases} p\cos\theta/4\pi\varepsilon_0 \varepsilon_r r^2 + (p\cos\theta(\varepsilon_r - 1)/((2\varepsilon_0 \varepsilon_r (2 + \varepsilon_r) \pi R^3)))r & r < R \\ (3p\cos\theta/(8\varepsilon_0 \pi + 4\varepsilon_0 \varepsilon_r \pi))r^{-2} & r > R \end{cases} \tag{79}$$

Fully simplifying gives us

$$\boxed{V(r, \theta) = \begin{cases} P\cos\theta(\gamma_1 r^{-2} + \gamma_2 r) & r \leq R \\ P\cos\theta(\gamma_3 r^{-2}) & r \geq R \end{cases}} \tag{80}$$

where

$$\gamma_1 := \frac{1}{4\pi\varepsilon_0 \varepsilon_r} \tag{81}$$

$$\gamma_2 := \frac{\varepsilon_r - 1}{2\varepsilon_0 \varepsilon_r (2 + \varepsilon_r) \pi R^3} \tag{82}$$

$$\gamma_3 := \frac{3}{8\varepsilon_0\pi + 4\varepsilon_0\varepsilon_r\pi} \tag{83}$$

To find the field outside, we first take the derivatives

$$\frac{\partial}{\partial r} P\text{cos}\theta(\gamma_3 r^{-2}) = -2P\text{cos}\theta\gamma_3 r^{-3} \tag{84}$$

$$\frac{1}{r}\frac{\partial}{\partial\theta} P\text{cos}\theta(\gamma_3 r^{-2}) = -P\text{sin}\theta\gamma_3 r^{-3} \tag{85}$$

Then, taking the gradient, we have

$$\vec{E} = -\vec{\nabla}V = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right)V \tag{86}$$

Hence, plugging in the derivatives, we have

$$\boxed{\vec{E} = \frac{p\gamma_3}{r^3}(2\text{cos}\theta\hat{r} + \text{sin}\hat{\theta})} \tag{87}$$

5. **A metal inside a shell**

A spherical conductor of radius R_1 is surrounded by a polarizable medium which extends from R_1 to R_2 with dielectric constant ε_r.

I. The conductor has charge Q.
(a) Find the energy of this configuration
(b) Find the electric field everywhere
(c) Show that the total polarization charge is zero

II. The conductor is grounded and the entire system is palced in a uniform electric field $\vec{E} = E_0\hat{z}$.
(a) Find the electrostatic potential everywhere.
(b) Find the electric field between R_1 and R_2.
(c) Determine how much charge is drawn up from ground to the conductor.

We are tasked to analyze a spherical metal conductor with thick dielectric shell. Let $R_1 = a$ and $R_2 = b$

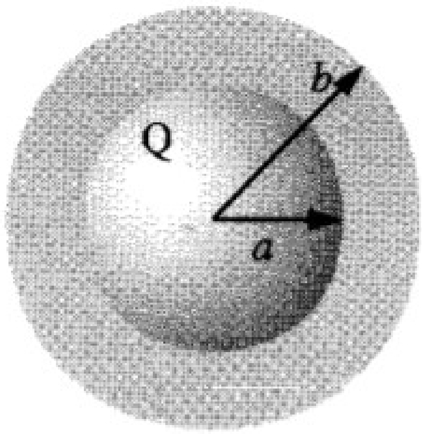

Figure 4: Image was taken from Griffiths on a problem with similar setup but different required quantities

To find the energy of the configuration, recall that the total work done to assemble a system wtih linear dielectrics is

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\ \mathrm{d}\tau \tag{88}$$

where this is to be integrated over all space. Hence, finding the total energy of the system is tantamount to finding the fields $\vec{E}$ and $\vec{D}$ everywhere. We already knew that $\vec{E}$ is zero inside a conductor. Hence,

$$\vec{E} = \vec{D} = \vec{P} = \vec{0} \qquad\qquad r < a \tag{89}$$

Since free charges only reside on the surface of the conductor, enclosing a concentric spherical Gaussian surface starting from the surface of the conductors onward and using the modified Gauss's law for displacement gives us

$$\oint \vec{D}\cdot\ \mathrm{d}\vec{a} = DA = Q \tag{90}$$

Hence, the displacement from $r = a$ outwards is

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r} \tag{91}$$

where we've tacked in a radial unit vector. From Eq. (46),

$$\vec{E} = \frac{D}{\varepsilon} = \frac{D}{\varepsilon_0 \varepsilon_r} \tag{92}$$

where, of course, the relative permittivity of vacuum is unity. Inside the dielectric,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_r\varepsilon_0}\frac{1}{r^2} \qquad a < r < b \tag{93}$$

while outside the dieletric,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2} \qquad r > b \tag{94}$$

Hence, summarizing the relevant fields at each region, we have

$$\boxed{\vec{D} = \begin{cases} \vec{0} & r < a \\ (Q/4\pi)(1/r^2)\hat{r} & a < r < b \\ (Q/4\pi)(1/r^2)\hat{r} & r > b \end{cases}} \qquad \boxed{\vec{E} = \begin{cases} \vec{0} & r < a \\ (Q/4\pi\varepsilon_r\varepsilon_0)(1/r^2)\hat{r} & a < r < b \\ (Q/4\pi\varepsilon_0)(1/r^2)\hat{r} & r > b \end{cases}} \tag{95}$$

With this, we can find the polarization of the dielectric

$$\vec{P} = \varepsilon_0\chi_e\vec{E} = \frac{Q(\varepsilon_r - 1)}{4\pi\varepsilon_r}\frac{1}{r^2}\hat{r} \tag{96}$$

Hence, we can find the polarization charges at both surfaces of dielectrics, $r = a$ and $r = b$. At $r = a$,

$$\sigma_b(a) = -\vec{P}\cdot\hat{n} = -\frac{Q(\varepsilon_r - 1)}{4\pi\varepsilon_r}\frac{1}{a^2} \tag{97}$$

where the negative sign comes from the fact that the normal vector of the inner surface is pointing inwards. Hence, by integrating, the polarization charge at the inner surface is

$$q_b(a) = \oint -\frac{Q(\varepsilon_r - 1)}{4\pi\varepsilon_r}\frac{1}{a^2}\,\mathrm{d}A = -\frac{Q(\varepsilon_r - 1)}{\varepsilon_r} \tag{98}$$

Similarly, at $r = b$,

$$\sigma_b(b) = \vec{P}\cdot\hat{n} = \frac{Q(\varepsilon_r - 1)}{4\pi\varepsilon_r}\frac{1}{b^2} \tag{99}$$

giving us the polarization charge by integration

$$q_b(b) = \oint \frac{Q(\varepsilon_r - 1)}{4\pi\varepsilon_r}\frac{1}{b^2}\,\mathrm{d}A = \frac{Q(\varepsilon_r - 1)}{\varepsilon_r} \tag{100}$$

Observe that the two polarization charge at both surfaces are additive inverses of each other. Recall that since $q_f \propto q_b$ and have no embedded free charge at the meat of the dielectric, q_b must be zero at the volume and polarization charge must reside solely on the surface. Hence, if the total polarization charge is given by $Q_b = q_b(a) + q_b(b) = 0$, we have

$$\boxed{Q_b = 0} \tag{101}$$

To find the energy of the system, we first find the relevant inner products. Inside the dielectric,

$$\vec{D}\cdot\vec{E} = \frac{Q}{4\pi r^2}\hat{r}\cdot\frac{Q}{4\pi\varepsilon_r\varepsilon_0 r^2}\hat{r} = \left(\frac{Q}{4\pi r^2}\right)^2\frac{1}{\varepsilon_0\varepsilon_r} \qquad a < r < b \tag{102}$$

while outside the dielectric,

$$\vec{D}\cdot\vec{E} = \frac{Q}{4\pi r^2}\hat{r}\cdot\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} = \left(\frac{Q}{4\pi r^2}\right)^2\frac{1}{\varepsilon_0} \qquad r > b \tag{103}$$

from Eq. (88), we then have

$$W = \frac{1}{2}\left(\int_a^b \left(\frac{Q}{4\pi r^2}\right)^2 \frac{1}{\varepsilon_r\varepsilon_0}\,\mathrm{d}\tau + \int_b^\infty \left(\frac{Q}{4\pi r^2}\right)^2 \frac{1}{\varepsilon_0}\,\mathrm{d}\tau\right) \tag{104}$$

Since the integrand is angularly independent, we can evaluate the angular terms of the volume integral separately first. Then, the integral reduces to a one-dimensional radial integration

$$W = \frac{1}{2}(2\pi)\left(\frac{Q}{4\pi}\right)^2\left(\int_a^b \frac{1}{\varepsilon_r\varepsilon_0}\frac{1}{r^2}\,\mathrm{d}r + \int_b^\infty \frac{1}{\varepsilon_0}\frac{1}{r^2}\,\mathrm{d}r\right) \tag{105}$$

$$= -2\pi\left(\frac{Q}{4\pi}\right)^2\left(\frac{1}{\varepsilon_r\varepsilon_0}\frac{1}{r}|_a^b + \frac{1}{\varepsilon_0}\frac{1}{r}|_b^\infty\right) \tag{106}$$

$$= -2\pi\left(\frac{Q}{4\pi}\right)^2\left(\frac{1}{\varepsilon_r\varepsilon_0}\left(\frac{1}{b}-\frac{1}{a}\right) - \frac{1}{\varepsilon_0}\frac{1}{b}\right) \tag{107}$$

Hence, the total energy of the system is

$$\boxed{W = \frac{Q^2}{8\pi}\left(\left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_0\varepsilon_r}\right)\frac{1}{b} + \frac{1}{\varepsilon_0\varepsilon_r}\frac{1}{a}\right)} \tag{108}$$

Now, we ground the conductor and turn on our constant $\vec{E}$ machine. We are tasked to find the electrostatic potential everywhere. Again, we solve Laplace's equation at three different regions and tie them together by the appropriate boundary conditions. Inside the conductor, the potential is grounded and zero. Outside the dielectric, potential must reduce to the given limiting case. That is,

$$V(r,\theta) = \begin{cases} 0 & r \le a \\ \sum_{l=0}^\infty \left(C_l r^l + \frac{D_l}{r^{l+1}}\right) P_l(\cos(\theta)) & a < r < b \\ -E_0 r\cos\theta + \sum_{l=0}^\infty (B_l/r^{l+1})P_l(\cos(\theta)) & r > b \end{cases} \tag{109}$$

Next, we construct our boundary conditions which is somehow similar to item 4 of problem set 3. However, in this case, we will be using a modified version of the derivative discontinuity equation. That is,

$$\varepsilon_r \frac{\partial V|}{\partial r}|b- = \frac{\partial V}{\partial r}|_{b+} \tag{110}$$

or

$$\varepsilon_r \sum_{l=0}^\infty \left(C_l l r^{l-1} - (l+1)\frac{D_l}{r^{l+2}}\right) P_l(\cos(\theta)) = -E_0\cos\theta + \sum_{l=0}^\infty (-(l+1))\frac{B_l}{r^{l+2}} P_l(\cos(\theta) \tag{111}$$

along with the two continuity equations, this gives us the boundary conditions

$$\begin{cases} 0 = \left(\sum_{l=0}^\infty \left(C_l r^l + \frac{D_l}{r^{l+1}}\right) P_l(\cos(\theta))\right)_a \\ \left(\sum_{l=0}^\infty \left(C_l r^l + \frac{D_l}{r^{l+1}}\right) P_l(\cos(\theta))\right)_b = -E_0 r\cos\theta + \left(\sum_{l=0}^\infty \frac{B_l}{r^{l+1}} P_l(\cos(\theta))\right)|_b \\ \varepsilon_r \sum_{l=0}^\infty \left(C_l l r^{l-1} - (l+1)\frac{D_l}{r^{l+2}}\right) P_l(\cos(\theta))|_b = -E_0\cos\theta + \sum_{l=0}^\infty (-(l+1))\frac{B_l}{r^{l+2}} P_l(\cos(\theta)|_b \end{cases} \tag{112}$$

The first condition gives us

$$C_l a^l + \frac{D_l}{a^{l+1}} = 0 \qquad \forall\, l \tag{113}$$

The second condition gives us

$$\begin{cases} \frac{B_l}{b^{l+1}} = \left(C_l b^l + \frac{D_l}{b^{l+1}}\right) \iff B_l = (b^{2l+1}C_l + D_l) & l \neq 1 \\ C_1 b + \frac{D_1}{b^2} = -E_0 b + \frac{B_1}{b^2} \end{cases} \tag{114}$$

The third condition gives us

$$\begin{cases} \varepsilon_r \left(C_l l b^{l-1} - (l+1)\frac{D_l}{b^{l+2}} \right)) = (-(l+1))\frac{B_l}{b^{l+2}} & l \neq 1 \\ \varepsilon_r (C_1 - 2\frac{D_1}{b^3}) = -E_0 - 2\frac{B_1}{b^3} \end{cases} \tag{115}$$

Summarizing, we have

$$\begin{cases} C_0 + \frac{D_0}{a} = 0 \\ C_1 b + \frac{D_1}{b^2} = -E_0 b + \frac{B_1}{b^2} \\ \varepsilon_r (C_1 - 2\frac{D_1}{b^3}) = -E_0 - 2\frac{B_1}{b^3} \\ C_l a^l + \frac{D_l}{a^{l+1}} = 0 & l \neq 0 \\ B_l = (b^{2l+1} C_l + D_l) & l \neq 1 \\ \varepsilon_r \left(C_l l b^{l-1} - (l+1)\frac{D_l}{b^{l+2}} \right)) = (-(l+1))\frac{B_l}{b^{l+2}} & l \neq 1 \end{cases} \tag{116}$$

Now, this is reminiscent of the system of equations we have for problem set 3 item 4 where we have to do extensive algebraic work to find solutions. Recall that we have analytically shown that the last 3 equations only have the zero vector as a solution. It is easy to see that these three equations form a homogeneous linear equation of form $A\vec{x} = \vec{0}$ where the matrix A is non-singular and is expressed as

$$A = \begin{bmatrix} 0 & a^l & 1/a^{l+1} \\ 1 & -2b^{2b+1} & -1 \\ (l+1)/b^{l+2} & \varepsilon_r l b^{l-1} & -(l+1)/b^{l+2} \end{bmatrix} \tag{117}$$

Hence, the system only has the zero vector as its unique solution and we have

$$\boxed{B_l = C_l = D_l = 0 \qquad l > 1} \tag{118}$$

Now, we are tasked to find the coefficients $B_0, C_0, D_0, B_1, C_1, D_1$. Observe that we have 6 linearly-independent equations that we can convert into 3 equations of 0-coefficients and 3 equations of 1-coefficients. Reducing the last 2 equations to an $l = 0$ case and the fourth equation to an $l = 1$ case, we have

$$\begin{cases} C_0 + D_0/a = 0 \\ B_0 = bC_0 + D_0 \\ \varepsilon_r (C_0 b - D_0/b^2) = -B_0/b^2 \end{cases} \qquad \begin{cases} C_1 a + D_1/a^2 = 0 \\ C_1 b + D_1/b^2 = -E_0 b + B_1/b^2 \\ \varepsilon_r (C_1 - 2D_1/b^3) = -E_0 - 2B_1/b^3 \end{cases} \tag{119}$$

observe that the $l = 0$ system is, again, another homogeneous equation with zero vector solution. Hence, we are left with the $l = 1$ case with matrix equation

$$\begin{bmatrix} 1/b^2 & -b & -1/b^2 \\ -2/b^3 & -\varepsilon_r & 2\varepsilon_r/b^3 \\ 0 & a & 1/a^2 \end{bmatrix} \begin{bmatrix} B_1 \\ C_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} E_0 b \\ E_0 \\ 0 \end{bmatrix} \tag{120}$$

Inverting gives us

$$\begin{bmatrix} B_1 \\ C_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} \gamma_b E_0 \\ \gamma_c E_0 \\ \gamma_d E_0 \end{bmatrix} \tag{121}$$

where the dimensionless parameters are

$$\gamma_b = \frac{b^3(a^3 - b^3 + \varepsilon_r(2a^3 + b^3))}{-2a^3 + 2b^3 + \varepsilon_r(2a^3 + b^3)} \tag{122}$$

$$\gamma_c = -\frac{3b^3}{-2a^3 + 2b^3 + \varepsilon_r(2a^3 + b^3)} \tag{123}$$

$$\gamma_d = \frac{3a^3 b^3}{-2a^3 + b^3 + \varepsilon_r(2a^3 + b^3)} \tag{124}$$

Plugging these to Eq. (109), the potential function in each region is, then,

$$V(r,\theta) = \begin{cases} 0 & r \leq a \\ \left(\gamma_c E_0 r + \frac{\gamma_d E_0}{r^2}\right)(\cos(\theta)) & a < r < b \\ -E_0 r \cos\theta + (\gamma_b E_0/r^2)(\cos(\theta)) & r > b \end{cases} \tag{125}$$

To find the electric field inside the dielectric, we simply take the gradient of the potential. Observe that all of the coefficients are position-independent. Taking the gradient inside the dielectric,

$$\vec{E} = -\vec{\nabla}V = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right)\left(\gamma_c E_0 r + \frac{\gamma_d E_0}{r^2}\right)(\cos(\theta)) \tag{126}$$

$$= -\cos(\theta)\frac{\partial}{\partial r}\left(\gamma_c E_0 r + \frac{\gamma_d E_0}{r^2}\right)\hat{r} - \left(\gamma_c E_0 + \frac{\gamma_d E_0}{r^3}\right)\frac{\partial}{\partial\theta}(\cos(\theta))\hat{\theta} \tag{127}$$

$$= -\left(\gamma_c E_0 - 2\frac{\gamma_d E_0}{r^3}\right)\cos(\theta)\hat{r} + \left(\gamma_c E_0 + \frac{\gamma_d E_0}{r^3}\right)\sin(\theta)\hat{\theta} \tag{128}$$

Hence, inside the dielectric, the electric field is

$$\boxed{\vec{E} = E_0\left(\left(2\frac{\gamma_d}{r^3} - \gamma_c\right)\cos(\theta)\hat{r} + \left(\gamma_c + \frac{\gamma_d}{r^3}\right)\sin(\theta)\hat{\theta}\right) \qquad a < r < b} \tag{129}$$

To find the total charge induced at the conductor, we first find the charge density and integrate it over the surface of the spherical conductor because that is where they reside. Recall that at $r = a$, the boundary condition reads

$$0 - \varepsilon\frac{\partial V}{\partial r} = -\sigma_f \tag{130}$$

Differentiating the potential inside the dielectric at $r = a$, we get

$$\sigma_f(\theta) = \varepsilon\left(2\frac{\gamma_d E_0}{a^3} - \gamma_c E_0\right)\cos\theta \tag{131}$$

Integrating at polar domain $[0, \pi]$, we have

$$Q = 2\pi\int_0^\pi \sigma_f(\theta)\ \mathrm{d}\theta = 2\pi\varepsilon\left(2\frac{\gamma_d E_0}{a^3} - \gamma_c E_0\right)\int_0^\pi \cos(\theta)\ \mathrm{d}\theta \tag{132}$$

Of course, the integral is zero and hence,

$$\boxed{Q = 0} \tag{133}$$

Hence, there is no net charge drawn up from the ground to the conductor.

2 Reflection

This problem set was different. At first, I thought I am now going to use Poisson's equation since it now includes calculation of fields within matter but I'm wrong. The properties of matter was instead quantified by its relative permittivity and there are analogous governing equations and boundary conditions. However, Laplace's equation still holds and is, in fact, used in almost all items in the problem set. A recurring theme is repeatedly solving Laplace's equation with the given boundary conditions. What's interesting is that, for most items, it always results to a specific linear combination of sine and cosine - and this is related to the field of a dipole. There is this ominous feeling that there are dipoles hidden everywhere in this problem set that I could not quite figure out.

- **Problem 1**
  The task was quite straightforward as I have already anticipated that method of images would work because of the infinite plane. I revised and followed the classic image problem chapter by Griffiths closely to aid my solutions. This is the shortest and the easiest of all problems to execute albeit not being the easiest conceptually.
- **Problem 2**
  We've already solved this countless of times. We merely have to modify the boundary conditions to account for the different dielectric constants. However, what confused me was the "convenient" modified boundary condition by Griffiths and the original one. A typical approach to eyeball coefficients in the end is the chaotic substitution approach which is a personal pet-peeve of mine. What I did was a more systematic and organized approach to solving linear equations which made my life much easier. What's bothering me the most is why does the field really look like a dipole at the region outside.
- **Problem 3** This item was somehow different to solve. It feels "discretized" to solve quantities one-by-one. Although the steps were arranged so that it flows smoothly towards solving the final step, it feels short and awkward. ~~It feels like solving a chemistry or an engineering problem~~. Conceptually, I think this aims to demonstrate the power of the modified Gauss's law and conservation of charge amidst the entire polarization process.
- **Problem 4** Again, yet another similar boundary value problem - now including a dipole in the center. Letting the dust settle in the end, we take back the field of a dipole - now scaled differently. In this case though, it is pretty obvious why would it conform to have a dipole field expression.
- **Problem 5** What best to end a problem set other than... of course, another boundary value problem. As always, although it is conceptually easy, it is computationally tedious to do. Observe that because I have extensively studied Griffiths, the quantities I've written was a instead of R_1 and b instead of R_2. This is where linear algebra really became my friend as it facilitated finding solutions for me. The invertible matrix equivalent theorems really gave me the ability to weed out the solutions such that I can solve it way faster. Again, the dipole expression popped up in the end.

The problem set was not conceptually difficult but is nonetheless computationally tedious and draining. I have to consult Griffiths book for all problems as I seem to commit a plethora of tiny mistakes in all of my calculations. Some remnants of these can be seen on footnote 1. I knew this because the dust sometimes won't settle in the end causing some inconsistencies with the units. Then there are these dipole expressions that I could not find out the links between these problems; I will continue to seek for answers. I also find that the analogous equations for electric fields in matter can be very convenient at times (directly taking Gauss's law by ignoring bound charges), but it requires getting used to its conditions and assumptions such as homogeneity and isotropy, or that free charges also must reside on the surface of an insulator, etc. I would rate the problem set as a yellow one - although I was confused by the conventional process on finding the coefficients of Laplace's solutions, I remedied it with linear algebra.

I wonder if the rest of electromagnetism are a conglomerate of boundary-value problems.