1 Outline

Silverman proves a version of mordell's theorem for curves having a point of order two. The goal of this project will be the following.

Theorem 1.1. *Let*

$$E : y^2 = x^3 + ax^2 + bx + c$$

be an elliptic curve containing a point of order 3, then $E(\mathbb{Q})$ is finitely generated.

Potentially we even get the following for free.

Corollary 1.2. *Let*

$$E : y^2 = x^3 + ax^2 + bx + c$$

be an elliptic curve containing a point of order 3, then $E(\mathbb{Q}(\sqrt{-3}))$ is finitely generated.

Our strategy will be to broadly follow Silverman's proof, namely we will use a specific case of the Descent Theorem from namely:

Theorem 1.3. *Let A be an abelian group. Suppose that there exists a function $h : A \to \mathbb{R}$ with the following properties.*

1. *Let $Q \in A$. There exists $C_1(A, Q)$, such that for all $P, Q \in A$*

$$h(P + Q) \leq 2h(P) + C_1(A, Q).$$

2. *There is $C_2(A)$, such that for all $P \in A$*

$$h(3P) \geq 9h(P) - C_2(A).$$

3. *For every $C_3 \in \mathbb{R}$ the set*

$$\{P \in A : h(P) \leq C_3\}$$

is finite.

If furthermore $A/3A$ is finite then A is finitely generated.

We shall prove this theorem and then prove all its hypotheses in order of increasing difficulty. Firstly, we shall define such a function $H : \mathbb{Q} \to \mathbb{R}$. And then generalise it to elliptic curves.

Definition 1.4. *Let $x = p/q \in \mathbb{Q}$ with $\gcd(p, q) = 1$. Then we define the height of x as $H(x) = \max\{|p|, |q|\}$.*

And now for Elliptic Curves. We shall understand E to be an Elliptic Curve.

Definition 1.5. *Define the height of a rational point $P = (x, y)$ on E as $H(P) := H(x)$.*

Now the lemmas.

Lemma 1.6. *For all constants $m \in \mathbb{R}$ we have that the set*

$$\{P \in E(\mathbb{Q}) : H(P) \leq m\}$$

is finite.

This is a result which holds more generally and its proof is straightforward.

Lemma 1.7. *Let $P_0 \in E(\mathbb{Q})$, then there exists $b \in \mathbb{R}$ such that for every $P \in E(\mathbb{Q})$*

$$h(P + P_0) \leq 2h(P) + b.$$

Which also holds more generally. The next result significantly deviates from Silverman's.

Lemma 1.8. *There is $k \in \mathbb{R}$ such that for every point $P \in E(\mathbb{Q})$ we have*

$$H(3P) \geq 9H(P) - k.$$

Which is harder to prove. And lastly:

Lemma 1.9. *Suppose $E(\mathbb{Q})$ has a point of order 3. Then the subgroup $3E(\mathbb{Q})$ has finite index.*