Trust-Region Kalman Mean Update for MacroIR (Variance Inflation to Enforce Probability Constraints)

1 Problem statement

In the MacroIR interval update, after propagating the mean and covariance to time t, we perform a scalar Kalman update using the interval-averaged current $\overline{y}^{\mathrm{obs}}_{0\to t}$.

Let:

- $\boldsymbol{\mu}^{\mathrm{prop}}(t) \in \mathbb{R}^K$: propagated per-channel mean occupancy at time t.
- $\boldsymbol{\Sigma}^{\mathrm{prop}}(t) \in \mathbb{R}^{K\times K}$: propagated per-channel covariance at time t.
- $\overline{y}^{\mathrm{pred}}_{0\to t}$: predictive mean interval current.
- σ^2_{pred}: predictive variance of the interval current, including intrinsic and measurement contributions.
- $\delta := \overline{y}^{\mathrm{obs}}_{0\to t} - \overline{y}^{\mathrm{pred}}_{0\to t}$: residual.
- $\mathbf{g} \in \mathbb{R}^K$: cross-covariance vector between the state at time t and the interval current (the MacroIR vector tilde $\widetilde{\gamma^\top \Sigma}$).

The standard scalar Gaussian conditioning (Kalman) update is

$$\boldsymbol{\mu}^{\mathrm{post}}(t) = \boldsymbol{\mu}^{\mathrm{prop}}(t) + \frac{\mathbf{g}\,\delta}{\sigma^2_{\mathrm{pred}}}, \tag{1}$$

$$\boldsymbol{\Sigma}^{\mathrm{post}}(t) = \boldsymbol{\Sigma}^{\mathrm{prop}}(t) - \frac{\mathbf{g}\mathbf{g}^\top}{\sigma^2_{\mathrm{pred}}}. \tag{2}$$

Issue. In this purely Gaussian update, the mean is unconstrained: a large residual $|\delta|$ can push some components of $\boldsymbol{\mu}^{\mathrm{post}}(t)$ outside the probability simplex, e.g. $\mu_i^{\mathrm{post}} < 0$ or $\mu_i^{\mathrm{post}} > 1$. This is unacceptable since occupancies must remain in $(0, 1)$ (or at least in a small open box inside the simplex).

On the other hand, in an exactly consistent joint Gaussian model the covariance update (2) *cannot* produce negative variances: it is the Schur complement of a positive semidefinite joint covariance, hence remains positive semidefinite (see Section 3). Any negative eigenvalues that arise in practice are due to approximation and floating-point effects, not to the algebra of Gaussian conditioning itself.

Goal. We want a minimal distortion of the Gaussian update that:

1. keeps all $\mu_i^{\text{post}}(t)$ in $[p_{\min}, p_{\max}]$, with $0 < p_{\min} \ll 1$;
2. preserves the standard Gaussian conditioning structure for the covariance (up to mild numerical regularisation);
3. has a clear interpretation at the log-likelihood level, since MacroIR fundamentally cares about the per-interval evidence.

We explicitly do not reparameterize the state (e.g. via logits); we stay in the "Gaussian-ish" regime in $\boldsymbol{\mu}$-space and only introduce a trust region on the mean update.

2 Solution (implementation-ready recipe)

We introduce a shrink factor $\alpha \in [0, 1]$ on the Kalman update and compute the maximal α that keeps the mean inside the safe region. This is exactly equivalent to an inflation of the predictive variance in the likelihood. The covariance update is scaled by the same α to preserve the usual Kalman algebra and the log-likelihood interpretation.

2.1 Parameters

We enforce probabilistic floors to prevent the update from leaving the simplex.

- $p_{\min} > 0$: minimum admissible occupancy per state (e.g. 10^{-10}). This prevents probabilities from reaching exact zero, which would create degenerate directions and numerical problems.
- $p_{\max} = 1 - (K-1)p_{\min}$: corresponding maximum occupancy ensuring that $\sum_i \mu_i = 1$ remains feasible.
- Optionally, a small variance floor $v_{\min} = \kappa\, p_{\min}(1 - p_{\min})$ with $\kappa \in [0.01, 0.1]$ (e.g. $\kappa = 0.03$) can be used *after* the update as a purely numerical safeguard if floating-point drift drives some diagonal entries slightly below this scale. This floor is not used to determine α.

2.2 Step 1: Compute the unconstrained update direction

Define

$$\Delta\boldsymbol{\mu} := \frac{\mathbf{g}\,\delta}{\sigma^2_{\text{pred}}},$$

so the unconstrained mean update would be

$$\boldsymbol{\mu}^{\text{post}}_{\text{unc}} = \boldsymbol{\mu}^{\text{prop}} + \Delta\boldsymbol{\mu}.$$

The corresponding (unscaled) covariance update is

$$\boldsymbol{\Sigma}^{\text{post}}_{\text{unc}} = \boldsymbol{\Sigma}^{\text{prop}} - \frac{\mathbf{g}\mathbf{g}^{\top}}{\sigma^2_{\text{pred}}}.$$

These define the direction of the update. The actual step length will be chosen by the trust-region constraints.

2.3 Step 2: Define a scaled update with factor α

Instead of taking the full step, use

$$\boldsymbol{\mu}^{\text{post}}(\alpha) = \boldsymbol{\mu}^{\text{prop}} + \alpha\,\Delta\boldsymbol{\mu}, \tag{3}$$

$$\boldsymbol{\Sigma}^{\text{post}}(\alpha) = \boldsymbol{\Sigma}^{\text{prop}} - \alpha\,\frac{\mathbf{g}\mathbf{g}^{\top}}{\sigma_{\text{pred}}^2}. \tag{4}$$

Here $\alpha \in [0,1]$. Both mean and covariance are updated with the same α, which will correspond to using an effective variance $\sigma_{\text{eff}}^2 = \sigma_{\text{pred}}^2/\alpha$ in a standard Kalman update (see Section 4.1).

2.4 Step 3: Constraints from probabilities

For each component $j \in \{1,\ldots,K\}$ we require

$$p_{\min} \le \mu_j^{\text{post}}(\alpha) \le p_{\max},$$

with

$$\mu_j^{\text{post}}(\alpha) = \mu_j^{\text{prop}} + \alpha\,\Delta\mu_j, \quad \Delta\mu_j = \frac{g_j\,\delta}{\sigma_{\text{pred}}^2}.$$

We only consider indices with $\Delta\mu_j \neq 0$, since the others do not move and therefore never constrain α.

Lower bound. If $\Delta\mu_j < 0$, the lower bound is potentially active:

$$\mu_j^{\text{prop}} + \alpha\,\Delta\mu_j \ge p_{\min} \quad\Longrightarrow\quad \alpha \le \alpha_j^{(\text{mean,low})} := \frac{\mu_j^{\text{prop}} - p_{\min}}{|\Delta\mu_j|}.$$

Upper bound. If $\Delta\mu_j > 0$, the upper bound is potentially active:

$$\mu_j^{\text{prop}} + \alpha\,\Delta\mu_j \le p_{\max} \quad\Longrightarrow\quad \alpha \le \alpha_j^{(\text{mean,high})} := \frac{p_{\max} - \mu_j^{\text{prop}}}{\Delta\mu_j}.$$

For each j with $\Delta\mu_j \neq 0$, define

$$\alpha_j^{(\text{mean})} := \begin{cases} \alpha_j^{(\text{mean,low})}, & \Delta\mu_j < 0, \\ \alpha_j^{(\text{mean,high})}, & \Delta\mu_j > 0. \end{cases}$$

The admissible interval for α is therefore

$$0 \le \alpha \le \alpha_{\text{mean}},$$

with

$$\alpha_{\text{mean}} := \min_{j:\,\Delta\mu_j \neq 0} \alpha_j^{(\text{mean})}.$$

If no component moves ($\Delta\boldsymbol{\mu} = \mathbf{0}$), we define $\alpha_{\text{mean}} := 1$; in that case the mean is already consistent with the observation and no variance inflation is needed.

If $\alpha_{\text{mean}} \le 0$ (which can happen if the unconstrained direction immediately leaves the safe box), we interpret this as a signal to strongly down-weight or ignore this observation (see below).

2.5 Step 4: Final shrink factor and variance inflation

We now define the actual shrink factor

$$\alpha^\star := \max\bigl(0, \min(1, \alpha_{\text{mean}})\bigr).$$

Two limiting cases are useful:

- If $\alpha^\star = 1$, we recover the standard Gaussian update with no variance inflation.
- If $\alpha^\star = 0$, we perform *no* Kalman correction: $\boldsymbol{\mu}^{\text{post}} = \boldsymbol{\mu}^{\text{prop}}$ and $\boldsymbol{\Sigma}^{\text{post}} = \boldsymbol{\Sigma}^{\text{prop}}$. In practice, one may treat such intervals as fully rejected outliers.

In implementation, it is often convenient to introduce a small threshold (e.g. $\alpha_{\min} = 10^{-3}$) and:

- if $\alpha^\star < \alpha_{\min}$, skip the update and set $\alpha^\star := 0$;
- otherwise use $\alpha^\star$ as computed.

This makes the "outlier" behaviour explicit: a sufficiently incompatible observation leaves the mean and covariance unchanged.

Define the (per-interval) variance inflation factor

$$\phi := \begin{cases} \dfrac{1}{\alpha^\star}, & \alpha^\star > 0, \\ 1, & \alpha^\star = 0 \end{cases} \qquad \text{so that} \quad \phi \geq 1.$$

The effective predictive variance used in the update and in the log-likelihood is

$$\sigma^2_{\text{eff}} := \phi\, \sigma^2_{\text{pred}} = \begin{cases} \dfrac{\sigma^2_{\text{pred}}}{\alpha^\star}, & \alpha^\star > 0, \\ \sigma^2_{\text{pred}}, & \alpha^\star = 0. \end{cases}$$

In the outlier case $\alpha^\star = 0$ we do not actually use σ^2_{eff} for an update, since the step is suppressed.

The implemented update is therefore

$$\boldsymbol{\mu}^{\text{post}} = \boldsymbol{\mu}^{\text{prop}} + \alpha^\star\, \Delta\boldsymbol{\mu}, \tag{5}$$

$$\boldsymbol{\Sigma}^{\text{post}} = \boldsymbol{\Sigma}^{\text{prop}} - \alpha^\star\, \frac{\mathbf{g}\mathbf{g}^\top}{\sigma^2_{\text{pred}}}, \tag{6}$$

or, equivalently, as a standard Kalman update with inflated variance $\sigma^2_{\text{eff}} = \sigma^2_{\text{pred}}/\alpha^\star$ whenever $\alpha^\star > 0$.

2.6 Step 5: Log-likelihood contribution for this interval

When we treat the update as a Kalman step with effective variance $\sigma^2_{\text{eff}} = \phi\, \sigma^2_{\text{pred}}$ ($\alpha^\star > 0$), the per-interval Gaussian log-likelihood is

$$\ell = \log p(\overline{y}^{\text{obs}}_{0\to t} \mid \text{state}) = -\frac{1}{2}\left[\log\bigl(2\pi\sigma^2_{\text{eff}}\bigr) + \frac{\delta^2}{\sigma^2_{\text{eff}}}\right] = -\frac{1}{2}\left[\log\bigl(2\pi\phi\sigma^2_{\text{pred}}\bigr) + \frac{\delta^2}{\phi\,\sigma^2_{\text{pred}}}\right].$$

If the interval is classified as a hard outlier ($\alpha^\star = 0$, update skipped), there are two consistent choices:

- either do not add any contribution to the evidence from this interval;
- or add the likelihood with the original predictive variance σ^2_{pred}, documenting that the observation has no effect on the state but still contributes to the evidence.

This choice is a modelling decision rather than part of the trust-region mechanism itself.

3 Why the covariance stays positive semidefinite

Consider the joint Gaussian model for the state $x \in \mathbb{R}^K$ and the scalar observation y:

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{xx} & g \\ g^\top & \sigma_y^2 \end{pmatrix} \right),$$

with:

- $\Sigma_{xx} = \mathbf{\Sigma}^{\text{prop}}$,
- $g = \text{Cov}(x, y) = \mathbf{g}$,
- $\sigma_y^2 = \sigma^2_{\text{pred}}$.

The conditional covariance of x given y is

$$\Sigma^{\text{post}} = \Sigma_{xx} - \frac{g\, g^\top}{\sigma_y^2},$$

i.e. exactly the rank-1 downdate in (2). This is the Schur complement of σ_y^2 in the joint covariance matrix.

If the joint covariance

$$\begin{pmatrix} \Sigma_{xx} & g \\ g^\top & \sigma_y^2 \end{pmatrix}$$

is positive semidefinite, then its Schur complement Σ^{post} is also positive semidefinite. In particular:

- all eigenvalues of Σ^{post} are nonnegative;
- all diagonal entries $\Sigma^{\text{post}}_{ii} \geq 0$.

At the level of individual components, consider the 2×2 marginal for (x_i, y):

$$\begin{pmatrix} \Sigma_{xx,ii} & g_i \\ g_i & \sigma_y^2 \end{pmatrix}.$$

Positive semidefiniteness of this matrix implies

$$g_i^2 \leq \Sigma_{xx,ii}\, \sigma_y^2,$$

which guarantees

$$\Sigma^{\text{post}}_{ii} = \Sigma_{xx,ii} - \frac{g_i^2}{\sigma_y^2} \geq 0.$$

Therefore, in an exact Gaussian model, the covariance update cannot produce negative variances. Any negative diagonals or small negative eigenvalues encountered in practice must arise from:

- approximate construction of $\Sigma_{xx}, g, \sigma_y^2$ (e.g. tilde operators, discretisation, model mismatch),
- floating-point errors and accumulated roundoff over many updates.

In MacroIR, those are handled as numerical hygiene:

- symmetrising the covariance $\mathbf{\Sigma}^{\text{post}} \leftarrow (\mathbf{\Sigma}^{\text{post}} + \mathbf{\Sigma}^{\text{post}\top})/2$,
- optionally clipping tiny negative eigenvalues in a PSD repair step,
- or enforcing a small floor $v_{\min}$ a posteriori if needed.

These steps are separate from the trust-region logic and do not influence α, which is purely determined by the probability constraints on $\boldsymbol{\mu}$.

4 Detailed deductions

For completeness, we summarise the algebraic steps that justify the formulas above.

4.1 Scaled update and its equivalence to variance inflation

Starting from the standard update:

$$\begin{aligned}\boldsymbol{\mu}^{\text{post}} &= \boldsymbol{\mu}^{\text{prop}} + \frac{\mathbf{g}\,\delta}{\sigma^2_{\text{pred}}},\\ \mathbf{\Sigma}^{\text{post}} &= \mathbf{\Sigma}^{\text{prop}} - \frac{\mathbf{g}\mathbf{g}^\top}{\sigma^2_{\text{pred}}},\end{aligned}$$

introduce $\alpha \in [0,1]$ and define

$$\begin{aligned}\boldsymbol{\mu}^{\text{post}}(\alpha) &= \boldsymbol{\mu}^{\text{prop}} + \alpha\,\frac{\mathbf{g}\,\delta}{\sigma^2_{\text{pred}}},\\ \mathbf{\Sigma}^{\text{post}}(\alpha) &= \mathbf{\Sigma}^{\text{prop}} - \alpha\,\frac{\mathbf{g}\mathbf{g}^\top}{\sigma^2_{\text{pred}}}.\end{aligned}$$

Set

$$\sigma^2_{\text{eff}} := \frac{\sigma^2_{\text{pred}}}{\alpha} \qquad \Longleftrightarrow \qquad \alpha = \frac{\sigma^2_{\text{pred}}}{\sigma^2_{\text{eff}}}.$$

Then

$$\begin{aligned}\boldsymbol{\mu}^{\text{post}}(\alpha) &= \boldsymbol{\mu}^{\text{prop}} + \frac{\mathbf{g}\,\delta}{\sigma^2_{\text{eff}}},\\ \mathbf{\Sigma}^{\text{post}}(\alpha) &= \mathbf{\Sigma}^{\text{prop}} - \frac{\mathbf{g}\mathbf{g}^\top}{\sigma^2_{\text{eff}}}.\end{aligned}$$

This is exactly the standard Kalman update with predictive variance σ^2_{eff}. Writing $\phi = \sigma^2_{\text{eff}}/\sigma^2_{\text{pred}} = 1/\alpha$, we can also say that we inflated the predictive variance by a factor ϕ.

4.2 Probability constraints

For each component,

$$\mu_j^{\text{post}}(\alpha) = \mu_j^{\text{prop}} + \alpha\, \Delta\mu_j, \quad \Delta\mu_j = \frac{g_j\, \delta}{\sigma_{\text{pred}}^2}.$$

We require

$$p_{\min} \le \mu_j^{\text{post}}(\alpha) \le p_{\max}.$$

If $\Delta\mu_j < 0$, the lower bound can be active and yields

$$\alpha \le \alpha_j^{(\text{mean,low})} := \frac{\mu_j^{\text{prop}} - p_{\min}}{|\Delta\mu_j|}.$$

If $\Delta\mu_j > 0$, the upper bound can be active and yields

$$\alpha \le \alpha_j^{(\text{mean,high})} := \frac{p_{\max} - \mu_j^{\text{prop}}}{\Delta\mu_j}.$$

If $\Delta\mu_j = 0$, there is no constraint from component j.

Taking the minimum over all moving components gives α_{mean}, and hence $\alpha^\star = \max(0, \min(1, \alpha_{\text{mean}}))$, as used in the implementation.

4.3 Log-likelihood with inflated variance

With effective variance $\sigma_{\text{eff}}^2 = \phi\sigma_{\text{pred}}^2$, the per-interval Gaussian log-likelihood is

$$\ell = -\frac{1}{2}\left[\log\bigl(2\pi\sigma_{\text{eff}}^2\bigr) + \frac{\delta^2}{\sigma_{\text{eff}}^2}\right] = -\frac{1}{2}\left[\log\bigl(2\pi\phi\sigma_{\text{pred}}^2\bigr) + \frac{\delta^2}{\phi\sigma_{\text{pred}}^2}\right].$$

As ϕ increases, the quadratic term decreases like $1/\phi$, while the normalisation term grows like $\log\phi$. Each time we inflate the variance to protect the state, we automatically pay a corresponding "price" in likelihood.