Statistical Causality

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Statistical Causality

1. The Problems of Causal Inference
2. Formal Frameworks for Statistical Causality
3. Graphical Representations and Applications
4. Causal Discovery

3. Graphical Representations and Applications

Graphical Representation

- Certain collections of CI properties can be described and manipulated using a DAG representation
  - *very far from complete*
- Each CI property *is represented by* a graphical separation property
  - *d-separation*
  - *moralization*

Moralization: 1

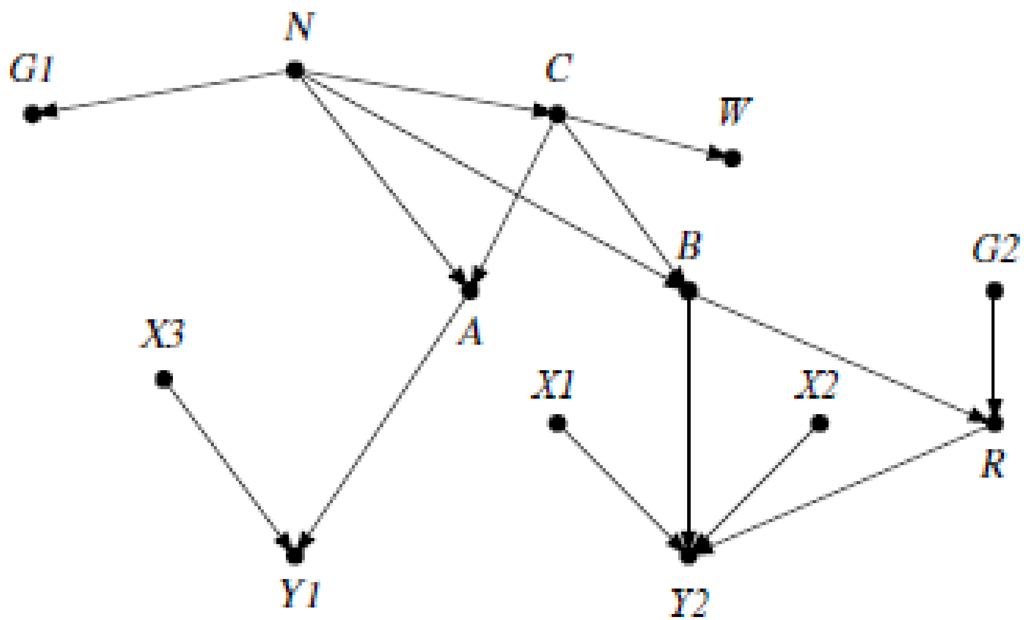


Figure 6.1: Directed graph $\mathcal{D}$ for criminal evidence

Moralization: 1

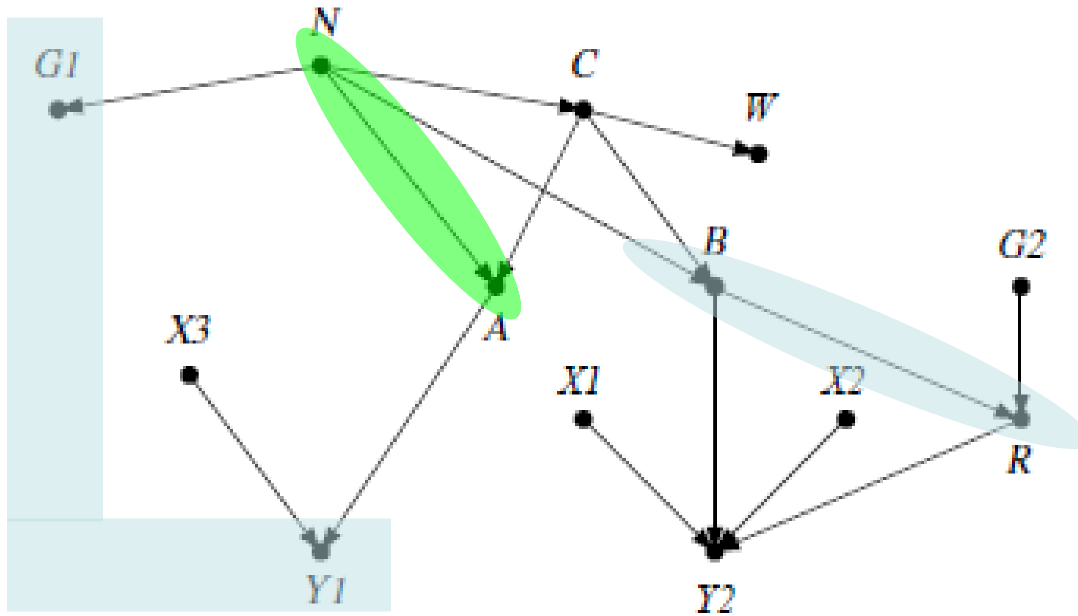


Figure 6.1: Directed graph $\mathcal{D}$ for criminal evidence

$$(B, R) \perp\!\!\!\perp (G1, Y1) \mid (A, N) \ ?$$

Moralization: 2

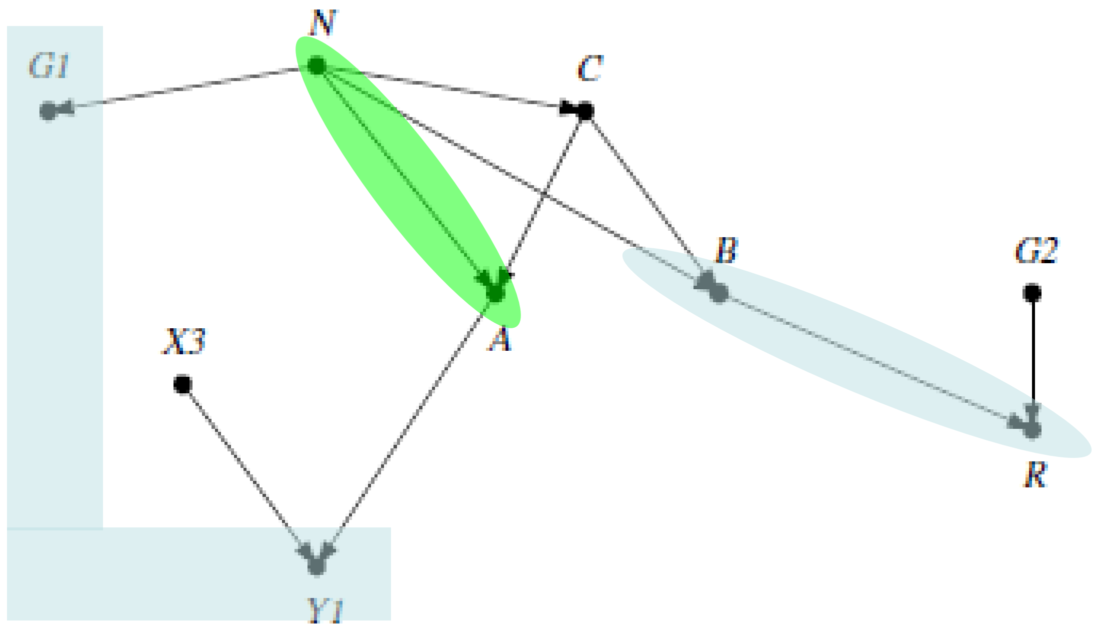


Figure 6.2: Ancestral subgraph $\mathcal{D}'$

$$(B, R) \perp\!\!\!\perp (G1, Y1) \mid (A, N) \ ?$$

Moralization: 3

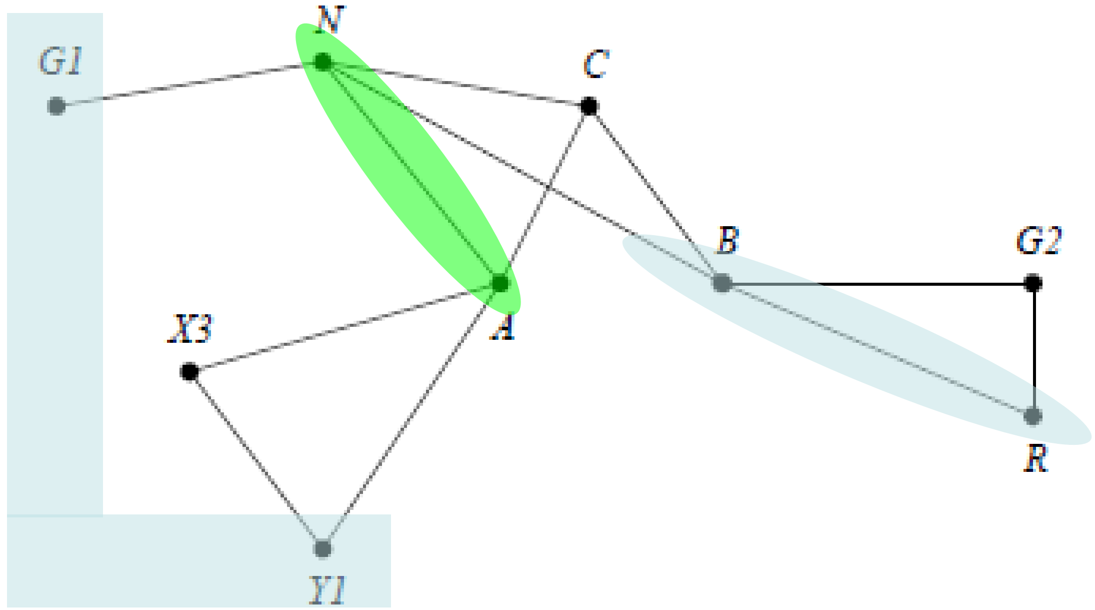


Figure 6.3: Moralized ancestral subgraph $\mathcal{G}'$

$$(B, R) \perp\!\!\!\perp (G1, Y1) \mid (A, N) \ ?$$

Extended Conditional Independence

Distribution of $Y \mid T$ the same in observational and experimental regimes:

$Y \mid (F_T, T)$ does not depend on value of F_T

Can express and manipulate using notation and theory of conditional independence:

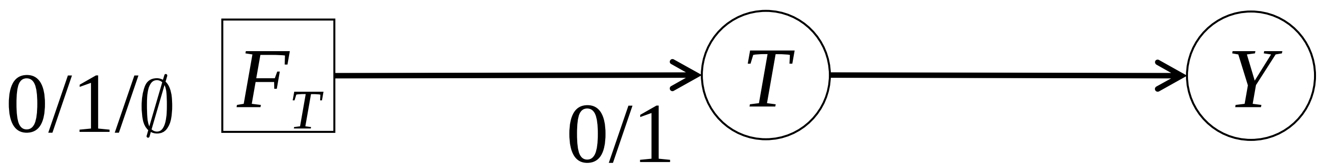
$$Y \perp\!\!\!\perp F_T \mid T$$

(even though F_T is not random)

Augmented DAG

– with random variables and intervention variables

– *probabilistic (not functional) relationships*

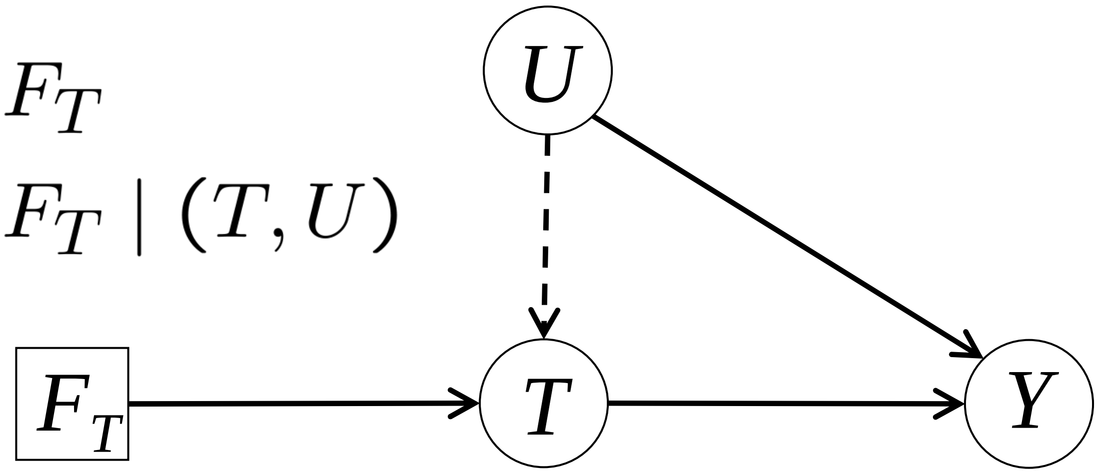


$T \mid (F_T = \emptyset) \sim P_T$ $\qquad$ $Y \mid T$

Absence of arrow $F_T \to Y$ expresses $Y \perp\!\!\!\perp F_T \mid T$

Sufficient Covariate
"(un)confounder"

$$U \perp\!\!\!\perp F_T$$
$$Y \perp\!\!\!\perp F_T \mid (T, U)$$



- Treatment assignment ignorable given U
  - (generally) *not* marginally ignorable
- If U is observed, can fit model (e.g. regression) for dependence of Y on (T,U)
  - causally meaningful

$$\mathrm{ACE}(u) := \mathrm{E}(Y \mid T = 1, U = u) - \mathrm{E}(Y \mid T = 0, U = u)$$

Sufficient covariate “(un)confounder”

$$U \perp\!\!\!\perp F_T$$

$$Y \perp\!\!\!\perp F_T \mid (T, U)$$

Can estimate ACE:

$$\begin{aligned}
\mathrm{E}(Y \mid F_T = t) &= \mathrm{E}\{\mathrm{E}(Y \mid U, F_T = t) \mid F_T = t)\} \\
&= \mathrm{E}\{\mathrm{E}(Y \mid U, F_T = t, T = t) \mid F_T = t)\} \\
&= \mathrm{E}\{\mathrm{E}(Y \mid U, T = t)\} \\
\mathrm{ACE} &= \mathrm{E}\{\mathrm{ACE}(U)\}
\end{aligned}$$

(“back-door” formula)

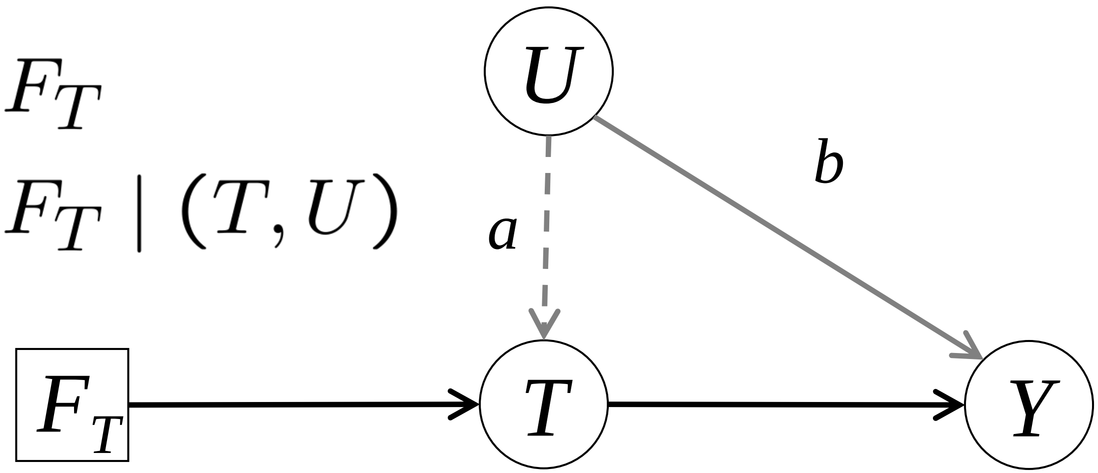
Similarly, whole interventional distribution:

$$p(y \mid F_T = t) = \int p(y \mid u, T = t)\, p(u)\, du$$

Non-confounding

$$U \perp\!\!\!\perp F_T$$

$$Y \perp\!\!\!\perp F_T \mid (T, U)$$



Treatment assignment ignorable given U

Ignorable marginally if either *a* or *b* is absent:

a $T \perp\!\!\!\perp U \mid F_T$

“randomization”

b $Y \perp\!\!\!\perp U \mid T$

“irrelevance”

–then need not even observe U

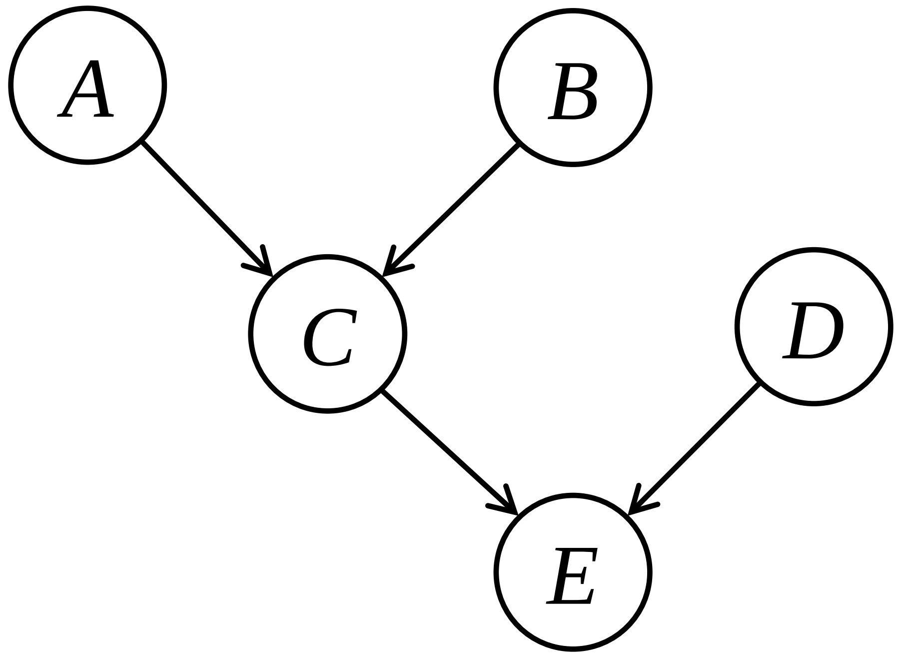
Pearlian DAG

- Envisage intervention on every variable in the system

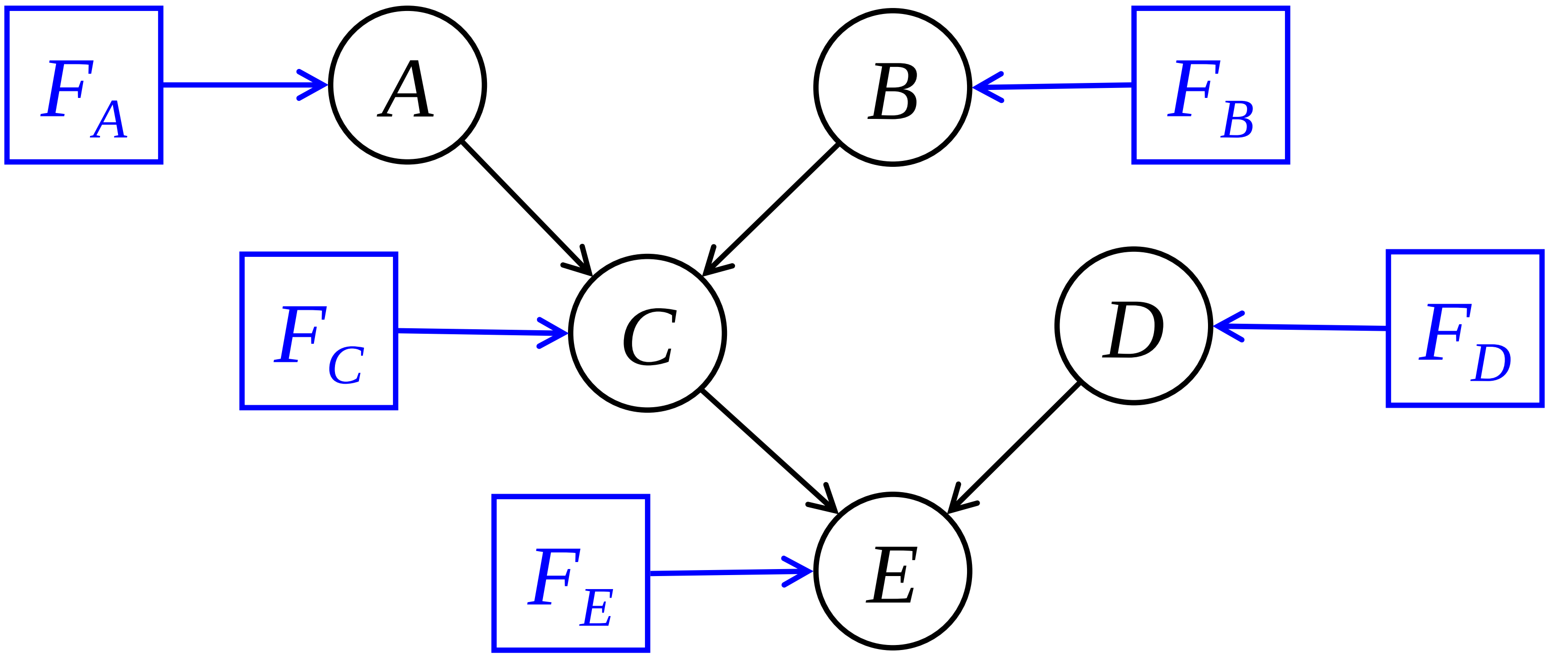
- Augmented DAG model
  - but with intervention indicators implicit

- Every arrow has a causal interpretation

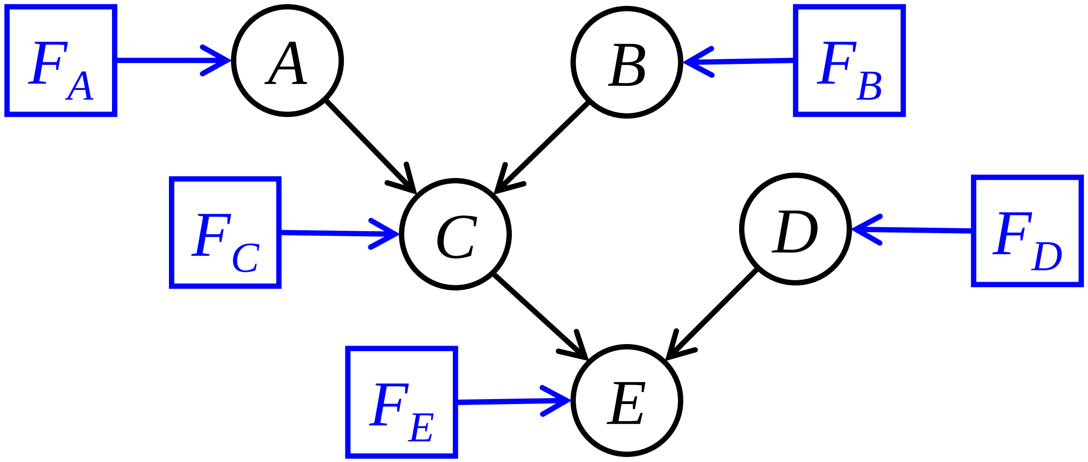
Pearlian DAG



Intervention DAG

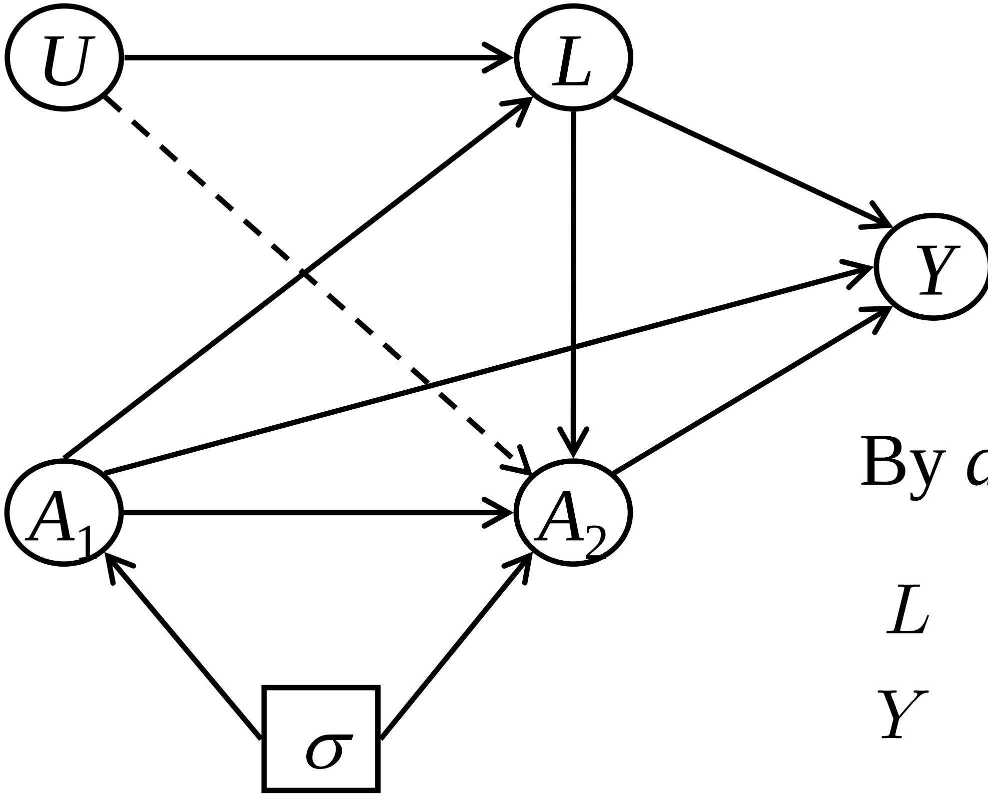


Intervention DAG



- *e.g.*, $E \perp\!\!\!\perp (A, B, F_A, F_B, F_C, F_D) \mid (C, D, F_E)$
- When E is not manipulated, its conditional distribution, given its parents C, D is unaffected by the values of A, B and by whether or not any of the other variables is manipulated
  - modular component

More complex DAGs



σ = treatment strategy

(influence diagram)

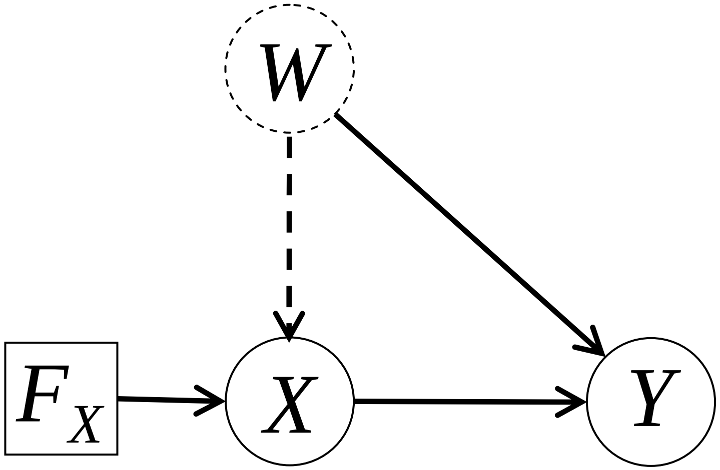
By *d*-separation:

$$L \perp\!\!\!\perp \sigma \mid A_1$$
$$Y \perp\!\!\!\perp \sigma \mid A_1, A_2, L$$

(would fail if *e.g.* $U \rightarrow Y$)

$$p(y \mid \sigma) = \int da_1\, dl\, da_2\; p_\sigma(a_1) p(l \mid a_1) p_\sigma(a_2 \mid a_1, l) p(y \mid a_2, a_2, l)$$

Instrumental Variable



$$W \perp\!\!\!\perp F_X$$
$$Y \perp\!\!\!\perp F_X \mid (X, W)$$

Linear model: $\mathrm{E}(Y \mid X{=}x,\, W,\, F_X = x\,) = f(W) + \beta\, x$

So $\mathrm{E}(Y \mid F_X = x) = \mathrm{E}\{f(W) \mid F_X = x\} + \beta\, x$

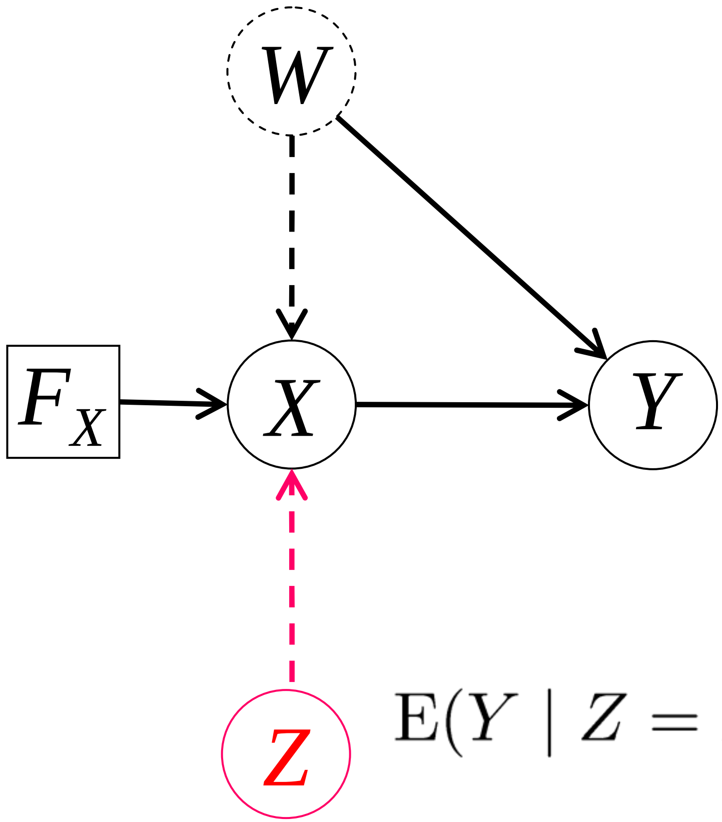
$$= \alpha + \beta\, x$$

- β is causal regression coefficient

– but not estimable from observational data:

$$\mathrm{E}(Y \mid X{=}x) = \mathrm{E}\{f(W) \mid X = x\} + \beta\, x$$

Instrumental Variable



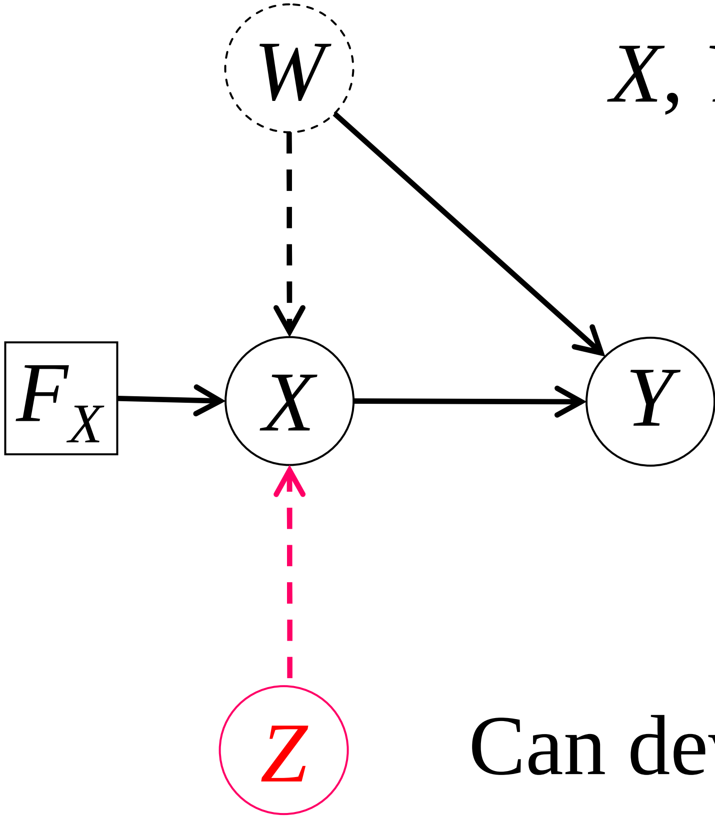
$$
\begin{array}{rcl}
W & \perp\!\!\!\perp & F_X \\
Y & \perp\!\!\!\perp & F_X \mid (X, W) \\
Z & \perp\!\!\!\perp & (W, F_X) \\
Y & \perp\!\!\!\perp & Z \mid (X, W; F_X)
\end{array}
$$

$$
\begin{aligned}
\mathrm{E}(Y \mid Z = z) &= \mathrm{E}\{\mathrm{E}(Y \mid Z, X, W) \mid Z = z\} \\
&= \mathrm{E}\{f(W) + \beta X \mid Z = z\} \\
&= \alpha + \beta\, \mathrm{E}(X \mid Z = z)
\end{aligned}
$$

–so can now identify β

Discrete case

X, Y, Z binary

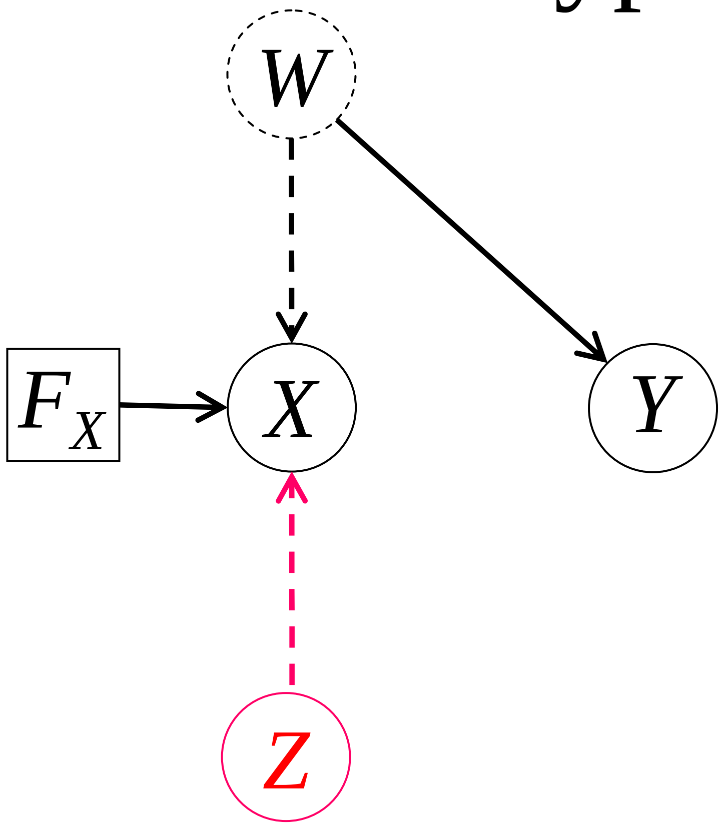


Can develop inequalities for ACE

$$E(Y \mid F_X = 1) - E(Y \mid F_X = 0)$$

in terms of estimable quantities

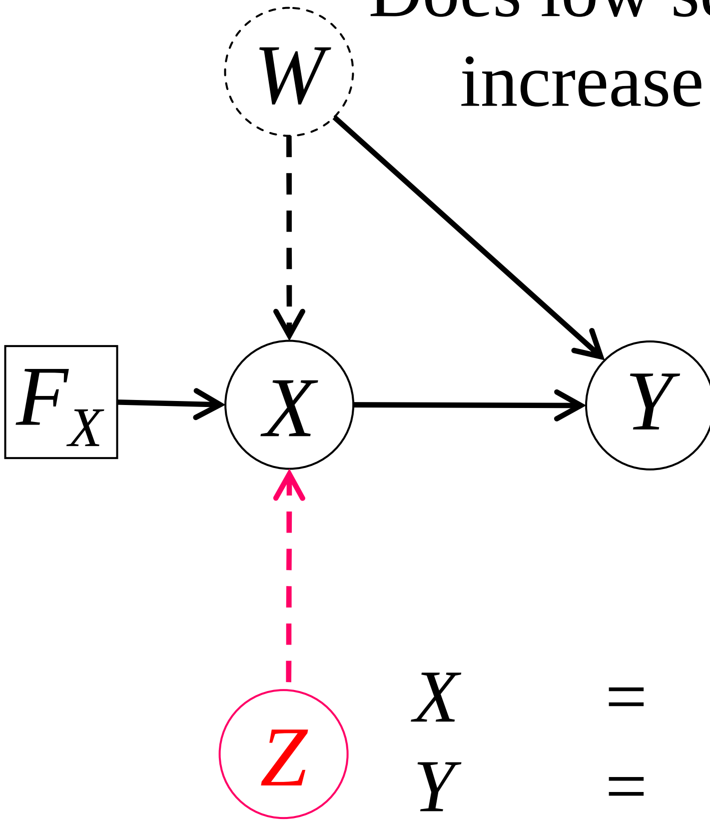
Hypothesis Test



If arrow $X \to Y$ missing, then $Y \perp\!\!\!\perp F_X$
(X has no causal effect on Y).
In this case $Y \perp\!\!\!\perp Z \mid F_X = \emptyset$—can test.

Mendelian Randomisation

Does low serum cholesterol level increase the risk of cancer?

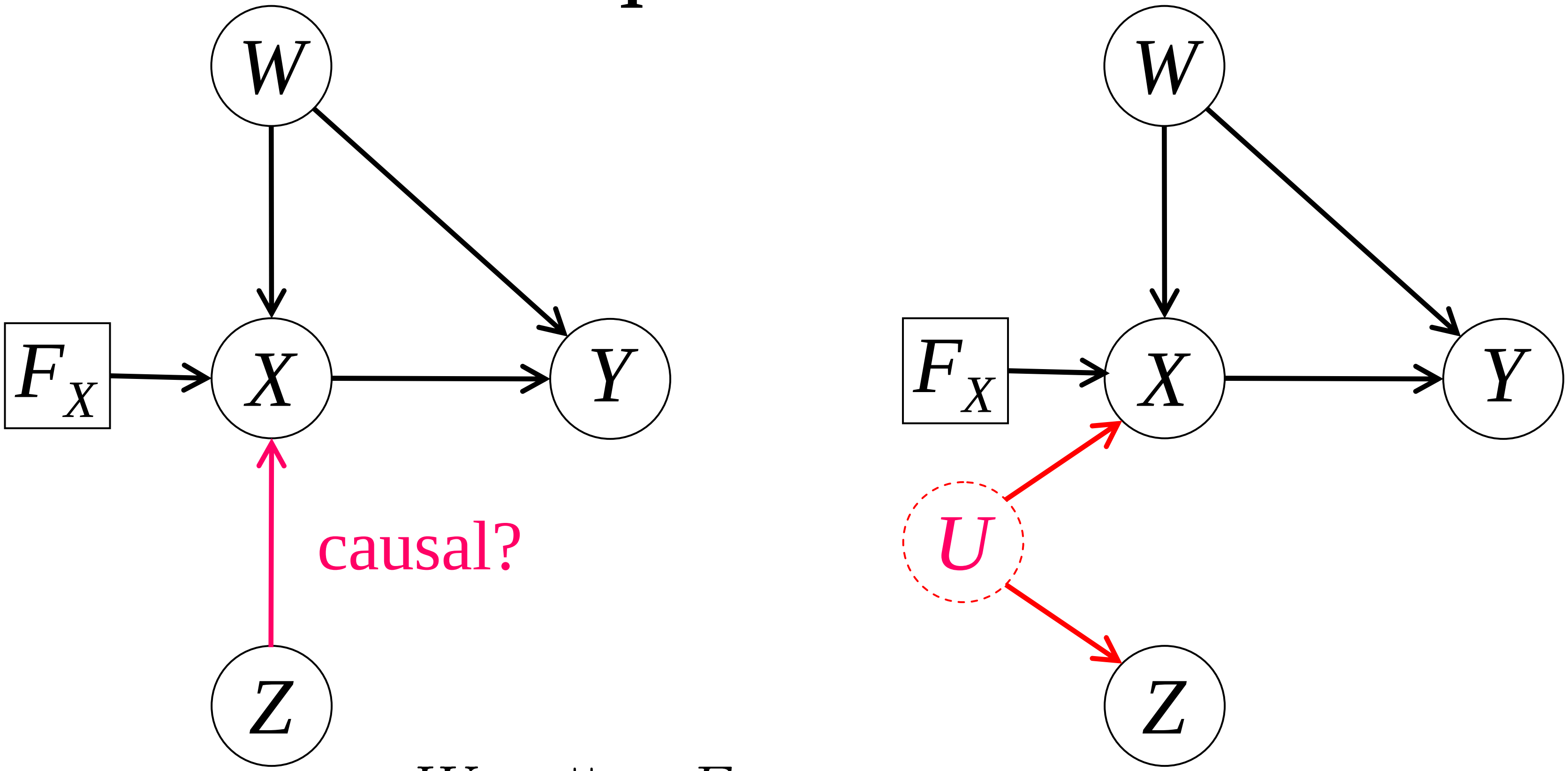


$$
\begin{aligned}
W &\perp\!\!\!\perp F_X \\
Y &\perp\!\!\!\perp F_X \mid (X, W) \\
{\color{red}Z} &{\color{red}\perp\!\!\!\perp (W, F_X)} \\
{\color{red}Y} &{\color{red}\perp\!\!\!\perp Z \mid (X, W; F_X)}
\end{aligned}
$$

X	=	serum cholesterol
Y	=	cancer
W	=	diet, smoking, hidden tumour,…
Z	=	APOE gene

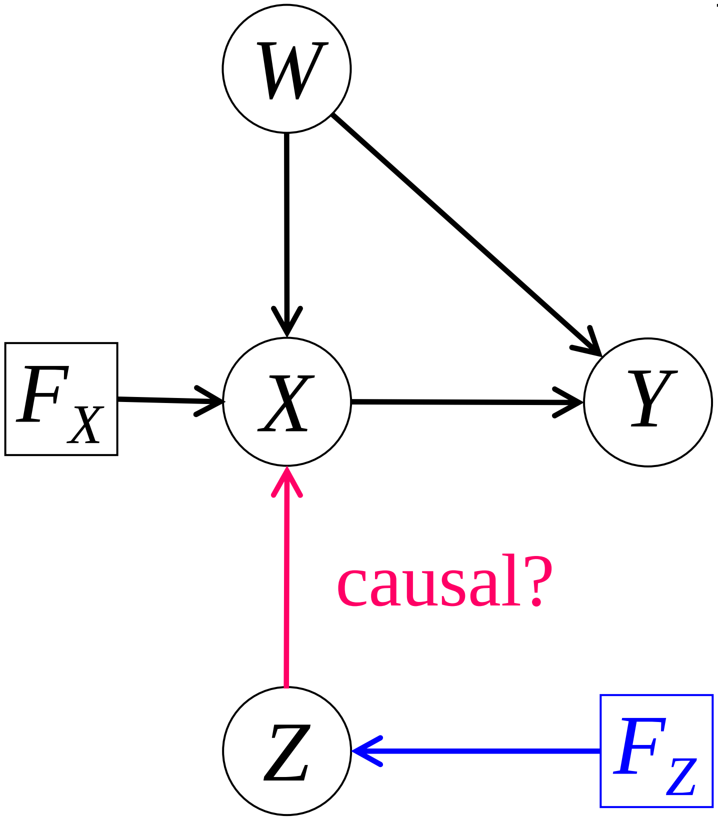
(E2 allele induces particularly low serum cholesterol)

Equivalence

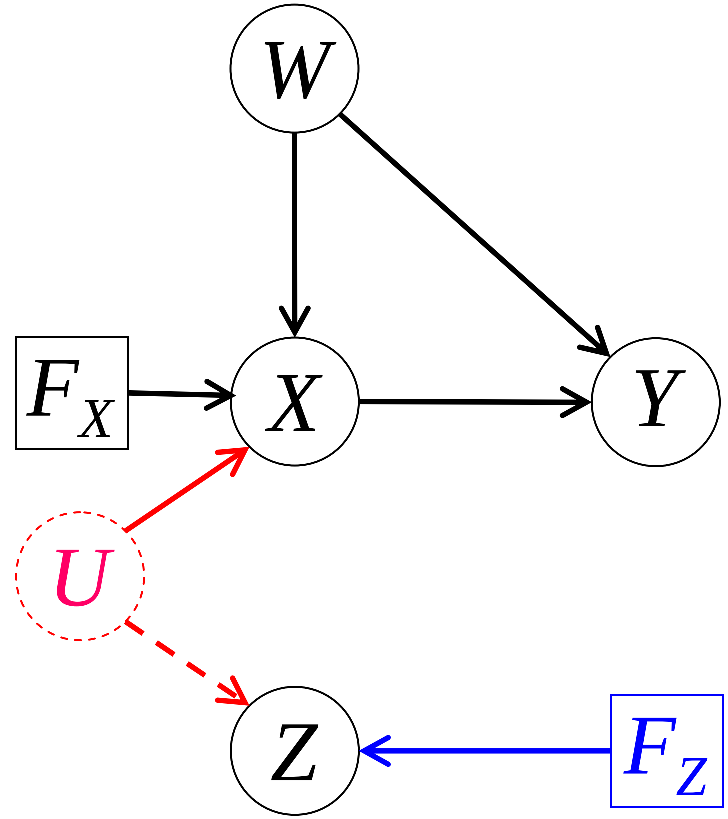


$$
\begin{aligned}
W &\perp\!\!\!\perp F_X \\
Y &\perp\!\!\!\perp F_X \mid (X, W) \\
Z &\perp\!\!\!\perp (W, F_X) \\
Y &\perp\!\!\!\perp Z \mid (X, W; F_X)
\end{aligned}
$$

Non-equivalence



$$X \not\perp\!\!\!\perp Z \mid F_Z \neq \emptyset$$

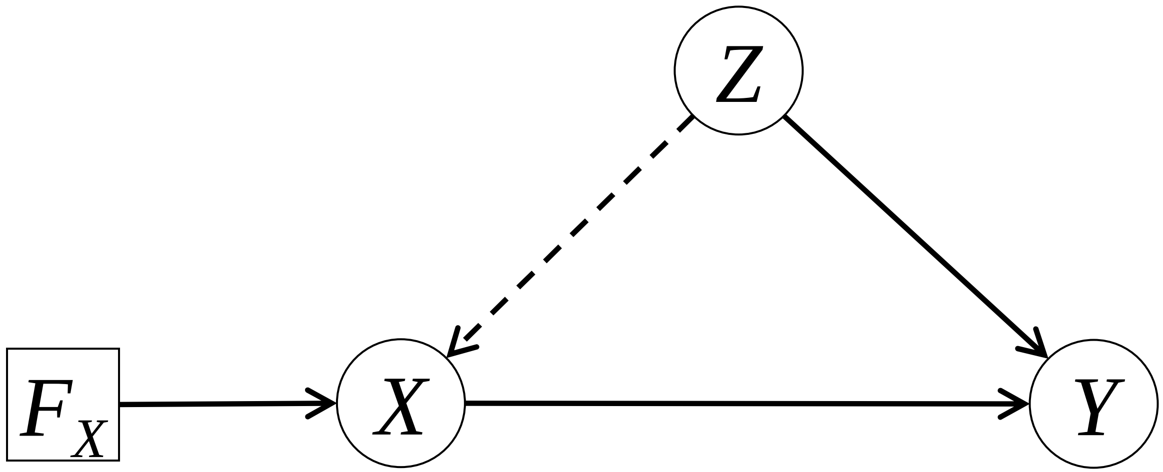


$$X \perp\!\!\!\perp Z \mid F_Z \neq \emptyset$$

Can we identify a causal effect from observational data?

- Model with domain and (explicit or implicit) intervention variables, specified ECI properties
  - e.g. augmented DAG, Pearlian DAG
- Observed variables $\mathcal{V}$, unobserved variables $\mathcal{U}$
- Can identify observational distribution over $\mathcal{V}$
- Want to answer causal query, e.g. $p(y \mid F_X = x)$
  - write as $p(y \mid \check{x})$
- When/how can this be done?

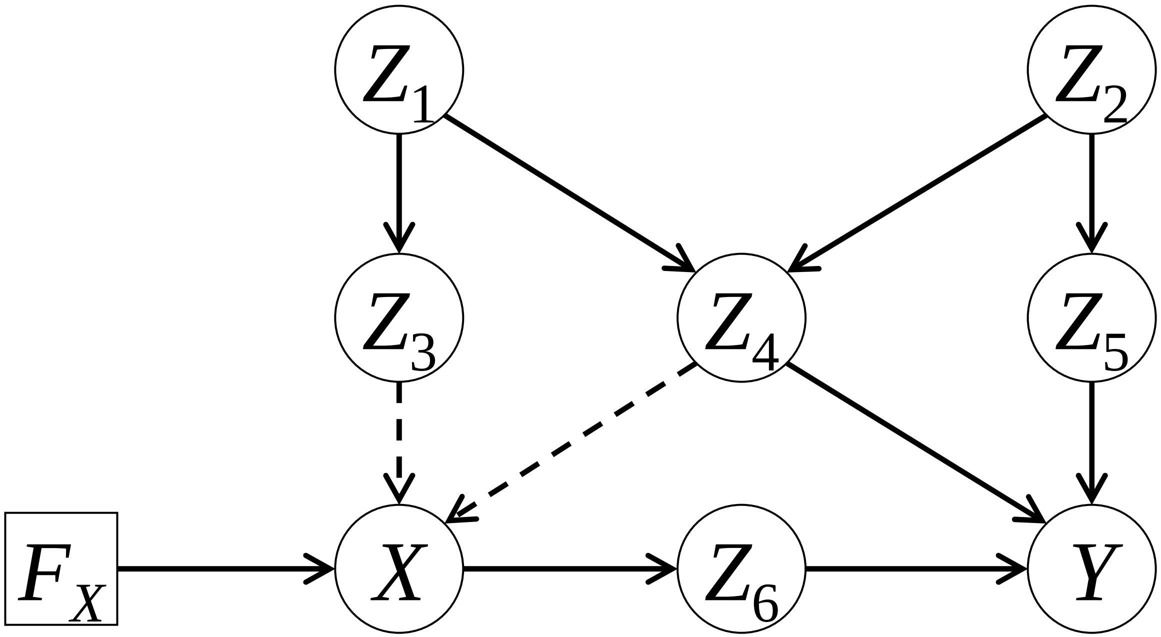
Example: “back-door formula”



$$Z \perp\!\!\!\perp F_X$$
$$Y \perp\!\!\!\perp F_X \mid (X, Z)$$

$$p(y \mid \check{x}) = \sum_z p(y \mid x, z)\, p(z)$$

Example: “back-door formula”

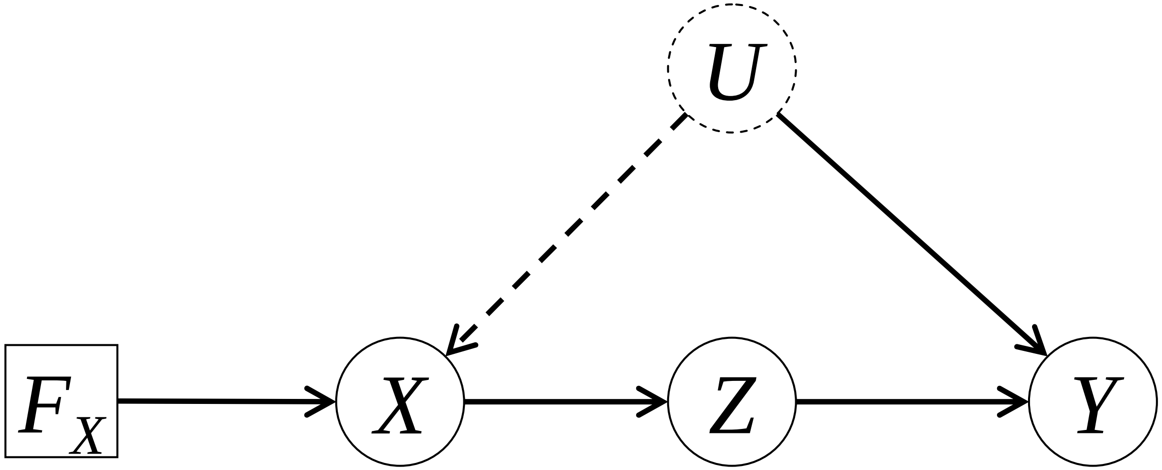


$$Z \perp\!\!\!\perp F_X$$

$$Y \perp\!\!\!\perp F_X \mid (X, Z)$$

Works for $Z = (Z_3, Z_4)$, and also for $Z = (Z_4, Z_5)$

Example: “front-door formula”



$$
\begin{aligned}
U &\perp\!\!\!\perp F_X \\
Z &\perp\!\!\!\perp (U, F_X) \mid X \\
Y &\perp\!\!\!\perp (F_X, X) \mid (Z, U)
\end{aligned}
$$

$$p(y \mid \check{x}) = \sum_z p(z \mid x) \sum_{x'} p(y \mid x', z)\, p(x').$$

do-calculus

Rule 1 (Insertion/deletion of observations)

If $Y \perp\!\!\!\perp Z \mid (X, F_X \neq \emptyset, W)$ then

$$p(y \mid \check{x}, z, w) = p(y \mid \check{x}, w)$$

Rule 2 (Action/observation exchange)

If $Y \perp\!\!\!\perp F_Z \mid (X, F_X \neq \emptyset, Z, W)$, then

$$p(y \mid \check{x}, \check{z}, w) = p(y \mid \check{x}, z, w)$$

Rule 3 (Insertion/deletion of actions)

If $Y \perp\!\!\!\perp F_Z \mid (X, F_X \neq \emptyset, W)$, then

$$p(y \mid \check{x}, \check{z}, w) = p(y \mid \check{x}, w)$$

do-calculus

For a problem modelled by a Pearlian DAG, the *do*-calculus is complete:

- We can tell whether a given causal effect is computable (from the observational distribution)
- Any computable causal effect can be computed by successive applications of rules 2 and 3
  - together with probability calculus, and property $F_T = t \Rightarrow T = t$ (delete dotted arrows)
- There exist algorithms to accomplish this

4. Causal Discovery

Probabilistic Causality

- *Intuitive concepts* of “cause”, “direct cause”,…
- Principle of the common cause:

  “Variables are independent, given their common causes”

- Assume *causal DAG* representation:
  - direct causes of *V* are its DAG parents
  - all “common causes” included

Probabilistic Causality

CAUSAL MARKOV CONDITION

– The causal DAG also represents the observational conditional independence properties of the variables

- WHEN??
- WHY??

- **CAUSAL FAITHFULNESS CONDITION**

– No extra conditional independencies

- WHY??

Causal Discovery

- An attempt to learn causal relationships from observational data
- Assume there is an underlying *causal DAG* (possibly including unobserved variables) satisfying the (faithful) Causal Markov Condition
- Use data to search for a DAG representing the observational independencies
  - ➢ *model selection*
- Give this a causal interpretation

Causal Discovery

Two main approaches:

- “Constraint-based”
  - Qualitative
  - Infer (patent or latent) conditional independencies between variables
  - Fit conforming DAG model(s)
- Statistical model selection
  - Quantitative
  - General approach, applied to DAG models
  - Need not commit to one model (model uncertainty)

Constraint-Based Methods (complete data)

- Identify/estimate conditional independencies holding between observed variables
- Assume sought-for causal DAG does not involve any variables other than those observed

Wermuth-Lauritzen algorithm

- Assume variables are "causally ordered" *a priori*:

  $(V_1, V_2, \ldots, V_N)$, s.t arrows can only go from lower to higher

- For each i, identify (smallest) subset S_i of

  $V^{i\text{-}1} := (V_1, V_2, \ldots, V_{i\text{-}1})$ such that

$$V_i \perp\!\!\!\perp V^{i-1} \mid S_i$$

- Draw arrow from each member of S_i to V_i

SGS algorithm (no prior ordering)

1. Start with complete undirected graph over V^N
2. Remove edges V–W s.t., for some S, $V \perp\!\!\!\perp W \mid S$
3. Orient any V–Z–W as $V \to Z \leftarrow W$ if:
   - no edge V–W
   - for each $S \subseteq V^N$ with $Z \in S$, $V \not\perp\!\!\!\perp W \mid S$
4. Repeat while still possible:

   i. if $V \to Z$ –W but not V–W, orient as $V \to Z \to W$

   ii. If $V \rightsquigarrow W$ and V–W, orient as $V \to W$

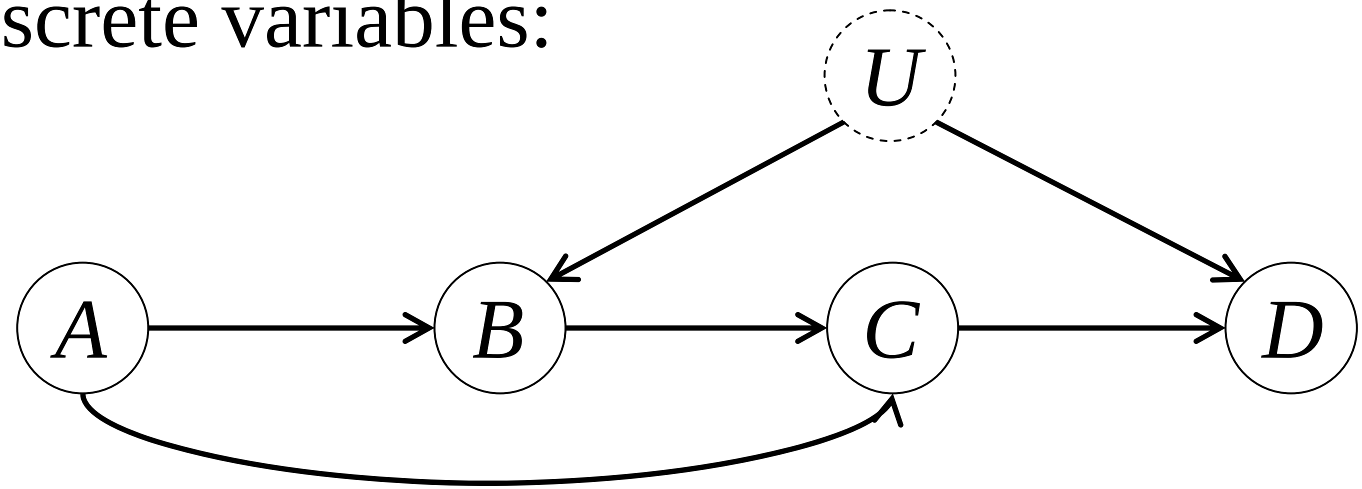
Comments

- Wermuth-Lauritzen algorithm
  - always finds a valid DAG representation
  - need not be faithful
  - depends on prior ordering
- SGS algorithm
  - may not succeed if there is no faithful DAG representation
  - output may not be fully oriented
  - computationally inefficient (too many tests)
  - better variations: PC, PC*

Constraint-Based Methods (incomplete data)

- Allow now for unobserved (latent) variables
- Can modify previous algorithms to work just with conditional independencies between observed variables
- But latent CI has other (quantitative) implications too…

Discrete variables:



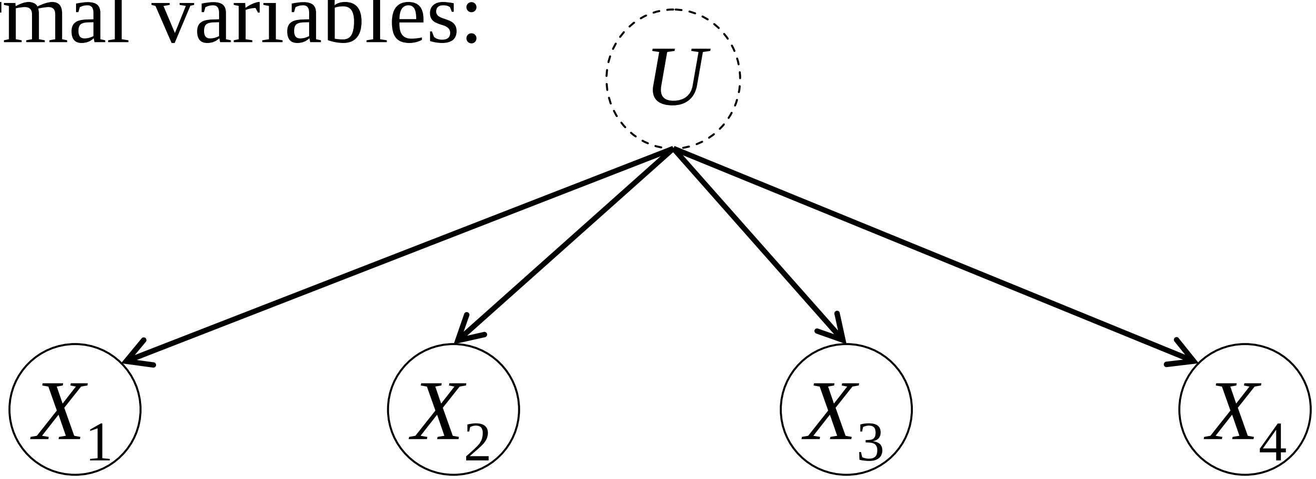
No CI properties between observables *A*, *B*, *C*, *D*.

But

$$\begin{aligned}\sum_b p(b \mid a)p(d \mid a, b, c) &= \sum_b p(b \mid a) \sum_u p(d \mid a, b, c, u)p(u \mid a, b, c) \\ &= \sum_u p(d \mid c, u)p(u \mid a)\end{aligned}$$

– does not depend on *a*

Normal variables:



No CI properties between observables X_1, X_2, X_3, X_4.

But $\rho_{13}\rho_{24} = \rho_{14}\rho_{23} = \rho_{12}\rho_{34}$

Such properties form basis of TETRAD II program

Bayesian Model Selection

- Consider collection $\mathcal{M} = \{M\}$ of models
- Have prior distribution $\pi_M(\boldsymbol{\theta}_M)$ for parameter $\boldsymbol{\theta}_M$ of model M
- Based on data $\boldsymbol{x}$, compute *marginal likelihood* for each model M:

$$L_M = \int p(\boldsymbol{x} \mid \boldsymbol{\theta}_M)\, \mathrm{d}\boldsymbol{\theta}_M$$

- Use as score for comparing models, or combine with prior distribution $\{w_M\}$ over models to get posterior:

$$w^*_M \propto w_M\, L_M$$

Bayesian Model Selection

- Algebraically straightforward for discrete or Gaussian DAG models, parametrised by parent-child conditional distributions, having conjugate priors (with local and global independence)
  - Zoubin Ghahramani's lectures
- Can arrange hyperparameters so that indistinguishable (Markov equivalent) models get same score

Mixed data

- Data from experimental and observational regimes
- Model-selection approach:
  - assume Pearlian DAG
  - ignore local likelihood contribution when the response variable is set
- Constraint-based approach?
  - base on ECI properties, e.g. $X \perp\!\!\!\perp F_Y \mid (W, F_Z)$

A Parting Caution

- We have powerful statistical methods for attacking causal problems
- But to apply them we have to make strong assumptions (e.g. ECI assumptions, relating distinct regimes)
- Important to consider and justify these in context
  - *e.g.*, Mendelian randomisation

"NO CAUSES IN, NO CAUSES OUT"

Thank you!

Further Reading

- A. P. Dawid (2007). *Fundamentals of Statistical Causality*. Research Report 279, Department of Statistical Science, University College London. 94 *pp*. http://www.ucl.ac.uk/Stats/research/reports/abs07.html#279
- R. E. Neapolitan (2003). *Learning Bayesian Networks*. Prentice Hall, Upper Saddle River, New Jersey.
- J. Pearl (2009). *Causality: Models, Reasoning and Inference* (second edition). Cambridge University Press.
- D. B. Rubin (1978). Bayesian inference for causal effects: the role of randomization. *Annals of Statistics* **6**, 34–68.
- P. Spirtes, C. Glymour, and R. Scheines (2000). *Causation, Prediction and Search* (second edition). Springer-Verlag, New York.
- P. Suppes (1970). *A Probabilistic Theory of Causality*. North Holland, Amsterdam.