1 Problem Set 6: Angular Momentum

1. **Angular Momentum Eigenstates**

Consider a wavefunction as a superposition of spherical harmonics

$$|\Psi(t)\rangle = \frac{1}{\sqrt{3}}|Y_1^0(t)\rangle + \frac{2}{\sqrt{3}}|Y_1^1(t)\rangle \tag{1}$$

where $Y_{lm}(\theta,\phi) = \langle\theta,\phi|Y_{lm}\rangle$ are the spherical harmonics as eigenstates of L^2 in angular basis.

a) Find the eigenvalues of L^2 and L_z

b) Find the expectation values of L_z and L_x

Recall that the general expression of the wavefunction is

$$|\Psi(t)\rangle = \sum_n\sum_l\sum_m c_{n,l,m}|\psi_{n,l,m}\rangle T_n(t) \tag{2}$$

where $T_n(t) := \exp(-iE_nt/\hbar)$ is the wiggle factor. In spherical coordinates,

$$\psi(r,\theta,\phi) = R_{n,l}(r)Y_l^m(\theta,\phi) \tag{3}$$

where $Y_l^m(\theta,\phi)$ are spherical harmonics. We can interpret the given wavefunction as

$$|\Psi(t)\rangle = \sum_n\sum_l\sum_m c_{n,l,m}|\psi_{n,l,m}\rangle T_n(t)\delta_{l1}\delta_{m0}\delta_{m1} \tag{4}$$

The survivors of the deltas are expressed as

$$|\Psi(t)\rangle = \left(\frac{1}{\sqrt{3}}|R_{n,1}Y_1^0\rangle + \frac{2}{\sqrt{3}}|R_{n,1}Y_1^1\rangle\right)\exp\left(\frac{-iE_nt}{\hbar}\right) \tag{5}$$

Of course, the vectors are simply the eigenfunctions of a Hermitian Hamiltonian operator. Thus, it must be orthonormal. Checking normalization, $\langle\Psi(t)|\Psi(t)\rangle$=1 must hold. That is,

$$\langle\Psi(t)|\Psi(t)\rangle = \left(\left(\frac{1}{\sqrt{3}}|R_{n,1}Y_1^0\rangle + \frac{2}{\sqrt{3}}|R_{n,1}Y_1^1\rangle\right)\exp\left(\frac{-iE_nt}{\hbar}\right)\right)^\dagger\left(\frac{1}{\sqrt{3}}|R_{n,1}Y_1^0\rangle + \frac{2}{\sqrt{3}}|R_{n,1}Y_1^1\rangle\right)\exp\left(\frac{-iE_nt}{\hbar}\right) \tag{6}$$

Temporal term is conjugated and must vanish to unity. Since the vectors are orthonomormal, it follows that

$$\langle\Psi(t)|\Psi(t)\rangle = \frac{1}{3}\langle R_{n,1}Y_1^0|R_{n,1}Y_1^0\rangle + \frac{4}{3}\langle R_{n,1}Y_1^1|R_{n,1}Y_1^1\rangle = \frac{1}{3}+\frac{4}{3} = \frac{5}{3} \tag{7}$$

As we can observe, the wavefunction is not normalized. We expect to find the normalization constant when solving for the radial equation specific for a given system. Now, we are tasked to find the allowable values of L^2 and L_z. We can express the wavefunction in simple terms as

$$|\Psi(t)\rangle = \left(\frac{1}{\sqrt{3}}|n10\rangle + \frac{2}{\sqrt{3}}|n11\rangle\right)\exp\left(\frac{-iE_nt}{\hbar}\right) \tag{8}$$

To compute for physically realizable results, we must first normalize it buy borrowing the normalization constant from R. For some reason, the non-inclusion of the radial part introduces some confusion. Hence, we will assume that the radial equation for this problem has a constant solution A for all n. [1]. With this, we can eliminate the radial dependence and directly interpret the given wavefunction as

[1] I don't know how would this make sense since this would imply a zero energy particle according to the radial equation but that is what we're given. On retrospect, the assumption is not needed since the product of all separable solutions still form an eigenfunction of the Hamiltonian - a Hermitian operator with orthonormal eigenfunctions. I'm simply too tired to rewrite the solutions

$$|\Psi(t)\rangle = A\left(\frac{1}{\sqrt{3}}|10\rangle + \frac{2}{\sqrt{3}}|11\rangle\right)\exp\left(\frac{-iE_nt}{\hbar}\right) \tag{9}$$

From Eq. (7), the normalization constant is

$$A = \sqrt{\frac{3}{5}} \tag{10}$$

Updating the coefficients, we now have a normalized wavefunction

$$|\Psi(t)\rangle = \left(\frac{1}{\sqrt{5}}|n10\rangle + \frac{2}{\sqrt{5}}|n11\rangle\right)\exp\left(\frac{-iE_nt}{\hbar}\right) \tag{11}$$

Recall that L_z and L^2 commute and are simulateneously diagonalizable. Their eigenvalues are expressed as

$$L^2Y_l^m = \hbar^2 l(l+1)Y_l^m \qquad L_zY_l^m = \hbar m Y_l^m \tag{12}$$

Observe in Eq. (8) that there are only two realizable states: $|n10\rangle$ corresponding to $l=1$ and $m=0$ and $|n11\rangle$ corresponding to $l=1$ and $m=1$. Hence, from Eq. (12), the only possible value of L^2 is

$$\boxed{L^2 = 3\hbar^2} \tag{13}$$

while the possible values for L_z are

$$\boxed{L_z = 0, \hbar} \tag{14}$$

We can sandwich the operators to find the expectation values as $\langle\Psi(t)|\, L_z\, |\Psi(t)\rangle$ and $\langle\Psi(t)|\, L^2\, |\Psi(t)\rangle$. However, note that we have already been given the probability amplitude for each states. Thus, we can directly solve for the individual expectation values using the statistical interpretation for observables

$$\langle Q\rangle = \sum_n q_n|c_n|^2 \tag{15}$$

With this, we have

$$\langle L^2\rangle = 3\hbar^2|\frac{1}{\sqrt{5}}|^2 + 3\hbar^2|\frac{2}{\sqrt{5}}|^2 = 3\hbar^2 \tag{16}$$

This makes sense because it is the only allowable value. What else could its average be. Finding the expectation value for L_z,

$$\langle L_z\rangle = 0|\frac{1}{\sqrt{5}}|^2 + \hbar|\frac{2}{\sqrt{5}}|^2 = \frac{4}{5}\hbar \tag{17}$$

For $\langle L_x\rangle$ however, the spherical harmonics are not its eigenstates. Finding the expectation values needs to be done by actually expressing it in terms of its differential form. Luckily, we have the tools to avoid that. Recall that the ladder operator is defined as

$$L_\pm = L_x \pm iL_y \tag{18}$$

From this, we have

$$L_x = \frac{1}{2}(L_+ + L_-) \qquad L_y = \frac{1}{2i}(L_+ - L_-) \tag{19}$$

To find $\langle L_x\rangle$, we sandwich it as $\langle\Psi(t)|\, L_x\, |\Psi(t)\rangle$. In terms of ladders,

$$\langle L_x\rangle = \langle\Psi(t)|\, \frac{1}{2}(L_+ + L_-)\, |\Psi(t)\rangle \tag{20}$$

Note that the wavefunction is a linear combination of spherical harmonics. It can be shown that we can climb the ladder via the following relations

$$L_+ Y_l^m = \hbar\sqrt{(l-m)(l+m+1)}Y_l^{m+1} \qquad L_- Y_l^m = \hbar\sqrt{(l+m)(l-m+1)}Y_l^{m-1} \tag{21}$$

For convenience, we ignore that temporal part as it would simply dissipate due to conjugation later. Recall that the wavefunction is

$$|\Psi\rangle = \left(\frac{1}{\sqrt{5}}|Y_1^0\rangle + \frac{2}{\sqrt{5}}|Y_1^1\rangle\right) \tag{22}$$

Then, using Eq. (21),

$$L_+ |Y_1^0\rangle = \sqrt{2}\hbar Y_1^1 \qquad L_- |Y_1^0\rangle = \sqrt{2}\hbar Y_1^{-1} \tag{23}$$

Similarly,

$$L_+ |Y_1^1\rangle = 0 \qquad L_- |Y_1^1\rangle = \sqrt{2}\hbar Y_1^0 \tag{24}$$

Expanding Eq, (20),

$$\langle L_x\rangle = \langle\Psi|\left(\frac{1}{2}L_+|\Psi\rangle + \frac{1}{2}L_-|\Psi\rangle\right) \tag{25}$$

Plugging in Eq. (22),

$$\langle\Psi|\left(\frac{1}{2}L_+\left(\frac{1}{\sqrt{5}}|Y_1^0\rangle + \frac{2}{\sqrt{5}}|Y_1^1\rangle\right) + \frac{1}{2}L_-\left(\frac{1}{\sqrt{5}}|Y_1^0\rangle + \frac{2}{\sqrt{5}}|Y_1^1\rangle\right)\right) \tag{26}$$

Invoking the ladder relations,

$$\langle\Psi|\hbar\left(\frac{\sqrt{2}}{2\sqrt{5}}|Y_1^1\rangle + \frac{\sqrt{2}}{2\sqrt{5}}|Y_1^{-1}\rangle + \frac{\sqrt{2}}{\sqrt{5}}|Y_1^0\rangle\right) = \frac{\hbar}{2}\sqrt{\frac{2}{5}}\langle\Psi|\left(|Y_1^1\rangle + |Y_1^{-1}\rangle + 2|Y_1^0\rangle\right) \tag{27}$$

Taking the inner product, note that the spherical harmonics are orthonormal. Hence, distributing the terms will all cross-terms leaving us with

$$\langle L_x\rangle = \left(\frac{\hbar}{2}\sqrt{\frac{2}{5}}\right)\left(\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}}\right) = \frac{2\sqrt{2}}{5}\hbar \tag{28}$$

Hence, compiling, we have

$$\boxed{\langle L_z\rangle = \frac{4}{5}\hbar} \tag{29}$$

and

$$\boxed{\langle L_x\rangle = \frac{2\sqrt{2}}{5}\hbar} \tag{30}$$

2. **A Pure Eigenstate**

Let

$$|\psi(t)\rangle = |Y_1^0(t)\rangle \tag{31}$$

a) Find $\langle L_x \rangle$
b) Find eigenvalues of L_z
c) Find eigenvalues of L_x. Explain.

Similar to the previous, we will use ladder operators. Sandwiching the operator,

$$\langle L_x \rangle = \langle \Psi | L_x | \Psi \rangle \tag{32}$$

Plugging in the wavefunction and using ladders,

$$\langle L_x \rangle = \langle Y_1^0 | \frac{1}{2}(L_+ + L_-) | Y_1^0 \rangle \tag{33}$$

Using the relations from Eq. (23),

$$\frac{1}{2} \langle Y_1^0 | (L_+ + L_-) | Y_1^0 \rangle = \frac{\sqrt{2}}{2} \langle Y_1^0 | \left(Y_1^{-1} + Y_1^1 \right) \tag{34}$$

Orthonormality dictates that the entire expression must be zero. Hence,

$$\boxed{\langle L_x \rangle = 0} \tag{35}$$

With similar reasoning, one can also show that $\langle L_y \rangle = 0$. Eigenvalues are measurable quantities for L_z. Since the state is a pure eigenstate of L_z (and, of course, of L^2), from Eq. (12),

$$L_z Y_1^0 = \hbar(0) Y_1^0 \tag{36}$$

Hence, the only possible values for L_z is zero.

$$\boxed{L_z = 0} \tag{37}$$

From the fundamental commutation relation, the commutator of L_x and L_z is

$$[L_z, L_x] = i\hbar L_y \tag{38}$$

Generalized uncertainty principle states that

$$\sigma_{L_z}^2 \sigma_{L_x}^2 \geq \left(\frac{1}{2i} \langle i\hbar L_y \rangle \right) = \frac{\hbar^2}{4} \langle L_y \rangle^2 \tag{39}$$

However, since $\langle L_y \rangle = 0$, the uncertainty vanishes

$$\sigma_{L_z}^2 \sigma_{L_x}^2 \geq 0 \tag{40}$$

This primes us that the angular momentum components can actually commute for some cases. Indeed, when the eigenvalue for L^2 is zero, there can be no other values for the components other than zero.[2] However, since for $l = 1$ and $m = 0$, $L^2 |\Psi\rangle \neq 0$, the components must have a generally non-zero value. Hence, since L_z and L_x generally does not commute,

$$\boxed{L_x \text{ is indeterminate}} \tag{41}$$

[2] Whether this is actually a physically realizable state, I still do not know yet.

3. **A Commutation Relation**

Show $[L_y, L_z] = i\hbar L_x$

We start by developing some relations for commutators. For any given operators A, B, and C and for any given scalar c,

$$[A, A] = 0 \tag{42}$$
$$[A, B] = -[B, A] \tag{43}$$
$$[A, c] = 0 \tag{44}$$
$$[A, cB] = c[A, B] \tag{45}$$
$$[A + B, C] = [A, C] + [B, C] \tag{46}$$
$$[A, BC] = [A, B]C + B[A, C] \tag{47}$$
$$[AB, CD] = A[B, C]D + [A, C]BD + CA[B, D] + C[A, B]B \tag{48}$$
$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{49}$$

Recall that the canonical commutation relation is expressed as

$$[r_i, p_j] = -[p_i, r_j] = i\hbar\delta_{ij} \qquad [r_i, r_j] = [p_i, p_j] = 0 \tag{50}$$

Since $L_y = zp_x - xp_z$ and $L_z = xp_y - yp_x$,

$$[L_y, L_z] = [zp_x - xp_z, xp_y - yp_x] \tag{51}$$

Using the identities from Eq. (43),(45), and (46) extensively,

$$[zp_x - xp_z, xp_y - yp_x] = [zp_x, xp_y] - [zp_x, yp_x] - [xp_z, xp_y] + [xp_z, yp_x] \tag{52}$$
$$\tag{53}$$

From the canonical commutation relation at Eq. (50) and the identity at (48), we can make a few observations to avoid as much algebra as possible. Study Eq. (48). The term can only survive if four of the resulting terms survive. That is, if all combinations of the element of a commutator survive, the term survives. A commutator in its primitive nature (composed of only position and momentum operator) can only survive if (i) it contains both position and momentum and (ii) it has the same component. We will call these "survivor pairs". For instance, $[p_x, x]$ must survive and is a survivor pair while $[x, y]$ or $[p_x, p_y]$ can't. With these conditions, it can be easily seen that the middle terms can't survive since it does not contain a survivor pair when expanded using (48). Extracting only the survivor pair of the first term and the fourth term, we have

$$[L_y, L_z] = z[p_x, x]p_y + y[x, p_x]p_z \tag{54}$$

Evaluating these survivors using the canonical commutation relation, we have

$$[x, p_x] = i\hbar \qquad p_x, x = -i\hbar \tag{55}$$

Hence,

$$[L_y, L_z] = -zi\hbar p_y + yi\hbar p_z = i\hbar(yp_z - zp_y) = i\hbar L_x \tag{56}$$

Hence,

$$\boxed{[L_y, L_z] = i\hbar L_x} \tag{57}$$

4. **Simultaneously Diagonalizable Trio**

a) Prove the following commutation relations:

- $[L_z, x] = i\hbar y$
- $[L_z, p_x] = i\hbar p_y$
- $[L_z, y] = -i\hbar x$
- $[L_z, p_y] = -i\hbar p_x$
- $[L_z, z] = 0$
- $[L_z, p_z] = 0$

b) Show $[L_z, L_x] = i\hbar L_y$.
c) Evaluate the commutators $[L_z, r^2]$ and $[L_z, p^2]$
d) Show that H commutes with all components of $\vec{L}$ provided that $V = V(r)$

Problem set instructed us to ignore item a as it is similar to the previous problem. However, I could not see any similarity with proving these commutation relations. On the other hand, it was problem b that is similar. Hence, we will be doing all of these. What we shall do is solve all of these problems in as little steps as possible by generalizations. Recall that L_z is expressed as

$$L_z = xp_y - yp_x \tag{58}$$

Then, we analyze the commutation relation for any component of position and momentum. Let q_i represent the ith component of either position or momentum. Hence

$$[L_z, r_i] = [xp_y - yp_x, q_i] \tag{59}$$

Using Eq. (46),

$$[L_z, q_i] = [xp_y, q_i] - [yp_x, q_i] = [q_i, yp_x] - [q_i, xp_y] \tag{60}$$

From Eq. (47),

$$[L_z, q_i] = [q_i, y]p_x + y[q_i, p_x] - [q_i, x]p_y - x[q_i, p_y] \tag{61}$$

From the earlier conditions for survival, we can see that $q_i = z$ and $q_i = p_z$ won't survive since it can't find its mate: p_z and z, respectively. Hence,

$$[L_z, z] = [L_z, p_z] = 0 \tag{62}$$

If $q_i = x$, it mates with the second term eliminating the rest and we get

$$[L_z, x] = y[x, p_x] = i\hbar y \tag{63}$$

With these arguments of "finding mates", we can see that $q_i = p_x$ mates with the third term (which is negative), $q_i = y$ mates with the fourth term (which is negative), and $q_i = p_y$ mates with the first term (which is positive). That is,

$$[L_z, p_x] = -[p_x, x]p_y = i\hbar p_y \tag{64}$$

$$[L_z, y] = -x[y, p_y] = -i\hbar x \tag{65}$$

$$[L_z, p_x] = [p_y, y]p_x = -i\hbar p_x \tag{66}$$

Hence, we have proven all six commutators. □

Solving for the commutator $[L_z, L_x] = i\hbar L_y$ is redundant to our previous problem but we'll find it anyway.

$$[L_z, L_x] = z[p_z, z]p_z + z[z, p_z]p_x \tag{67}$$

Evaluating these survivors using the canonical commutation relation, we have

$$[z, p_z] = i\hbar \qquad p_z, z = -i\hbar \tag{68}$$

Hence,

$$[L_z, L_x] = -xi\hbar p_z + zi\hbar p_x = i\hbar(zp_x - xp_z) = i\hbar L_y \tag{69}$$

Hence,

$$\boxed{[L_z, L_x] = i\hbar L_y} \tag{70}$$

Now, we are tasked to find $[L_z, r^2]$ and $[L_z, p^2]$. Of course, we will still use the survival arguments because why not. Let q_i represent either position or momentum. Hence, the commutators become

$$[xp_y - yp_x, q_1^2 + q_2^2 + q_3^2] \tag{71}$$

where q_1 is either x or p_x, and so on. Expanding gives us

$$[xp_y - yp_x, q_1^2] + [xp_y - yp_x, q_2^2] + [xp_y - yp_x, q_3^2] \tag{72}$$

Expanding some more,

$$[xp_y, q_1^2] - [yp_x, q_1^2] + [xp_y, q_2^2] - [yp_x, q_2^2] + [xp_y, q_3^2] - [yp_x, q_3^2] \tag{73}$$

Expressing the squares explicitly, we have

$$[xp_y, q_1q_1] - [yp_x, q_1q_1] + [xp_y, q_2q_2] - [yp_x, q_2q_2] + [xp_y, q_3q_3] - [yp_x, q_3q_3] \tag{74}$$

This is as far as we can go without choosing the appropriate form for q_i. Letting q_i represent position, we have

$$[xp_y, xx] - [yp_x, xx] + [xp_y, yy] - [yp_x, yy] + [xp_y, zz] - [yp_x, zz] \tag{75}$$

Eyeballing the survivors using Eq. (48), only the second and third term surives. Updating,

$$[L_z, r^2] = [xp_y, yy] - [yp_x, xx] \tag{76}$$

Extracting the surivors again from each of the two commutators usng Eq. (48),

$$x[p_y, y]y + yx[p_y, y] - y[p_x, x]x - xy[p_x, x] = i\hbar(yx + xy - xy - yx) = 0 \tag{77}$$

Hence,

$$\boxed{[L_z, r^2] = 0} \tag{78}$$

For q_i representing momentum, we have

$$[xp_y, p_xp_x] - [yp_x, p_xp_x] + [xp_y, p_yp_y] - [yp_x, p_yp_y] + [xp_y, p_zp_z] - [yp_x, p_zp_z] \tag{79}$$

The first and fourth term survives. Hence,

$$[L_z, r^2] = [xp_y, p_xp_x] - [yp_x, p_yp_y] \tag{80}$$

Extracting the survivors similarly as Eq. (77)

$$[x, p_x]p_yp_x + p_x[x, p_x]p_y - [y, p_y]p_xp_y - p_y[y, p_y]p_x = i\hbar(p_yp_x + p_yp_x - p_xp_y - p_yp_x) = 0 \tag{81}$$

Hence,

$$\boxed{[L_z, p^2] = 0} \tag{82}$$

We are only a few steps away in proving our ultimate goal, the commutativity of Hamiltonian for all components of the angular momentum. We have already proven L_z and p^2 commutes. It can actually be shown that the exact same arguments holds true for L_y and L_z. Explicitly, Eq. (77) takes the form of

$$[L_x, r^2] = [yp_z, zz] - [zp_y, yy] = i\hbar(zy + yz - yz - zy) = 0 \tag{83}$$

and

$$[L_y, r^2] = [zp_x, xx] - [xp_z, zz] = i\hbar(xz + zx - zx - xz) = 0 \tag{84}$$

while (81) takes the form of

$$[L_x, p^2] = [yp_z, p_z[_z] - [zp_y, p_y p_y] = i\hbar(p_z p_y + p_y p_z - p_y p_z - p_z p_y) = 0 \tag{85}$$

and

$$[L_y, p^2] = [zp_x, p_x p_x] - [xp_z, p_z p_z] = i\hbar(p_x p_z + p_z p_x - p_z p_x - p_x p_z) = 0 \tag{86}$$

We now have all the ingredients. Proving $[H, L_i]$ for any i=1,2,3,

$$[H, L_i] = \frac{1}{2m}[p^2, L_i] + [V, L_i] \tag{87}$$

Nowm since p^2 commutes for any component of $\vec{L}$ and V is simply a scalar, both terms must commute for all i![3]. Hence,

$$\boxed{[H, L_i] = 0 \qquad\qquad i = 1, 2, 3} \tag{88}$$

Hence, we can simultaneously measure H, L^2, and L_z without getting bamboozled by uncertainties.

[3]The exclamation point signifies that the writer is exhausted for the semester