Novel well-balanced arbitrary high order CIP stabilizations for continuous FEM/RD

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Universität Zürich

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Inria Bordeaux sud-ouest

Outline

- Governing equations and motivation
- Continuous FEM
- Well-balancing
  - Spatial part
  - Stabilization, continuous interior penalty
- Numerical results

Section 1

Governing equations and motivation

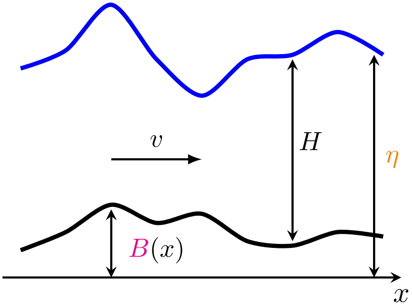
Shallow water equations

$$\frac{\partial}{\partial t}\boldsymbol{u} + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{S}(x,\boldsymbol{u}), \quad (x,t) \in \Omega \times [0,T], \; \Omega := (x_L, x_R)$$

$$\boldsymbol{u} := \begin{pmatrix} H \\ Hv \end{pmatrix}$$

$$\boldsymbol{F}(\boldsymbol{u}) := \begin{pmatrix} Hv \\ Hv^2 + g\frac{H^2}{2} \end{pmatrix}$$

$$\boldsymbol{S}(x,\boldsymbol{u}) := -\begin{pmatrix} 0 \\ gH\frac{\partial}{\partial x}B(x) \end{pmatrix}$$



$$\eta := H + B$$

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Well-balancing (in particular $\frac{\partial}{\partial t}\boldsymbol{u} \equiv 0 \;\Leftrightarrow\; \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{S}(x, \boldsymbol{u})$)

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$$\boldsymbol{u}_{eq} := \begin{pmatrix} \eta - B \\ 0 \end{pmatrix}$$

$$\eta = H + B \equiv const$$

Figure: Lake of Zürich at rest

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- We look for $\boldsymbol{u}_h(x,t) := \sum_{j=1}^{I} \boldsymbol{c}_j(t)\varphi_j(x)$ s.t.

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Section 3

Well-balancing

Subsection 1

Spatial part

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$$\left[\frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}_h)-\boldsymbol{S}(x,\boldsymbol{u}_h)\right]_h=\frac{\partial}{\partial x}\boldsymbol{F}_h(x)-\boldsymbol{S}_h(x)$$

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Special WB discretization[1]

$$\left[\frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}_h)-\boldsymbol{S}(x,\boldsymbol{u}_h)\right]_h = \left[\frac{\partial}{\partial x}\begin{pmatrix} Hv \\ Hv^2 + g\frac{H^2}{2}\end{pmatrix} + \begin{pmatrix} 0 \\ gH\frac{\partial}{\partial x}B\end{pmatrix}\right]_h$$

$$= \frac{\partial}{\partial x}\left[\begin{pmatrix} Hv \\ Hv^2\end{pmatrix}\right]_h + \begin{pmatrix} 0 \\ gH_h\frac{\partial}{\partial x}\left(H_h + B_h\right)\end{pmatrix}$$

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Subsection 2

Stabilization, continuous interior penalty

Stabilization part

$$\int_\Omega \varphi_i(x)\frac{\partial}{\partial t}\boldsymbol{u}_h d\boldsymbol{x} + \int_\Omega \varphi_i(x)\left[\frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}_h) - \boldsymbol{S}(x,\boldsymbol{u}_h)\right]_h + \mathsf{ST}_i(\boldsymbol{u}_h) = \mathbf{0} \quad \forall i$$

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CIP[2] stabilization terms (**jump c**)

$$\mathsf{ST}_i(\boldsymbol{u}_h) := \sum_f \alpha_f \int_f [\![\nabla_{\boldsymbol{x}}\varphi_i]\!][\![\nabla_{\boldsymbol{x}}\boldsymbol{u}_h]\!] d\sigma$$
$$= \sum_f \alpha_f \left[\!\!\left[\frac{\partial}{\partial x}\varphi_i\right]\!\!\right]\left[\!\!\left[\frac{\partial}{\partial x}\boldsymbol{u}_h\right]\!\!\right]$$

[2] J. Douglas Jr and T. Dupont Interior Penalty Procedures for Elliptic and Parabolic Galerkin Methods, 1976

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- **jump t** (total height)

$$\mathsf{ST}_i(\boldsymbol{u}_h) := \sum_f \alpha_f \left[\!\left[\frac{\partial}{\partial x}\varphi_i \right]\!\right] \left[\!\left[\frac{\partial}{\partial x} \begin{pmatrix} \eta \\ Hv \end{pmatrix} \right]\!\right]$$

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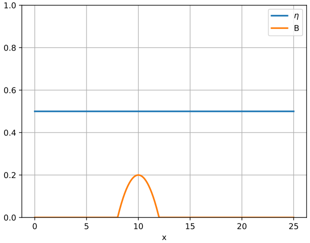
- **jump r** (residual)

$$\mathsf{ST}_i(\boldsymbol{u}_h) := \sum_f \alpha_f \left[\!\left[\boldsymbol{J} \frac{\partial}{\partial x}\varphi_i \right]\!\right] |\boldsymbol{J}|^{-1} \left[\!\left[\boldsymbol{J} \frac{\partial}{\partial x}\boldsymbol{u} - \boldsymbol{S} \right]\!\right], \quad \boldsymbol{J} := \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}$$

Section 4

Numerical results

Well-balancing



Framework	L^1 error H	L^1 error Hv
WB-HS jc	3.007E-004	5.963E-004
WB-HS jt	9.403E-013	4.418E-012
WB-HS je	9.396E-013	4.415E-012
WB-HS jr	9.409E-013	4.415E-012

Table: PGL4 with 100 elements at $T_f = 10$

Same with Pn and Bn

Small perturbation of lake at rest

Same domain, non-smooth bathymetry, only C^0

$$\eta(x) = \begin{cases} \eta_{eq} + A \exp\left(1 - \frac{1}{1-\left(\frac{x-6}{0.5}\right)^2}\right) & 5.5 < x < 6.5 \\ \eta_{eq} & otherwise \end{cases}$$

with $A = 5 \cdot 10^{-5}$.

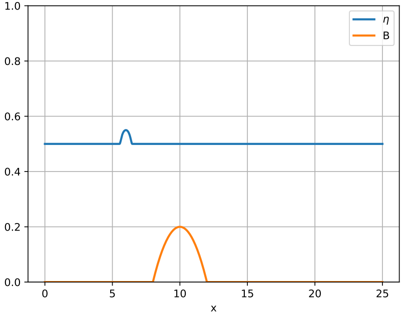
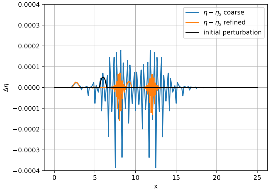


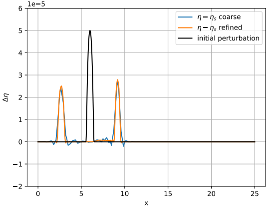
Figure: Perturbation amplified by 1000

Small perturbation of lake at rest

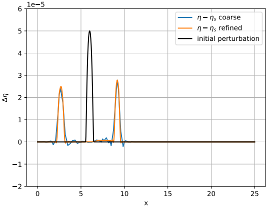
PGL4; coarse: 30 elements; refined: 128 elements



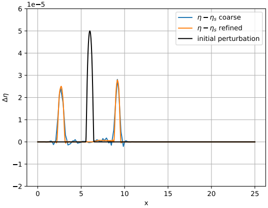
(a) Jump c



(b) Jump t



(c) Jump e

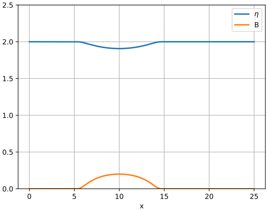


(d) Jump r

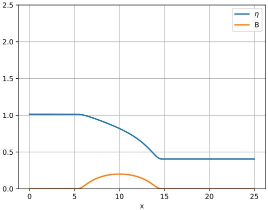
Question

What about the other steady states?

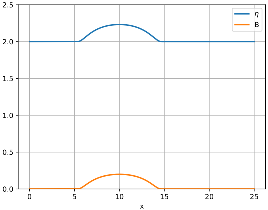
Convergence, PGL4



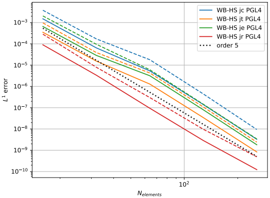
(a) Sub, $Hv \equiv 4.42$

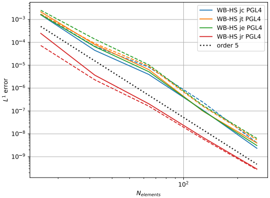


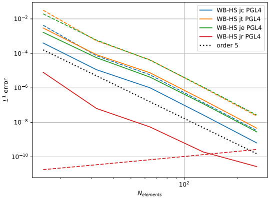
(b) Trans, $Hv \equiv 1.53$



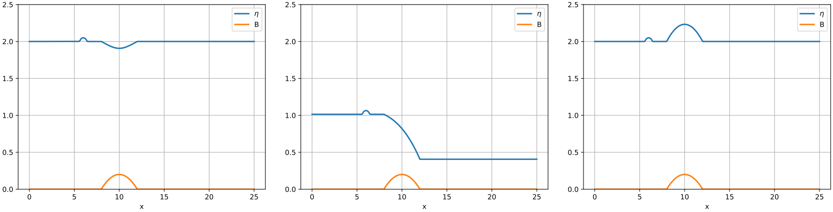
(c) Super, $Hv \equiv 24$





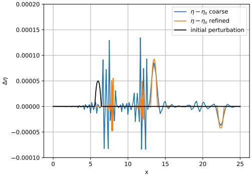


Continuous H, Dashed Hv

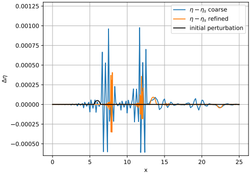


(a) Subcritical (b) Transcritical (c) Supercritical

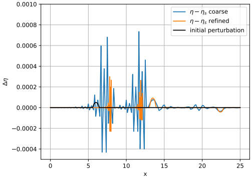
Figure: Perturbation amplified by 1000



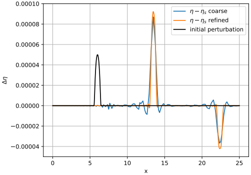
(a) Jump c



(b) Jump t

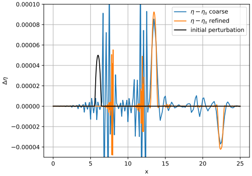


(c) Jump e



(d) Jump r

Supercritical



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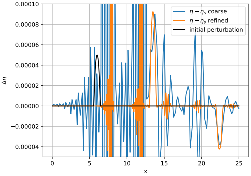
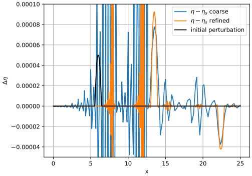
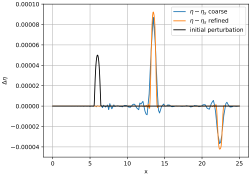


(b) Jump t



(c) Jump e

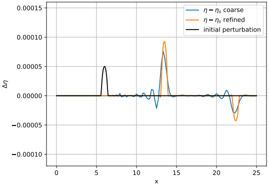


(d) Jump r

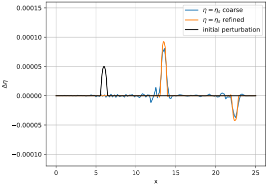
Analogous results

Analogous results for...

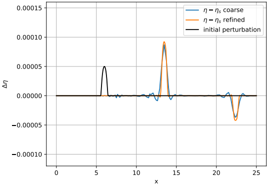
- for other basis functions (Bn, Pn and PGLn)



(a) PGL2



(b) PGL3

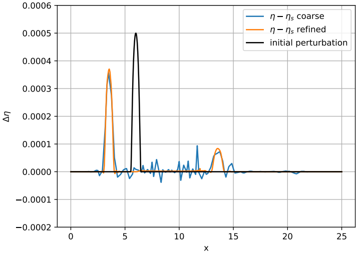


(c) PGL4

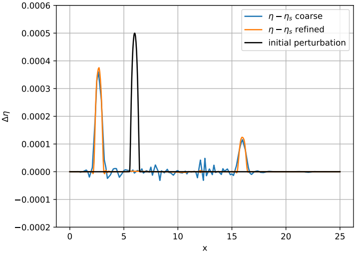
Supercritical: "fair" comparison with 60, 40 and 30 elements

Analogous results

- for other tests



(a) Transcritical



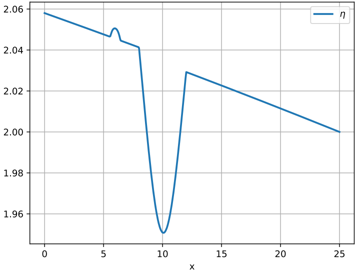
(b) Subcritical

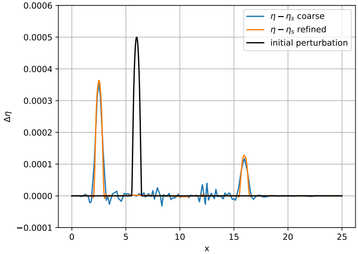
(NB: No limiting at all)

Analogous results

- with friction

$$\boldsymbol{S}(x,\boldsymbol{u}) = -\begin{pmatrix} 0 \\ gH\frac{\partial}{\partial x}B \end{pmatrix} - g\frac{n^2|Hv|}{H^{\frac{7}{3}}}\begin{pmatrix} 0 \\ Hv \end{pmatrix}$$





Subcritical: perturbation amplified by 100

(Same for supercritical)

Further

Other WB space discretizations and stabilizations: Global flux

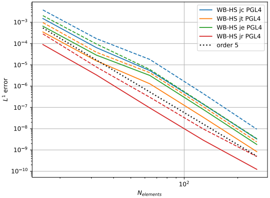
$$\frac{\partial}{\partial t}\boldsymbol{u} + \frac{\partial}{\partial x}\boldsymbol{F} = \boldsymbol{S}$$

$$\Downarrow \;\; \boldsymbol{R} = -\int_{x_L}^{x} \boldsymbol{S}, \quad \boldsymbol{G} = \boldsymbol{F} + \boldsymbol{R}$$

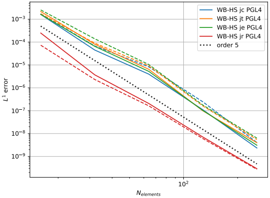
$$\frac{\partial}{\partial t}\boldsymbol{u} + \frac{\partial}{\partial x}\boldsymbol{G} = \boldsymbol{0}$$

$$\mathbf{ST}_i(\boldsymbol{u}_h) = \sum_f \alpha_f \left[\!\left[\boldsymbol{J} \frac{\partial}{\partial x}\varphi_i \right]\!\right] |\boldsymbol{J}|^{-1} \left[\!\left[\frac{\partial}{\partial x}\boldsymbol{G} \right]\!\right]$$

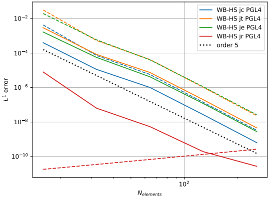
Convergence



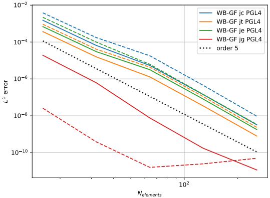
(a) Subcritical

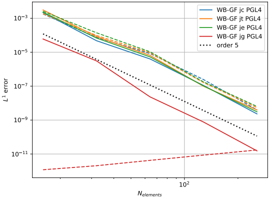


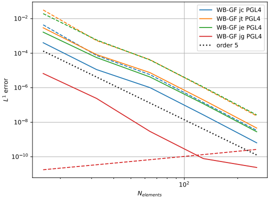
(b) Transcritical



(c) Supercritical







Section 5

Conclusions

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- WB space discretization (w.r.t. lake at rest)

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- Three new well-balanced CIP stabilizations
  - jt (total height)
  - je (entropy variables)
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Conclusions

- WB space discretization (w.r.t. lake at rest)
- Three new well-balanced CIP stabilizations
  - jt (total height)
  - je (entropy variables)
  - jr (residual)
- The numerical results confirm
  - Exact well-balancing for lake at rest
  - HO accuracy
  - Ability of jr to handle general steady states

The End

Thank you[3][4]

[3] Micalizzi, Ricchiuto, Abgrall, Novel well-balanced continuous interior penalty stabilizations, 2023

[4] Looking for a postdoc in the U.S.