# Spectral Lagrangian methods (in particular for SW)

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# Outline

- Governing equations and motivation
- Continuous FEM
- Well-balancing
  - Spatial part
  - Stabilization, continuous interior penalty
- Numerical results

Section 1

# Governing equations and motivation

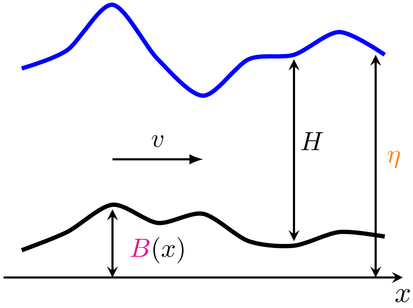
# Shallow water equations, Eulerian

$$\frac{\partial}{\partial t}\boldsymbol{u} + div_{\boldsymbol{x}}\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{u}), \quad (\boldsymbol{x}, t) \in \Omega \times [0, T]$$

$$\boldsymbol{u} := \begin{pmatrix} H \\ H\boldsymbol{v} \end{pmatrix}$$

$$\boldsymbol{F}(\boldsymbol{u}) := \begin{pmatrix} H\boldsymbol{v} \\ H\boldsymbol{v}\otimes\boldsymbol{v} + g\frac{H^2}{2} \end{pmatrix}$$

$$\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{u}) := -\begin{pmatrix} 0 \\ gH\nabla_{\boldsymbol{x}}B(x) \end{pmatrix}$$




$$\eta := H + B$$

# Shallow water equations, simplification of Euler (with gravity)

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$$H \sim \rho$$

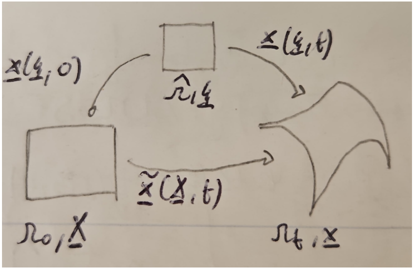
$$p \sim g\frac{H^2}{2}$$

$$B(x) \sim \phi(x)$$

Neglecting the energy equation

# Shallow water equations, Lagrangian

$$\Omega_0 \ni \boldsymbol{X} \longrightarrow \boldsymbol{x} = \widetilde{\boldsymbol{x}}(\boldsymbol{X}, t) \in \Omega_t$$
$$\widehat{\Omega} \ni \boldsymbol{\xi} \longrightarrow \boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi}, t) \in \Omega_t$$
$$\boldsymbol{X} = \boldsymbol{x}(\boldsymbol{\xi}, 0)$$



- **Motion**

$$\frac{d}{dt}\boldsymbol{x}(\boldsymbol{\xi},t) = \boldsymbol{v}(\boldsymbol{x}(\boldsymbol{\xi},t),t);$$

- **Water height/density (SMC)**

$$H(\boldsymbol{x}(\boldsymbol{\xi},t),t) = \frac{\widehat{H}(\boldsymbol{\xi})}{det\boldsymbol{J}(\boldsymbol{\xi},t)}, \quad \boldsymbol{J}(\boldsymbol{\xi},t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}(\boldsymbol{\xi},t);$$

- **Velocity**

$$\begin{aligned} H(\boldsymbol{x},t)\frac{d}{dt}\boldsymbol{v}(\boldsymbol{x},t) &= -\nabla_{\boldsymbol{x}} p - gH\nabla_{\boldsymbol{x}} B \\ &= -\nabla_{\boldsymbol{x}}\left(\frac{gH^2}{2}\right) - gH\nabla_{\boldsymbol{x}} B \end{aligned}$$

$$\frac{d}{dt}\boldsymbol{v}(\boldsymbol{x},t) = -g\nabla_{\boldsymbol{x}}(H+B)$$

# Shallow water equations, FEM discretization

$\widehat{\varphi}_i \in \mathbb{P}_{M+1}$ continuous
$\widehat{\psi}_i \in \mathbb{P}_M$ discontinuous

$$\varphi_i(\boldsymbol{x}(\xi, t), t) = \widehat{\varphi}_i(\xi)$$

$$\psi_i(\boldsymbol{x}(\xi, t), t) = \widehat{\psi}_i(\xi)$$

$$\boldsymbol{x}_h(\boldsymbol{\xi}, t) = \sum_i \boldsymbol{x}_i(t)\widehat{\varphi}_i(\xi)$$

$$\boldsymbol{v}_h(\boldsymbol{\xi}, t) = \sum_i \boldsymbol{v}_i(t)\widehat{\varphi}_i(\xi)$$

$$H_h(\boldsymbol{\xi}, t) = \sum_i H_i(t)\widehat{\psi}_i(\xi)$$

$$\boldsymbol{v}_h(\boldsymbol{x}, t) = \sum_i \boldsymbol{v}_i(t)\widehat{\varphi}_i(\xi(\boldsymbol{x}, t))$$

$$H_h(\boldsymbol{x}, t) = \sum_i H_i(t)\widehat{\psi}_i(\xi(\boldsymbol{x}, t))$$

# Shallow water equations, discretized equations

- **Motion**

$$\frac{d}{dt}\boldsymbol{x}_i(t) = \boldsymbol{v}_i(t);$$

- **Water height/density (SMC) $\Rightarrow$ PP**

$$H_i(t) = \frac{\widehat{H}_i}{det\boldsymbol{J}(\boldsymbol{\xi}_i, t)}, \quad \boldsymbol{J}(\boldsymbol{\xi}, t) = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}(\boldsymbol{\xi}, t);$$

- **Velocity**

$$\sum_{K \in K_i} \sum_{\boldsymbol{x}_j \in K} \left( \int_K \varphi_i \varphi_j d\boldsymbol{x} \right) \frac{d}{dt}\boldsymbol{v}_j(t)$$
$$= -g \sum_{K \in K_i} \left[ \int_K \varphi_i \nabla_{\boldsymbol{x}}(H_h + B_h) d\boldsymbol{x} + \boldsymbol{CT}_i^K \right] + \boldsymbol{ST}_i$$

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$$\boldsymbol{CT}_i^K =$$

$$\mathsf{ST}_i(\boldsymbol{u}_h) := \sum_f \alpha_f \left[\!\left[\frac{\partial}{\partial x}\varphi_i\right]\!\right]\left[\!\left[\frac{\partial}{\partial x}\begin{pmatrix}\eta\\ Hv\end{pmatrix}\right]\!\right]$$

## Motivation

**Well-balancing** (in particular $\frac{\partial}{\partial t}\boldsymbol{u} \equiv 0 \;\Leftrightarrow\; \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{S}(x, \boldsymbol{u})$)

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- Well-balancing $\Rightarrow$ Not easy

$$\boldsymbol{u}_{eq} := \begin{pmatrix} \eta - B \\ 0 \end{pmatrix}$$

$$\eta = H + B \equiv const$$

Figure: Lake of Zürich at rest

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But in general unknown

Section 2

# Continuous FEM

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- We look for $\boldsymbol{u}_h(x,t) := \sum_{j=1}^{I} \boldsymbol{c}_j(t)\varphi_j(x)$ s.t.

$$\int_\Omega \left( \frac{\partial}{\partial t}\boldsymbol{u}_h + \left[ \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}_h) - \boldsymbol{S}(x,\boldsymbol{u}_h) \right]_h \right) \varphi_i(x) + \mathsf{ST}_i(\boldsymbol{u}_h) = \mathbf{0} \quad \forall i$$

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Section 3

# Well-balancing

Subsection 1

# Spatial part

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$$\left[\frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}_h) - \boldsymbol{S}(x, \boldsymbol{u}_h)\right]_h = \frac{\partial}{\partial x}\boldsymbol{F}_h(x) - \boldsymbol{S}_h(x)$$

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Special WB discretization[1]

$$\left[\frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}_h)-\boldsymbol{S}(x,\boldsymbol{u}_h)\right]_h = \left[\frac{\partial}{\partial x}\begin{pmatrix} Hv \\ Hv^2 + g\frac{H^2}{2}\end{pmatrix} + \begin{pmatrix} 0 \\ gH\frac{\partial}{\partial x}B\end{pmatrix}\right]_h$$

$$= \frac{\partial}{\partial x}\left[\begin{pmatrix} Hv \\ Hv^2\end{pmatrix}\right]_h + \begin{pmatrix} 0 \\ gH_h\frac{\partial}{\partial x}\left(H_h + B_h\right)\end{pmatrix}$$

[1] Ricchiuto and Bollermann. Stabilized residual distribution for shallow water simulations, 2009

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Subsection 2

## Stabilization, continuous interior penalty

# Stabilization part

$$\int_\Omega \varphi_i(x)\frac{\partial}{\partial t}\boldsymbol{u}_h d\boldsymbol{x} + \int_\Omega \varphi_i(x)\left[\frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}_h) - \boldsymbol{S}(x,\boldsymbol{u}_h)\right]_h + \mathsf{ST}_i(\boldsymbol{u}_h) = \boldsymbol{0} \quad \forall i$$

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CIP[2] stabilization terms (**jump c**)

$$\mathsf{ST}_i(\boldsymbol{u}_h) := \sum_f \alpha_f \int_f [\![\nabla_{\boldsymbol{x}}\varphi_i]\!][\![\nabla_{\boldsymbol{x}}\boldsymbol{u}_h]\!]d\sigma$$
$$= \sum_f \alpha_f \left[\!\!\left[\frac{\partial}{\partial x}\varphi_i\right]\!\!\right]\left[\!\!\left[\frac{\partial}{\partial x}\boldsymbol{u}_h\right]\!\!\right]$$

[2] J. Douglas Jr and T. Dupont Interior Penalty Procedures for Elliptic and Parabolic Galerkin Methods, 1976

# Stabilization part

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&= \sum_f \alpha_f \left[\!\!\left[\frac{\partial}{\partial x}\varphi_i\right]\!\!\right]\left[\!\!\left[\frac{\partial}{\partial x}\boldsymbol{u}_h\right]\!\!\right] \\
&= \sum_f \alpha_f \left[\!\!\left[\frac{\partial}{\partial x}\varphi_i\right]\!\!\right]\left[\!\!\left[\frac{\partial}{\partial x}\begin{pmatrix} H_h \\ (Hv)_h \end{pmatrix}\right]\!\!\right] \neq \mathbf{0}
\end{aligned}$$

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# Stabilization part, novel well-balanced jumps

- **jump t** (total height)

$$\mathsf{ST}_i(\boldsymbol{u}_h) := \sum_f \alpha_f \left[\!\left[ \frac{\partial}{\partial x}\varphi_i \right]\!\right] \left[\!\left[ \frac{\partial}{\partial x} \begin{pmatrix} \eta \\ Hv \end{pmatrix} \right]\!\right]$$

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- **jump e** (entropy variables)

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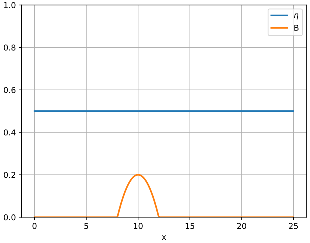
- **jump r** (residual)

$$\mathsf{ST}_i(\boldsymbol{u}_h) := \sum_f \alpha_f \left[\!\left[ \boldsymbol{J}\frac{\partial}{\partial x}\varphi_i \right]\!\right] |\boldsymbol{J}|^{-1} \left[\!\left[ \boldsymbol{J}\frac{\partial}{\partial x}\boldsymbol{u} - \boldsymbol{S} \right]\!\right], \quad \boldsymbol{J} := \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{u}}$$

Section 4

# Numerical results

# Well-balancing




| Framework | $L^1$ error $H$ | $L^1$ error $Hv$ |
|---|---|---|
| WB-HS jc | 3.007E-004 | 5.963E-004 |
| WB-HS jt | 9.403E-013 | 4.418E-012 |
| WB-HS je | 9.396E-013 | 4.415E-012 |
| WB-HS jr | 9.409E-013 | 4.415E-012 |

Table: PGL4 with $100$ elements at $T_f = 10$

Same with P$n$ and B$n$

# Small perturbation of lake at rest

Same domain, non-smooth bathymetry, only $C^0$

$$\eta(x) = \begin{cases} \eta_{eq} + A\exp\left(1 - \frac{1}{1-\left(\frac{x-6}{0.5}\right)^2}\right) & 5.5 < x < 6.5 \\ \eta_{eq} & otherwise \end{cases}$$

with $A = 5 \cdot 10^{-5}$.

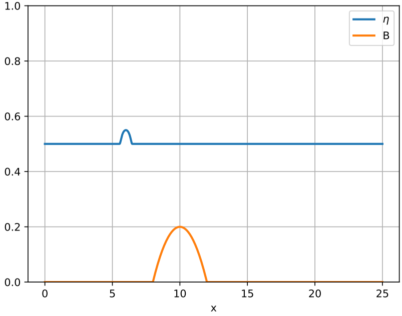
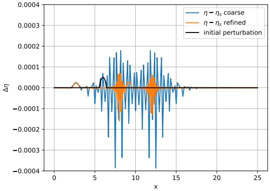
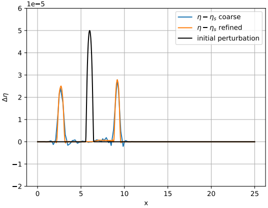



Figure: Perturbation amplified by 1000

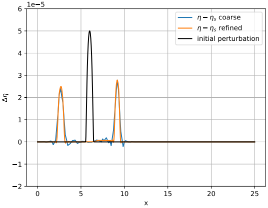
PGL4; coarse: $30$ elements; refined: $128$ elements



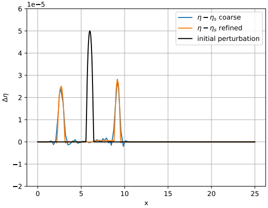

(a) Jump c




(b) Jump t



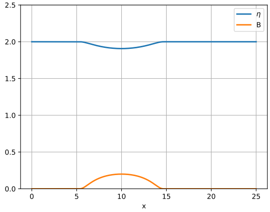

(c) Jump e



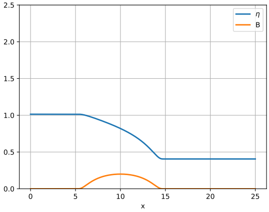

(d) Jump r

# Question

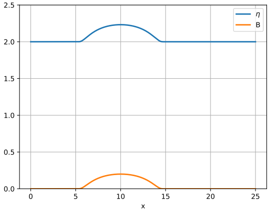
What about the other steady states?



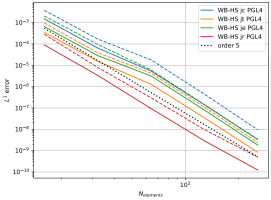

(a) Sub, $Hv \equiv 4.42$

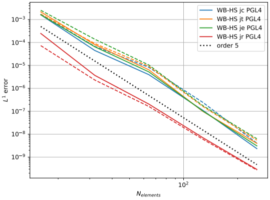


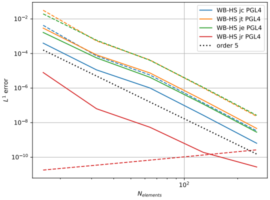

(b) Trans, $Hv \equiv 1.53$




(c) Super, $Hv \equiv 24$

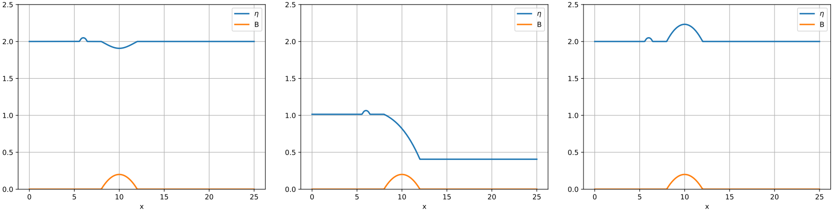



Continuous $H$, Dashed $Hv$

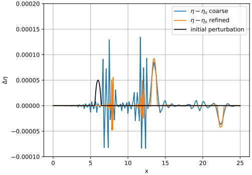
# Small perturbation



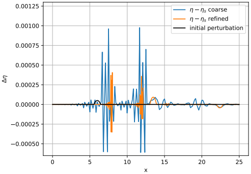

(a) Subcritical (b) Transcritical (c) Supercritical

Figure: Perturbation amplified by $1000$

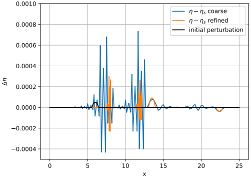
# Supercritical



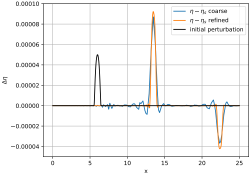

(a) Jump c




(b) Jump t

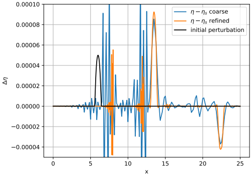



(c) Jump e

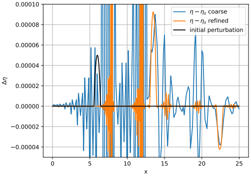



(d) Jump r

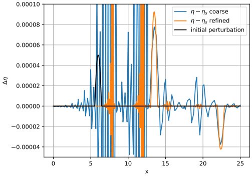
# Supercritical



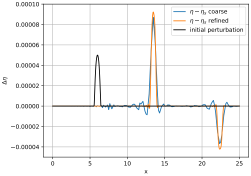

(a) Jump c




(b) Jump t




(c) Jump e



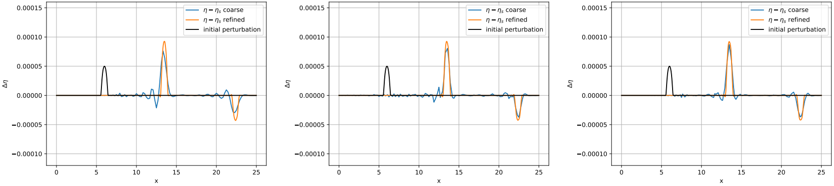

(d) Jump r

# Analogous results

Analogous results for...

# Analogous results

- for other basis functions (Bn, Pn and PGLn)



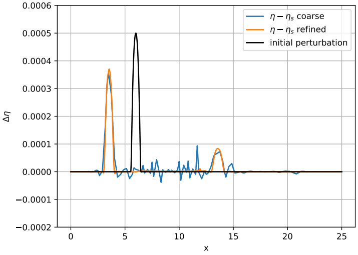

(a) PGL2 (b) PGL3 (c) PGL4

Supercritical: "fair" comparison with $60$, $40$ and $30$ elements

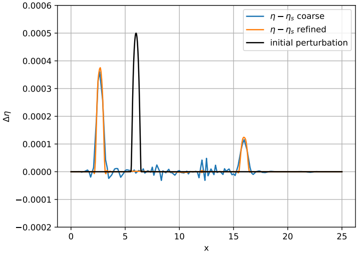
(Constant number of DoFs)

# Analogous results

- for other tests




(a) Transcritical



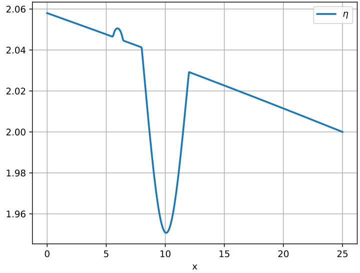

(b) Subcritical

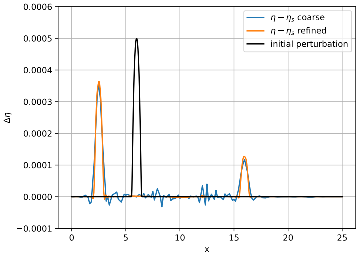
(NB: No limiting at all)

# Analogous results

- with friction

$$\boldsymbol{S}(x,\boldsymbol{u}) = -\begin{pmatrix} 0 \\ gH\frac{\partial}{\partial x}B \end{pmatrix} - g\frac{n^2|Hv|}{H^{\frac{7}{3}}}\begin{pmatrix} 0 \\ Hv \end{pmatrix}$$

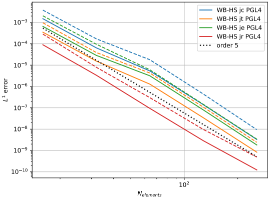


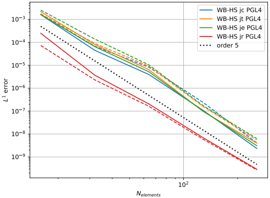

Subcritical: perturbation amplified by 100

(Same for supercritical)

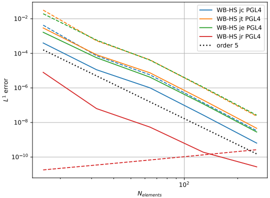
# Global flux, $\boldsymbol{G} = \boldsymbol{F} - \int_{x_L}^{x} \boldsymbol{S}$



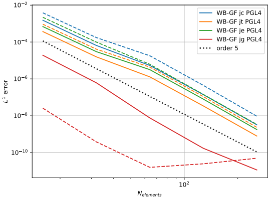

(a) Subcritical

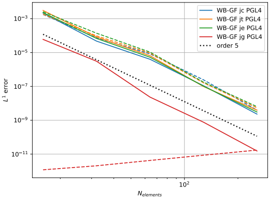


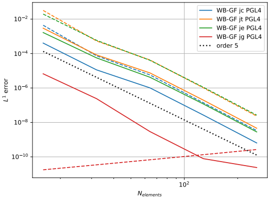

(b) Transcritical




(c) Supercritical



Section 5

# Conclusions

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- WB space discretization (w.r.t. lake at rest)

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- Three new well-balanced CIP stabilizations
  - jt (total height)
  - je (entropy variables)
  - jr (residual)

# Conclusions

- WB space discretization (w.r.t. lake at rest)
- Three new well-balanced CIP stabilizations
  - jt (total height)
  - je (entropy variables)
  - jr (residual)
- The numerical results confirm
  - Exact well-balancing for lake at rest
  - HO accuracy
  - Ability of jr to handle general steady states

# *Thank you*[3]

Novel well-balanced continuous interior penalty stabilizations;
Micalizzi, Ricchiuto, Abgrall; 2023

[3]Looking for a postdoc position in the U.S.

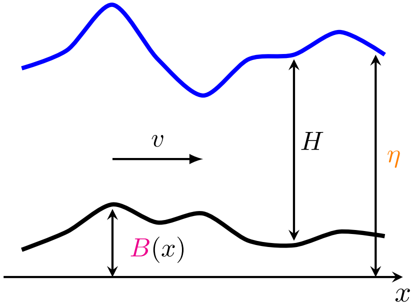
# Shallow water equations, Eulerian

$$\frac{\partial}{\partial t}\boldsymbol{u} + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{S}(x,\boldsymbol{u}), \quad (x,t) \in \Omega \times [0,T],\ \Omega := (x_L, x_R)$$

$$\boldsymbol{u} := \begin{pmatrix} H \\ Hv \end{pmatrix}$$

$$\boldsymbol{F}(\boldsymbol{u}) := \begin{pmatrix} Hv \\ Hv^2 + g\frac{H^2}{2} \end{pmatrix}$$

$$\boldsymbol{S}(x,\boldsymbol{u}) := -\begin{pmatrix} 0 \\ gH\frac{\partial}{\partial x}B(x) \end{pmatrix}$$




$$\eta := H + B$$

# Shallow water equations, simplification of Euler (with gravity)

$$\frac{\partial}{\partial t}\boldsymbol{u} + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{u}) = \boldsymbol{S}(x, \boldsymbol{u}), \quad (x,t) \in \Omega \times [0,T],\ \Omega := (x_L, x_R)$$

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$$\boldsymbol{S}(x, \boldsymbol{u}) := -\begin{pmatrix} 0 \\ gH\frac{\partial}{\partial x}B(x) \end{pmatrix}$$

$$H \sim \rho$$

$$p \sim g\frac{H^2}{2}$$

$$B(x) \sim \phi(x)$$

Neglecting the energy equation