1 Problem Set 5: Magnetostatics

1. **A moving charge near a long wire**

> Analyze the motion of a particle (charge q, mass m) in the magnetic field of a long straight wire carrying a steady current I.
> (a) Is its kinetic energy conserved?
> (b) Find the force on the particle, in cylindrical coordinates, with I along the z axis
> (c) Obtain the equations of motion
> (d) Suppose $\partial_t z$ is constant. Describe its motion.

We are tasked to analyze the motion of a particle subject to the magnetic field of a straight wire configuration. In this problem set, we assume that $\vec{F}$ is solely from magnetic field (unless specified that there exists an electrostatic interaction). Obviously, since

$$\vec{F} \propto \vec{v} \times \vec{B} \iff \vec{F} \cdot \vec{v} \propto (\vec{v} \cdot \vec{B}) \cdot \vec{v} = 0 \tag{1}$$

Thus, magnetic fields must do no work and from Work-KE theorem,

$$\boxed{\text{Kinetic energy is conserved}}$$

Recall that for both electric and magnetic, it is divided into two parts: finding the cause (field calculations) and effect (force calculations). The latter is governed by Lorentz force law telling us that the force exerted by a magnetic field on a particle is expressed as

$$\vec{F} = q(\vec{v} \times \vec{B}) \tag{2}$$

Now, our task is to express vectors $\vec{v}$ and $\vec{B}$ in cylindrical coordinates. The position vector in this coordinate system is expressed

$$\vec{r} = s\hat{s} + z\hat{z} \tag{3}$$

Differentiating gives us the velocity

$$\dot{\vec{r}} = \vec{v} = \frac{\mathrm{d}}{\mathrm{d}t}(s\hat{s} + z\hat{z}) = \dot{s}\hat{s} + \dot{z}\hat{z} + s\frac{\mathrm{d}\hat{s}}{\mathrm{d}t} + z\frac{\mathrm{d}\hat{z}}{\mathrm{d}t} \tag{4}$$

However, recall that the coordinate transformation from z-cartesian coordinate to z-cylindrical coordinate is an identity transformation. Moreover, all cartesian basis vectors are constant. Hence, $D_t\hat{z} = 0$. Differentiating the coordinate transformation of $\hat{s}$ into cartesian,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{s} = \frac{\mathrm{d}}{\mathrm{d}t}(\cos\,\phi\hat{x} + \sin\,\phi\hat{y}) = \dot{\phi}\hat{\phi} \tag{5}$$

Then, the velocity in cylindrical coordinates is expressed as

$$\vec{v} = \dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \tag{6}$$

In anticipation to find the equation of motion later, we differentiate further to find the acceleration in cylindrical coordinates

$$\vec{a} = \ddot{s}\hat{s} + \dot{s}\frac{\mathrm{d}\hat{s}}{\mathrm{d}t} + \dot{s}\dot{\phi}\hat{\phi} + s\ddot{\phi}\hat{\phi} + s\dot{\phi}\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}t} + \ddot{z}\hat{z} \tag{7}$$

Using similar unit vector derivatives, we have

$$\vec{a} = (\ddot{s} - s\dot{\phi}^2)\hat{s} + (s\ddot{\phi} + 2\dot{s}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z} \tag{8}$$

Now, we proceed to find the expression of the magnetic field produced by a straight wire configuration in cylindrical coordinates. Recall that we can use Biot-Savart law to easily find the magnetic field from a long wire. However, in building up the tools needed from proper execution of Ampere's law (compare with

Gauss's law), we have already accumulated results from different configuration prototypes: lines, planes, solenoids, and toroids. Assuming that the line is long enough, the magnetic field is expressed as

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi} \tag{9}$$

To find the equation of motion, we start by finding the force on the particle. Using Lorentz force law,

$$\vec{F} = q(\vec{v} \times \vec{B}) = \frac{\mu_0 Iq}{2\pi s}(\dot{s}\hat{s} + s\dot{\phi}\hat{\phi} + \dot{z}\hat{z} \times \hat{\phi}) = \frac{\mu_0 Iq}{2\pi s}\begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ \dot{s} & s\dot{\phi} & \dot{z} \\ 0 & 1 & 0 \end{vmatrix} \tag{10}$$

Evaluation of the determinant gives us

$$\boxed{\vec{F} = \frac{\mu_0 Iq}{2\pi s}(-\dot{z}\hat{s} + \dot{s}\hat{z})} \tag{11}$$

Using Newton's second law, $\vec{F} = m\vec{a}$, gives us a vector equation

$$\frac{\mu_0 Iq}{2\pi s}(-\dot{z}\hat{s} + \dot{s}\hat{z}) = m((\ddot{s} - s\dot{\phi}^2)\hat{s} + (s\ddot{\phi} + 2\dot{s}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z}) \tag{12}$$

Equality of vector implies equality of coordinates in a given basis. This gives us three differential equations

$$\boxed{\ddot{s} - s\dot{\phi}^2 + \frac{\mu_0 Iq}{2\pi ms}\dot{z} = 0} \qquad \boxed{s\ddot{\phi} + 2\dot{s}\dot{\phi} = 0} \qquad \boxed{\ddot{z} - \frac{\mu_0 Iq}{2\pi ms}\dot{s} = 0} \tag{13}$$

These are the equations of motion of a particle near the infinite wire. Not only are the differential equations coupled but are also nonlinear. Dimensional analysis tells us that the parameter $\mu_0 Iq/2\pi ms$ has dimensions $[[1/T]]$. Assuming that this parameter is small, one may find a perturbative solution to the system. One can also analyze the Jacobian at fixed points for a dynamical system approach of analysis or find a cheap solution via RK4. [1] Nonetheless, the three unsolved differential equations already constitute a valid set of equations of motion so I assume we can stop here. What we can do, though, is to assume that $\ddot{z} = 0$. This gives us the following simplifying implications: $\dot{s} = 0$, $\ddot{s} = 0$, $s =$ constant. While the third one became a tautology, we are left with two (equivalent) equations of motion,

$$\dot{\phi} = \frac{1}{s}\sqrt{\frac{\mu_0 Iq}{2\pi ms}\dot{z}} \qquad \ddot{\phi} = 0 \tag{14}$$

Observe that the RHS of the first equation are all constants. Since there is no radial motion, but has constant angular rotation and a constant azimuthal translation, the particle will inevitably undergo a helical motion. Thus, under the assumption that $\ddot{z} = 0$,

$$\boxed{\text{The magnetic field from a long straight wire subjects the particle to a helical path.}} \tag{15}$$

[1]These are left as an exercise for the writer after the semester ends. If this was left undone, the writer have enjoyed their holidays.

2. **Calculating with vector potential**

Find the magnetic vector potential of a finite segment of straight wire carrying a current I.

One way to find the magnetic vector potential is to solve the three-dimensional Poisson's equation

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \tag{16}$$

However, since $\vec{A}$ vanishes at infinity since we are dealing with a finite segment of straight wire. This licenses us to avoid solving the differential equation and simply use its" solution" in integral form

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r})'}{|\vec{r} - \vec{r}'|} \, \mathrm{d}\tau' \tag{17}$$

Of course, we use linear density for a line current. Integrating from a finite interval $[z_1, z_2]$ and noting that $\hat{z}$ is a constant unit vector,

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_{z_1}^{z_2} \frac{1}{|\vec{r} - \vec{r}'|} \, \mathrm{d}\vec{l}' = \frac{\mu_0 I}{4\pi} \hat{z} \int_{z_1}^{z_2} \frac{1}{(r^2 + r'^2 - 2rr' \cos\gamma)^{1/2}} \, \mathrm{d}z' \tag{18}$$

Note that we ultimately need to parametrize the result with the polar angle. With this, we shall move the origin to the field point instead, switch coordinates from cylindrical to spherical, then analyze the vector potential at the origin. With this system, the position vectors in spherical basis are simply $\vec{r} = (0, 0, 0)^T$ and $\vec{r}' = (r', 0, 0)^T$. From the figure, r runs from $z_1/\cos\,\theta_1$ to $z_2/\cos\,\theta_2$. Moreover, $z = s\,\tan\,\theta$. Hence, r run from $s\,\tan\,\theta_1\,\sec\,\theta_1$ to $s\,\tan\,\theta_2\,\sec\,\theta_2$. However, in this coordinate system, we are integrating along a space curve in two-dimensions. [2].

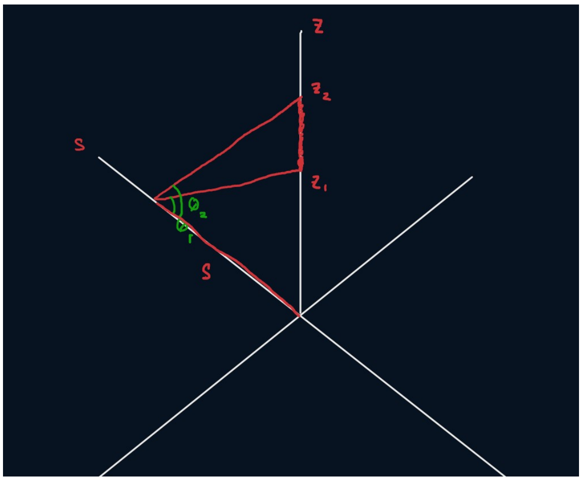

Figure 1: Observe that we are only dealing with $\phi = 0$

There is a more direct way to do this. Upon inspection of Griffiths Eq. 5.37, one can see that the derivation rests on the assumption that point P lies on the xy plane although it was not clearly stated. Same distance s but displaced from the xy plane must result to a different B-field; a single variable s alone can not fully specify the B-field and an azimuthal distance must be specified (in this case, field point is taken at $z = 0$). Hence, we are actually dealing with a constraint that field points and source points are always perpendicular to each other, $\cos\,\gamma = 0$, where $\vec{r} = (s, 0, 0)^T$ and $\vec{r}' = (0, 0, z)^T$ in cylindrical basis. If our task is ultimately to conform with the results of Eq. 5.67, we can lose some generality for the sake of simplification. The integral now becomes

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \hat{z} \int_{z_1}^{z_2} \frac{1}{(s^2 + z'^2)^{1/2}} \, \mathrm{d}z' \tag{19}$$

This is a one-dimensional integral that is trivial to evaluate. Trigonometric substitution $z = s\,\tan\theta$, $\mathrm{d}z = s\,\sec^2\theta$, $(s^2 + z'^2)^{1/2} = s\,\sec\theta$ transforms the integral to

[2] I'm not sure if this would work as I haven't tried this approach although it is probably similar to the one-dimensional method.

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\hat{z}\int_{\theta_1}^{\theta_2} \sec\,\theta'\ \mathrm{d}\theta' \tag{20}$$

Observe that the trigonometric substitution angle here is not merely a mathematical convenience but a physical angular quantity. In fact, it is the polar angle θ from Figure 1. Hence, the limits simply range from θ_1 to θ_2 eliminating the need for limit transformations. Evaluating the integral gives us the vector potential parametrized in terms of the angle made by the ends of the wire to the field point

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\hat{z}\frac{1}{2}\ \ln|\frac{(1+\sin\theta_2)(1-\sin\theta_1)}{(1-\sin\theta_2)(1+\sin\theta_1)}|} \tag{21}$$

where $\theta_2 = \tan^{-1}(z_2/s)$ and $\theta_1 = \tan^{-1}(z_1/s)$ since $z = s\ \tan\theta$. By trigonometric identity, we have $\sin\theta = z/\sqrt{z^2+s^2}$. Observe that the vector potential expression is actually a function of s only. Hence, taking the curl is cylindrical coordinates means taking only the partial derivative with respect to s. By chain rule,

$$\frac{\partial A}{\partial s} = \frac{\partial A}{\partial \sin\theta_1}\frac{\partial \sin\theta_1}{\partial s} + \frac{\partial A}{\partial \sin\theta_2}\frac{\partial \sin\theta_2}{\partial s} \tag{22}$$

Finding the s derivatives,

$$\frac{\partial \sin\theta_1}{\partial s} = -s\frac{\sin\theta_1}{s^2+z_1^2} \qquad \frac{\partial \sin\theta_2}{\partial s} = -s\frac{\sin\theta_2}{s^2+z_2^2} \tag{23}$$

Then,

$$\frac{\partial A}{\partial s} = -s\left(\frac{\sin\theta_1}{s^2+z_1^2}\frac{\partial A}{\partial \sin\theta_1} + \frac{\sin\theta_2}{s^2+z_2^2}\frac{\partial A}{\partial \sin\theta_2}\right) \tag{24}$$

Taking the sine derivatives and letting the dust settle, the partial derivatives eventually simplifies to

$$\frac{\partial A}{\partial \sin\theta_1} = \frac{2}{-1+\sin^2\theta_1} \qquad \frac{\partial A}{\partial \sin\theta_2} = -\frac{2}{-1+\sin^2\theta_2} \tag{25}$$

Then,

$$\frac{\partial A}{\partial s} = -2s\left(\frac{\sin\theta_1}{s^2+z_1^2}\frac{1}{-1+\sin^2\theta_1} - \frac{\sin\theta_2}{s^2+z_2^2}\frac{1}{-1+\sin^2\theta_2}\right) \tag{26}$$

Taking the curl to find the magnetic field,

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\partial A}{\partial s}\hat{\phi} = \frac{\mu_0 I}{4\pi}s\left(\frac{\sin\theta_1}{s^2+z_1^2}\frac{1}{-1+\sin^2\theta_1} - \frac{\sin\theta_2}{s^2+z_2^2}\frac{1}{-1+\sin^2\theta_2}\right)\hat{\phi} \tag{27}$$

Observe that

$$\frac{s}{s^2+z^2}\frac{1}{-1+\sin^2\theta} = \frac{s}{s^2+z^2}\frac{1}{(-\cos^2\theta)} = -\frac{1}{s}\left(\frac{s^2}{\sqrt{s^2+z^2}^2}\frac{1}{\cos^2\theta}\right) = -\frac{1}{s} \tag{28}$$

Plugging this in, the magnetic field simplifies to

$$\boxed{\vec{B} = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)\hat{\phi}} \tag{29}$$

which gives us the expression from Eq. 5.37 of Griffiths.

3. **A spinning charged shell**

A spherical shell with radius R, carrying a uniform surface charge σ, is set spinning at angular velocity ω
(a) Find the vector potential it produces at point a. That is when $r < R$ and when $r > R$.
(b) Use your result to find the magnetic field inside the solid sphere of uniform charge density ρ and radius R, that is rotating at a constant angular velocity ω.
(c) Calculate the magnetic force of attraction between the northern and southern hemispheres of a spinning charged spherical shell.

We are tasked to find the vector potential of a spinning spherical shell. The configuration is physically realistic and does not extend to infinity. This licenses us to use the integral formulation of finding the vector potential. The configuration is given below.

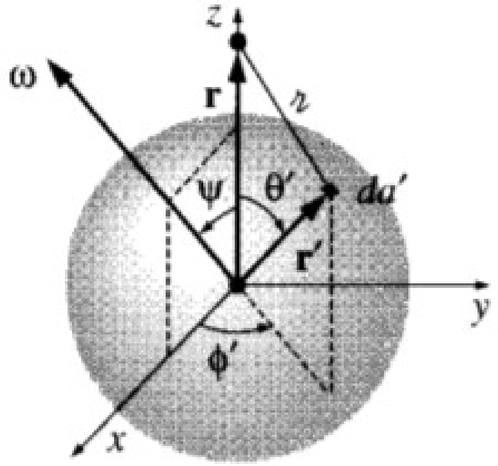

Figure 2: Taken from Griffiths which gave us some guidance for proper alignment of angles

Recall that for a surface current,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{|\vec{r}-\vec{r}'|}\,\mathrm{d}a' \tag{30}$$

If the surface charge is uniform, the surface current density depends only on the tangential velocity of the charge. Using the cosine law for the unit vector and using a spherical Jacobian, the integral becomes

$$\vec{A}(\vec{r}) = \frac{\mu_0\sigma}{4\pi}\int \frac{\vec{v}}{(R^2+r^2-2rr'\cos\gamma)^{1/2}}R^2\,\sin\theta'\mathrm{d}\theta'\mathrm{d}\phi' \tag{31}$$

Note that the source charges are only located at $r' = R$. With the existence of the $\vec{v}$, there is a lot of angles of interest - separation angle, angles of position vectors, and angle of the angular velocity vector. However, Griffith's already provided us with a smart way to align the system such that we can avoid as many integrals and calculations as possible. Echoing the arguments of Griffiths, we must choose an orientation where $\gamma = \theta'$ by orienting the field point at the z-axis. With this, the velocity vector is given by

$$\vec{v} = \vec{\omega}\times\vec{r}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\theta' & R\sin\theta'\sin\theta' & R\cos\theta' \end{vmatrix} \tag{32}$$

Griffiths gives us the insight that integrating individual trigonometric terms at interval $[0, 2\pi]$ would give us zero. Hence, the entire integral reduces to

$$\vec{A}(\vec{r}) = -\frac{\mu_0 R^3\sigma\omega\sin\psi}{2}\left(\int_0^{\pi}\frac{\cos\theta'\sin\theta'}{(R^2+r'^2-2Rr\cos\theta')^{1/2}}\mathrm{d}\theta'\right)\hat{y} \tag{33}$$

Leaving us with a single variable to integrate. This is easily integrable by the substitution $u := \cos\theta'$, $\mathrm{d}u = -\sin\theta'\mathrm{d}\theta'$. The limits transform from $u = -1$ to $u = 1$ and the integral becomes

$$\int_{-1}^{1} \frac{u}{(R^2 + r^2 - 2Rru)^{1/2}}\mathrm{d}u \tag{34}$$

Recall that

$$\int \frac{x}{\sqrt{ax+b}}\mathrm{d}x = \frac{2(ax-2b)}{3a^2}\sqrt{ax+b} \tag{35}$$

Plugging in $a = -2rR$ and $b = R^2 + r^2$ and evaluating the limits gives us,

$$\begin{aligned}
\int_{-1}^{1} \frac{u}{(R^2 + r^2 - 2Rru)^{1/2}}\mathrm{d}u &= -\frac{(R^2 + r^2 + Rru)}{3R^2r^2}(R^2 + r^2 - 2rRu)^{1/2}|_{-1}^{1} && (36)\\
&= -\frac{1}{3R^2r^2}((R^2 + r^2 + Rr|R-r| - (R^2 + r^2 - Rr)(R+r) && (37)\\
&= \begin{cases} 2r/3R^2 & r < R \\ 2R/3r^2 & r > R \end{cases} && (38)
\end{aligned}$$

Tacking in the constant factor, the vector potential is

$$\vec{A}(\vec{r}) = -\frac{\mu_0 R^3 \sigma\omega\sin\psi}{2}\left(\begin{cases} 2r/3R^2 & r < R \\ 2R/3r^2 & r > R \end{cases}\right)\hat{y} = \begin{cases} -\mu_0 R^2 \sigma\omega r\sin\psi\hat{y}/3 & r < R \\ -\mu_0 R^4 \sigma\omega r\sin\psi\hat{y}/3r^3 & r > R \end{cases} \tag{39}$$

It is insightful to convert $\hat{y}$ back to its original expression from the cross product: $\vec{\omega} \times \vec{r} = -\omega r\sin\psi\hat{y}$. Then, the vector potential is

$$\vec{A}(\vec{r}) = \begin{cases} (\mu_0 R\sigma/3)(\vec{\omega} \times \vec{r}) & r < R \\ (\mu_0 R^4\sigma/3r^3)(\vec{\omega} \times \vec{r}) & r > R \end{cases} \tag{40}$$

Reorienting the sphere such that it is rotating again with respect to the z-axis, ψ becomes the polar angle θ. By right hand rule, the normal direction for both $\vec{\omega}$ and $\vec{r}$ is $\hat{\phi}$. In this coordinate alignment, the cross product is then $\vec{\omega} \times \vec{r} = \omega r\sin\theta\hat{\phi}$. This gives us the vector potential expression

$$\boxed{\vec{A}(\vec{r}) = \begin{cases} (\mu_0 R\sigma/3)(\omega r\sin\theta)\hat{\phi} & r \le R \\ (\mu_0 R^4\sigma/3r^3)(\omega r\sin\theta)\hat{\phi} & r \ge R \end{cases}} \tag{41}$$

Now that we have an expression for the vector potential of a spherical shell, we can use it to integrate thin spherical shells into a solid sphere. That is, $\vec{A} = \int \mathrm{d}\vec{A}$. In integration, note that R now varies. To emphasize this, we redefine $R = \bar{r}$ and denote the radius of the solid sphere to be R_0. One can imagine a growing sphere of radius $\bar{r}$ and adding up vector potentials at increments $\mathrm{d}\bar{r}$. Finding the differential vector potential,

$$\mathrm{d}\vec{A}(\vec{r}) = \begin{cases} (\mu_0 \bar{r}\sigma/3)(\omega r\sin\theta)\hat{\phi} & r \le R \\ (\mu_0 \bar{r}^4\sigma/3r^3)(\omega r\sin\theta)\hat{\phi} & r \ge R \end{cases} = \rho\frac{\mu_0\omega\sin\theta}{3}\begin{cases} \bar{r}r & r > \bar{r} \\ \bar{r}^4/r^2 & r > \bar{r} \end{cases} \tag{42}$$

Fortunately, the charge density ρ is uniform. Hence, for a spherical shell of charge,

$$\rho = \frac{Q}{V}\mathrm{d}\bar{r} \tag{43}$$

Plugging this in,

$$\mathrm{d}\vec{A}(\vec{r}) = \frac{Q}{V}\frac{\mu_0\omega\sin\theta}{3}\hat{\phi}\begin{cases} \bar{r}r\mathrm{d}\bar{r} & r < \bar{r} \\ (\bar{r}^4/r^2)\mathrm{d}\bar{r} & r > \bar{r} \end{cases} \tag{44}$$

To find the vector potential inside the solid sphere at r distance from the center, imagine the following. Let there be an imaginary spherical boundary S of radius r. Imagine a growing concentric sphere S' of radius $\bar{r}$. Before crossing S, the vector potential at S' is $\bar{r}^4/r^2$. After crossing S, the vector potential is $\bar{r}r$. Executing the integral,

$$\vec{A}(\vec{r}) = \frac{Q}{V}\frac{\mu_0\omega \sin\theta}{3}\hat{\phi}\left(\int_0^r \frac{\bar{r}^4}{r^2}\mathrm{d}\bar{r} + \int_r^{R_0} \bar{r}r\mathrm{d}\bar{r}\right) \tag{45}$$

Integration is trivial and evaluation gives us the vector potential inside a solid rotating charged sphere

$$\vec{A}(\vec{r}) = \frac{Q}{V}\frac{\mu_0\omega \sin\theta}{3}\left(\frac{rR_0^2}{2} - \frac{3r^3}{10}\right)\hat{\phi} \tag{46}$$

Using the volume of a sphere, $V = (4/3)\pi R^3$,

$$\vec{A}(\vec{r}) = \frac{Q}{4\pi R_0^3}\mu_0\omega \sin\theta\left(\frac{rR_0^2}{2} - \frac{3r^3}{10}\right)\hat{\phi} \tag{47}$$

Before executing the curl, a few observation can be made. There are no r and θ components. Moreover, the sole ϕ component is actually ϕ independent. This simplifies the curl operator into

$$\vec{\nabla} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta)\right)\hat{r} - \frac{1}{r}\left(\frac{\partial}{\partial r}(r)\right)\hat{\phi} \tag{48}$$

which is less hideous. Finding out the relevant derivatives,

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta) = \frac{2}{r}\cos\theta \qquad \frac{1}{r}\left(\frac{\partial}{\partial r}r\left(\frac{rR_0^2}{2} - \frac{3r^3}{10}\right)\right) = R_0^2 - \frac{12r^2}{10} \tag{49}$$

Plugging these in gives us the curl of the vector potential as the magnetic field inside the rotating solid sphere with radius R_0

$$\boxed{\vec{B} = \vec{\nabla}\times\vec{A} = \frac{Q}{4\pi R_0^3}\mu_0\omega\left(\left(R_0^2 - \frac{3r^2}{5}\right)\cos\theta\hat{r} - \left(R_0^2 - \frac{6r^2}{5}\right)\sin\theta\hat{\theta}\right)} \tag{50}$$

Our last task is to find the magnetic attraction between the northern and southern hemispheres of a spherical shell. Here's the plan. To find the magnetic interaction, we will use the expression

$$\vec{F} = \int \vec{K}\times\vec{B}\ \mathrm{d}a \tag{51}$$

where $\vec{K}$ is, of course, $\sigma\vec{v}$. However, Griffiths tells us some caveat. Due to $\vec{B}$ discontinuity at the surface, we need to use the average $\vec{B}$ in the force expression. This is done by taking the average of the field inside and outside the sphere where we have already previously conveniently derived the expressions for the vector potential. We have also already derived the velocity $\vec{v} = \omega\times\vec{r}$. Hence,

$$\vec{v} = \omega R\sin\theta\hat{\phi} \iff \vec{K} = \sigma\vec{v} = \sigma\omega R\sin\theta\hat{\phi} \tag{52}$$

Using Eq. (41),

$$\vec{B} = \vec{\nabla}\times\left(\begin{cases}(\mu_0 R\sigma/3)(\omega r\sin\theta)\hat{\phi} & r\le R\\ (\mu_0 R^4\sigma/3r^3)(\omega r\sin\theta)\hat{\phi} & r\ge R\end{cases}\right) \tag{53}$$

Executing the curl,

$$\vec{B} = \left(\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta)\right)\hat{r} - \frac{1}{r}\left(\frac{\partial}{\partial r}(r)\right)\hat{\phi}\right)\begin{cases}(\mu_0 R\sigma/3)(\omega r\sin\theta) & r\le R\\ (\mu_0 R^4\sigma/3r^3)(\omega r\sin\theta) & r\ge R\end{cases} \tag{54}$$

$$= \begin{cases}\left(\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta)\right)\hat{r} - \frac{1}{r}\left(\frac{\partial}{\partial r}(r)\right)\hat{\phi}\right)(\mu_0 R\sigma/3)(\omega r\sin\theta) & r\le R\\ \left(\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta)\right)\hat{r} - \frac{1}{r}\left(\frac{\partial}{\partial r}(r)\right)\hat{\phi}\right)(\mu_0 R^4\sigma/3r^3)(\omega r\sin\theta) & r\ge R\end{cases} \tag{55}$$

Finding out the relevant derivatives,

$$\frac{1}{r\,\sin\,\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta) = 2\cos\theta \qquad \frac{1}{r\,\sin\,\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{r^2}\sin^2\theta\right) = \frac{2\cos\theta}{r^3} \tag{56}$$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}r^2\sin\theta\right) = 2\sin\theta \qquad \frac{1}{r}\left(\frac{\partial}{\partial r}\frac{1}{r}\sin\theta\right) = -\frac{1}{r^3}\sin\theta \tag{57}$$

Plugging these in gives us

$$\vec{B} = \frac{1}{3}\sigma\mu_0 R\omega \begin{cases} 2(\cos\theta\hat{r} - \sin\theta\hat{\theta}) & r \leq R \\ (R^3/r^3)(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) & r \geq R \end{cases} \tag{58}$$

Finding the average at $r = R$,

$$\vec{B}_{\text{av}} = \frac{1}{6}\sigma\mu_0 R\omega(2(\cos\theta\hat{r} - \sin\theta\hat{\theta}) + (2\cos\theta\hat{r} + \sin\theta\hat{\theta})) = \frac{\sigma\mu_0 R\omega}{6}(4\cos\theta\hat{r} - \sin\theta\hat{\theta}) \tag{59}$$

Forming our interaction integral using Eqs. (51) and (52),

$$\vec{F} = \int \sigma\omega R\sin\theta\hat{\phi} \times \frac{\sigma\mu_0 R\omega}{6}(4\cos\theta\hat{r} - \sin\theta\hat{\theta})\ \mathrm{d}a \tag{60}$$

$$= (\sigma\omega R\sin\theta)\frac{\sigma\mu_0 R\omega}{6}\int \hat{\phi} \times (4\cos\theta\hat{r} - \sin\theta\hat{\theta})\ \mathrm{d}a \tag{61}$$

$$= \frac{\mu_0\sigma^2\omega^2R^2}{6}\int (4\cos\theta\hat{r} + \sin\theta\hat{\theta})\sin\theta\ \mathrm{d}a \tag{62}$$

Our task is to only find the hemispherical force. Hence, we only consider the z-component. Dotting this with $\hat{z}$ in spherical coordinates,

$$F_z = \frac{\mu_0\sigma^2\omega^2R^2}{6}\int ((4\cos\theta\hat{r} + \sin\theta\hat{\theta})\sin\theta) \cdot (\cos\theta\hat{r} - \sin\theta\hat{\theta})\ \mathrm{d}a \tag{63}$$

$$= -\frac{\mu_0\sigma^2\omega^2R^2}{2}\int_0^{2\pi}\int_0^{\pi/2} R^2\sin^3\theta\ \cos\theta\ \mathrm{d}\theta\mathrm{d}\phi \tag{64}$$

ϕ integrates to 2π. U-substitution of $u = \sin\theta$ makes the θ integration trivial which gives us

$$F_z = -\frac{\mu_0\sigma^2\omega^2R^4}{8}(2\pi)\ \sin^4\theta|_0^{\pi/2} \tag{65}$$

Hence, evaluation gives us the magnitude of the attraction of the northern and southern hemispheres

$$\boxed{F = \frac{\mu_0\pi\sigma^2\omega^2R^4}{4}} \tag{66}$$

4. **Magnetic Dipole**

A circular loop of wire, with radius R, lies in the xy plane (centered at the origin) and carries a current I running counter clockwise as viewed from the positive z-axis.
(a) What is its magnetic dipole moment?
(b) What is its (approximate) magnetic field at points far from the origin?
(c) Show consistency with Ex. 5.6

Recall that the dipole expansion of the vector potential is expressed as

$$\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \tag{67}$$

where the magnetic dipole moment is

$$\vec{m} := I\int \mathrm{d}\vec{a} = I\vec{A} \tag{68}$$

Finding the magnetic dipole moment for the configuration is trivial - simply calculate the loop area and use right-hand rule to determine direction. That is,

$$\vec{m} = I\pi R^2\hat{z} \tag{69}$$

Next, we are tasked to find the approximate magnetic field far from the origin. As we go further away from the source, we can simply ignore higher-order terms for the multipole expansion and the behavior conforms to a dipole potential. Hence, we simply use the vector potential from Eq. (67). Recall that an expression of the magnetic dipole moment in coordinate free form is expressed as

$$\vec{B}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}(3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}) \tag{70}$$

Plugging the magnetic dipole moment of the loop gives us

$$\vec{B}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}(3(I\pi R^2\hat{z}\cdot\hat{r})\hat{r} - I\pi R^2\hat{z}) = \frac{\mu_0}{4\pi}\frac{I\pi R^2}{r^3}(3(\hat{z}\cdot\hat{r})\hat{r} - \hat{z}) \tag{71}$$

Transforming $\hat{z}$ in spherical coordinates, $\hat{z} = \cos\theta\hat{r} - \sin\hat{\theta}$, $(\hat{z}\cdot\hat{r})\hat{r} = \cos\,\theta\hat{r}$. This gives us

$$\vec{B}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I\pi R^2}{r^3}(3\cos\theta\hat{r} - (\cos\theta\hat{r} - \sin\theta\hat{\theta})) \tag{72}$$

Hence, the approximate magnetic field far from the origin is

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I\pi R^2}{r^3}(2\cos\theta\hat{r} - \ \sin\theta\hat{\theta})} \tag{73}$$

At the limiting case, $\theta \to 0$, $\hat{r} \to \hat{z}$, and $r \to z$ Then, the magnetic field reduces to

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi}\frac{I\pi R^2}{z^3}\hat{z}} \tag{74}$$

Recall that from Example 6 of Griffiths, the magnetic field of a circular loop is expressed as

$$B(\vec{r}) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}} = \frac{\mu_0 I}{2}\frac{R^2}{|z|^3(1+R^2/z^2)^{3/2}} \tag{75}$$

At the limiting case,

$$B(\vec{r}) = \frac{\mu_0 I}{2}\frac{R^2}{|z|^3} \tag{76}$$

which is the magnitude of the result we got at Eq. (74). Hence, the result is consistent with Example 6.
□

5. **A spinning charged solid sphere**

A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.

(a) What is the magnetic dipole moment of the sphere?
(b) Find the average magnetic field within the sphere.
(c) Find the approximate vector potential at a point (r, θ) where $r >> R$.
(d) Find the exact potential at a point outside the sphere, and check that it is consistent with (c)
(e) Find the magnetic field at a point (r, θ) inside the sphere and check that it is consistent with (b).

To find the dipole moment of a solid sphere, we start by breaking it up into thin spherical shells, and then to series of infinitesimal rings. Hence, we start with the dipole moment of a ring, then integrate to find the dipole moment of a shell, then integrate again to find the dipole moment of a solid sphere. Recall that the dipole moment of a ring is expressed as

$$\vec{m} = I\vec{A} \tag{77}$$

Breaking up into elements and using the result in the previous item,

$$\mathrm{d}\vec{m} = \mathrm{d}I\vec{A} = \mathrm{d}I\pi R^2\hat{z} \tag{78}$$

However, the radius R of the ring depends on the polar angle θ. The relationship is expressed as $R = r\ \sin\ \theta$ where r is the radius of the sphere. Updating the differential dipole moment,

$$\mathrm{d}\vec{m} = \mathrm{d}I\vec{A} = \mathrm{d}I\pi r^2\sin^2\theta\hat{z} \tag{79}$$

Recall that the surface current density was defined as

$$\vec{K} := \frac{\mathrm{d}\vec{I}}{\mathrm{d}l_\perp} \tag{80}$$

where $\mathrm{d}l_\perp$ is the width of the strip, the infinitesimal arc length $r\mathrm{d}\theta$. Hence,

$$\mathrm{d}I = Kr\mathrm{d}\theta \tag{81}$$

As we have derived a couple of times already,

$$K = \sigma v = \sigma|\vec{\omega} \times \vec{r}| = \sigma\omega r\sin\theta \tag{82}$$

Collecting these gives us the dipole moment ring element as

$$\mathrm{d}\vec{m} = \pi\sigma\omega R^4\sin^3\mathrm{d}\theta\hat{z} \tag{83}$$

where $r = R$ for a shell. Integrating across all polar angles gives us the magnetic dipole moment of a spherical shell

$$\vec{m} = \pi\sigma\omega R^4\int_0^\pi \sin^3\theta\ \mathrm{d}\theta \tag{84}$$

$$= \pi\sigma\omega R^4\left(\frac{1}{3}\cos^3(x) - \cos(x)\right)|_0^\pi \tag{85}$$

Evaluation gives us

$$\vec{m} = \frac{4\pi\sigma\omega R^4}{3}\hat{z} \tag{86}$$

Now, we collapse the shell into a point, let it slowly grow by increments $\mathrm{d}r$, and add up dipole moments from each increment. The volume density would then be $\sigma = \rho\mathrm{d}r$. Finding the infinitesimal shell dipole moment element,

$$d\vec{m} = \frac{4\rho\pi\omega\bar{r}^4}{3}d\bar{r}\hat{z} \tag{87}$$

where we have converted $R = \bar{r}$ in anticipation for integration. Integrating from the point singularity $\bar{r} = 0$ to the radius of the solid sphere $\bar{r} = R$,

$$\vec{m} = \frac{4\rho\pi\omega}{3}d\bar{r}\hat{z}\int_0^R \bar{r}^4 \, d\bar{r} \tag{88}$$

$$= \frac{4\rho\pi\omega}{3}d\bar{r}\hat{z}\frac{R^5}{5} \tag{89}$$

Lastly, since the density is uniform, $\rho = (4/3)Q\pi R^3$, plugging this in gives us the dipole moment of a solid sphere,

$$\boxed{\vec{m} = \frac{1}{5}Q\omega R^2\hat{z}} \tag{90}$$

Our next task is to find the average magnetic field within the sphere. Conveniently, there is a direct relation between the magnetic dipole moment and the average B-field within the sphere expressed as

$$\vec{B}_{\text{av}} = \frac{\mu_0}{4\pi}\frac{2\vec{m}}{R^3} \tag{91}$$

Plugging in our derived dipole moment expression,

$$\boxed{\vec{B}_{\text{av}} = \frac{\mu_0}{4\pi}\frac{2}{R^3}\frac{1}{5}Q\omega R^2\hat{z}} \tag{92}$$

To find the approximate vector potential far from the sphere, we can use a dipole expansion approximation for the potential expressed as

$$\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \tag{93}$$

Plugging in the dipole moment

$$\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi r^2}\frac{1}{5}Q\omega R^2\hat{z}\times\hat{r} \tag{94}$$

Noting that $\hat{z}\times\hat{r} = \sin\theta\hat{\phi}$, we have the approximate expression of the vector potential when $r >> R$ expressed as

$$\boxed{\vec{A} = \frac{\mu_0 Q\omega R^2 \sin\theta}{20\pi}\frac{1}{r^2}\hat{\phi}} \tag{95}$$

Do these results make sense? Continuing our unfinished business earlier of finding the vector potential of a solid sphere of radius R_0 outside the sphere by revising the integral at Eq. (45), (read the arguments again leading up to (45)) and placing the S boundary at infinity. Hence, our test radius $\bar{r}$ shall never cross it and we don't have to integrate outside due to zero density. Redoing the integral,

$$\vec{A}(\vec{r}) = \frac{Q}{V}\frac{\mu_0\omega\sin\theta}{3}\hat{\phi}\left(\int_0^{R_0}\frac{\bar{r}^4}{r^2}d\bar{r}\right) \tag{96}$$

Plugging in the volume $V = (4/3)\pi R^3$ and noting, of course, that the antiderivative is $\bar{r}^5/5$, evaluating the limits gives us

$$\boxed{\vec{A} = \frac{\mu_0 Q\omega R^2 \sin\theta}{20\pi}\frac{1}{r^2}\hat{\phi}} \tag{97}$$

which is, surprisingly, the same as an ideal dipole. It is important to note that we took the limiting condition of $r >> R$ by placing S' such that we never have to cross it (i.e. the second integral at Eq. (45) vanishes). Conveniently, we have already derived the expression for the B-field of a solid sphere from (58). Recall that we have already derived an expression of the magnetic field inside the solid sphere at Eq. (50)

$$\vec{B} = \frac{Q}{4\pi R_0^3}\mu_0\omega\left(\left(R_0^2 - \frac{3r^2}{5}\right)\cos\theta\hat{r} - \left(R_0^2 - \frac{6r^2}{5}\right)\sin\theta\hat{\theta}\right) \tag{98}$$

Finally, we simply have to integral over the entire spherical volume divided by the sphere's volume to find the average field. It is important to note that, due to symmetry, the average must point towards $\hat{z}$. Hence, we simply should dot the expression with $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$. This simply squares the trigonometric terms and flips the $\hat{\theta}$ component. Building the triple integral,

$$\vec{B}_{\text{av}} = \frac{Q}{4\pi R_0^3 V}\mu_0\omega\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\left(\left(R_0^2 - \frac{3r^2}{5}\right)\cos^2\theta\hat{r} + \left(R_0^2 - \frac{6r^2}{5}\right)\sin\theta^2\hat{\theta}\right)r^2\sin\theta\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi \tag{99}$$

Distributing the r^2 to the integrand, integrating with respect to r and separately and simultaneously integrating the ϕ part gives us the remaining θ integral to evaluate. Moreover, plugging in $V = (4/3)\pi R^3$,

$$\frac{\mu_0\omega Q}{200\pi R}\int_0^{\pi}(9\cos^2\theta + 7)\sin\theta\ \mathrm{d}\theta \tag{100}$$

where u-substitution $u = \sin\theta$, $\mathrm{d}u = \cos\theta\mathrm{d}\theta$ gives us

$$-\frac{\mu_0\omega Q}{200\pi R}(3\cos^3\theta + 7\cos\theta)|_0^{\pi}\hat{z} \tag{101}$$

which, finally, evaluates to

$$\frac{\mu_0\omega Q}{10\pi R}\hat{z} \tag{102}$$

which is equal to our derived average B-field from Eq. (91)

2 Reflection

Even though there are a lot of parallels to the electrostatic versions, magnetic fields are personally generally harder to do. For one, the potential itself if vectorial in nature. It is surprising that the problem set have not included an amperian loop problem. As a meta-reflection, even though this is one of the shortest problem set I have written, the lines of code is still the same to an average problem set. This speaks of the density of the calculations in each problem. Although it is significantly faster to do the problems on paper, I find that I can express my thoughts more and internalize what's happening if I type my reasoning out. This is also to teach my dumber future self.

Admittedly, I have to do the problems with the Griffith's book for guidance. For the past problem sets, I have attempted to do the problems first from the ground up. I decided not to do is this time since I want to enjoy the holiday season and grab the opportunity to rest (how little time it is). Moreover, the typhoon caused recurring blackouts hindering my ability to typeset my solutions efficiently.

- **Problem 1** The first item is very straightforward. One only needs to use Lorentz force law to find the equations of motion. I think this is a good review in handling unit vectors, above else. I haven't attempted to solve the equations of motion since I already lack the energy to do so.
- **Problem 2** Although the solution was short, this problem took me some time to finish. I keep finding an appropriate coordinate to evaluate my integral with respect to. Moreover, I also keep finding appropriate parameter to express my solutions with so that it would be convenient to conform it with Griffith's solution in the end. I have read Griffith's arguments to find out one simplifying orthogonality constraint to make the integral simple.
- **Problem 3** This item has the longest solution of all items. Essentially, we have to analyze a spherical shell and use the results to derive a higher-dimensional solid sphere. It took some time to construct the arguments. It was straightforward, but long.
- **Problem 4** This item was the shortest one to write and also the most fluid one to follow. The sub-items were continuous to do, building up the results of the previous ones. Above all, of course, it does not contain nasty integrals and curls (in spherical coordinates).
- **Problem 5**

  Luckily for this problem, some expressions have already been derived in the previous problem (Problem 3) such as finding the the two sanity checks in the end. Otherwise, this might have been a beast in the given problem set. This was the second longest item to construct solutions to.

Admittedly, as of the typing of this problem set, I have not pondered too deeply about magnetism in general. I merely solved items for the sake of it - finding out relevant fundamental expressions to use and building from there. I plan to continue studying after the break ends but this is it for now. There was no apparent conceptual roadblock that halted me but sheer length of the solutions. Since I'm not really sure if I have some major conceptual barrier outside the problem set, I can say that the material is a green one.