1 Problem Set 3: Potentials

1. **Infinitely long wires**

Two infinitely long wires running parallel to the x-axis carry uniform charge densities. Find the potential at any point (x, y, z) using the origin as reference

The task is to find the potential from a given charge distribution. Observe that we can exploit a cylindrical symmetry in this problem. The plan is to find the field first and then derive the potential. The charge distribution is given below.

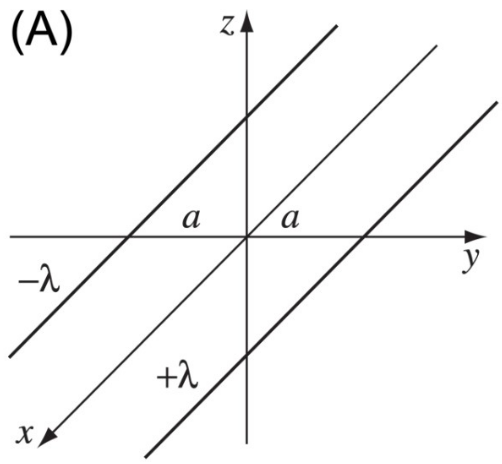

Figure 1: Essentially, the problem is two dimensional withholding z at a constant.

We start with finding the potential of a line of charge. Recall that the potential can be derived from the field from

$$V = -\int \vec{E} \cdot \, \mathrm{d}\vec{l} \tag{1}$$

Observe that we are dealing with an infinitely-long wire. This licenses us to assume full cylindrical symmetry along each axis. To find $\vec{E}$, we use Gauss's exploitation of symmetry by surrounding the line charge with a concentric cylinder.

$$\oint \vec{E} \cdot \, \mathrm{d}\vec{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \iff |\vec{E}| \oint \mathrm{d}\vec{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \tag{2}$$

$$\tag{3}$$

Hence, using cylindrical surface area $A = 2\pi s_+ x$ and $Q_{\text{enc}} = \lambda_+ x$, the field of around the line of charge is

$$\vec{E}_+ = \frac{\lambda_+}{2\pi\varepsilon_0 s_+}\hat{s}_+ \tag{4}$$

where s_+ is the radial distance from the axis of the positive line. Similarly, for the negative line,

$$\vec{E}_- = \frac{\lambda_-}{2\pi\varepsilon_0 s_-}\hat{s}_- \tag{5}$$

By superposition,

$$\vec{E} = \frac{\lambda_+}{2\pi\varepsilon_0 s_+}\hat{s}_+ + \frac{\lambda_-}{2\pi\varepsilon_0 s_-}\hat{s}_- \tag{6}$$

Now, note that $\hat{s}_+ = \hat{s}_-$ since they point in the same direction, $\lambda_- = \lambda = -\lambda_-$ which simplifies the expression into

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{1}{s_+} - \frac{1}{s_-}\right)\hat{s} \tag{7}$$

Integrating over the radial distance to find the potential, we have

$$V = -\int \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{1}{s_+} - \frac{1}{s_-}\right)\hat{s}\cdot\hat{s}\ \mathrm{d}s = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s_-}{s_+}\right) \tag{8}$$

where, of course, it is reference independent since we're essentially taking a potential difference. Now, we convert s_+ and s_- in Cartesian coordinates with reference to the origin. Refer to the following figure.

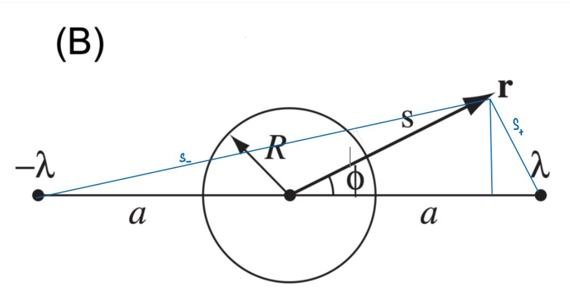

Figure 2: Blue lines imply an attempt to individually use Gauss's law for each line charge.

From the diagrams, we see that $s_+ = \sqrt{(a-y)^2 + z^2}$ and $s_- = \sqrt{(a+y)^2 + z^2}$. With reference to the origin, the field is, then, the potential of two infinite lines is

$$\boxed{V = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{(a+y)^2 + z^2}{(a-y)^2 + z^2}\right)} \tag{9}$$

In cylindrical coordinates,

$$\boxed{V = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{a^2 + s^2 + 2as\cos(\theta)}{a^2 + s^2 - 2as\cos(\theta)}\right)} \tag{10}$$

Now, we proceed to find the image of a cylinder. First, we consider only the left negative leg of the pair of line charge. The potential is given as

$$V = -\frac{\lambda}{2\pi\varepsilon_0}\ln(\sqrt{a^2 + s^2 - 2as\cos(\theta)} + c_1 \tag{11}$$

where c_1 is an arbitrary constant. We find the image of the cylinder as a line charge placed at $y = b$. The potential now is

$$V = -\frac{\lambda}{2\pi\varepsilon_0}(\ln(\sqrt{a^2 + s^2 - 2as\cos(\theta)} + c_1 + \lambda'\ln(\sqrt{b^2 + s^2 - 2as\cos(\theta)} + c_2) \tag{12}$$

Then, we invoke the boundary condition at the surface of the conductor, we recall that the tangential component of the field at $s = R$ must be zero - $-\partial_\theta V = 0$. That is,

$$\frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda(2aR\sin(\theta)}{a^2 + R^2 - 2ar\sin(\theta)} + \frac{\lambda'(2bR\sin(\theta)}{b^2 + R^2 - 2br\sin(\theta)}\right) = 0 \tag{13}$$

Cross multiplication gives us

$$\lambda(2aR\sin(\theta)(b^2 + R^2 - 2bR\cos(\theta) = -\lambda'(2bR\sin(\theta)(a^2 + R^2 - 2aR\cos(\theta) \tag{14}$$

Expanding and comparing coefficients, we have

$$2aR\lambda(b^2 + R^2) = -2bR\lambda'(a^2 + R^2) \tag{15}$$

and also

$$-2\lambda abR = 2\lambda' abR \iff \lambda = -\lambda' \tag{16}$$

Plugging the density relation to Eq. (15), we see that

$$b = \frac{R^2}{a} \tag{17}$$

and thus we now have solved the auxillary problem. It is interesting to note that is is also the image of a spherical conductor from Griffiths. We simply apply the same process in the other leg and arrive at a similar result. Hence, we see that we should place a mirror of the charge configuration at distance $y = R^2/a$ from the origin. Lastly, we simply add up all potentials by superposition. Adding up the mirror term,

$$V = \frac{\lambda}{2\pi\varepsilon_0}\left(\ln\left(\frac{\sqrt{(a+y)^2+z^2}}{\sqrt{(a-y)^2+z^2}}\right) - \ln\left(\frac{\sqrt{(b+y)^2+z^2}}{\sqrt{(b-y)^2+z^2}}\right)\right) \tag{18}$$

Plugging in the value for b and invoking logarithm rules, we have

$$\boxed{V(\vec{r}) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{((a+y)^2+z^2)(((R^2/a)-y)^2+z^2)}{((a-y)^2+z^2)(((R^2/a)+y)^2+z^2)}\right)} \tag{19}$$

where $\vec{r} = (x, y, z)$ in Cartesian basis. In cylindrical coordinates,

$$\boxed{V(\vec{r}) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{(a^2+s^2+2as\cos(\theta))((R^2/a)^2+s^2-2as\cos(\theta))}{(a^2+s^2-2as\cos(\theta))((R^2/a)^2+s^2+2as\cos(\theta))}\right)} \tag{20}$$

where $\vec{r} = (s, 0, x)$ in cylindrical basis.

2. **Long rectangular metal pipe**

An infinitely long rectangular metal pipe sides a at xy plane and b at zx plane is grounded while one end at $x = 0$ is maintained at a specified potential $V_0(y, z)$. Find V and $\vec{E}$ inside this pipe.

Our first task is to convert the word problem into a purely mathematical boundary-value problem. Of course, we start with visualizing what the system is. The phrase "infinitely long rectangular metal pipe" contains loads of information. First, it is infinite. Hence, we can't do our usual calculation of potentials by integration of charge distribution since it breaks down if the latter extends to infinity. Rectangular means we will be using Cartesian coordinates. Metal pipe means we will be dealing with equipotentials and might as well set them to zero. Using the given dimensions, we proceed to sketch the figure as follows:

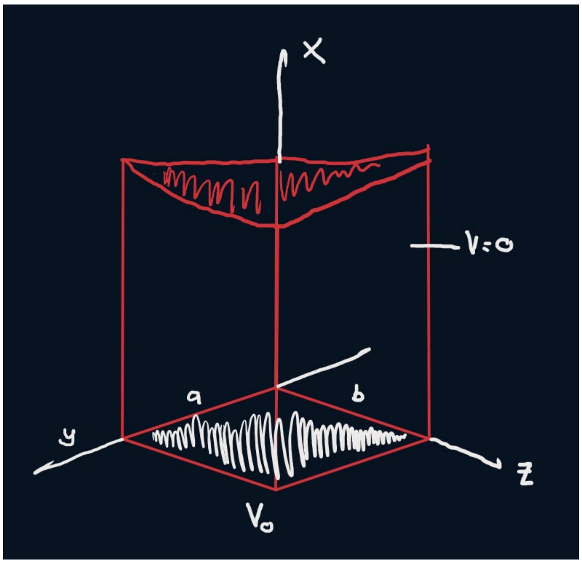

Figure 3: A scetion of the infinitely long pipe.

As always, the potential can be uniquely determined by the boundary conditions. The boundary conditions can be extracted directly from the figure above plus an additional constraint that V must vanish at $x \to \infty$. Compiling these boundary conditions:

$$\begin{cases} V(x, 0, z) = 0 \\ V(x, a, z) = 0 \\ V(x, y, 0) = 0 \\ V(x, y, b) = 0 \\ V(\infty, y, z) = 0 \\ V(0, y, z) = V_0(y, z) \end{cases} \tag{21}$$

These boundary conditions will uniquely determine the specific solution of Laplace's equation that describes our system. As always, we use separation of variables by introducing the substitution $V(x, y, z) = X(x)Y(y)Z(z)$ and isolating the terms to have

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = 0 \tag{22}$$

as of our usual separation of variables arguments, each of these terms can be separately varied without affecting (i) the other terms and (ii) the sum which is zero. Hence, each of these terms are constant and can be separated as

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = \alpha \qquad \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} = \beta \qquad \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = \gamma \tag{23}$$

where $\alpha + \beta + \gamma = 0$. To further set restrictions to the future general solution, we must first observe the boundary conditions. Observe that $V(x, a, z) = 0$ and $V(x, 0, z) = 0$. Also, $V(x, y, b) = 0$ and $V(x, y, 0) = 0$. Both need to vanish at two points and thus both needs a sinusoidal solution. Meanwhile,

$V(\infty, y, z) = 0$ and so X needs an exponential solution. Sinusoidal solutions must have negative separation constants while exponential solutions must have positive separation constants. Hence,

$$\alpha = k^2 + l^2 \qquad \beta = -k^2 \qquad \gamma = -k^2 \tag{24}$$

Plugging these in gives us

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = (k^2 + l^2)X \qquad \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = -k^2 Y \qquad \frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = -l^2 Z \tag{25}$$

with solutions

$$\begin{cases} X(x) = A\ \exp(\sqrt{k^2 + l^2}x) + B\ \exp(-\sqrt{k^2 + l^2}x) \\ Y(y) = C\ \sin(ky) + D\ \cos(ky) \\ Z(z) = E\ \sin(lz) + F\ \cos(lz) \end{cases} \tag{26}$$

Hence, the general separable solution is

$$V(x, y, z) = (A\ \exp(\sqrt{k^2 + l^2}x) + B\ \exp(-\sqrt{k^2 + l^2}x)(C\ \sin(ky) + D\ \cos(ky))(E\ \sin(lz) + F\ \cos(lz)) \tag{27}$$

This separable solution yields the basis for our solution set. To describe the system, we now fit our boundary conditions. $V(\infty, y, z) = 0$ implies $A = 0$. $V(x, 0, z) = 0$ implies $D = 0$, $V(x, y, 0) = 0$ implies $F = 0$, $V(x, a, z) = 0$ implies $k = n\pi/a$, and lastly, $V(x, y, b) = 0$ implies $l = n\pi/b$. We've used five of our boundary conditions and this leaves us with

$$V(x, y, z) = C\exp(-\pi\sqrt{(n/a)^2 + (m/b)^2}x)\ \sin(n\pi y/a)\ \sin(m\pi z/b) \tag{28}$$

Observe that we have two free indices, m and n so we have to sum over all possible combinations of both. That is,

$$V(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\exp(-\pi\sqrt{(n/a)^2 + (m/b)^2}x)\ \sin(n\pi y/a)\ \sin(m\pi z/b) \tag{29}$$

With this solution, we can express the remaining boundary condition as

$$V(0, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\ \sin(n\pi y/a)\ \sin(m\pi z/b) = V_0(y, z) \tag{30}$$

As usual, to find the remaining coefficient, we employ Fourier's trick by exploiting orthonormality. Applying the following operator $\mathbb{L}$,

$$\mathbb{L} = \int_0^a \int_0^b\ \mathrm{d}y\ \mathrm{d}z\ \sin(n'\pi y/a)\ \sin(m'\pi z/b) \tag{31}$$

the middle part of the equation gives us

$$\int_0^a \int_0^b\ \mathrm{d}y\ \mathrm{d}z\ \sin(n'\pi y/a)\ \sin(m'\pi z/b)\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\ \sin(n\pi y/a)\ \sin(m\pi z/b)\right) \tag{32}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \int_0^a\ \sin(n\pi y/a)\ \sin(n'\pi y/a)\ \mathrm{d}y \int_0^b\ \sin(m\pi z/b)\ \sin(m'\pi z/b)\ \mathrm{d}z \tag{33}$$

Recall that the orthonormality relation of sine is

$$\int_0^a\ \sin(n\pi y/a)\ \sin(n'\pi y/a)\ \mathrm{d}y = \frac{a}{2}\delta(n - n') \tag{34}$$

Plugging this into Eq. (33), this gives us

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\left(\frac{a}{2}\delta(n-n')\frac{b}{2}\delta(m-m')\right) \tag{35}$$

$$= \frac{ab}{4}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\delta(n-n')\delta(m-m') \tag{36}$$

$$= \frac{ab}{4}C_{n',m'} \tag{37}$$

Forming the equation by similarly applying $\mathbb{L}$ on $V_0(y,z)$,

$$\frac{ab}{4}C_{n',m'} = \mathbb{L}\{V_0(y,z)\} \iff C_{n',m'} = \frac{4}{ab}\mathbb{L}\{V_0(y,z)\} := C_{n,m} \tag{38}$$

where we simply rename the dummy indices back to its original form. Plugging this in the general solution at Eq. (29), we have now our solution

$$\boxed{V(x,y,z) = \frac{4}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbb{L}\{V_0(y,z)\}\exp(-\pi\sqrt{(n/a)^2+(m/b)^2}x)\ \sin(n\pi y/a)\ \sin(m\pi z/b)} \tag{39}$$

This is as far as we can go without explicit expression for $V_0(y,z)$. If the surface at $x=0$ is a conducting metal, we could go further to assume the the potential is constant at $V_0(y,z) = V_0$, then the solution is

$$V(x,y,z) = \frac{4V_0}{ab}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbb{L}\{1\}\exp(-\pi\sqrt{(n/a)^2+(m/b)^2}x)\ \sin(n\pi y/a)\ \sin(m\pi z/b) \tag{40}$$

Finding $\mathbb{L}\{1\}$,

$$\mathbb{L}\{1\} = \int_0^a \sin(n'\pi y/a)\ \mathrm{d}y \int_0^b \sin(m'\pi z/b)\ \mathrm{d}z \tag{41}$$

$$= \frac{a}{n'\pi}\int_0^{n'\pi}\sin(u)\ \mathrm{d}u\ \frac{b}{m'\pi}\int_0^{m'\pi}\sin(v)\ \mathrm{d}v \tag{42}$$

$$= \frac{ab}{n'm'\pi^2}((-\cos(u)|_0^{n'\pi})(-\cos(v)|_0^{m'\pi}) \tag{43}$$

$$= \begin{cases} \frac{4ab}{nm\pi^2} & \text{n or m is even} \\ 0 & \text{n and m are odd} \end{cases} \tag{44}$$

noting that $n' = n$ and $m' = m$ since these are the only survivors. With a constant potential condition at $x = 0$, we have the solution

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\exp(-\pi\sqrt{(n/a)^2+(m/b)^2}x)\ \sin(n\pi y/a)\ \sin(m\pi z/b)} \tag{45}$$

To find the field, simply take the gradient of the potential.

$$\vec{E} = -\vec{\nabla}V \tag{46}$$

$$= -\vec{\nabla}\left(\frac{16V_0}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\exp(-\pi\sqrt{(n/a)^2+(m/b)^2}x)\ \sin(n\pi y/a)\ \sin(m\pi z/b)\right) \tag{47}$$

$$= -\frac{16V_0}{\pi^2}\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\frac{\partial}{\partial x}\left(\exp(-\pi\sqrt{(n/a)^2+(m/b)^2}x)\right)\ \sin(n\pi y/a)\ \sin(m\pi z/b)\right)\hat{x} \tag{48}$$

$$-\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\exp(-\pi\sqrt{(n/a)^2+(m/b)^2}x)\frac{\partial}{\partial y}(\ \sin(n\pi y/a))\ \sin(m\pi z/b)\right)\hat{y}$$

$$-\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\exp(-\pi\sqrt{(n/a)^2+(m/b)^2}x)\ \sin(n\pi y/a)\frac{\partial}{\partial z}(\ \sin(m\pi z/b))\right)\hat{z}$$

Hence, the field is

$$\vec{E} = \frac{16V_0}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\lambda_{n,m}}{nm}\exp(-\lambda_{n,m}x)\ \sin(\nu_n y)\ \sin(\mu_m z)\hat{x} \tag{49}$$

$$-\frac{16V_0}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\nu_n}{nm}\exp(-\lambda_{n,m}x)\ \cos(\nu_n y)\ \sin(\mu_m z)\hat{y} \tag{50}$$

$$-\frac{16V_0}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\mu_m}{nm}\exp(-\lambda_{n,m}x)\ \sin(\nu_n y)\ \cos(\mu_m z)\hat{z} \tag{51}$$

where $\lambda_{n,m} := \pi\sqrt{(n/a)^2+(m/b)^2}$, $\nu_n := n\pi y/a$, and $\mu_m := m\pi z/b$.

3. **Cylindrical coordinates**

Solve Laplace's equation in cylindrical coordinates with no z-dependence.

In cylindrical coordinates (using a cylindrical Laplacian), Laplace's equation reads[1]:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0 \tag{52}$$

For this derivation, we assume no z-dependence (think of analyzing infinitely-long cylinders). Then, Laplace's equation reads

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}=0 \tag{53}$$

Plugging in our separable ansatz $V(r,\phi,z)=R(r)\Phi(\phi)Z_0$. We ignore Z_0 now and simply tack it in later at the end. To compactify expression, we suppress the independent parameter. This gives us

$$\frac{\Phi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{R}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2}=0 \tag{54}$$

Multiplying by $r^2/(R\Phi)$, we have

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}=0 \tag{55}$$

As usual with our constant terms argument, of course, both terms are constant. Since ϕ is oscillatory in nature, it needs a negative separation constant which means that the r equation must have a positive separation constant. Then,

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)=k^2 \qquad \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2}=-k^2 \tag{56}$$

We start with the more familiar ϕ equation. Rearranging gives us

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2}=-k^2\Phi \tag{57}$$

which has, of course, a sinusoidal solution of form

$$\boxed{\Phi(\phi)=A_k\sin(k\phi)+B_k\cos(k\phi)} \tag{58}$$

Now, we proceed to find the solution for the radial equation. Rearranging gives us

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right)=k^2\frac{R}{r} \tag{59}$$

Executing the product rule and rearranging,

$$r\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{\mathrm{d}R}{\mathrm{d}r}-\frac{k^2}{r}R=0 \iff r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+r\frac{\mathrm{d}R}{\mathrm{d}r}-k^2R=0 \tag{60}$$

Observe the Eq. (60) is simply a Cauchy-Euler equation with already well-known solution so we could stop here just like what we did at Eq. (58) and what Griffith's did for the spherical case. However, deriving its solution is straightforward so we will. We try an ansatz of form $R=r^m$. The derivatives are then

$$\frac{\mathrm{d}R}{\mathrm{d}r}=mr^{m-1} \qquad \frac{\mathrm{d}^2R}{\mathrm{d}r^2}=m(m-1)r^{m-2} \tag{61}$$

Plugging these gives us

[1] I aplogize for using r instead of s as radial distance

$$r^m(k^2 - m^2) = 0 \iff m = \pm k \tag{62}$$

If $k \neq 0$, then we automatically have two solutions

$$R(r) = C_k r^k + D_k r^{-k} \tag{63}$$

If $k = 0$, then we have a single solution

$$R(r) = C_0 \tag{64}$$

Nothing's stopping us from finding another solution via reduction of order. Observe that Eq. (60) reduces to

$$r^2 \frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} = 0 \tag{65}$$

The second solution, then, has the form $R(r) = v(r)C_0$. We plug this ansatz into the differential equation to find

$$r\frac{\mathrm{d}^2 v}{\mathrm{d}r^2} + \frac{\mathrm{d}v}{\mathrm{d}r} = 0 \iff r\frac{\mathrm{d}w}{\mathrm{d}r} + w = 0 \tag{66}$$

where we have defined $w := \mathrm{d}v/\mathrm{d}t$. Lastly, we perform separation of variables to find

$$\frac{\mathrm{d}w}{w} = -\frac{1}{r} \iff \ln(w) = -D\ \ln(r) \tag{67}$$

Exponentiating gives us

$$w(r) = \frac{D}{r} \tag{68}$$

where we have retained the arbitrary constant and simply rename them later. Lastly, integrating gives

$$v(r) = D\ \ln(r) + E \tag{69}$$

where the second solution has the same form (just multiplied by a constant using $R(r) = v(r)C_0$) Combining this with Eq. (64) and renaming constants, we now have the general solution for the $k = 0$ case as

$$R(r) = C_0 + D_0\ \ln(r) \tag{70}$$

Then, the general solution for the r-equation is

$$R(r) = \begin{cases} R(r) = C_0 + D_0\ \ln(r) & k = 0 \\ R(r) = C_k r^k + D_k r^{-k} & k \neq 0 \end{cases} \tag{71}$$

Using these separable solutions for $R(r)$ and $\Phi(\phi)$ as basis, we can now form the general solution as

$$\begin{aligned} V(r,\phi,z) &= \sum_{k=0}^{\infty} Z_k(z) R_k(r) \Phi_k(\phi) && (72) \\ &= \sum_{k=0}^{\infty} Z_0 \left(\begin{cases} R(r) = C_0 + D_0\ \ln(r) & k = 0 \\ R(r) = C_k r^k + D_k r^{-k} & k \neq 0 \end{cases} \right) (A_k \sin(k\phi) + B_k \cos(k\phi)) && (73) \end{aligned}$$

Moving the index, renaming the coefficients, and distributing terms, we have the general solution for Laplace's equation in cylindrical coordinates as

$$\boxed{V(r,\phi,z) = Z_0\left(A_0 + B_0\,\ln(r) + \sum_{k=1}^{\infty} r^k(A_k\sin(k\phi) + B_k\cos(k\phi)) + r^{-k}(C_k\sin(k\phi) + D_k\cos(k\phi))\right)} \tag{74}$$

where we can simply absorb Z_0 to the other coefficients inside. An infinite line charge is not only z-symmetric but also ϕ symmetric. Hence, the all ϕ-dependent terms in the summation must all vanish. This leaves us with

$$V(r,\phi,z) = A_0 + B_0\,\ln(r) \tag{75}$$

which is the potential of a infinite line of charge using Gauss's law where we have absorbed Z_0. Now we proceed to solve a boundary value problem

> An infinitely long metal cylindrical pipe or radius R is immersed perpendicularly into an otherwise uniform electric field $\vec{E} = E_0\hat{x}$. Find V, $\vec{E}$, and induced charge.

Now, we can use our cylindrical solution to Laplace's equation. As usual, we first convert the word problem into boundary conditions. The metal cylindrical pipe is an equipotential so we set $V = 0$ at its surface. Integrating the constant field $\vec{E} = E_0\hat{x}$ using from the definition of potential : $V = -\int \vec{E}\cdot\, \mathrm{d}\vec{l}$, we get $V = -E_0x + C$. However, by cylindrical symmetry, the entire yz plane must be zero which means $V = -E_0x$. Then, boundary conditions are

$$\begin{cases} V(R,\phi,z) = 0 \\ V(\infty,\phi,z) = -E_0 s\cos(\phi) \end{cases} \tag{76}$$

Before directly applying the boundary conditions, we should clean the expression up first from some observations. Observe that we only need the $\cos((1)\phi)$ as a basis and dispose the rest. Disposing unusable terms such as $\sin(56\phi)$ or $\cos(133\phi)$, we need that

$$A_0 = B_0 = A_k = C_k = 0 \tag{77}$$

also, sparing the $k = 1$ case, for $k \neq 1$,

$$B_k = D_k = 0 \tag{78}$$

The expression reduces to

$$V(r,\phi) = (B_1 r + D_1 r^{-1})\cos(\phi) \tag{79}$$

The first condition tells us that

$$V(R,\phi) = (B_1 R + D_1 R^{-1})\cos(\phi) = 0 \iff D_1 = -B_1R^2 \tag{80}$$

Updating the potential expression,

$$V(r,\phi) = B_1(1 - R^2 r^{-1})\cos(\phi) \tag{81}$$

The second condition tells us that

$$V(\infty,\phi) = B_1\cos(\phi) = -E_0\cos(\phi) \iff B_1 = -E_0 \tag{82}$$

Hence, $B_1 = -E_0$ and $D_1 = E_0R^2$. Plugging this into Eq. (79), we now have our specific solution

$$\boxed{V(r,\phi) = \left(-E_0 r + \frac{E_0R^2}{r}\right)\cos(\phi)} \tag{83}$$

To find the induced surface charge, simply find $-\varepsilon_0\partial_r V$. That is,

$$\sigma = -\varepsilon_0 \frac{\partial}{\partial r}\left(-E_0 r + \frac{E_0 R^2}{r}\right)\cos(\phi)|r = R = \varepsilon_0 E_0(1+1)\cos(\phi) \tag{84}$$

Hence, the induced surface charge at the conductor is

$$\boxed{\sigma(\phi) = 2\varepsilon_0 E_0 \cos(\phi)} \tag{85}$$

The source of $\vec{E}$ is turned off leaving $\sigma(\phi) = a\sin(5\phi)$ on the surface. Find V and $\vec{E}$ inside and outside the pipe.

Again, we start from the general solution

$$V(r,\phi,z) = A_0 + B_0\ \ln(r) + \sum_{k=1}^{\infty} r^k(A_k \sin(k\phi) + B_k\cos(k\phi)) + r^{-k}(C_k\sin(k\phi) + D_k\cos(k\phi)) \tag{86}$$

To find the V inside and outside, we utilize no explosion boundary condition, continuity and derivative boundary condition - the amount of discontinuity $\vec{E}$ experiences at the surface. Inside, to avoid explosions at $r = 0$, we throw away $\ln(r)$ and r^{-1}. Hence,

$$V(r,\phi,z) = A_0 + \sum_{k=1}^{\infty} r^k(A_k\sin(k\phi) + B_k\cos(k\phi)) \tag{87}$$

Outside, we also avoid explosion at infinity by throwing away $\ln(r)$ and r. Hence,

$$V(r,\phi,z) = A_0' + \sum_{k=1}^{\infty} r^{-k}(C_k\sin(k\phi) + D_k\cos(k\phi)) \tag{88}$$

The potential expression can be summarized as

$$V = (r,\phi) = \begin{cases} V(r,\phi,z) = A_0 + \sum_{k=1}^{\infty} r^k(A_k\sin(k\phi) + B_k\cos(k\phi)) & r < R \\ V(r,\phi,z) = A_0' + \sum_{k=1}^{\infty} r^{-k}(C_k\sin(k\phi) + D_k\cos(k\phi)) & r > R \end{cases} \tag{89}$$

Recall that the surface charge density is given by

$$\sigma = -\varepsilon_0\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right) \tag{90}$$

Differentiating gives us

$$\sigma = \sin(5\phi) = -\varepsilon_0\left(-\frac{k}{R^{k+1}}(C_k\sin(k\phi) + D_k\cos(k\phi)) - kR^{k-1}(A_k\sin(k\phi) + B_k\cos(k\phi))\right) \tag{91}$$

Likewise, from our previous arguments, we throw away all terms except the $\sin(5\phi)$ term. Hence,

$$V = (r,\phi) = \begin{cases} A_0 + r^5 A_5\sin(5\phi) & r < R \\ A_0' + r^{-5}C_5\sin(5\phi) & r > R \end{cases} \tag{92}$$

The derivative condition gives us

$$a = -\varepsilon_0\left(-\frac{5}{R^6}C_5 - 5R^4 A_5\right) = 5\varepsilon_0\left(\frac{1}{R^6}C_5 + R^4 A_5\right) \tag{93}$$

Continuity gives us

$$A_0 + R^5 A_5 = A_0' + R^{-5}C_5 \tag{94}$$

Due to linear independence, we have $A_0 = A_0'$ and we see that $C_5 = R^{10}A_5$. Plugging this into the derivative condition,

$$a = 5\varepsilon_0 \left(R^4 A_5 + R^6 A_5\right) \iff A_5 = \frac{a}{10\varepsilon_0 R^4} \tag{95}$$

and hence

$$C_5 = \frac{aR^6}{10\varepsilon_0} \tag{96}$$

For convenience, we set $A_0 = A_0' = 0$. Finally, the potential is given as

$$\boxed{V = (r,\phi) = \frac{a\sin(5\phi)}{10\varepsilon_0}\begin{cases}(r^5/R^4) & r < R\\ (R^6/r^5) & r > R\end{cases}} \tag{97}$$

Taking the gradient immediately gives us $\vec{E}$.

$$\vec{\nabla}\left(\frac{a}{10\varepsilon_0 R^4} r^5 \sin(5\phi)\right) = \frac{5ar^4}{10\varepsilon_0 R^4}(\sin(5\phi)\hat{r} + \cos(5\phi)\hat{\phi}) \tag{98}$$

also

$$\vec{\nabla}\left(\frac{aR^6}{10\varepsilon_0} r^{-5} \sin(5\phi)\right) = \frac{5aR^6}{10\varepsilon_0 r^6}(\sin(5\phi)\hat{r} + \cos(5\phi)\hat{\phi}) \tag{99}$$

Hence, the $\vec{E}$ is

$$\boxed{\vec{E}(r,\phi) = \frac{5a}{10\varepsilon_0}(\sin(5\phi)\hat{r} + \cos(5\phi)\hat{\phi})\begin{cases}(r/R)^4 & r < R\\ (R/r)^6 & r > R\end{cases}} \tag{100}$$

4. **A metal inside a shell**

A conducting sphere of radius a at potential V_0 is surrounded by a thin concentric spherical shell of radius b over which someone has glued a surface charge $\sigma(\theta) = k\cos(\theta)$ where k is a constant and θ is the polar angle.
a) Find V in each region
b) Find $\vec{E}$ in each region
c) Find the induced surface charge $\sigma(\theta)$ on the conductor
d) What is the total charge of this system? Check the limits.

Due to an obvious spherical symmetry, we will be using spherical coordinate solution the Laplace's equation. Specifically, we can use the azimuthally symmetric solution. That is,

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta)) \tag{101}$$

Next task is to identify the boundary conditions of the system. Solving Laplace's equation is akin to solving Schrodinger equation in various regions of a finite well, throwing away non-normalizable terms, and then tying them with cute little knots in each boundaries. We start from the innermost region ($r < a$). Of course, this is a conductor implying a constant potential region. Hence,

$$V(r < a, \theta) = V_0 \tag{102}$$

At the region in between, we have

$$V(a < r < b, \theta) = \sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos(\theta)) \tag{103}$$

Lastly, outside the thin shell, we want the potential to vanish at infinity. $A_l r^l$ explodes at infinity so we will throw it away. This is remiscent to throwing away non-normalizable terms in the wavefunction. Hence,

$$V(r > b, \theta) = \sum_{l=0}^{\infty} \frac{D_l}{r^{l+1}} P_l(\cos(\theta)) \tag{104}$$

Collecting them, we have

$$V(r,\theta) = \begin{cases} V_0 & r < a \\ \sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos(\theta)) & a < r < b \\ \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos(\theta)) & r > b \end{cases} \tag{105}$$

Observe that we have three arbitrary parameters to "spend". Recall from solving the wavefunction that we have to "tie" this together at the boundaries. Hence, we require them to be continuous at $r = a$ and $r = b$. However, they need not be smooth like in QM as $\vec{E}$ are generally discontinuous at the boundaries. This discontinuity (of the normal component) can be properly quantified: $E_+ - E_- = \sigma(\theta)/\varepsilon_0$. In terms of potentials, $\partial_r V_+ - \partial_r V_- = -\sigma(\theta)/\varepsilon_0$. Luckily, spherical nature of the surface assures that the field is entirely radial. Differentiating,

$$\frac{\partial}{\partial r} \left(\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos(\theta)) \right) - \frac{\partial}{\partial r} \left(\sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l(\cos(\theta)) \right) = -\frac{\sigma(\theta)}{\varepsilon_0} \tag{106}$$

$$\sum_{l=0}^{\infty} (-(l+1)) \frac{B_l}{r^{l+2}} P_l(\cos(\theta) - \sum_{l=0}^{\infty} \left(C_l l r^{l-1} - (l+1) \frac{D_l}{r^{l+2}} \right) P_l(\cos(\theta)) = -\frac{k}{\varepsilon_0} P_1(\cos(\theta)) \tag{107}$$

In summary, our boundary conditions are

$$\begin{cases} V_0|_a = \left(\sum_{l=0}^{\infty}\left(C_l r^l + \frac{D_l}{r^{l+1}}\right)P_l(\cos(\theta))\right)_a \\ \left(\sum_{l=0}^{\infty}\left(C_l r^l + \frac{D_l}{r^{l+1}}\right)P_l(\cos(\theta))\right)_b = \left(\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos(\theta))\right)|_b \\ \sum_{l=0}^{\infty}(-(l+1))\frac{B_l}{r^{l+2}}P_l(\cos(\theta) - \sum_{l=0}^{\infty}\left(C_l l r^{l-1} - (l+1)\frac{D_l}{r^{l+2}}\right)P_l(\cos(\theta)) = -\frac{k}{\varepsilon_0}P_1(\cos(\theta)) \end{cases} \tag{108}$$

Observe that we have to analyze three general cases: $l = 0$, $l = 1$, and $l > 1$. Now, we solve for the coefficients in Eq. (105) subject to the conditions in Eq. (108). As advised by Griffiths, we can do away with Fourier's trick and use "eyeballing" instead. Observe that the summation only "turns" on for a single value of l for the first and third condition. This tells us that we need to analyze different cases of l. From eyeballing, the first conditions tells us

$$V_0 = \sum_{l=0}^{\infty}\left(C_l a^l + \frac{D_l}{a^{l+1}}\right)P_l(\cos(\theta)) \tag{109}$$

Legendre polynomials can only be a constant when $l = 0$. Hence, this kills of all terms except $l = 0$ giving us two cases:

$$\begin{cases} C_0 a^0 + \frac{D_0}{a} = V_0 & l = 0 \\ C_l a^l + \frac{D_0}{a^{l+1}} = 0 & l \neq 0 \end{cases} \tag{110}$$

The second condition is more general posing no separate cases since same basis can be seen on both sides

$$\frac{B_l}{b^{l+1}} = C_l b^l + \frac{D_l}{b^{l+1}} \iff B_l = b^{2l+1}C_l + D_l \tag{111}$$

The third condition gives us

$$\begin{cases} (-(l+1))\frac{B_l}{r^{l+2}}P_l(\cos(\theta) - \left(C_l l r^{l-1} - (l+1)\frac{D_l}{r^{l+2}}\right)P_l(\cos(\theta)) = 0 & l \neq 1 \\ C_1 + \frac{2}{b^3}(B_1 - D_1) = \frac{k}{\varepsilon_0} & l = 1 \end{cases} \tag{112}$$

The conditions now read

$$\begin{cases} C_0 a^0 + \frac{D_0}{a} = V_0 & l = 0 \\ C_l a^l + \frac{D_0}{a^{l+1}} = 0 & l \neq 0 \\ B_l = b^{2l+1}C_l + D_l & \\ (-(l+1))\frac{B_l}{r^{l+2}}P_l(\cos(\theta) - \left(C_l l r^{l-1} - (l+1)\frac{D_l}{r^{l+2}}\right)P_l(\cos(\theta)) = 0 & l \neq 1 \\ C_1 + \frac{2}{b^3}(B_1 - D_1) = \frac{k}{\varepsilon_0} & l = 1 \end{cases} \tag{113}$$

We will refer to the first two equations as "C-D" equations and the last two as "B-C-D" equations. The first three conditions can be combined to form "B-C" equations

$$\begin{cases} B_l = (b^{2l+1} - a^{2l+1})C_l & l \neq 0 \\ B_0 = (b - a)C_0 + aV_0 \end{cases} \tag{114}$$

Plugging the general B-C equation and C-D equation into the general B-C-D equation gives us

$$(2l+1)C_l = 0 \iff C_l = 0 \tag{115}$$

From B-C and D-C equations, it follows that

$$\boxed{B_l = C_l = C_l = 0 \qquad l > 1} \tag{116}$$

Hence, we conclude that l can only take two values: $l = 0$ and $l = 1$. Plugging the B-C equation and C-D equation into the general B-C-D equation for $l = 1$ gives us

$$C_1 + 2C_1 = k \tag{117}$$
$$D_1 = -a^3 C_1 \tag{118}$$
$$B_1 = (b^3 - a^3)C_1 \tag{119}$$

which, then, gives us

$$\boxed{B_1 = \frac{(b^3 - a^3)k}{3\varepsilon_0} \qquad C_1 = \frac{k}{3\varepsilon_0} \qquad D_1 = \frac{-a^3 k}{3\varepsilon_0}} \tag{120}$$

Doing the similar substitutions (B-C and C-D into B-C-D) and letting the dust settle, we arrive at

$$\boxed{B_0 = aV_0 \qquad C_0 = 0 \qquad D_0 = aV_0} \tag{121}$$

Plugging all of these results into (105), we have

$$\boxed{V(r,\theta) = \begin{cases} V_0 & r < a \\ \frac{aV_0}{r} + \frac{k}{3\varepsilon_0}\left(r - \frac{a^3}{r^2}\right)\cos(\theta) & a < r < b \\ \frac{aV_0}{r} + \frac{b^3 - a^3}{3r^2\varepsilon_0}\cos(\theta) & r > b \end{cases}} \tag{122}$$

To find $\vec{E}$, one can simply take the gradient of V. Taking the gradient in $r \leq a$, the constant vanishes. This makes sense as there can be no field inside a conductor. Taking the gradient at $a < r < b$,

$$\vec{E} = -\vec{\nabla}\left(\frac{aV_0}{r} + \frac{k}{3\varepsilon_0}r - \frac{k}{3\varepsilon_0}\frac{a^3}{r^2}\cos(\theta)\right) \tag{123}$$
$$= -\vec{\nabla}\frac{aV_0}{r} - \vec{\nabla}\frac{k}{3\varepsilon_0}r\cos(\theta) + \vec{\nabla}\frac{k}{3\varepsilon_0}\frac{a^3}{r^2}\cos(\theta) \tag{124}$$
$$= -\frac{\partial}{\partial r}\frac{aV_0}{r}\hat{r} - \frac{\partial}{\partial r}\frac{k}{3\varepsilon_0}r\cos(\theta)\hat{r} + \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}\right)\frac{k}{3\varepsilon_0}\frac{a^3}{r^2}\cos(\theta) \tag{125}$$
$$= \left(\frac{aV_0}{r^2} - \frac{k}{3\varepsilon_0}\left(1 + \frac{2a^3}{r^3}\right)\cos(\theta)\right)\hat{r} - \left(\frac{ka^3}{3\varepsilon_0 r}\sin(\theta)\right)\hat{\theta} \tag{126}$$

Similarly for $r > b$,

$$\vec{E} = -\vec{\nabla}\left(\frac{aV_0}{r} + \frac{b^3 - a^3}{3r^2\varepsilon_0}\cos(\theta)\right) \tag{127}$$
$$= -\frac{\partial}{\partial r}\frac{aV_0}{r} - \left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta}\right)\frac{b^3 - a^3}{3r^2\varepsilon_0}\cos(\theta) \tag{128}$$
$$= \left(\frac{aV_0}{r^2} + \frac{2(b^3 - a^3)}{3r^3\varepsilon_0}\cos(\theta)\right)\hat{r} - \frac{b^3 - a^3}{3r^2\varepsilon_0}\sin(\theta)\hat{\theta} \tag{129}$$

Hence,

$$\boxed{V(r,\theta) = \begin{cases} 0 & r < a \\ \left(\frac{aV_0}{r^2} - \frac{k}{3\varepsilon_0}\left(1 + \frac{2a^3}{r^3}\right)\cos(\theta)\right)\hat{r} - \left(\frac{ka^3}{3\varepsilon_0 r}\sin(\theta)\right)\hat{\theta} & a < r < b \\ \left(\frac{aV_0}{r^2} + \frac{2(b^3 - a^3)}{3r^3\varepsilon_0}\cos(\theta)\right)\hat{r} - \frac{b^3 - a^3}{3r^2\varepsilon_0}\sin(\theta)\hat{\theta} & r > b \end{cases}} \tag{130}$$

Note that at $r = a$,

$$-\frac{\partial}{\partial r} = \frac{aV_0}{a^2} - \frac{k}{3\varepsilon_0}\left(1 + \frac{2a^3}{a^3}\right)\cos(\theta) = \frac{V_0}{a} - \frac{k}{\varepsilon_0}\varepsilon_0 \tag{131}$$

Hence, since the induced charge on a conductor is $\sigma = -\varepsilon_0 \partial_r V$,

$$\boxed{\sigma(\theta) = V_0 \frac{\varepsilon_0}{a} - k\cos(\theta)} \tag{132}$$

Lastly, the total induced charge is

$$Q = \int \sigma(\theta)\ \mathrm{d}a = \int V_0 \frac{\varepsilon_0}{a}\ \mathrm{d}a - \int k\cos(\theta)\ \mathrm{d}a \tag{133}$$

Of course, integrating a cosine function gives a sine function and evaluating from 0 to π gives zero. Hence, integrating over a sphere,

$$Q = V_0 \frac{\varepsilon_0}{a} \int\ \mathrm{d}a = V_0 \frac{\varepsilon_0}{a}(4\pi a^2) \tag{134}$$

Then, the total induced charge is

$$\boxed{Q = 4\pi a \varepsilon_0 V_0} \tag{135}$$

Plugging in the potential of a spherical shell (which looks like a point charge), one can observe that this is simply the total charge. Observing that in the limiting case of large r, from Eq. (122),

$$V(r) = \frac{aV_0}{r} \tag{136}$$

since the second term vanishes faster. To confirm, we analyze the limiting case where we can express the potential of a sphere exactly like a point charge. Then,

$$V(r) = \frac{1}{4\pi\varepsilon_0} \frac{4\pi a \varepsilon_0 V_0}{r} = \frac{aV_0}{r} \tag{137}$$

$\boxed{\text{Indeed, the charge induced is the total charge.}}$

5. **A uniformly distributed charge**

A charge +Q is distributed uniformly along the z-axis from z=-a to z=+a. Derive the electric potential up to the 5th term.

Recall that the multipole expansion is given by

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos(\theta))\rho(\vec{r}')\ \mathrm{d}\tau' \tag{138}$$

We need the first five terms of the expansion corresponding the five integral evaluations. The charge density was given by $\lambda = Q/L = Q/2a$ where our integration path ranges from $z = -a$ to $z = a$. Moreover, $r' = z$ only since $\vec{r}' = (0, 0, z)$ in cylindrical basis. The expression then reduces to

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{P_n(\cos(\theta))\lambda}{r^{(n+1)}}\int_{-a}^{a} z^n\ \mathrm{d}z \tag{139}$$

Integration is trivial

$$\int_{-a}^{a} z^n \mathrm{d}z = \frac{z^{n+1}}{n+1}\Big|_{-a}^{a} = \frac{a^{n+1}}{n+1} - \frac{(-a)^{n+1}}{n+1} \tag{140}$$

Observe that when n is odd, the exponent is even and the two terms cancel out while they combine when n is even. Hence,

$$\int_{-a}^{a} z^n \mathrm{d}z = \begin{cases} \frac{2a^{n+1}}{n+1} & \text{n is even} \\ 0 & \text{n is odd} \end{cases} \tag{141}$$

Updating our potential expression,

$$V(\vec{r}) = \frac{\lambda}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{P_n(\cos(\theta))}{r^{(n+1)}}\begin{cases} \frac{2a^{n+1}}{n+1} & \text{n is even} \\ 0 & \text{n is odd} \end{cases} \tag{142}$$

Letting $\lambda = Q/2a$ and reindexing to ignore odd indices,

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\sum_{n=0,2,4\ldots}^{\infty}\frac{P_n(\cos(\theta))}{r^{(n+1)}}\frac{2a^n}{n+1} \tag{143}$$

Of course, from our previous experience for line charges, they approximate a point charge potential from large distance. We make our expression more suggestive by factoring out $1/r$ and also to neat out expression inside the sum. This gives us with a neat expression

$$\boxed{V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\sum_{n=0,2,4\ldots}^{\infty}\left(\frac{a}{r}\right)^n\frac{P_n(\cos(\theta))}{n+1}} \tag{144}$$

To find a five-term approximation, we simply have to instantiate five summand terms.

$$\boxed{V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\left(1 + \left(\frac{a}{r}\right)^2\frac{P_2(\cos(\theta))}{3} + \left(\frac{a}{r}\right)^4\frac{P_4(\cos(\theta))}{5} + \left(\frac{a}{r}\right)^6\frac{P_6(\cos(\theta))}{7} + \left(\frac{a}{r}\right)^8\frac{P_8(\cos(\theta))}{9}\right)} \tag{145}$$

where we have resolve the first Legendre term to be more suggestive that the other terms are corrective approximations.

6. **A pendulum charge**

An ideal electric dipole is situated at the origin and points in the z direction. An electric charge is released from rest at a point in the xy plane. Show that it swings back and forth in a semi-circular arc as though it were a pendulum supported at the origin.

Recall that an ideal dipole pointing at the z-direction produces a field given by

$$\vec{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}) \tag{146}$$

and hence, the force on that particle with charge q is

$$\vec{F}(r,\theta) = \frac{qp}{4\pi\varepsilon_0 r^3}(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}) \tag{147}$$

Consider the force on the pendulum

$$\vec{F} = -mg\hat{z} - T\hat{r} \tag{148}$$

To conform with the given basis, we transform $\hat{z}$ in spherical coordinates

$$\hat{z} = \cos(\theta)\hat{r} - \sin(\theta)\hat{\theta} \tag{149}$$

Updating the pendulum force, we have

$$\vec{F} = -mg(\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}) - T\hat{r} = -(T + mg\cos(\theta))\hat{r} + mg\sin(\theta)\hat{\theta} \tag{150}$$

Lastly, we realize that T must be angle dependent and proceed to derive $T = T(\theta)$. Given the diagram below,

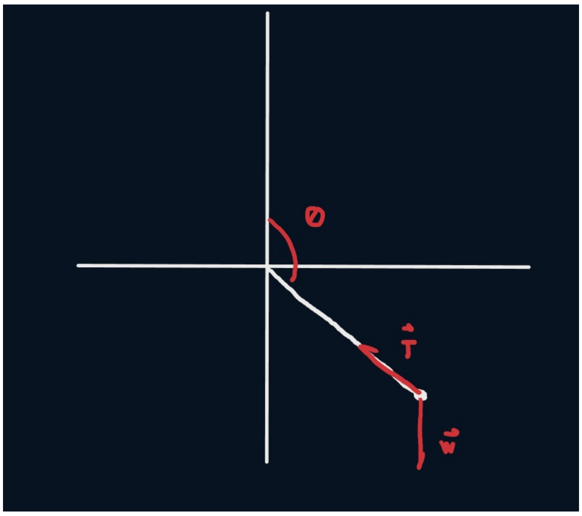

Figure 4: Free body diagram of a pendulum

By conservation of energy, in the domain $\theta \in [\pi/2, -\pi/2]$,

$$mgl\cos(180-\theta) = \frac{1}{2}mv^2 \iff v^2(\theta) = -2gl\cos(\theta) \tag{151}$$

we see that the radial part of the weight is $-mg\cos(180-\theta) = mg\cos(\theta)$ and since tension is entire radial, the centripetal force is

$$T + mg\cos(\theta) = \frac{mv^2}{l} \tag{152}$$

Plugging in the expression for $V(\theta)$. we then get

$$T(\theta) = -3mg\cos(\theta) \tag{153}$$

Finally, plugging this into Eq. (151), we get

$$\boxed{\vec{F}(r,\theta) = mg(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta})} \tag{154}$$

which is mathematically similar to Eq. (147) aside from the physical constants thus exhibiting the same behavior.

2 Reflection

The problem was certainly more interesting than the previous one but it also certainly was more time consuming. Finally, I get to solve electrostatic problems using differential form - mainly Laplace's equation. In summary, to solve electrostatics problem in a vacuum, we simply solve Laplace's equation and invoke boundary conditions specific for the problem. The solution is unique, and hence, can be exploited in some cases by conjuring up images of the system that induces similar boundary conditions.

- **Problem 1**
  Personally, the first problem was the hardest. Although finding the potentials of the line of charge was straightforward by using Gauss's law, it is in the next problem that took me some time to finish. The task is to utilize method of images to find the potential when a conducting cylinder is placed between two lines of charge. The problem lies in the auxillary problem that I couldn't easily solve. Griffiths made no attempt to show how the R^2/a term was derived in the spherical case. It turns out that this was also the distance of the image in the cylindrical case. I wonder if there is an underlying principle covering both.
- **Problem 2**
  The second problem is straightforward. Solving Laplace's equation in Cartesian coordinate is simple. However, the separation constant must be chosen so that it could fit the stability of the boundary condition (oscillating, exponential). Lastly, Fourier's trick will directly give us the coefficients. I have been using Fourier's trick for my quantum mechanics problem sets in terms of Dirac notation so I have attempted to use them. I find them really intuitive to use as you can really see the exploitation of orthonormality. However, I can't seem to recast them into such notation.
- **Problem 3** Again, separation of variables is relatively straightforward to do. Axial symmetry lets us resolve only two equations: polar and radial. The polar equation is the trivial SHO. Although the radial equation was intimidating at first sight, one can quickly observe that it is simply a Cauchy-Euler equation. Putting the solutions together, a general solution was derived to solve the follow-up problem. This problem showcased that fact that we don't need to use Fourier's trick all the time as boundary conditions can oftentimes be "eyeballed".
- **Problem 4** Although this problem was relatively long, this was the most fulfilling for me to solve. Currently, my two main courses are QM and EM. This problem really allowed me to make connections and generalize methods by analogies. Solving Laplace's equation is similar to solving Schrodinger equation. The boundary conditions, both Dirichlet and Neumann, can be found in both field. Assuring normalizability, ensuring continuity, and invoking smoothness conditions, these have direct analogies to the problem.
- **Problem 5** This problem was the easiest of all. It made me somehow anxious that it has a great amount of bounty attached to it. I find multipole expansion really cool. This is the first time I have expanded something not in terms of traditional basis such as $x, x^2, ...$ or trigonometric/exponential ones, but in terms of special function in a physical scenario. I look forward to applying this as an approximation method for complicated distributions in the future.
- **Problem 6** I rate this to be as easy as problem 5 so it might also be the easiest. The task was also straightforward, short, and simple. However, I find it interesting. This added to my collection of somehow abstract phenomena related to classical stuffs such as the quantum harmonic oscillator.

Overall, the tasks were relatively straightforward except for the auxillary problem in Problem 1 which took me some time to derive the R^2/a condition. I'll look for some general methods and symmetry properties for methods of images as I find it the most unintuitive topic in the chapter. The problem set was excellently curated and deepened my understandings in EM. Although this took some time to finish, it was worth it.