Bachelor Project
An Elementary Proof of Mordell's Theorem for Elliptic Curves with a point of Order 3

Levi Moes

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Abstract

Tate-Silverman proves Mordell's theorem for curves with a point of order 2, we show this proof generalises to curves with a point of order 3.

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1 Introduction

2 Elliptic Curves

The general notion of an elliptic curve shall be the centre of focus for this thesis. One might define an Elliptic Curve as follows.

Definition 2.1. *Let $f(x)$ be a 3rd degree monic 3rd polynomial having distinct roots. An Elliptic Curve is a curve*

$$E : y^2 = f(x).$$

If K is a field, we denote for a given Elliptic Curve

$$E(K) := \left\{(x, y) \in K \times K : y^2 = f(x)\right\}.$$

Example 2.2. Take $K = \mathbb{R}$, $f(x) = x^3 + px^2 + 1$, where we let $r \in \{-6, \ldots, 6\}$. This yields a sequence of Elliptic Curves

2.1 Elliptic Curves over the Rationals

2.1.1 The Group law

2.1.2 Projective Geometry and the Identity

2.2 Elliptic Curves over an Arbitrary Field

3 Points of Finite Order

4 Mordell's Theorem

5 The 3-Descent Theorem

6 Finding a Height Function

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References

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