SPECTRAL STABILITY OF GASEOUS DIFFUSION FRONTS IN GLASSY POLYMERS

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Abstract. This paper studies the spectral stability of a traveling wave solution that models the diffusion of a penetrant gas in a glassy polymer where the concentration an the diffusion parameter are coupled by the evolution system proposed by Cohen and Stanley [CS83]. It is established, by energy estimates, that the linear perturbation equation around Cohen & Stanley's solution determines an spectral problem which is spectrally stable in the exponentially weighted space $L^2_\alpha \times L^2_\alpha$ when the propagation speed is greater that a threshold speed, determined by the model parameters.

1. Introduction

Glassy polymers have gain a lot attention in the last decades due to their applications that range from home appliance and 3D printing to industry usages due to their price, durability and versatility. A common issue in these materials is the diffusion of a penetrant, commonly in a gaseous phase, in the bulk of the glassy polymer, where high concentrations of the penetrant modify the expected behavior, stress and deformation, of the glassy polymer []. For example at 3D printing the accumulation of big concentration of moisture in the filament provokes the generation of bobbles, and most of the time the tip printer cannot reach correct temperature at printing []. Penetrant diffusion in a glassy polymers is a complex problem because the polymer tends to swells in the presence of the penetrant. The bulk swelling is generated because the polymer adsorbs and absorbs the penetrant. The amount of sorbed penetrant depends on the class of penetrant, the class of glassy polymer and in the penetrant concentration C. Thus, the diffusion coefficient D is not longer constant but it satisfies an evolution equation depending on the concentration C. The most-well accepted model to described the diffusion of a penetrant gas in a glassy polymer was proposed by Cohen and Stanley [CS83]: [1]

$$C_\tau = (DC_\chi)_\chi + R(\chi, \tau, C, C_\tau, C_\chi) \tag{1.1a}$$

$$D_\tau = F'(C)C_\tau + \alpha(C)[G(C) - D] \tag{1.1b}$$

where $F(C)$ and $G(C)$ represent the instantaneous and equilibrium diffusion functions, respectively, $\alpha(C)$ is a rate-controlling function, and $R(\chi, \tau, C, C_\tau, C_\chi)$ is the kinetic-like term which takes into account the interaction between the polymer and the penetrant. This interaction can be interpreted as filling microvoids in the bulk and both kinds of sorptions []. System (1.1) can model many diffusion process by choosing properly these functions. For example, if $R(\chi, \tau, C, C_\tau, C_\chi)$ and $\alpha(C)$ are

2020 *Mathematics Subject Classification.* ?????

Key words and phrases. ?????

[1] A preliminary model that only considers swelling was given by Crank, this system is almost the same as (1.1), but it lacks of the kinetic-like term $R(\chi, t, C, C_t, C_\chi)$.

identically zero, the system (1.1) reduces to a classical diffusion equation which models standard diffusion without interaction. Also, if only $R(\chi,\tau,C,C_\tau,C_\chi)=0$, D has two asymptotic behaviors depending on C. Indeed, $D\sim F(C)$ for small values of $\alpha(C)$ and $D\sim G(C)$ if $\alpha(C)\to\infty$. Hence, (1.1b) turns into a diffusion equation where the diffusion function $D(C)$ evolves. This type of diffusion is experimentally observed in glassy polymers, []. The expression for the kinetic term $R(\chi,\tau,C,C_\tau,C_\chi)$ can be very complex but we specializes our analysis in the case where $R(\chi,\tau,C,C_\tau,C_\chi)=\mu C(k-C)$ for certain material constants $\mu,k>0$. Following Cohen & Stanley [CS83], one can write (1.1) in dimensionless variables as

$$c_t =[(g(c)-w)c_x]_x + c(1-c), \tag{1.2a}$$

$$w_t =-\beta(c)w + h(c)c_t, \tag{1.2b}$$

where

$$\begin{array}{cccc} t=\nu\tau, & x=\sqrt{\nu\gamma\chi}, & c=k^{-1}C, & w=\gamma(G(C)-D), \\ g(c)=\gamma G(C) & f(c)=\gamma F(C) & \beta(c)=\nu^{-1}\alpha(C), & h(c)=g'(c)-f'(c), \end{array} \tag{1.3}$$

with $\nu:=\mu k$, and $\gamma=1/G(0)$. The evolution system (1.2) turns hard to analyzed due to the coupling of different orders in both equation. Indeed, by assuming that $c(x,t)$ ann the initial conditions on $w(x,t)$ are known, we can solve for the latter variable from equation (1.2b) and substitute the result in equation (1.2a). This process leads to the following evolution equation

$$c_t=\left[\left(g(c)-w_0(x)e^{-\int_0^t\beta\circ c(x,\tau)\,d\tau}-\int_0^t e^{-\int_s^t\beta\circ c(x,\tau)\,d\tau}\partial_s(H\circ c(x,s))\,ds\right)c_x\right]_x+c(1-c), \tag{1.4}$$

where $H:=g-h$. Equation (1.4) is an integro-differential equation on the relative concentration c, where the diffusion coefficient has "memory". The existence of solutions of this type of equations is hard to show and just a few methods are available []. Also, the stability of these solutions is not and easy task []. The simplest type of solutions for a complex evolution equation in unbounded domains are traveling waves solutions. This ansatz on the solutions allows us to transform the evolution equation into a dynamical system where a bast analysis toolbox is available. In [CS83] the authors assumed the following conditions

- H1 $\beta(0)>0$,
- H2 $f(0)=g(0)=1$, and $g\geq f$,
- H3 the functions f,g, and β are monotonically nondecreasing,
- H4 $h(c)\geq 0$ for $c\geq 0$,
- H5 the derivatives f', g', and the function β satisfy a Lipschitz condition for $c\geq 0$,

to stablish the existence of traveling wave solutions to system (1.2) if the wave speed $a>0$ is large enough. We summarize Cohen & Stanley 's result in the following

`Check redaction and hypothesis! If possible, determine` a^* `explicitely from [CS83].`

Theorem 1.1. *There exists $\mu^*>0$ such that for every $\mu>\mu^*$ there exists $a^*>0$ such that for every $a>a^*$ there exists a function $(\mathbb{c},\mathbb{w}):\mathbb{R}\to\mathbb{R}^2$ where $(\mathbb{c}(-(x+$*

$at)), \mathbb{w}(-(x+at)))$ is a solution to (1.2). Moreover, $\mathbb{c}(\xi) \in (0,1)$ for every $\xi \in \mathbb{R}$ and

(1) $-\mathbb{c}'(\xi) \in (0, \mu\mathbb{c}(\xi)(1-\mathbb{c}(\xi)))$,
(2) $\mathbb{w}(\xi) \leq (g-f)\circ\mathbb{c}(\xi)$.

In addition, the curve $\xi \mapsto (\mathbb{c}(\xi), -\mathbb{c}(\xi)', \mathbb{w}(\xi))^\intercal$ satisfies

$$\lim_{\xi\to-\infty}\begin{pmatrix}\mathbb{c}(\xi)\\-\mathbb{c}'(\xi)\\\mathbb{w}(\xi)\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \text{and} \quad \lim_{\xi\to\infty}\begin{pmatrix}\mathbb{c}(\xi)\\-\mathbb{c}'(\xi)\\\mathbb{w}(\xi)\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Despite of this existence result was given 1983, there is no further analysis pertaining with the stability of this traveling wave solution. This type of study are important for industry and applications because it gives parameter regimens to preserve, or destroy, the traveling wave when it is present in a glassy polymer. In this work we deep on the properties of Cohen & Stanley's traveling wave such as regularity and exponential decay to asymptotic values in order to stablish sufficient conditions for spectral stability of this family of solutions, *i.e.*, we find conditions on the model functions f, g, β and wave speed $a > 0$ to guarantee that the linearization of (1.2) around Cohen & Stanley's solution is a linear evolution problem with and spatial linear operator whose spectrum is not contained in $\{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda > 0\}$. The complexity of this spectral analysis lies in many aspects of the problems that we list in the sequel

(1) The model functions f, g, and β are quite arbitrary because they are require to be monotonically nondecreasing functions satisfy hypothesis H1 to H5 and no more. This wide family of model functions allows (1.2) to model a wide variety of gas diffusion in glassy polymers with logistic kinetic term, but at the same time it makes harder the analysis since no more information can be deduced for the traveling wave or the eigenvalue problem (3.5).
(2) the eigenvalue problem (3.5) has ODEs with different orders for each variable. Indeed, the variable v, related with a perturbation in the diffusion coefficient, satisfies a nonhomogeneous first order differential equation that may generate unbounded solutions for bounded initial conditions.
(3) At first glance, it looks that the point spectrum of (4.8) can accumulate on the imaginary axis for $\operatorname{Re}\lambda > 0$, implying the spectral instability of model (1.2). Indeed, it is not difficult to show that any eigenvalue $\lambda \in \mathbb{C}$ of (3.5) satisfy that $\operatorname{Re}\lambda < M$ for a fix $M > 0$ but no uniform bound on $\operatorname{Im}\lambda$ is already established when $\operatorname{Re}\lambda > 0$ is assume. Moreover, It is not possible to invoke classical arguments such as permanence of exponential dichotomy under small L^∞-perturbations, see [San02, LZ21], to bound $|\lambda|$ when $\operatorname{Re}\lambda > 0$ because the limiting system does not exhibits exponential dichotomy when $|\lambda| \to \infty$ and $\operatorname{Re}\lambda$ can be arbitrarily small. This fact can be easily seen from the associated dynamical system (4.1), with $\alpha = 0$, which has a singular leading term in $|\lambda|$.
(4) The model (1.2) has an integro-differential equation linked to the concentration equation *i.e.*, a memory effect. These attributes also permeate to the the eigenvalue system (4.8).

From the mathematical view point, spectral stability of the traveling wave is the minimal requirement if we want to stablish the nonlinear stability of the solution *i.e.* the stability of Cohen & Stanley solution to whole nonlinear model (1.2) for any perturbation in an appropriate Banach space. This necessity is exposed in many seminal works in the field such as [].

Heuristically, one expects that any slight perturbations to Cohen & Stanley's solution may evolve, following (1.2), into a superposition of a dominant traveling wave plus some other "smaller" traveling waves, also known as convective instabilities [San02], and dissipative waves. This behavior is expected since the linear perturbation equation for (1.2) around Cohen & Stanley's solution, see (3.2), behaves as a constant-coefficient evolution equation for large values of the Galilean variable $\xi = -x - at$ and the associated dispersion equations assign distinct wave speeds to different spatial modes, see (4.2) and (4.3) for $\alpha = 0$. Thus, any "small" traveling waves should move to a different speed than that of the main traveling wave. Therefore, a suitable chosen perturbation space will allow us to prove spectral stability.

Plan of the paper. The remainder of the paper is structured as follows. In order to be able to state our main result (see Theorem 4.8 below), Section 2 collects must relevant information given in [CS83] about its traveling wave solution and the dynamical system associated. Also, we exploit the hyperbolicity of the ODE's equilibria to stablish the minimal regularity of this solution as well as its asymptotic behavior. The linear and nonlinear perturbation equations and the associated eigenvalue problem are presented in Section 3. Also, the clossedness of block operator for the eigenvalue problem is established in the same section. In Section 4 I give sufficient conditions to obtain essential spectrum stability on the exponentially weighted space $L^2_\alpha \times L^2_\alpha$ by the correct choice of the weight α. This exponential weight is also used to guaranteed point spectral stability. Finally, Section **??** discuses limitations and capabilities of this work as well as possible new investigation lines.

Notations. We denote the spaces $L^2(\mathbb{R}, \mathbb{C})$, $H^1(\mathbb{R}, \mathbb{C})$ and $H^2(\mathbb{R}, \mathbb{C})$ of complex-valued functions as L^2, H^1 and H^2. Their real-valued counterparts are denoted as $L^2(\mathbb{R})$, $H^1(\mathbb{R})$ and $H^2(\mathbb{R})$, respectively. For any number or complex-valued function, the operation $(\cdot)^*$ denotes complex conjugation. For any linear, closed, and densely defined operator operator, $\mathcal{L} : D(\mathcal{L}) \subset X \to Y$, with X, Y Banach spaces and domain $D(\mathcal{L}) \subset X$, the resolvent set, $\rho(\mathcal{L})$, is defined as the set of complex numbers $\lambda \in \mathbb{C}$ such that $\mathcal{L} - \lambda$ is injective and onto, and $(\mathcal{L} - \lambda)^{-1}$ is a bounded operator. The spectrum of $\mathcal{L}$ is the complex complement of the resolvent, $\sigma(\mathcal{L}) = \mathbb{C} \backslash \rho(\mathcal{L})$.

2. Cohen & Stanley solution resume

We recall that the traveling wave $(\mathbb{c}, \mathbb{w})$ depends on the fix speed a, we do not make explicit this dependence to keep the notation as simple as possible. Because its nature, $(\mathbb{c}, \mathbb{w})$ satisfies the following pair of ordinary differential equations

$$\begin{aligned} [(g(\mathbb{c}) - \mathbb{w})\mathbb{c}']' + a\mathbb{c}' &= -\mathbb{c}(1 - \mathbb{c}) \\ \mathbb{w}' - h(\mathbb{c})\mathbb{c}' &= \frac{\beta(\mathbb{c})}{a}\mathbb{w}. \end{aligned} \tag{2.1}$$

By calling $\mathbb{u} = -\mathbb{c}'$, we readily get the following first order differential system

$$\begin{aligned} \mathbb{c}' &= -\mathbb{u} \\ [(g(\mathbb{c}) - \mathbb{w})\mathbb{u}]' &= -a\mathbb{u} + \mathbb{c}(1-\mathbb{c}) \\ \mathbb{w}' &= \frac{\beta(\mathbb{c})}{a}\mathbb{w} - h(\mathbb{c})\mathbb{u}, \end{aligned} \tag{2.2}$$

with $X_0 := (0,0,0)^\intercal$ and $X_1 := (1,0,0)^\intercal$ as equilibrium points. Indeed, standard ODE theory implies that the local dynamics of (2.2) is described by the equation $X' = JF_{X_i}(X - X_i)$, for $i = 0$ or 1, where

$$JF_{X_0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -a & 0 \\ 0 & -h(0) & \beta(0)/a \end{pmatrix}, \quad \text{and} \quad JF_{X_1} = \begin{pmatrix} 0 & -1 & 0 \\ -1/g(1) & -a/g(1) & 0 \\ 0 & -h(1) & \beta(1)/a \end{pmatrix}.$$

It is easily proved that both equilibrium points are hyperbolic for $a > 2$. Indeed, the eigenvalues for JF_{x_0} and JF_{x_1} are

$$\nu_0 = \frac{\beta(0)}{a}, \quad \nu_\pm = \frac{-a \pm \sqrt{a^2 - 4}}{2}, \tag{2.3}$$

and

$$\zeta_0 = \frac{\beta(1)}{a}, \quad \zeta_\pm = \frac{-a \pm \sqrt{a^2 + 4g(1)}}{2g(1)}, \tag{2.4}$$

respectively. Hence, JF_{x_0} has two negative eigenvalues meanwhile JF_{x_1} has two positive eigenvalues. Moreover, the negative eigenvalues of JF_{x_0} are well localized. Indeed, if $a > 2$ then Taylor expansion implies that

$$\nu_- = -a + \frac{1}{a} + \frac{1}{a^3} + O\left(\frac{1}{a^5}\right), \quad \text{and} \quad \nu_+ = -\frac{1}{a} - \frac{1}{a^3} + O\left(\frac{1}{a^5}\right), \tag{2.5}$$

Also, by using that $x \le \sqrt{x} \le (x+1)/2$ for $0 \le x \le 1$, we get that

$$\frac{1}{a} - a \le \nu_- \le \frac{2}{a} - a, \quad \text{and} \quad \frac{-2}{a} \le \nu_+ \le \frac{-1}{a}.$$

It is well known from the qualitative theory of ODEs, see *e.g.* chapter 9 of [Tes12], that if $\Phi(t;x)$ is the flux of a vector field $F : M \to T_xM$ on a manifold M and $\gamma(x)_\pm := \{\Phi(t;x) \in M \,|\, 0 \le \pm t < \pm T_\pm(x)\}$ denotes the forward and backward orbits from $x \in M$, defined on its maximal interval, then, for any hyperbolic equilibrium point $x_0 \in M$ there exists a neighborhood $U(x_0)$ where

$$\begin{aligned} &\left\{ x \in M \,|\, \lim_{t\to\pm\infty} |\Phi(t;x) - x_0| = 0 \right\} \\ &\qquad = \bigcup_{\alpha>0} \left\{ x | \gamma_\pm(x) \subset U(x_0), \ \& \ \sup_{\pm t > 0} e^{\pm\alpha t} |\Phi(t;x) - x_0| < \infty \right\}. \end{aligned}$$

Because of Theorem 1.1, the pulse $(\mathbb{c}(\xi), \mathbb{u}(\xi), \mathbb{w}(\xi))^\intercal$ goes exponentially fast to X_1 and X_0 as $\xi \to -\infty$ and $\xi \to \infty$, respectively. This observation yields the following

Lemma 2.1. *Let β, f and g be real-valued functions such that hypotheses 1 to 1 are satisfied. Then, the Cohen & Stanley's traveling wave satisfies that $\mathbb{c} \in L^2_{loc}(\mathbb{R})$ and $(\mathbb{c}', \mathbb{w}) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$. Moreover, $(\mathbb{c}(\xi), -\mathbb{c}'(\xi), \mathbb{w}(\xi))^\intercal = O(e^{\nu_+\xi})$ for $\xi \to \infty$.*

Proof. First, we prove that $\mathbb{c} \in L^2_{loc}$. Let $\phi := (\mathbb{c}, \mathbb{u}, \mathbb{w})^{\intercal}$, then $\phi \in \mathbb{C}^1(\mathbb{R}, \mathbb{R}^3)$ by construction. Thus, for any compact set $K \subset \mathbb{R}$ there holds that $\|\phi \mathbf{1}_K(\xi)\|^2_{L^2} < \|\phi\|^2_\infty |K| < \infty$, and in particular, $\phi \in L^2_{loc}(\mathbb{R})$ due to the arbitrariness of K.

Second, we prove that $(\mathbb{c}', \mathbb{w}) \in H^2 \times H^2$. Since the equilibrium points of (2.2) are hyperbolic and there holds that $\lim_{\xi \to -\infty} \phi(\xi) = X_1$ and $\lim_{\xi \to \infty} \phi(\xi) = X_0$, we conclude that there exist two radius $r_\pm > 0$, two exponents $\alpha_\pm > 0$ and a constant $\Xi > 0$ such that for any point $\phi_0 \in \{\phi(\xi) \,|\, \xi \in \mathbb{R}\}$ there holds that

$$\phi(\xi) = \begin{cases} X_1 + e^{\alpha_- \xi} \ell^-(\xi; \phi_0), & \text{if } \phi_0 \in B_{r_-}(X_1) \text{ and } \xi < -\Xi, \\ X_0 + e^{-\alpha_+ \xi} \ell^+(\xi; \phi_0), & \text{if } \phi_0 \in B_{r_+}(X_0) \text{ and } \xi > \Xi, \end{cases} \tag{2.6}$$

where $\ell^\pm$ satisfy that $\ell^+(0, \phi_0) = \phi_0 - X_0$, $\ell^-(0, \phi_0) = \phi_0 - X_1$, and $\sup_{\pm\xi > \Xi} |\ell^\pm(\xi, \phi_0)| < \infty$. Since ϕ is L^∞-bounded, the relations (2.6) and young's inequality readily imply that

$$\begin{aligned} \|\mathbb{u}\|^2_{L^2} &= \|\mathbb{u}\mathbf{1}_{[-\Xi,\Xi]}(\xi)\|^2_{L^2} + \|\mathbb{u}\mathbf{1}_{\{|\xi|>\Xi\}}(\xi)\|^2_{L^2} \\ &\leq 2\Xi\|\phi\|^2_\infty + \|\mathbb{u}\mathbf{1}_{\{|\xi|>\Xi\}}(\xi)\|^2_{L^2} \\ &\leq 2\Xi\|\phi\|^2_\infty + \|e^{\alpha_-\xi}\ell^-(\xi;\phi_0)\mathbf{1}_{\{\xi<-\Xi\}}(\xi)\|^2_{L^2} + \|e^{-\alpha_+\xi}\ell^+(\xi;\phi_0)\mathbf{1}_{\{\xi>\Xi\}}(\xi)\|^2_{L^2} \\ &\leq 2\Xi\|\phi\|^2_\infty + \sup_{\xi<-\Xi}|\ell^-(\xi,\phi_0)|^2 \int_{-\infty}^{-\Xi} e^{2\alpha_-\xi}\,d\xi + \sup_{\xi>\Xi}|\ell^+(\xi,\phi_0)|^2 \int_{\Xi}^{\infty} e^{-2\alpha_+\xi}\,d\xi \\ &< \infty. \end{aligned}$$

Therefore, $\mathbb{u} \in L^2(\mathbb{R})$. A straightforward modification of the previous inequalities implies that $\|\mathbb{w}\|_{L^2} < \infty$. Also, $\mathbb{c}(1-\mathbb{c}) \in L^2(\mathbb{R})$, due to $\mathbb{c} \in (0,1)$. In addition, $\mathbb{u}'$ and $\mathbb{w}'$ belong to $L^2(\mathbb{R})$ because (2.2) and the facts that $g(\mathbb{c}) - \mathbb{w} > 1$ and β, g', and h belong to $L^\infty(\mathbb{R})$. Hence $(\mathbb{u}, \mathbb{w}) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Third, by differentiating the system (2.2), we have

$$\begin{aligned} \mathbb{c}'' &= -\mathbb{u}' \\ \mathbb{u}'' &= \frac{(1-2\mathbb{c})\mathbb{c}' + (3g'(\mathbb{c})\mathbb{u} + 2\mathbb{w}' - a)\mathbb{u}' + \mathbb{w}''\mathbb{u} - g''(\mathbb{c})\mathbb{u}^3}{g(\mathbb{c}) - \mathbb{w}} \\ \mathbb{w}'' &= \left[\frac{\beta^2(\mathbb{c})}{a^2} - \frac{\beta'(\mathbb{c})}{a}\mathbb{u}\right]\mathbb{w} + \left[h'(\mathbb{c})\mathbb{u} - \frac{\beta(\mathbb{c})}{a}h(\mathbb{c})\right]\mathbb{u} - [h(\mathbb{c})]\mathbb{u}'. \end{aligned}$$

From the last expression, we easily see that the terms in squared brackets belong to $L^\infty(\mathbb{R})$. Hence, by Young's inequality, $\|\mathbb{w}''\|^2_{L^2} \leq C(\|\mathbb{w}\|^2_{L^2} + \|\mathbb{u}\|^2_{H^1}) < \infty$ and $\mathbb{w} \in H^2$. Also, if we take L^2norms in the equation for $\mathbb{u}''$ the Sobolev and triangle inequalities yield

$$\|\mathbb{u}''\|^2_{L^2} \leq \frac{1}{4}\|\mathbb{c}'\|^2_{L^2} + \|3g'(\mathbb{c})\mathbb{u} + 2\mathbb{w}' - a\|^2_{L^\infty}\|\mathbb{u}'\|^2_{L^2} + \|\mathbb{u}\|^2_{L^\infty}\|\mathbb{w}''\|^2_{L^2} + \|g''(\mathbb{c})\mathbb{u}^2\|^2_{L^\infty}\|\mathbb{u}\|^2_{L^2},$$

proving that $\mathbb{u} \in H^2$.

Third, we show that $\phi = O(e^{\xi\nu_+})$ for $\xi \to \infty$. In the particular case of JF_{X_0}, the associated eigenvectors are

$$Y_{\nu_0} = (0,0,1)^{\intercal}, \quad \text{and} \quad Y_{\nu_\pm} = \left(1, -\nu_\pm, \frac{h(0)\nu_\pm}{\nu_\pm - \nu_0}\right)^{\intercal}$$

Since $\phi(\xi)$ is a bounded solution that converge to X_0 as $\xi \to \infty$ then,

$$\phi(\xi) = e^{\xi JF_{X_0}}P(0)\phi(0) + \int_0^\infty e^{(\xi - r)JF_{X_0}}P(\xi - r)[F(\phi(r)) - JF_{X_0}\phi(r)]dr, \tag{2.7}$$

where $P(x)$ is the spectral projector along the stable subspace of JF_{X_0} for $x \geq 0$ and the spectral projector along the unstable subspace of JF_{X_0} for $x < 0$. Let $R(\phi)$ to denote the integral term in (2.7) then,

$$\phi(\xi) = e^{\xi\nu_-}\phi_{\nu_-}Y_{\nu_-} + e^{\xi\nu_+}\phi_{\nu_+}Y_{\nu_+} + R(\phi(\xi)),$$

for some $\phi_{\nu_\pm} \in \mathbb{R}$ such that $P(0)\phi(0) = \phi_{\nu_-}Y_{\nu_-} + \phi_{\nu_+}Y_{\nu_+}$. If $R_i(\phi)$ denotes the i-entry of $R(\phi)$, the expressions for $Y_{\nu_\pm}$ and ϕ yield

$$-\frac{\mathbb{c}'(\xi)}{\mathbb{c}(\xi)} = \frac{-\nu_+e^{\xi\nu_+}\phi_{\nu_+} - \nu_-e^{\xi\nu_-}\phi_{\nu_-} + R_2(\phi(\xi))}{e^{\xi\nu_+}\phi_{\nu_+} + e^{\xi\nu_-}\phi_{\nu_-} + R_1(\phi(\xi))}$$

By assuming that $R(\phi) = o(e^{\xi JF_{X_0}}P(0)\phi(0))$, the relations in (2.5) imply that $\phi_{\nu_+} \neq 0$, otherwise $\mathbb{c}'$ can not satisfy condition (1) in Theorem 1.1 for any speed $a > a^*$. Therefore, $\phi(\xi) = O(e^{\nu_+\xi})$ for $\xi \to \infty$.

Finally, we prove that $R(\phi) = o(e^{\xi JF_{X_0}}P(0)\phi(0))$ due to an appropriate choice of the initial condition. More precisely, if the initial condition is chosen such that $\sup\{|\phi(\xi)| : \xi > 0\} < \delta$ then, $R(\phi)$ is of order $o(\delta)$. Indeed, since P projects along the stable space of JF_{X_0} if $\xi > 0$ and along the unstable subspace of JF_{X_0} if $\xi < 0$, there holds that

$$\left|e^{(\xi-r)JF_{X_0}}P(\xi-r)[F(\phi(r)) - JF_{X_0}\phi(r)]\right| \leq e^{-\alpha|\xi|}|F(\phi(r)) - JF_{X_0}\phi(r)|,$$

for every $\alpha < \min\{-\nu_\pm, \nu_0\}$. Also, because the function $F(X) - JF_{X_0}X$ and its derivative vanish at X_0, it follows that for every $\epsilon > 0$ there exists $\delta > 0$ such that $|F(X) - JF_{X_0}X| < \epsilon|X|$ if $|X| < \delta$, then

$$\begin{aligned} I :=& \left|\int_0^\infty e^{(\xi-r)JF_{X_0}}P(\xi-r)[F(\phi(r)) - JF_{X_0}\phi(r)]dr\right| \\ \leq& \int_0^\infty \left|e^{(\xi-r)JF_{X_0}}P(\xi-r)[F(\phi(r)) - JF_{X_0}\phi(r)]\right| dr \\ \leq& \int_0^\infty e^{-\alpha|\xi-r|}|F(\phi(r)) - JF_{X_0}\phi(r)|dr \\ \leq& \int_0^\infty e^{-\alpha|\xi-r|}\epsilon|\phi(r)|dr \\ \leq& \frac{2\epsilon}{\alpha}\delta \end{aligned}$$

□

Along this work, we rather to work with an equivalent system of PDE's which follows from (1.2) and (2.1). To this end, we let $H := g - f$ and define

$$v = w - H(c), \quad \text{and} \quad \mathbb{v} = \mathbb{w} - H(\mathbb{c}). \tag{2.8}$$

Notice that $\mathbb{v} \in L^2_{loc}$ but it does not belong to L^2 because $\mathbb{v}$ has the following asymptotic limits

$$\lim_{\xi\to-\infty} \mathbb{v}(\xi) = -H(1) \leq 0 \quad \text{and} \quad \lim_{\xi\to\infty} \mathbb{v}(\xi) = 0.$$

In addition, the Lemma 2.1 implies that $\mathbb{v}' \in H^1$, at least. Finally, equations (1.2) and (2.1) implies that these new variables satisfy the following evolution equations

$$\begin{aligned} c_t &= [(f(c) - v)c_x]_x + c(1-c), \\ v_t &= -\beta(c)(v + H(c)), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} [(f(\mathbb{c}) - \mathbb{v})\mathbb{c}']' + a\mathbb{c}' &= -\mathbb{c}(1-\mathbb{c}), \\ a\mathbb{v}' &= \beta(\mathbb{c})(\mathbb{v} + H(\mathbb{c})). \end{aligned} \tag{2.10}$$

we highlight that (2.10) is the ODE system gotten from (2.9) by assuming solutions $(c,v)^{\intercal}(x,t) = (\mathbb{c},\mathbb{v})^{\intercal}(-x-at)$ as before. This variable change modify the equilibrium points but the eigenvalues of the matrices which determine the dynamics around them are the same.

Remark 2.2. If we make the asumption that $\mathbb{u} = -\mathbb{c}'$, the associated firs order ODE system (2.10) has $Y_0 = (0,0,0)^{\intercal}$ and $Y_1 = (1,0,-H(1))^{\intercal}$ as equilibrium points. Despite of $Y_1 \neq X_1$, the eigenvalues of the Jacobian matrices JF_{Y_0} and JF_{Y_1} that describe the local dynamics of (2.10) around Y_0 and Y_1 has exactly the eigenvalues given in (2.3) and (2.4). This follows trivially from the following expression.

$$JF_{Y_1} = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{g(1)} & -\frac{a}{g(1)} & 0 \\ \frac{\beta(1)H'(1)}{a} & 0 & \frac{\beta(1)}{a} \end{pmatrix}, \quad \text{and} \quad JF_{Y_0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -a & 0 \\ \frac{\beta(0)H'(0)}{a} & 0 & \frac{\beta(0)}{a} \end{pmatrix}.$$

3. The perturbation equation and the eigenvalue problem

Despite of the traveling solution $(\mathbb{c},\mathbb{v})^{\intercal}$ has not bounded L^2 mass, we want to determine it is stable under $L^2 \times L^2$ perturbations. To this end, we determine in this section its perturbation equation, which is the nonlinear evolution equation satisfied by any admissible perturbation and determine the eigenvalue equation obtained from the perturbation equation.

To stablish the perturbation equation, we assume that any solution $(c(x,t), v(x,t))$ to the system (2.9) can be written as $(c(x,t), v(x,t)) = (\mathbb{c}(\xi), \mathbb{v}(\xi)) + (c(\xi,t), v(\xi,t))$ where $\xi = -(x+at)$ and $a > 0$ is a fixed velocity such that Theorem 1.1 holds. It is straightforward that

$$\begin{aligned} c_t &= -a\mathbb{c}' - ac_\xi + c_t, \qquad & c_x &= -\mathbb{c}' - c_\xi, \\ v_t &= -a\mathbb{v}' - av_\xi + v_t, \qquad & v_x &= -\mathbb{v}' - v_\xi. \end{aligned}$$

A substitution of the previous identities in (1.2a) yields

$$\begin{aligned} c_t &= a\mathbb{c}' + ac_\xi + [(f(\mathbb{c}+c) - \mathbb{v} - v)(\mathbb{c}' + c_\xi)]_\xi + (\mathbb{c}+c)(1-\mathbb{c}-c)), \\ v_t &= a\mathbb{v}' + av_\xi - \beta(\mathbb{c}+c)[\mathbb{v} + v + H(\mathbb{c}+c)]. \end{aligned}$$

By virtue of (2.10), the last system reduces to

$$\begin{aligned} c_t &= ac_\xi + [(f(\mathbb{c}+c) - \mathbb{v} - v)(\mathbb{c}' + c_\xi)]_\xi - [(f(\mathbb{c}) - \mathbb{v})\mathbb{c}']' + c(1 - 2\mathbb{c} - c), \\ v_t &= av_\xi - \beta(\mathbb{c}+c)[\mathbb{v} + v + H(\mathbb{c}+c)] + \beta(\mathbb{c})[\mathbb{v} + H(\mathbb{c})], \end{aligned}$$

and by the chain rule and the linearity of differentiation, we obtain the *nonlinear perturbation equations*

$$\begin{aligned} c_t &= ac_\xi + [(f(\mathbb{c}+c) - f(\mathbb{c}) - v)\mathbb{c}']_\xi + [(f(\mathbb{c}+c) - v - \mathbb{v})c_\xi]_\xi + c(1-2\mathbb{c}-c), \\ v_t &= av_\xi - (\beta(\mathbb{c}+c) - \beta(\mathbb{c}))[\mathbb{v} + v + H(\mathbb{c}+c)] - \beta(\mathbb{c})[v + H(\mathbb{c}+c) - H(\mathbb{c})]. \end{aligned} \tag{3.1}$$

This pair of equations is recast as the sum of a linear contribution and a nonlinear term as

$$\frac{d}{dt}\begin{pmatrix} c \\ v \end{pmatrix} = \mathcal{A}\begin{pmatrix} c \\ v \end{pmatrix} + \mathcal{N}\begin{pmatrix} c \\ v \end{pmatrix} \tag{3.2}$$

where

$$\mathcal{A} := \begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix}, \quad \text{and} \quad \mathcal{N}\begin{pmatrix} c \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{N}_1(c,v) \\ \mathcal{N}_2(c,v) \end{pmatrix},$$

are the linear and nonlinear operators resulting from (3.1) whose entries are

$$\mathcal{L}_1 c := [(a + f(\mathbb{c})')c + (f(\mathbb{c}) - \mathbb{v})c_\xi]_\xi + (1 - 2\mathbb{c})c, \tag{3.3a}$$

$$\mathcal{L}_2 v := -[\mathbb{c}' v]_\xi, \tag{3.3b}$$

$$\mathcal{L}_3 c := -[\beta'(\mathbb{c})\mathbb{v} + (\beta H)'(\mathbb{c})]c, \tag{3.3c}$$

$$\mathcal{L}_4 v := a v_\xi - \beta(\mathbb{c}) v. \tag{3.3d}$$

$$\begin{aligned}
\mathcal{N}_1(c,v) =& [(f(\mathbb{c}+c) - f(\mathbb{c}) - f'(\mathbb{c})c)\mathbb{c}]_\xi + [(f(\mathbb{c}+c) - f(\mathbb{c}) - v)c_\xi]_\xi - c^2, \\
\mathcal{N}_2(c,v) =& -(\beta(\mathbb{c}+c) - \beta(\mathbb{c}))[\mathbb{v} + v + H(\mathbb{c}+c)] + \beta'(\mathbb{c})\mathbb{v} + \beta(\mathbb{c})H'(\mathbb{c})c \\
& - \beta(\mathbb{c})[H(\mathbb{c}+c) - H(\mathbb{c}) - H'(\mathbb{c})c].
\end{aligned}$$

We regard $\mathcal{A}$ as block operator defined in the perturbation space $L^2 \times L^2$, which is a Hilbert space, implying that $\mathcal{L}_i$ is a linear operator in L^2 for each $i = 1, 2, 3, 4$. Also, if we are interested in perturbations whose velocity (c_t, v_t) has also finite L^2-mass, we must, at least, set $\mathcal{D}(\mathcal{L}_1) = H^2$, $\mathcal{D}(\mathcal{L}_2) = \mathcal{D}(\mathcal{L}_4) = H^1$, and $\mathcal{D}(\mathcal{L}_3) = L^2$, but we recall the reader that we are interested in the solutions to an evolution system. Thus, we assume that $\mathcal{D}(\mathcal{A}) := H^2 \times H^1$. This assumption makes all the linear operators densely defined.

The *linear perturbation equation* is gotten when we drop out the nonlinear terms from (3.2). The linearized version of (3.2) generates a genuine eigenvalue problem by specializing the linear perturbation equation to solutions of the form $(c(\xi,t), v(\xi,t))^\top = e^{\lambda t}(c(\xi), v(\xi))^\top$. For this model, the eigenvalue problem is

$$\lambda \begin{pmatrix} c \\ v \end{pmatrix} = \mathcal{A}\begin{pmatrix} c \\ v \end{pmatrix}, \quad \text{with} \quad \mathcal{A} := \begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix}, \tag{3.4}$$

or equivalently,

$$\begin{aligned}
\lambda c =& (f(\mathbb{c}) - \mathbb{v})c'' + (a + 2f(\mathbb{c})' - \mathbb{v})c' + (f(\mathbb{c})' + 1 - 2\mathbb{c})c - \mathbb{c}'' v - \mathbb{c}' v', \\
\lambda v =& a v' - \beta(\mathbb{c}) v - [\beta'(\mathbb{c})\mathbb{v} + (\beta H)'(\mathbb{c})]c.
\end{aligned} \tag{3.5}$$

To properly set equation (3.4) as an eigenvalue problem in Banach spaces, we need to prove that $\mathcal{A}$ is a closed (or at least closable) operator in $L^2 \times L^2$. Before proving this property of $\mathcal{A}$, we recall the closedness definition, resolvent and spectrum definitions as well as relative boundedness definition. Also we recover two auxiliary results given in [KP13] pp 40 and in [Tre08] pp 100-101, respectively. This two results deals with the closedness property of scalar differential operators and block operators. For brevity, we skip their proofs.

Definition 3.1. Let X and Y be a Banach spaces and $\mathcal{A} : X \to Y$ be a linear operator with domain $\mathcal{D}(\mathcal{A})$. We say that $\mathcal{A}$ is a *closed operator* if for every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{A})$, in X, such that $\{\mathcal{A}x_n\}_{n\in\mathbb{N}}$ also is Cauchy in Y, there holds that $x := \lim x_n \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}x = \lim \mathcal{A}x_n$.

Definition 3.2. Let $\mathcal{A} : X \to Y$ be a closed linear operator from X to Y, Banach spaces. The resolvent $\rho(\mathcal{A})$, the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{A})$, and the essential spectrum

$\sigma_{\rm ess}(\mathcal{A})$ are defined as:

$$\begin{aligned}
\rho(\mathcal{A}) &:= \{\lambda \in \mathbb{C} \, : \, \mathcal{A}-\lambda \text{ is injective and onto}\},\\
\sigma_{\rm pt}(\mathcal{A}) &:= \{\lambda \in \mathbb{C} \, : \, \mathcal{A}-\lambda \text{ is Fredholm with index zero and has a non-trivial kernel}\},\\
\sigma_{\rm ess}(\mathcal{A}) &:= \{\lambda \in \mathbb{C} \, : \, \mathcal{A}-\lambda \text{ is either not Fredholm or has index different from zero}\}.
\end{aligned}$$

The spectrum of $\mathcal{A}$ is the set $\sigma(\mathcal{A}) := \sigma_{\rm ess}(\mathcal{A}) \cup \sigma_{\rm pt}(\mathcal{A})$. Every $\lambda \in \mathbb{C}$ such that $\mathcal{A}-\lambda$ has no trivial kernel is referred as an *eigenvalue*. Hence, $\sigma_{\rm pt}(\mathcal{A})$ is contained in the set of eigenvalues but, not the other way around.

Lemma 3.3. *Let $\mathcal{L} : L^2 \to L^2$, with domain $\mathcal{D}(\mathcal{L}) = H^m(\mathbb{R})$, be the following linear operator:*

$$\mathcal{L} := \partial_x^m + b_{m-1}\partial_x^{m-1} + \cdots + b_0\mathbb{I}.$$

If the coefficients $\{b_j\}_{j=0}^{m-1} \in W^{1,\infty}(\mathbb{R})$, then the operator $\mathcal{L} : H^m \subset L^2 \to L^2$ is closed.

Definition 3.4. Let $\mathcal{T}$ and $\mathcal{S}$ two linear operator from X to Y, Banach spaces. It is said that $\mathcal{S}$ is relatively bounded with respect to $\mathcal{T}$ (or simply $\mathcal{T}$-bounded) if $D(\mathcal{T}) \subset D(\mathcal{S})$ and

$$\|\mathcal{S}u\| \le a\|u\| + b\|\mathcal{T}u\|, \qquad \forall\, u \in D(\mathcal{T}),$$

where a, b are non-negative constants. The greatest lower bound b_0 of all possible constants b is called the relative bound of $\mathcal{S}$ with respect to $\mathcal{T}$ (or simply the $\mathcal{T}$-bound of $\mathcal{S}$).

Lemma 3.5. *Let H_1 and H_2 be two Banach spaces and assume four linear $\mathcal{L}_1 : \mathcal{D}(\mathcal{L}_1) \subset H_1 \to H_1$, $\mathcal{L}_2 : \mathcal{D}(\mathcal{L}_2) \subset H_2 \to H_1$, $\mathcal{L}_3 : \mathcal{D}(\mathcal{L}_3) \subset H_1 \to H_2$, and $\mathcal{L}_4 : \mathcal{D}(\mathcal{L}_4) \subset H_2 \to H_2$ such that $\mathcal{L}_3$ is $\mathcal{L}_1$-bounded with relative bound δ_o and $\mathcal{L}_2$ is $\mathcal{L}_4$-bounded with relative bound δ_e. Also, let $H := H_1 \times H_2$ and define $\mathcal{A} : H \to H$ such that*

$$\mathcal{A} := \begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix}$$

with it natural domain $\mathcal{D}(\mathcal{A}) := (\mathcal{D}(\mathcal{L}_1) \cap \mathcal{D}(\mathcal{L}_3)) \times (\mathcal{D}(\mathcal{L}_2) \cap \mathcal{D}(\mathcal{L}_4))$.

If either $\delta_o^2(1+\delta_e^2) < 1$, $\delta_e^2(1+\delta_o^2) < 1$, or $\max\{\delta_o, \delta_e\} < 1$ then, $\mathcal{A}$ is a closable operator. In addition, if $\mathcal{L}_1$ and $\mathcal{L}_4$ are closed operator then, $\mathcal{A}$ is a closed operator.

Previous Definitions and Lemmata allow us to prove that $\mathcal{A}$ is a closed operator.

Corollary 3.6. *The block operator $\mathcal{A} : L^2 \times L^2 \to L^2 \times L^2$ with domain $\mathcal{D} = H^2 \times H^1$ is closed.*

Proof. First, we claim that $\mathcal{L}_1$ and $\mathcal{L}_4$ are closed operators. In fact, we know that $(f(\mathbb{c}) - \mathbb{v})^{-1}\mathcal{L}_1 : H^2 \subset L^2 \to L^2$ and $\frac{1}{a}\mathcal{L}_1 : H^1 \subset L^2 \to L^2$ are closed operators due to the Lemma 3.3. In addition, the operators $[f(\mathbb{c}) - \mathbb{v}]\mathbb{I} : L^2 \to L^2$ and $a\mathbb{I} : L^2 \to L^2$ are bounded with a bounded inverse. This two observations readily yields the claim, see e.g. page 164 in [Kat80].

Second, we claim that $\mathcal{L}_3$ is $\mathcal{L}_1$-bounded with vanishing relative bound. Indeed, for every $\gamma > 0$, there holds that

$$\|\mathcal{L}_3 c\| \le \|\mathbb{v}\beta'(\mathbb{c}) + (H\beta)'(\mathbb{c})\|_{L^\infty}\|c\|_{L^2} + \gamma\|\mathcal{L}_1 c\|_{L^2}.$$

Previous relations yields $\delta_o = 0$. Finally, we prove that $\mathcal{L}_2$ is $\mathcal{L}_4$-bounded with finite relative bound. In fact, $\mathcal{L}_2$ is written as:

$$\mathcal{L}_2 = \frac{-\mathbb{c}'}{a}\mathcal{L}_4 - \left[\frac{\beta(\mathbb{c})}{a}\mathbb{c}' + \mathbb{c}''\right]\mathbb{I}$$

and from triangle inequality there holds that

$$\|\mathcal{L}_2 v\|_{L^2} \leq \frac{\|\mathbb{c}'\|_{L^\infty}}{a}\|\mathcal{L}_4 v\|_{L^2} + \left\|\frac{\beta(\mathbb{c})}{a}\mathbb{c}' + \mathbb{c}''\right\|_{L^\infty}\|v\|_{L^2}$$

for every $v \in H^1$, which immediately implies that $\delta_e \leq \|\mathbb{c}'\|_{L^\infty}/a$.

Therefore, the block operator $\mathcal{A}$ is closed in $L^2 \times L^2$ is by Lemma 3.5. □

4. Spectral stability on weighted spaces

In this section we determine sufficient conditions to ensure spectral stability of the block operator $\mathcal{A}$. We decided to work on exponentially weighted spaces because it can be prove that $\lambda = 0$ is an eigenvalue embedded in the essential spectrum when $\mathcal{A}$ is defined in $L^2 \times L^2$. This affirmation is follows by analyzing the spectrum of the asymptotic operator $\mathcal{A}_\infty := \mathbf{1}_{(-\infty,0)}(x)\mathcal{A}_- + \mathbf{1}_{(0,\infty)}(x)\mathcal{A}_+$ in $L^2 \times L^2$ obtained from $\mathcal{A}$ by replacing the variable coefficients, at each $\mathcal{L}_i$, with its limits at $\pm\infty$. Since $\mathcal{A}$ is a relatively compact perturbation[2] of $\mathcal{A}_\infty$, then both share the same essential spectrum, see [KP13]. Thus two options emerges, either $\mathcal{A}_\infty$ is not a Fredholm operator yielding that $0 \in \sigma_{\text{ess}}(\mathcal{A})$, or $\mathcal{A}_\infty$ is Fredholm with nonzero index because its Fredholm index is the difference of the Morse indices of the matrices JF_{Y_1} and JF_{Y_0}. This last condition also implies that $\lambda = 0$ belongs to $\sigma_{\text{ess}}(\mathcal{A})$. Therefore, the spectral stability of the operator $\mathcal{A}$ is only expected on exponentially weighted spaces, where a carefully choice if the exponential weight allows to kills some convective instabilities that cloud exist in the whole domain $L^2 \times L^2$.

4.1. Essential spectrum stability.

We define the exponentially weighted Sobolev spaces as follows. For each $k \geq 0$ and $\alpha \in \mathbb{R}$, we define the following set of functions:

$$H^k_\alpha := \{\tilde{w} : \mathbb{R} \to \mathbb{C} \,|\, \|e^{\alpha\xi}\tilde{w}\|_{H^k} < \infty\},$$

equipped with the norm $\|\tilde{w}\|_{H^k_\alpha} := \|e^{\alpha\xi}\tilde{w}\|_{H^k}$. It is known that H^k_α is a Hilbert space such that $\tilde{w} \in H^k_\alpha$ if and only if $w := e^{\alpha\xi}\tilde{w} \in H^k$. These spaces allows that w grows, at most, exponentially for $\xi < 0$ and decays, exponentially fast for $\xi > 0$ if $\alpha > 0$. The symmetric behavior must be satisfy by w in case of negative α.

The eigenvalue equation (3.4) for the the operator $\mathcal{A}$ in the restricted space $L^2_\alpha \times L^2_\alpha$ with dense domain $H^2_\alpha \times H^1_\alpha$ is equivalent to the following eigenvalue equation on $L^2 \times L^2$:

$$e^{\alpha\xi}\mathcal{A}e^{-\alpha\xi}\begin{pmatrix} c \\ v \end{pmatrix} = \lambda\begin{pmatrix} c \\ v \end{pmatrix} \in H^2 \times H^1.$$

Thus, By letting $\mathcal{A}_\alpha := e^{\alpha\xi}\mathcal{A}e^{-\alpha\xi}$ it is straightforward that $\sigma(\mathcal{A})|_{L^2_\alpha \times L^2_\alpha} = \sigma(\mathcal{A}_\alpha)|_{L^2 \times L^2}$. The following lemma establishes the range of values for α where $\sigma_{\text{ess}}(\mathcal{A}_\alpha) \subset \{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda < 0\}$.

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[2]This follows because the traveling wave $(\mathbb{c}, \mathbb{v})$ has enough regularity, see Lemma 2.1 and comments below equation (2.10), and it (exponential) converges to some fix limits.

Lemma 4.1 (Essential spectrum stability)**.** *Let $a > 2$ be fix and assume that $\nu_\pm$ and $\zeta_\pm$ are as in (2.3) and (2.4), respectively. Also let $\mathcal{S} \subset [1, \infty]$ be given by*

$$\mathcal{S} = \begin{cases} (-\nu_+, -\zeta_-) & \text{if } g(1) \in (a^2 - a\sqrt{a^2-4} - 1, a^2 + a\sqrt{a^2-4} - 1), \\ (-\nu_+, -\nu_-) & \text{if } g(1) \in [1, a^2 - a\sqrt{a^2-4} - 1], \end{cases}$$

Then, $\sigma_{\mathrm{ess}}(\mathcal{A}_\alpha) \subset \{\lambda \in \mathbb{C} \,|\, \operatorname{Re}\lambda < 0\}$ for any $\alpha \in \mathcal{S}$.

Remark 4.2. Notice that for any function g such that $g(1) > a^2 + a\sqrt{a^2-4} - 1$ there is not α-weight that move the essential spectrum to the left of the imaginary axis. Hence $\lambda = 0$ is not an isolated eigenvalue of $\mathcal{A}_\alpha$ for any α. Despite of this point, the least upper bound for the range of values for $g(1)$ where $\mathcal{S} \neq \emptyset$, is of order $2a^2 - 3$.

Proof. First we get explicit expression of $\mathcal{A}_\alpha$. Since $\mathcal{A}$ is a block operator given in terms of the operators $\mathcal{L}_i$ for $i = 1, 2, 3, 4$, see equations (3.3a) - (3.3d), we complain that

$$\mathcal{A}_\alpha \begin{pmatrix} c \\ w \end{pmatrix} = \begin{pmatrix} \mathcal{L}_1^\alpha & \mathcal{L}_2^\alpha \\ \mathcal{L}_3^\alpha & \mathcal{L}_4^\alpha \end{pmatrix} \begin{pmatrix} c \\ w \end{pmatrix}$$

where:

$$\begin{aligned} \mathcal{L}_1^\alpha c &:= e^{\alpha\xi}\mathcal{L}_1(e^{-\alpha\xi}c) \\ &= e^{\alpha\xi}(f(\mathbb{c}) - \mathbb{v})(e^{-\alpha\xi}c)_{\xi\xi} + e^{\alpha\xi}(a + 2f(\mathbb{c})' - \mathbb{v}')(e^{-\alpha\xi}c)_\xi + [f(\mathbb{c})'' + 1 - 2\mathbb{c}]c \\ &= (f(\mathbb{c}) - \mathbb{v})(c_{\xi\xi} - 2\alpha c_\xi + \alpha^2 c) + (a + 2f(\mathbb{c})' - \mathbb{v}')(c_\xi - \alpha c) + [f(\mathbb{c})'' + 1 - 2\mathbb{c}]c \\ &= (f(\mathbb{c}) - \mathbb{v})c_{\xi\xi} + [a + 2f(\mathbb{c})' - \mathbb{v}' - 2\alpha(f(\mathbb{c}) - \mathbb{v})]c_\xi \\ &\qquad + [(f(\mathbb{c}) - \mathbb{v})\alpha^2 - (a + 2f(\mathbb{c})' - \mathbb{v}')\alpha + f(\mathbb{c})'' + 1 - 2\mathbb{c}]c, \\ \mathcal{L}_2^\alpha v &:= e^{\alpha\xi}\mathcal{L}_2(e^{-\alpha\xi}v) = -\mathbb{c}''v - \mathbb{c}'e^{\alpha\xi}(e^{-\alpha\xi}v)_\xi = -\mathbb{c}'v_\xi + [\alpha\mathbb{c}' - \mathbb{c}'']v, \\ \mathcal{L}_3^\alpha c &:= e^{\alpha\xi}\mathcal{L}_3(e^{-\alpha\xi}c) = \mathcal{L}_3 c, \\ \mathcal{L}_4^\alpha v &:= e^{\alpha\xi}\mathcal{L}_4(e^{-\alpha\xi}v) = av_\xi - [a\alpha + \beta(\mathbb{c})]v. \end{aligned}$$

Second, we determine the eigenvalue system for the block operator $\mathcal{A}_\alpha$. Indeed, from the equations for $\mathcal{L}_i^\alpha$'s, we achieve

$$\begin{aligned} \lambda c &= \mathcal{L}_1^\alpha c + \mathcal{L}_2^\alpha v \\ &= (f(\mathbb{c}) - \mathbb{v})c_{\xi\xi} + [a + 2f(\mathbb{c})' - \mathbb{v}' - 2\alpha(f(\mathbb{c}) - \mathbb{v})]c_\xi \\ &\qquad + [(f(\mathbb{c}) - \mathbb{v})\alpha^2 - (a + 2f(\mathbb{c})' - \mathbb{v}')\alpha + f(\mathbb{c})'' + 1 - 2\mathbb{c}]c - \mathbb{c}'v_\xi + [\alpha\mathbb{c}' - \mathbb{c}'']v, \end{aligned}$$

$$\begin{aligned} \lambda v &= \mathcal{L}_3^\alpha c + \mathcal{L}_4^\alpha v \\ &= -[\beta'(\mathbb{c})\mathbb{v} + (\beta H)'(\mathbb{c})]c + av_\xi - [a\alpha + \beta(\mathbb{c})]v \end{aligned}$$

Third, we obtain the first-order dynamical system associated to the eigenvalue problem $\lambda(c, v)^\intercal = \mathcal{A}_\alpha(c, v)^\intercal$. We arrive to this dynamical system by introducing the auxiliary variable $u = c'$, indeed,

$$\frac{d}{d\xi}\begin{pmatrix} c \\ -c' \\ w \end{pmatrix} = A_\alpha(\xi, \lambda)\begin{pmatrix} c \\ -c' \\ w \end{pmatrix}, \tag{4.1}$$

where

$$A_\alpha(\xi,\lambda)=\begin{pmatrix} 0 & -1 & 0\\ K & -\frac{a+2f(\mathbb{c})'-\mathbb{v}'-2\alpha(f(\mathbb{c})-\mathbb{v})}{f(\mathbb{c})-\mathbb{v}} & -\frac{(\beta(\mathbb{c})+\lambda)\mathbb{c}'+a\mathbb{c}''}{a(f(\mathbb{c})-\mathbb{v})}\\ \frac{\beta'(\mathbb{c})\mathbb{v}+(\beta H)'(\mathbb{c})}{a} & 0 & \frac{a\alpha+\beta(\mathbb{c})+\lambda}{a}\end{pmatrix},$$

and

$$K=-\frac{(\beta'(\mathbb{c})\mathbb{v}+(\beta H)'(\mathbb{c}))\mathbb{c}'+a[\lambda-1+2\mathbb{c}-f(\mathbb{c})''-(f(\mathbb{c})-\mathbb{v})\alpha^2+(a+2f(\mathbb{c})'-\mathbb{v}')\alpha]}{a(f(\mathbb{c})-\mathbb{v})}.$$

Fourth, we determine the *dispersion curves, i.e.*, the boundary of the essential spectrum $\sigma_{\rm ess}(L_\alpha)$. These curves are made of those points where the asymptotic matrices $A^\pm_\alpha:=\lim_{\xi\to\pm}A_\alpha(\xi,\lambda)$ loose their hyperbolicity, see [KP13].

By letting $\beta_i:=\beta(i)$ and $g_i:=g(i)$ for $i=0$ or 1, we get that

$$A^-_\alpha(\lambda)=\begin{pmatrix} 0 & -1 & 0\\ \alpha^2-\frac{\lambda+1+a\alpha}{g_1} & 2\alpha-\frac{a}{g_1} & 0\\ \frac{\beta_1 H'(1)}{a} & 0 & \frac{a\alpha+\beta_1+\lambda}{a}\end{pmatrix},$$

and

$$A^+_\alpha(\lambda)=\begin{pmatrix} 0 & -1 & 0\\ \alpha^2-a\alpha+1-\lambda & 2\alpha-a & 0\\ \frac{\beta_0 H'(0)}{a} & 0 & \frac{a\alpha+\beta_0+\lambda}{a}\end{pmatrix}.$$

Their characteristic polynomials are

$$P^-_\alpha(\mu;\lambda):=\left(\frac{\beta_1+\lambda}{a}-(\mu-\alpha)\right)\left[(\mu-\alpha)^2+\frac{a}{g_1}(\mu-\alpha)-\frac{\lambda+1}{g_1}\right],\tag{4.2}$$

and

$$P^+_\alpha(\mu;\lambda):=\left(\frac{\beta_0+\lambda}{a}-(\mu-\alpha)\right)[(\mu-\alpha)^2+a(\mu-\alpha)+1-\lambda],\tag{4.3}$$

respectively. an their eigenvalues are

$$\begin{aligned}\mu^-_1(\lambda;\alpha)&=\alpha+\tfrac{\beta(1)+\lambda}{a}, & \mu^+_1(\lambda;\alpha)&=\alpha+\tfrac{\beta(0)+\lambda}{a},\\ \mu^-_2(\lambda;\alpha)&=\alpha+\tfrac{-a-\sqrt{a^2+4g(1)(1+\lambda)}}{2g(1)}, & \mu^+_2(\lambda;\alpha)&=\alpha+\tfrac{-a-\sqrt{a^2-4(1-\lambda)}}{2},\\ \mu^-_3(\lambda;\alpha)&=\alpha+\tfrac{-a+\sqrt{a^2+4g(1)(1+\lambda)}}{2g(1)}, & \mu^+_3(\lambda;\alpha)&=\alpha+\tfrac{-a+\sqrt{a^2-4(1-\lambda)}}{2}.\end{aligned}\tag{4.4}$$

Thus, the matrices $A^\pm_\alpha(\lambda)$ loose their hyperbolicity on the set

$$\{P_\pm(i\xi,\lambda)=0\,|\,\xi\in\mathbb{R}\}.\tag{4.5}$$

From the expressions of $P^\pm_\alpha(\mu;\lambda)$ is immediate that the set in equation (4.5) is made of the following four dispersion curves

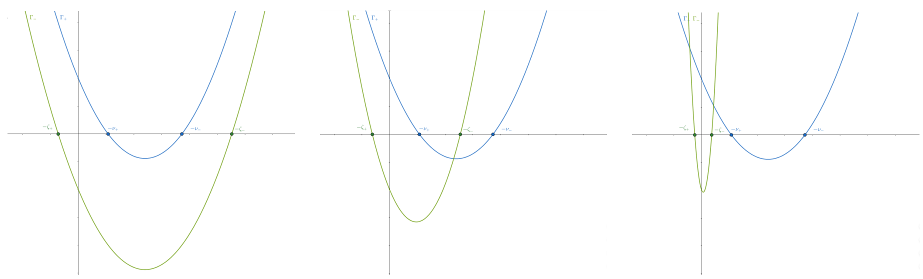
$$\begin{aligned}\mathcal{C}^-_1&:=\{-a\alpha-\beta_1+ia\xi\,|\,\xi\in\mathbb{R}\}.\\ \mathcal{C}^-_2&:=\{\alpha^2g_1-1-a\alpha-g_1\xi^2+i(a-2\alpha g_1)\xi\,|\,\xi\in\mathbb{R}\}.\\ \mathcal{C}^+_1&:=\{-a\alpha-\beta_0+ia\xi\,|\,\xi\in\mathbb{R}\}.\\ \mathcal{C}^+_2&:=\{\alpha^2-a\alpha+1-\xi^2+i(a-2\alpha)\xi\,|\,\xi\in\mathbb{R}\}.\end{aligned}$$

Notice that if $\alpha>0$ then, $\operatorname{Re}\lambda<0$ for every $\lambda\in\mathcal{C}^-_1\cup\mathcal{C}^+_1$ due to $\beta_1\geq\beta_0$. We see that $\mathcal{C}^\pm_2$ are parabolæ, with vertices at $\Gamma_\pm(\alpha)$, where

$$\Gamma_-(\alpha):=g_1\alpha^2-a\alpha-1\geq\operatorname{Re}\lambda_-\quad\text{and}\quad\Gamma_+(\alpha):=\alpha^2-a\alpha+1\geq\operatorname{Re}\lambda_+,\tag{4.6}$$

for every $\lambda_\pm \in \mathcal{C}_2^\pm$. We pursuit the set $D \subset \mathbb{R}$ such that $\max\{\Gamma_-(\alpha), \Gamma_+(\alpha)\} < 0$, for any $\alpha \in D$. Using the expressions for $\nu_\pm$ and $\zeta_\pm$, see (2.3) and (2.4), we complain that

$$\Gamma_-(\alpha) = (\alpha + \zeta_-)(\alpha + \zeta_+) \quad \text{and} \quad \Gamma_+(\alpha) = (\alpha + \nu_-)(\alpha + \nu_+).$$



(a) (b) (c)

FIGURE 1. Show the possible location of curves $\Gamma_\pm$. Figure (a), (b), and (c) pertrain the cases $g_1 \in [1, -2a\nu_+ - 1]$, $g_1 \in (-2a\nu_+ - 1, -2a\nu_- - 1)$, and $g_1 \in [-2a\nu_- - 1, \infty)$, respectively.

Since $-\zeta_+ < 0 < -\zeta_-$ and $0 < -\nu_+ < -\nu_-$ we conclude that

$$D = \begin{cases} \emptyset & \text{if } \zeta_- \geq \nu_+ \\ (-\nu_+, -\zeta_-) & \text{if } \nu_+ > \zeta_- > \nu_- \\ (-\nu_+, -\nu_-) & \text{if } \nu_- \geq \zeta_- \end{cases}$$

Standard algebra implies that any pair of constants $(a, g_1) \in [2, \infty] \times [1, \infty]$ such that either $\zeta_- \geq \nu_+$ or $\nu_- \geq \zeta_-$ also reduces to the simpler condition

$$(g_1 + 1)^2 - 2a^2(g_1 - 1) \geq 0. \tag{4.7}$$

Certainly, the relations $\zeta_- \geq \nu_+$ and $\nu_- \geq \zeta_-$ are equivalent to

$$\frac{a + \sqrt{a^2 + 4g_1}}{2g_1} \leq \frac{a - \sqrt{a^2 - 4}}{2} \quad \text{and} \quad \frac{a + \sqrt{a^2 - 4}}{2} \leq \frac{a + \sqrt{a^2 + 4g_1}}{2g_1},$$

respectively. Hence, by solving for $\sqrt{a^2 + 4g_1}$, one gets

$$\sqrt{a^2 + 4g_1} \leq a(g_1 - 1) - g_1\sqrt{a^2 - 4} \quad \text{and} \quad a(g_1 - 1) + g_1\sqrt{a^2 - 4} \leq \sqrt{a^2 + 4g_1}.$$

Now, by taking squares on both inequalities and solving for $(g_1 - 1)a\sqrt{a^2 - 4}$ in the former and for $-(g_1 - 1)a\sqrt{a^2 - 4}$ in the latter, we obtain that

$$(g_1 - 1)a\sqrt{a^2 - 4} \leq (g_1 - 1)a^2 - 2(g_1 + 1) \quad \text{and} \quad (g_1 - 1)a^2 - 2(g_1 + 1) \leq -(g_1 - 1)a\sqrt{a^2 - 4},$$

respectively. Because $g_1 > 1$ we conclude that at each of the previous relation both terms have the same sign. Indeed, both terms are positive in the former and both terms are negative in the latter. Hence, by taking squares at both relations yield relation (4.7) for both inequalities.

Since inequality (4.7) is equivalent to

$$[g_1 + 1 + 2a\nu_+]\,[g_1 + 1 + 2a\nu_-] \geq 0,$$

we conclude that either $g_1 \geq -2a\nu_- - 1$ or $-2a\nu_+ - 1 \geq g_1$. These curves, in the (a, g_1)-plane, determine the boundaries of the sets of tuples (a, g_1) where either $-\zeta_- \leq -\nu_+$ or $-\nu_- \leq -\zeta_-$ hold. By inspection, it follows that

$$\{(a, g_1) \in [2,\infty] \times [1,\infty] \,|\, \zeta_- \geq \nu_+\} = \{(a, g_1) \in [2,\infty] \times [1,\infty] \,|\, g_1 \geq -2a\nu_- - 1\},$$
$$\{(a, g_1) \in [2,\infty] \times [1,\infty] \,|\, \nu_- \geq \zeta_-\} = \{(a, g_1) \in [2,\infty] \times [1,\infty] \,|\, -2a\nu_+ - 1 \geq g_1\}.$$

Finally, we conclude that

$$D = \begin{cases} \emptyset & \text{if } g_1 \in (-2a\nu_- - 1, \infty), \\ (-\nu_+, -\zeta_-) & \text{if } g_1 \in (-2a\nu_+ - 1, -2a\nu_- - 1), \\ (-\nu_+, -\nu_-) & \text{if } g_1 \in [1, -2a\nu_+ - 1], \end{cases}$$

and the proof is complete. □

Remark 4.3. In the case where the function g and the velocity a satisfy that $\mathcal{S} \neq \emptyset$, the case $\lambda = 0$ turns to be relevant since both asymptotic matrices $A^{\pm}_{\alpha}(0)$ have one negative eigenvalue and two positive eigenvalues. Thus, the Fredholm index of the operator $\mathcal{A}_\alpha$ is zero, making $\lambda = 0$ an isolated eigenvalue of $\mathcal{A}_\alpha$. This change in de Fredholm index is because the negative eigenvalue ν_+ of $A_+(\lambda) = JF_{Y_0}$ before the shifting ($\alpha = 0$) moved to the positive eigenvalue $\alpha + \nu_+$ for $\alpha \in \mathcal{S}$. Also, letting $\alpha \in \mathcal{S}$ implies that $e^{\alpha\xi}(\mathbb{c}', \mathbb{v}')^\intercal$ is no longer a eigenvector with $\lambda = 0$ for the operator $\mathcal{A}_\alpha$, despite it formally satisfies $\mathcal{A}_\alpha e^{\alpha\xi}(\mathbb{c}', \mathbb{v}')^\intercal = 0$. This follows because $e^{\alpha\xi}(\mathbb{c}', \mathbb{v}')$ is not square-integrable since it is $O(e^{(\alpha+\nu_+)\xi})$ for $\xi \to \infty$.

Remark 4.4. When the block operator $\mathcal{A}$ defined in the unweighted space $L^2 \times L^2$, or equivalently $\alpha = 0$, satisfies that $|\lambda|$ is big enough[3] and $\mathcal{A} - \lambda\mathbb{I}$ is Fredholm, then its Fredholm index is equal to zero, *i.e.*, λ either belongs to the resolvent set or it is an eigenvalue of $\mathcal{A}$. This affirmation follows because the Morse indexes associated to $A^{\pm}_0(\lambda)$ satisfy that $i_- := \dim \mathbb{E}^u_- = 2 = \dim \mathbb{E}^u_+ =: i_+$. Hence every eigenfunction $(c, v)^\intercal$, solution to (3.4) must be a bounded solution of the dynamical system (**??**) with $\alpha = 0$, but due to exponential decay of $A_0(\xi, \lambda)$ to $A^{\pm}_0(\lambda)$ as $\xi \to \pm\infty$, there holds that for $\lambda > 0$ big enough, there holds that $(c, -c', v)^\intercal = O(e^{\mu_2^+(\lambda;0)\xi})$ for $\xi \to \infty$, and $(c, -c', v)^\intercal = o(e^{\mu_2^-(\lambda;0)\xi})$ for $\xi \to -\infty$, see Proposition 1.6 in [PW92]. In the case where $\mathcal{A}$ is defined in the weighted space $L^2_\alpha \times L^2_\alpha$, for some $\alpha \in \mathcal{S}$, there holds a similar decaying behavior on $(c, -c', v)^\intercal$ for every λ with positive real part. Indeed, $(c, -c', v)^\intercal = O(e^{\mu_2^+(\lambda;\alpha)\xi})$ for $\xi \to \infty$, and $(c, -c', v)^\intercal = o(e^{\mu_2^-(\lambda;\alpha)\xi})$ for $\xi \to -\infty$. In this case the set $\{\lambda : \operatorname{Re}\lambda > 0\}$ belongs to the natural domain of the Evans function.

4.2. **Point spectral stability.** At this point, we have proved essential spectral stability on exponentially weighted spaces for a suitable choices of the weight α, see Lemma 4.1. Now, we pursuit point spectral stability by using energy estimates on the admissible eigenvalues of the operator $\mathcal{A}_\alpha$, namely,

$$\begin{aligned} \lambda c &= D_1 c'' + [D_2 - 2\alpha D_1]c' + [\alpha^2 D_1 - \alpha D_2 + D_3]c - \mathbb{c}' v' + [\alpha\mathbb{c}' - \mathbb{c}'']v, \\ \lambda v &= a v' - [a\alpha + \beta(\mathbb{c})]v - [\beta'(\mathbb{c})\mathbb{v} + (\beta H)'(\mathbb{c})]c, \end{aligned} \tag{4.8}$$

[3]It is enough that $|\lambda| > 2$ with $\operatorname{Re}\lambda > 0$ to guarantee that the Fredholm index of the operator $\mathcal{A} - \lambda\mathbb{I}$ is zero, this follows by noticing that all of the dispersion curves $\mathcal{C}^{\pm}_i$ of the the operator $A^{\pm}_0$, belongs to the set $\{\lambda : \operatorname{Re}\lambda < 0\} \cup \{\lambda : |\lambda| \leq 2\}$, see Figure 2, yielding that Fredholm index must be constant in the complement, see [KP13].

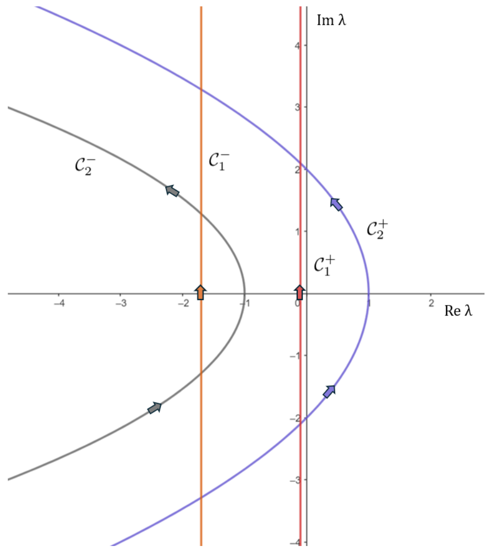


FIGURE 2. Shows the Fredholm borders that bound the essential spectrum and determine the change of the Fredholm index for the operator $A - \lambda\mathbb{I}$. The curves $\mathcal{C}_1^-$ is on the left of $\mathcal{C}_1^+$ because β is non-decreasing. The same happends with the curves $\mathcal{C}_2^-$ and $\mathcal{C}_2^+$ but, they do not intersect each other since g is also increasing. Along any of the four curves, its orientation is pointed out by an arrow. We recall that transitions from right to left along the curves $\mathcal{C}_i^\pm$ change ± 1 the Fredholm index of $A - \lambda\mathbb{I}$, see [].

where

$$D_1 := f(\mathbb{c}) - \mathbb{v}, \quad D_2 := a + 2f(\mathbb{c})' - \mathbb{v}', \quad \text{and} \quad D_3 := 1 - 2\mathbb{c} + f(\mathbb{c})'' \tag{4.9}$$

To this end, we propose the variable change $c(\xi) = r(\xi)e^{\theta(\xi)}$ and $\mathbb{c}' := \mathbb{r}e^{\theta}$, with

$$\theta(\xi) := \alpha(\xi - \xi_0) - \frac{1}{2}\int_{\xi_0}^{\xi} \frac{D_2 - D_1'}{D_1}\, d\eta = \alpha(\xi - \xi_0) - \frac{1}{2}\int_{\xi_0}^{\xi} \frac{a + f(\mathbb{c})'}{f(\mathbb{c}) - \mathbb{v}}\, d\eta. \tag{4.10}$$

We readily see that

$$\begin{aligned}
c'e^{-\theta} &= r' + \left(\alpha - \frac{D_2 - D_1'}{2D_1}\right) r, \\
c''e^{-\theta} &= r'' + 2\left(\alpha - \frac{D_2 - D_1'}{2D_1}\right) r' + \left[\left(\alpha - \frac{D_2 - D_1'}{2D_1}\right)^2 - \left(\frac{D_2 - D_1'}{2D_1}\right)'\right] r,
\end{aligned}$$

thus, we multiply the first equation of (4.8) by $e^{-\theta}$ and substitute these expressions to obtain

$$
\begin{aligned}
\lambda r =& \frac{D_2 - D_1'}{2D_1}\mathbb{r} v - (\mathbb{r} v)' + D_1 r'' + D_1' r' \\
&+ \frac{1}{2D_1}\left[\tfrac{1}{2}(2\alpha D_1 - D_2 + D_1')^2 - D_1(D_2' - D_1'') + (D_2 - D_1')D_1' \right.\\
&\qquad \left. +(D_2 - 2\alpha D_1)(2\alpha D_1 - D_2 + D_1') + 2\alpha^2 D_1^2 - 2\alpha D_1 D_2 + 2D_1 D_3\right] r \\
=&(D_1 r')' + \frac{1}{2D_1}\left[\tfrac{1}{2}(D_1'^2 - D_2^2) - D_1(D_2' - D_1'') + (D_2 - D_1')D_1' + 2D_1 D_3\right] r \\
&\qquad + \frac{D_2 - D_1'}{2D_1}\mathbb{r} v - (\mathbb{r} v)'.
\end{aligned}
$$

The later equation is simplified when we substitute the values of coefficients D_i, Indeed the full eigen system is

$$
\begin{aligned}
\lambda r =& [(f(\mathbb{c}) - \mathbb{v})r']' + \left[1 - 2\mathbb{c} + \frac{f(\mathbb{c})''}{2} - \frac{(a + f(\mathbb{c})')^2}{4(f(\mathbb{c}) - \mathbb{v})}\right] r + \frac{a + f(\mathbb{c})'}{2(f(\mathbb{c}) - \mathbb{v})}\mathbb{r} v - (\mathbb{r} v)', \\
\lambda v =& a v' - [a\alpha + \beta(\mathbb{c})]v - [\beta'(\mathbb{c})\mathbb{v} + (\beta H)'(\mathbb{c})]e^{\theta} r.
\end{aligned}
\tag{4.11}
$$

Lemma 4.5. *Let $a > 0$, g, and f be such that the set $\mathcal{S}$ in Lemma 4.1 is not empty and $a > \|f(\mathbb{c})'\|_{L^\infty}$. Assume $\alpha \in \mathcal{S}$ and let θ be as in (4.10). Then, $\theta = O(\xi)$ for $|\xi| \to \infty$. In addition, if*

$$
\frac{a}{2g_1} \le \alpha \le \frac{a}{2} \tag{4.12}
$$

then the function e^{θ} is bounded.

Proof. We recall that θ grows, at most, linearly for $|\xi|$ big enough. Certainly, by the boundedness of $\mathbb{v}$ and f, we know that

$$
k_0 := \|f\|_{L^\infty} - \inf_{\mathbb{R}}\{\mathbb{v}(\xi)\} \ge f(\mathbb{c}) - \mathbb{v} \ge 1.
$$

Also, because $\mathbb{c}' < 0$ and f is increasing, we conclude that $f(\mathbb{c})' \le 0$. These two conclusions show that

$$
\frac{a - 2k_0\alpha}{2k_0}(\xi - \xi_0) + \frac{f(\mathbb{c}(\xi)) - f(\mathbb{c}(\xi_0))}{2} \le -\theta(\xi) \le \frac{a - 2\alpha}{2}(\xi - \xi_0) + \frac{f(\mathbb{c}(\xi)) - f(\mathbb{c}(\xi_0))}{2k_0}, \tag{4.13}
$$

proving that $\theta = O(\xi)$ for $|\xi| \to \infty$.

Now, we prove e^{θ} is bounded for α as in (4.12). It is easy to see that e^{θ} is a continuous function, hence $e^{\theta} \in L^2_{loc}$. Thus, it is bounded if and only if it remains bounded when $|\xi| \to \infty$. Because we show that $\theta = O(\xi)$, we will find the envelope of the the tangent line $\ell_{\pm}(\xi) := m_{\pm}\xi + b_{\pm}$ to the function θ when $\xi \to \pm\infty$. From basic calculus, and L'Hospital rule, we know that

$$
m_{\pm} = \lim_{\xi \to \pm\infty} \frac{\theta(\xi)}{\xi} = \alpha - \frac{1}{2}\lim_{\xi \to \pm\infty} \frac{1}{\xi}\int_{\xi_0}^{\xi} \frac{a + f(\mathbb{c})'}{f(\mathbb{c}) - \mathbb{v}}\, d\eta = \begin{cases} \alpha - \frac{a}{2} & \text{if } \xi \to \infty, \\ \alpha - \frac{a}{2g_1} & \text{if } \xi \to -\infty. \end{cases},
$$

and

$$
b_{\pm} := \lim_{\xi \to \pm} \theta(\xi) - m_{\pm}\xi.
$$

Notice that $b_\pm \in \mathbb{R}$ because the integrant is nonnegative and there holds (4.13).

$$\lim_{\xi\to\pm\infty} e^{\theta}(\xi) = \lim_{\xi\to\pm\infty} e^{m_\pm \xi}\left(\lim_{\xi\to\pm\infty} e^{b_\pm} e^{\theta(\xi)-\ell_\pm(\xi)}\right).$$

Due to $b_\pm \in \mathbb{R}$, the term in parenthesis has a limit in $\mathbb{R}$. Moreover, the whole expression has finite limits if and only if

$$\frac{a}{2g_1} \le \alpha \le \frac{a}{2},$$

proving the final assertion. □

Remark 4.6. Since $a/(2g_1) = -(\zeta_- + \zeta_+)/2$, and $a/(2) = -(\nu_- + \nu_+)/2$, and $g_1 \ge 1$, we know that

$$\frac{a}{2g_1} = -\frac{\zeta_+ + \zeta_-}{2} < -\zeta_- \quad \text{and} \quad -\frac{\zeta_+ + \zeta_-}{2} \le \frac{a}{2} = -\frac{\nu_+ + \nu_-}{2} < -\nu_-,$$

These two inequalities imply that

$$\frac{a}{2g_1} \le \min\{-\nu_-, -\zeta_-\}.$$

Therefore,

$$\left\|e^{\theta}\right\|_{L^\infty} < \infty \text{ for any } \alpha \in \begin{cases} [-\nu_+, a/2], & \text{if } a/(2g_1) \le -\nu_+ \text{ and } a/2 \le -\zeta_-, \\ [a/(2g_1), a/2], & \text{if } a/(2g_1) > -\nu_+ \text{ and } a/2 \le -\zeta_-, \\ [-\nu_+, -\zeta_-], & \text{if } a/(2g_1) \le -\nu_+ \text{ and } a/2 > -\zeta_-, \\ [a/(2g_1), -\zeta_-], & \text{if } a/(2g_1) > -\nu_+ \text{ and } a/2 > -\zeta_-. \end{cases}$$

Also, we remark that any of the previous sets belong to $\mathcal{S}$.

The following lemma shows that r has enough regularity to give sense to equation (**??**) due to the exponentially decay of the vector $(c, -c', v)^\intercal$.

Lemma 4.7. *For each pair of eigenvalue $\lambda \in \mathbb{C}$ and eigenvector $(c, v)^\intercal \in H^2 \times H^1$, there holds that the function $r = ce^{-\theta} \in H^2$.*

Proof. First, we prove that $r \in L^2$. By Remark 4.4, we know that $(c, -c', v)^\intercal = O(e^{\operatorname{Re}\mu_2^+(\lambda;\alpha)\xi})$ for $\xi \to \infty$, and $(c, -c', v)^\intercal = o(e^{\operatorname{Re}\mu_2^-(\lambda;\alpha)\xi})$ for $\xi \to -\infty$, and because $\mu_2^\pm(\lambda;\alpha) = \mu_2^\pm(\lambda;0) + \alpha$, we obtain that

$$r(\xi) = c(\xi)e^{-\theta(\xi)} = \begin{cases} O\left(e^{\left(\frac{a}{2} + \operatorname{Re}\mu_2^+(\lambda;0)\right)\xi}\right), & \text{for } \xi \to \infty, \\ o\left(e^{\left(\frac{a}{2k_0} + \operatorname{Re}\mu_2^-(\lambda;0)\right)\xi}\right), & \text{for } \xi \to -\infty. \end{cases}$$

Thus, we conclude that $r(\xi)$ vanishes exponentially fast as $|x| \to \infty$ for $a > 0$ big enough such that $a/2 + Re\,\mu_2^+(\lambda;0) < 0$, but this holds trivially due to relations (4.4).

Second, we show that $r \in H^1$. Certainly, the previous step can be modify to show that $c'(\xi)e^{-\theta(\xi)} \to 0$ exponentially fast as $|\xi| \to \infty$. Therefore, the exponential decay of $c'(\xi)e^{-\theta(\xi)}$ and $c(\xi)e^{-\theta(\xi)}$ imply that $r \in H^1$ since $\theta'(\xi)$ is bounded.

Finally, we prove that $r \in H^2$. If we derive twice the equation for $r(\xi)$ and we solve for $\mathcal{c}''$ in terms of $\mathcal{c}$, $\mathcal{c}'$, and v, from the relations in (4.8), we obtain that

$$\begin{aligned}
r'' =& (\theta'^2 - \theta'')ce^{-\theta} - 2\theta' c' e^{-\theta} + c'' e^{-\theta},\\
=& \left(\theta'^2 - \theta'' - \alpha^2 - \frac{D_3 - \alpha D_2 - \lambda + a^{-1}[\beta(\mathbb{c})'\mathbb{v} + (\beta H(\mathbb{c}))']}{D_1}\right) ce^{-\theta}\\
&- \left(2(\theta' - \alpha) + \frac{D_2}{D_1}\right) c' e^{-\theta} + \left(\frac{a\mathbb{c}'' + \mathbb{c}'(\lambda + \beta(\mathbb{c}))}{aD_1}\right) ve^{-\theta}.
\end{aligned}$$

Since f' is Lipschitz, we conclude that θ' and θ'' are bounded, and in general, all of the coefficients inside any parenthesis are bounded. Therefore, we conclude that $r \in H^2$ due to the exponential decay of the whole vector $e^{-\theta(\xi)}(\mathcal{c}, -\mathcal{c}', v)^\intercal$ as $|\xi| \to \infty$. $\square$

Now, we are able to prove spectral stability for the operator $\mathcal{A}_\alpha$ or equivalently, spectral stability for the operator $\mathcal{A}$ regarded as an operator in $L^2_\alpha \times L^2_\alpha$.

Theorem 4.8. *[Spectral stability] Let $a > 0$, f, and g be such that Lemma 4.5 holds. Also assume that there exist constants $\epsilon_1 \in (0, 1)$ and $c_0 > 0$ such that $\beta(\mathbb{c}) - \frac{1}{4\epsilon_1}\mathbb{r}^2 \geq c_0 > 0$. If*

$$a \geq \left\| \frac{4D_1^2}{D_1 - \epsilon_1} \left(\tfrac{1}{4c_0}[\beta'(\mathbb{c})\mathbb{v} + (\beta H)'(\mathbb{c})]^2 e^{2\theta} + 1 - 2\mathbb{c} + \tfrac{1}{2} f(\mathbb{c})'' \right) \right\|_{L^\infty}^{1/2} + \|f(\mathbb{c})'\|_{L^\infty}, \tag{4.14}$$

then $\sigma(\mathcal{A})|_{L^2_\alpha \times L^2_\alpha} \subset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\,\lambda \leq 0\}$.

First, we notice that $\mathcal{S} \neq$ because the conditions in Lemma 4.5 hold. Then, we conclude essential spectrum stability on $L^2_\alpha \times L^2_\alpha$ for any $\alpha \in \mathcal{S}$, see lemma 4.1.

Second, we assume $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{A})|_{L^2_\alpha \times L^2_\alpha}$. Thus, there exists $(r, v) \in H^2 \times H^1$ solution to the system (4.11), which is written as

$$\begin{aligned}
\lambda r &= (D_1 r')' + (1 - 2\mathbb{c} + \tfrac{1}{2} f(\mathbb{c})'' - D_1(\theta' - \alpha)^2) r - (\theta' - \alpha)\mathbb{r} v - (\mathbb{r} v)',\\
\lambda v &= a v' - \beta(\mathbb{c}) v - [\beta'(\mathbb{c})\mathbb{v} + (\beta H)'(\mathbb{c})] e^\theta r.
\end{aligned} \tag{4.15}$$

By applying the L^2-functionals $\langle \cdot, r\rangle_{L^2}$ and $\langle \cdot, w\rangle_{L^2}$ on the first and second equations, respectively, and using integration by parts in the first and last terms of $\langle \lambda r, r\rangle_{L^2}$, we get

$$\begin{aligned}
\lambda \|r\|^2_{L^2} =& -\int_{\mathbb{R}} D_1 \left(|r'|^2 + (\theta' - \alpha)^2 |r|^2\right) d\xi + \int_{\mathbb{R}} \left(1 - 2\mathbb{c} + \tfrac{1}{2} f(\mathbb{c})''\right) |r|^2 \, d\xi\\
&+ \langle \mathbb{r} v, r' - (\theta' - \alpha) r\rangle_{L^2},\\
\lambda \|v\|^2_{L^2} =& \, a \langle v', v\rangle_{L^2} - \int_{\mathbb{R}} \beta(\mathbb{c}) |v|^2 \, d\xi - \left\langle [\beta'(\mathbb{c})\mathbb{v} + (\beta H)'(\mathbb{c})] e^\theta r, v\right\rangle_{L^2}.
\end{aligned} \tag{4.16}$$

Now, we add both relations and take the real part of the resulting expression leading us to the following L^2-energy equation

$$\begin{aligned}
\mathrm{Re}\,\lambda \left(\|r\|^2_{L^2} + \|v\|^2_{L^2}\right) =& -\int_{\mathbb{R}} D_1 \left(|r_\xi|^2 + (\theta' - \alpha)^2 |r|^2\right) d\xi - \int_{\mathbb{R}} \beta(\mathbb{c}) |v|^2 \, d\xi\\
&+ \int_{\mathbb{R}} \left(1 - 2\mathbb{c} + \tfrac{1}{2} f(\mathbb{c})''\right) |r|^2 \, d\xi + \mathrm{Re}\, \langle \mathbb{r} v, r' - (\theta' - \alpha) r\rangle_{L^2}\\
&- \mathrm{Re}\, \left\langle [\beta'(\mathbb{c})\mathbb{v} + (\beta H)'(\mathbb{c})] e^\theta r, v\right\rangle_{L^2}.
\end{aligned} \tag{4.17}$$

From equation (4.17), we use Young's inequality at each inner product. Hence, for every $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there holds that

$$\begin{aligned}\operatorname{Re}\lambda\left(\|r\|_{L^2}^2+\|v\|_{L^2}^2\right) \leq & -\int_{\mathbb{R}} D_1\left(|r_\xi|^2+(\theta'-\alpha)^2|r|^2\right)\,d\xi-\int_{\mathbb{R}}\beta(\mathbb{c})|v|^2\,d\xi\\ &+\int_{\mathbb{R}}\left(1-2\mathbb{c}+\tfrac{1}{2}f(\mathbb{c})''\right)|r|^2\,d\xi+\frac{1}{4\epsilon_1}\|\mathbb{r}v\|_{L^2}^2+\epsilon_1\|r'-(\theta'-\alpha)r\|_{L^2}^2\\ &+\frac{1}{4\epsilon_2}\left\|[\beta'(\mathbb{c})\mathbb{v}+(\beta H)'(\mathbb{c})]e^{\theta}r\right\|_{L^2}^2+\epsilon_2\|v\|_{L^2}^2,\end{aligned}$$

and because $\|r'-(\theta'-\alpha)r\|_{L^2}^2 \leq \|r'\|_{L^2}^2+\|(\theta'-\alpha)r\|_{L^2}^2$, we obtain

$$\begin{aligned}\operatorname{Re}\lambda\left(\|r\|_{L^2}^2+\|v\|_{L^2}^2\right) \leq & -\int_{\mathbb{R}}(D_1-\epsilon_1)\left(|r_\xi|^2+(\theta'-\alpha)^2|r|^2\right)\,d\xi-\int_{\mathbb{R}}\left[\beta(\mathbb{c})-\frac{1}{4\epsilon_1}\mathbb{r}^2-\epsilon_2\right]|v|^2\,d\xi\\ &+\int_{\mathbb{R}}\left(1-2\mathbb{c}+\tfrac{1}{2}f(\mathbb{c})''+\tfrac{1}{4\epsilon_2}[\beta'(\mathbb{c})\mathbb{v}+(\beta H)'(\mathbb{c})]^2e^{2\theta}\right)|r|^2\,d\xi.\end{aligned}$$

This upper bound is easily recast as

$$\begin{aligned}\operatorname{Re}\lambda\left(\|r\|_{L^2}^2+\|v\|_{L^2}^2\right) \leq & -\int_{\mathbb{R}}(D_1-\epsilon_1)|r_\xi|^2\,d\xi-\int_{\mathbb{R}}[\beta(\mathbb{c})-\frac{1}{4\epsilon_1}\mathbb{r}^2-\epsilon_2]|v|^2\,d\xi\\ &-\int_{\mathbb{R}}\left[(D_1-\epsilon_1)(\theta'-\alpha)^2-\tfrac{1}{4\epsilon_2}[\beta'(\mathbb{c})\mathbb{v}+(\beta H)'(\mathbb{c})]^2e^{2\theta}-\left(1-2\mathbb{c}+\tfrac{1}{2}f(\mathbb{c})''\right)\right]|r|^2\,d\xi\end{aligned} \tag{4.18}$$

From this inequality, it is straightforward that the first term on the righthand side of equation (4.18) is non positive because $D_1-\epsilon_1 > 1-\epsilon_1 \geq 0$, see Hypothesis (H2) and property 2 in Theorem 1.1. Moreover, by letting $\epsilon_2 \in (0, c_0)$, we also have that the second term on the righthand side of equation (4.18) is non positive, since $\beta(\mathbb{c})-(4\epsilon_1)^{-1}\mathbb{r}^2 \geq c_0 > 0$. These observations lead us to conclude that

$$\operatorname{Re}\lambda\left(\|r\|_{L^2}^2+\|v\|_{L^2}^2\right) \leq -\int_{\mathbb{R}}\left[(D_1-\epsilon_1)(\theta'-\alpha)^2-\tfrac{1}{4\epsilon_2}[\beta'(\mathbb{c})\mathbb{v}+(\beta H)'(\mathbb{c})]^2e^{2\theta}-\left(1-2\mathbb{c}+\tfrac{1}{2}f(\mathbb{c})''\right)\right]|r|^2\,d\xi, \tag{4.19}$$

Finally, if a satisfies (4.14) we conclude that

$$\begin{aligned}(a+f(\mathbb{c})')^2 \geq & \left\|\frac{4D_1^2}{D_1-\epsilon_1}\left(\tfrac{1}{4c_0}[\beta'(\mathbb{c})\mathbb{v}+(\beta H)'(\mathbb{c})]^2e^{2\theta}+1-2\mathbb{c}+\tfrac{1}{2}f(\mathbb{c})''\right)\right\|_{L^\infty}\\ \geq & \frac{4D_1^2}{D_1-\epsilon_1}\left(\tfrac{1}{4c_0}[\beta'(\mathbb{c})\mathbb{v}+(\beta H)'(\mathbb{c})]^2e^{2\theta}+1-2\mathbb{c}+\tfrac{1}{2}f(\mathbb{c})''\right),\end{aligned}$$

because $f(\mathbb{c}) \leq 0$. Therefore,

$$\begin{aligned}(D_1-\epsilon_1)(\theta'-\alpha)^2 = & \frac{(a+f(\mathbb{c})')^2}{4D_1^2}(D_1-\epsilon)\\ \geq & \left(\tfrac{1}{4c_0}[\beta'(\mathbb{c})\mathbb{v}+(\beta H)'(\mathbb{c})]^2e^{2\theta}+1-2\mathbb{c}+\tfrac{1}{2}f(\mathbb{c})''\right),\end{aligned}$$

and the integral on the righthand side of (4.19) is not positive. Thus, we achieve point spectral stability.

Acknowledgements

The author gratefully thanks Professor R. G. Plaza for all delightful discussions and comments while preparing the present work.

Summary Statement

Funding declarations. This research was supported by was supported by CONAHCyT, México, through the Programs "Estancias Postdoctorales por México 2022" and "Programa Ciencia de Frontera", grant CF-2023-G-122.

Conflict of interest. Not applicable.

Author contributions. Not applicable.

Ethics declaration. Not applicable.

References

[CS83] Donald S. Cohen and E. Ann Stanley, *Gaseous diffusion in glassy polymers*, SIAM J. Appl. Math. **43** (1983), no. 4, 949–970. MR MR709748 (85g:82078)

[Kat80] Tosio Kato, *Perturbation theory for linear operators*, Second ed., Classics in Mathematics, Springer-Verlag, New York, 1980.

[KP13] Todd Kapitula and Keith Promislow, *Spectral and dynamical stability of nonlinear waves*, Applied Mathematical Sciences, vol. 185, Springer, New York, 2013. MR 3100266

[LZ21] Corrado Lattanzio and Delyan Zhelyazov, *Spectral analysis of dispersive shocks for quantum hydrodynamics with nonlinear viscosity*, Mathematical Models and Methods in Applied Sciences **31** (2021), no. 09, 1719–1747.

[PW92] Robert L Pego and Michael I Weinstein, *Eigenvalues, and instabilities of solitary waves*, Philosophical Transactions of the Royal Society of London. Series A: Physical and Engineering Sciences **340** (1992), no. 1656, 47–94.

[San02] Björn Sandstede, *Stability of travelling waves*, Handbook of dynamical systems, Vol. 2 (B. Fiedler, ed.), North-Holland, Amsterdam, 2002, pp. 983–1055.

[Tes12] Gerald Teschl, *Ordinary differential equations and dynamical systems*, vol. 140, American Mathematical Soc., 2012.

[Tre08] Christiane Tretter, *Spectral theory of block operator matrices and applications*, Imperial college press, 2008.

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