Quantum Mechanics I (141) Problem Set 1

Lyle Kenneth Geraldez

I. Solutions overview

This is an overview of what the document contains

II. Wavefunction Normalization

The task is to normalize $\Psi(x) = Ae^{(-\alpha x^2)}$. That is,

$$\int_{-\infty}^{\infty} \Psi^*(x)\Psi(x)\mathrm{d}x = 1 \tag{1}$$

Letting A, α be real, equation (1) gives us

$$A^2 \int_{-\infty}^{\infty} e^{(-2\alpha x^2)}\mathrm{d}x = 1 \tag{2}$$

A. Gamma Function Approach

Defining, for convenience, $B \equiv A^2, \beta \equiv 2\alpha$,

$$B \int_{-\infty}^{\infty} e^{(-\beta x^2)}\mathrm{d}x = 1 \tag{3}$$

Executing a change of variable $t = \beta x^2$, $\mathrm{d}t = 2\beta x \mathrm{d}x$. Observe that there are two branches of solutions stemming from the quadratic nature of x. That is, $x = \pm\sqrt{\frac{t}{\beta}}$. To constrict x into the real domain, it must be that $\alpha \geq 0$. Before completing t substitution, we must divide the integral

$$B\left(\int_{-\infty}^{0} e^{(-\beta x^2)}\mathrm{d}x + \int_{0}^{\infty} e^{(-\beta x^2)}\mathrm{d}x\right) = 1 \tag{4}$$

As x approaches $-\infty$ and $+\infty$, t approaches $+\infty$ while both approaches zero together. Changing variables and limits,

$$\frac{B}{2\beta}\left(\int_{\infty}^{0} -\sqrt{\frac{\beta}{t}}e^{-t}\mathrm{d}t + \int_{0}^{\infty} \sqrt{\frac{\beta}{t}}e^{-t}\mathrm{d}t\right) = 1 \tag{5}$$

Switching limits of the first term and combining the integrals,

$$\frac{B}{\beta}\left(\int_{0}^{\infty} \sqrt{\frac{\beta}{t}}e^{-t}\mathrm{d}t\right) = \frac{B}{\sqrt{\beta}}\int_{0}^{\infty} t^{1/2-1}e^{-t}\mathrm{d}t = 1 \tag{6}$$

The integral in Eq. (6) can be simplified into a Gamma function expression

$$\frac{B}{\sqrt{\beta}}\int_{0}^{\infty} t^{1/2-1}e^{-t}\mathrm{d}t = \frac{B}{\sqrt{\beta}}\Gamma\left(\frac{1}{2}\right) = 1 \tag{7}$$

Hence,

$$B = \sqrt{\frac{\beta}{\pi}} \tag{8}$$

Evaluating back to original forms A and α,

$$A = \left(\frac{2\alpha}{\pi}\right)^{1/4}, \qquad \alpha \geq 0 \tag{9}$$

Therefore, the normalized wavefunction with unit magnitude is

$$\boxed{\Psi(x) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{(-\alpha x^2)}} \tag{10}$$

Observe that even if α is generally complex, the imaginary parts will cancel out leaving us with Eq. (2) to solve.

B. Integration via Polar Coordinates

We proceed on evaluating the integral on Eq. (2) using polar coordinates. Multiplying it with itself and changing the other factor from x into y, we get

$$A^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(-2\alpha x^2)}e^{(-2\alpha y^2)}\mathrm{d}x\mathrm{d}y = 1 \tag{11}$$

$$A^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(-2\alpha(x^2+y^2))}\mathrm{d}\Omega = 1 \tag{12}$$

where $\mathrm{d}\Omega$ represents the area element. To evaluate, we use the polar coordinate system and use the substitution $r^2 = x^2+y^2$ transforming the area element using the polar Jacobian: $\mathrm{d}\Omega = r\mathrm{d}r\mathrm{d}\phi$. Integral becomes

$$A^4 \int_{0}^{2\pi}\int_{0}^{\infty} e^{(-2\alpha(r^2))}r\mathrm{d}r\mathrm{d}\phi = 1 \tag{13}$$

Making a substitution $u = -2\alpha r^2$, $\mathrm{d}u = -4\alpha r\mathrm{d}r$, u approaches $-\infty$ as r approaches ∞ and both approaches zero together. The integral becomes

$$-\frac{A^4}{4\alpha}\int_{0}^{2\pi}\int_{0}^{-\infty} e^{u}\mathrm{d}u\mathrm{d}\phi = 1 \tag{14}$$

The double integral is trivial and direct evaluation yields

$$\frac{A^4\pi}{2\alpha} = 1 \tag{15}$$

This gives us the same normalization constant as Eq. (9).

$$A = \left(\frac{2\alpha}{\pi}\right)^{1/4}, \qquad \alpha \geq 0 \tag{16}$$

III. Proof of Continuity equation

Without loss of content, we can restructure the problem into verifying the continuity equation.