Derivation of the MacroIR Interval Update
A Pedagogical Tutorial

0. Overview

This note derives, step by step, the formulas collected in the *MacroIR Interval Update: Unified Boundary-State and Tilde Operator Spec* document.

The aim is pedagogical: a new student should be able to recover all of the following from first principles:

- Propagation of the macroscopic mean and covariance: $\boldsymbol{\mu}^{\text{prop}}(t)$, $\boldsymbol{\Sigma}^{\text{prop}}(t)$.
- The boundary-state covariance and its compressed representation via the *tilde* operator.
- The predictive mean and variance of the interval-averaged current.
- The cross-covariance vector $\mathbf{g} = \widetilde{\gamma^\top \Sigma}$.

Throughout we work at the level of *counts* of channels, and then normalise by N_{ch} to define the macroscopic statistics $(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$.

The notation is aligned with the specification document but this tutorial is otherwise self-contained.

1 Basic ensemble model and notation

We consider an ensemble of N_{ch} independent Markov channels, each with K microscopic states.

1.1 State counts and macroscopic statistics

Let

$$\mathbf{N}_0 = \begin{pmatrix} N_{0,1} \\ \vdots \\ N_{0,K} \end{pmatrix} \in \mathbb{N}^K$$

denote the *counts* of channels in each state at time 0.

We define *per-channel* macroscopic statistics as

$$\boldsymbol{\mu}_0 = \frac{1}{N_{\text{ch}}}\,\mathbb{E}[\mathbf{N}_0], \tag{1}$$

$$\boldsymbol{\Sigma}_0 = \frac{1}{N_{\text{ch}}}\,\text{Cov}(\mathbf{N}_0). \tag{2}$$

Thus

$$\mathbb{E}[\mathbf{N}_0] = N_{\text{ch}}\,\boldsymbol{\mu}_0, \qquad \operatorname{Cov}(\mathbf{N}_0) = N_{\text{ch}}\,\boldsymbol{\Sigma}_0.$$

This convention is crucial: all covariance matrices in the spec ($\boldsymbol{\Sigma}_0, \boldsymbol{\Sigma}^{\text{prop}}(t)$, etc.) are *per-channel* covariances, i.e. total covariances divided by N_{ch}.

1.2 Markov dynamics

Each channel follows a continuous-time Markov chain with generator $\mathbf{Q}$ and transition matrix

$$\mathbf{P}(t) = e^{\mathbf{Q}t} \in \mathbb{R}^{K\times K}, \qquad P_{i_0\to i_t}(t) = [\mathbf{P}(t)]_{i_0 i_t}.$$

We adopt the convention that $\mathbf{P}(t)$ is *row-stochastic*:

$$\sum_{i_t} P_{i_0\to i_t}(t) = 1 \quad \text{for each } i_0.$$

1.3 Instantaneous and interval currents

At any time τ, a single channel in state k carries an instantaneous current (or conductance-based current) $y(\tau)$ with

$$\mathbb{E}[y(\tau) \mid \text{state}(\tau) = k] = \gamma_k,$$

plus additional fast fluctuations we will later summarise by an intrinsic interval variance.

Over an interval $[0,t]$, the *interval-averaged* current of a single channel is

$$\overline{y}_{0\to t} = \frac{1}{t}\int_0^t y(\tau)\,\mathrm{d}\tau.$$

The key objects from the spec are the *boundary-conditioned* mean and variance of this interval current:

$$\overline{\Gamma}_{i_0\to i_t} = \mathbb{E}[\overline{y}_{0\to t}\,|\,\text{start in } i_0\text{, end in } i_t]\,, \tag{3}$$

$$\overline{V}_{i_0\to i_t} = \operatorname{Var}(\overline{y}_{0\to t}\,|\,\text{start in } i_0\text{, end in } i_t)\,. \tag{4}$$

We assume these are precomputed from microscopic physics (e.g. via a Bessel filter or similar); we do not derive them here.

We collect them in $K\times K$ matrices

$$\overline{\boldsymbol{\Gamma}} = (\overline{\Gamma}_{i_0\to i_t}), \qquad \overline{\mathbf{V}} = (\overline{V}_{i_0\to i_t}).$$

2 Boundary counts and their distribution

2.1 Definition of boundary counts

For the ensemble, define the *boundary counts*

$$N_{i_0\to i_t} = \text{number of channels that start in } i_0 \text{ and end in } i_t$$

over the interval $[0,t]$.

We collect them into a K^2-dimensional vector

$$\mathbf{B} = \big(N_{i_0\to i_t}\big)_{i_0,i_t}.$$

Conditioned on $\mathbf{N}_0$, the counts in different rows i_0 are independent. For a fixed start state i_0,

$$\big(N_{i_0\to 1},\ldots,N_{i_0\to K}\big)\,\big|\,N_{0,i_0} \sim \text{Multinomial}\big(N_{0,i_0},\,\mathbf{P}_{i_0\cdot}(t)\big),$$

where $\mathbf{P}_{i_0\cdot}(t)$ is the i_0-th row of $\mathbf{P}(t)$.

2.2 Mean of boundary counts

From the multinomial definition,

$$\mathbb{E}[N_{i_0\to i_t} \mid \mathbf{N}_0] = N_{0,i_0}\, P_{i_0\to i_t}(t).$$

Taking expectation over $\mathbf{N}_0$,

$$\begin{aligned}\mathbb{E}[N_{i_0\to i_t}] &= \mathbb{E}\big[\,\mathbb{E}[N_{i_0\to i_t} \mid \mathbf{N}_0]\,\big] = \mathbb{E}[N_{0,i_0}]\, P_{i_0\to i_t}(t) \\ &= N_{\text{ch}}\, \mu_{0,i_0}\, P_{i_0\to i_t}(t).\end{aligned} \tag{5}$$

2.3 Covariance of boundary counts

We compute $\text{Cov}(N_{i_0\to i_t}, N_{j_0\to j_t})$ via the law of total covariance:

$$\text{Cov}(X,Y) = \mathbb{E}[\text{Cov}(X,Y \mid Z)] + \text{Cov}(\mathbb{E}[X \mid Z], \mathbb{E}[Y \mid Z]).$$

Set $X = N_{i_0\to i_t},\ Y = N_{j_0\to j_t},\ Z = \mathbf{N}_0$.

Step 1: conditional covariance.

- If $i_0 \neq j_0$, then different rows of the multinomial construction are conditionally independent, so
$$\text{Cov}(N_{i_0\to i_t}, N_{j_0\to j_t} \mid \mathbf{N}_0) = 0.$$
- If $i_0 = j_0$, we use standard multinomial covariances:
$$\text{Cov}(N_{i_0\to i_t}, N_{i_0\to j_t} \mid \mathbf{N}_0) = N_{0,i_0}\big[P_{i_0\to i_t}(t)\,\delta_{i_t j_t} - P_{i_0\to i_t}(t)P_{i_0\to j_t}(t)\big],$$
where δ_{ij} is the Kronecker delta.

Thus

$$\text{Cov}(N_{i_0\to i_t}, N_{j_0\to j_t} \mid \mathbf{N}_0) = \delta_{i_0 j_0}\, N_{0,i_0}\big[P_{i_0\to i_t}(t)\,\delta_{i_t j_t} - P_{i_0\to i_t}(t)P_{i_0\to j_t}(t)\big]. \tag{6}$$

Taking expectation over $\mathbf{N}_0$ and using $\mathbb{E}[N_{0,i_0}] = N_{\text{ch}}\,\mu_{0,i_0}$,

$$\mathbb{E}\big[\text{Cov}(N_{i_0\to i_t}, N_{j_0\to j_t} \mid \mathbf{N}_0)\big] = \delta_{i_0 j_0}\, N_{\text{ch}}\mu_{0,i_0}\big[P_{i_0\to i_t}(t)\,\delta_{i_t j_t} - P_{i_0\to i_t}(t)P_{i_0\to j_t}(t)\big]. \tag{7}$$

Step 2: covariance of conditional means. We also have

$$\mathbb{E}[N_{i_0\to i_t} \mid \mathbf{N}_0] = N_{0,i_0}P_{i_0\to i_t}(t), \qquad \mathbb{E}[N_{j_0\to j_t} \mid \mathbf{N}_0] = N_{0,j_0}P_{j_0\to j_t}(t).$$

Therefore

$$\begin{aligned}\text{Cov}\big(\mathbb{E}[N_{i_0\to i_t} \mid \mathbf{N}_0], \mathbb{E}[N_{j_0\to j_t} \mid \mathbf{N}_0]\big) &= P_{i_0\to i_t}(t)P_{j_0\to j_t}(t)\,\text{Cov}(N_{0,i_0}, N_{0,j_0}) \\ &= P_{i_0\to i_t}(t)P_{j_0\to j_t}(t)\big(N_{\text{ch}}\,\Sigma_{0,i_0 j_0}\big),\end{aligned} \tag{8}$$

using $\text{Cov}(\mathbf{N}_0) = N_{\text{ch}}\boldsymbol{\Sigma}_0$.

Step 3: combine both terms. Adding (**??**) and (**??**),

$$\begin{aligned}\mathrm{Cov}(N_{i_0\to i_t}, N_{j_0\to j_t}) = \delta_{i_0 j_0}\, N_{\mathrm{ch}}\mu_{0,i_0}\big[P_{i_0\to i_t}(t)\,\delta_{i_t j_t} - P_{i_0\to i_t}(t)P_{i_0\to j_t}(t)\big] \\ + N_{\mathrm{ch}}\Sigma_{0,i_0 j_0} P_{i_0\to i_t}(t)P_{j_0\to j_t}(t).\end{aligned} \tag{9}$$

It is convenient to define the *per-channel boundary covariance*

$$\Sigma^{\mathrm{prior}}_{0\to t,(i_0\to i_t)(j_0\to j_t)} := \frac{1}{N_{\mathrm{ch}}}\,\mathrm{Cov}(N_{i_0\to i_t}, N_{j_0\to j_t}).$$

Then (**??**) becomes

$$\begin{aligned}\Sigma^{\mathrm{prior}}_{0\to t,(i_0\to i_t)(j_0\to j_t)} = P_{i_0\to i_t}(t)\big[\Sigma_{0,i_0 j_0} - \delta_{i_0 j_0}\mu_{0,i_0}\big]P_{j_0\to j_t}(t) \\ + \delta_{i_0 j_0}\delta_{i_t j_t}\mu_{0,i_0}P_{i_0\to i_t}(t).\end{aligned} \tag{10}$$

This is exactly the boundary covariance stated (componentwise) in the specification document.

3 Propagation of mean and covariance to time tt

We next derive the macroscopic mean $\boldsymbol{\mu}^{\mathrm{prop}}(t)$ and covariance $\boldsymbol{\Sigma}^{\mathrm{prop}}(t)$ at the end of the interval.

3.1 Mean propagation

Let $\mathbf{N}(t)$ denote the state counts at time t, with components $N_t(a)$. Conditional on $\mathbf{N}_0$,

$$\mathbb{E}[N_t(a) \mid \mathbf{N}_0] = \sum_{i_0} N_{0,i_0}P_{i_0\to a}(t) = \big(\mathbf{P}(t)^\top \mathbf{N}_0\big)_a.$$

Taking expectation,

$$\mathbb{E}[\mathbf{N}(t)] = \mathbf{P}(t)^\top\,\mathbb{E}[\mathbf{N}_0] = \mathbf{P}(t)^\top\,\big(N_{\mathrm{ch}}\boldsymbol{\mu}_0\big). \tag{11}$$

Dividing by N_{ch}, we obtain

$$\boldsymbol{\mu}^{\mathrm{prop}}(t) := \frac{1}{N_{\mathrm{ch}}}\mathbb{E}[\mathbf{N}(t)] = \mathbf{P}(t)^\top\boldsymbol{\mu}_0.$$

This is the mean-propagation formula

$$\boxed{\boldsymbol{\mu}^{\mathrm{prop}}(t) = \mathbf{P}(t)^\top\boldsymbol{\mu}_0.}$$

3.2 Covariance propagation via boundary counts

We now derive the covariance of $\mathbf{N}(t)$ using the boundary covariance (**??**). By definition,

$$N_t(a) = \sum_{i_0} N_{i_0\to a},$$

so

$$\mathrm{Cov}(N_t(a), N_t(b)) = \sum_{i_0,j_0}\mathrm{Cov}(N_{i_0\to a}, N_{j_0\to b}).$$

Dividing by N_{ch} and substituting $\Sigma^{\mathrm{prior}}_{0\to t} = \mathrm{Cov}(\mathbf{B})/N_{\mathrm{ch}}$ from (**??**),

$$\begin{aligned}\Sigma^{\mathrm{prop}}_{ab}(t) &:= \frac{1}{N_{\mathrm{ch}}}\,\mathrm{Cov}(N_t(a), N_t(b)) \\ &= \sum_{i_0, j_0} \Sigma^{\mathrm{prior}}_{0\to t,(i_0\to a)(j_0\to b)}. \end{aligned} \tag{12}$$

Insert (**??**):

$$\begin{aligned}\Sigma^{\mathrm{prop}}_{ab}(t) = &\sum_{i_0,j_0} P_{i_0\to a}(t)\big[\Sigma_{0,i_0j_0} - \delta_{i_0j_0}\mu_{0,i_0}\big]P_{j_0\to b}(t) \\ &+ \sum_{i_0,j_0} \delta_{i_0j_0}\delta_{ab}\mu_{0,i_0}P_{i_0\to a}(t). \end{aligned} \tag{13}$$

The second sum simplifies immediately:

$$\sum_{i_0,j_0} \delta_{i_0j_0}\delta_{ab}\mu_{0,i_0}P_{i_0\to a}(t) = \delta_{ab}\sum_{i_0}\mu_{0,i_0}P_{i_0\to a}(t) = \delta_{ab}\,\mu^{\mathrm{prop}}_a(t),$$

since $\boldsymbol{\mu}^{\mathrm{prop}}(t) = \mathbf{P}(t)^\top\boldsymbol{\mu}_0$.

The first sum can be recognised as a matrix product:

$$\begin{aligned}&\sum_{i_0,j_0} P_{i_0\to a}(t)\big[\Sigma_{0,i_0j_0} - \delta_{i_0j_0}\mu_{0,i_0}\big]P_{j_0\to b}(t) \\ &\quad= \big[\mathbf{P}(t)^\top\big(\boldsymbol{\Sigma}_0 - \mathrm{diag}(\boldsymbol{\mu}_0)\big)\mathbf{P}(t)\big]_{ab}. \end{aligned} \tag{14}$$

Thus, in matrix form,

$$\boxed{\boldsymbol{\Sigma}^{\mathrm{prop}}(t) = \mathbf{P}(t)^\top\big(\boldsymbol{\Sigma}_0 - \mathrm{diag}(\boldsymbol{\mu}_0)\big)\mathbf{P}(t) + \mathrm{diag}\big(\boldsymbol{\mu}^{\mathrm{prop}}(t)\big).}$$

This is the covariance propagation formula used in the spec.

4 Start-conditioned interval current statistics

We now connect the boundary counts to the interval-averaged current.

4.1 Boundary-conditioned single-channel current

For a single channel with boundary state (i_0, i_t), the interval current $\overline{y}_{0\to t}$ has:

- mean $\overline{\Gamma}_{i_0\to i_t}$ (from (**??**)),
- intrinsic variance $\overline{V}_{i_0\to i_t}$ (from (**??**)).

We do *not* need to know the full path distribution; all path dependence is summarised in these boundary-conditioned statistics.

4.2 Start-conditioned mean and variance

Define the start-conditioned mean interval current:

$$(\overline{\gamma}_0)_{i_0} := \mathbb{E}[\overline{y}_{0\to t} \,|\, \text{start in } i_0]\,.$$

Conditioning on the end state i_t,

$$(\overline{\gamma}_0)_{i_0} = \sum_{i_t} \mathbb{E}[\overline{y}_{0\to t} \,|\, i_0, i_t]\, \mathbb{P}(i_t \mid i_0) = \sum_{i_t} \overline{\Gamma}_{i_0\to i_t} P_{i_0\to i_t}(t). \tag{15}$$

In matrix form, if we define

$$\mathbf{G} := \overline{\mathbf{\Gamma}} \circ \mathbf{P}(t), \qquad G_{i_0 i_t} = \overline{\Gamma}_{i_0\to i_t} P_{i_0\to i_t}(t),$$

then

$$\overline{\boldsymbol{\gamma}}_0 = \mathbf{G}\mathbf{1},$$

where $\mathbf{1}$ is the all-ones column vector.

Similarly, the start-conditioned intrinsic variance is

$$(\sigma^2_{\overline{\gamma}_0})_{i_0} := \mathrm{Var}(\overline{y}_{0\to t} \,|\, \text{start in } i_0) = \sum_{i_t} \overline{V}_{i_0\to i_t} P_{i_0\to i_t}(t).$$

Defining

$$\mathbf{V} := \overline{\mathbf{V}} \circ \mathbf{P}(t), \qquad V_{i_0 i_t} = \overline{V}_{i_0\to i_t} P_{i_0\to i_t}(t),$$

we can write

$$\boldsymbol{\sigma}^2_{\overline{\gamma}_0} = \mathbf{V}\mathbf{1}.$$

These quantities appear in the predictive mean and variance of the ensemble current.

5 Predictive mean of the interval current

Let $\overline{y}^{\text{tot}}_{0\to t}$ denote the total interval-averaged current of the ensemble (sum over all channels). Ignoring instrument noise for the moment, we can write

$$\overline{y}^{\text{tot}}_{0\to t} \approx \sum_{i_0, i_t} \overline{\Gamma}_{i_0\to i_t} N_{i_0\to i_t}.$$

Taking expectation and using (**??**),

$$\begin{aligned}
\mathbb{E}[\overline{y}^{\text{tot}}_{0\to t}] &= \sum_{i_0, i_t} \overline{\Gamma}_{i_0\to i_t} \mathbb{E}[N_{i_0\to i_t}] \\
&= \sum_{i_0, i_t} \overline{\Gamma}_{i_0\to i_t} N_{\text{ch}} \mu_{0,i_0} P_{i_0\to i_t}(t) \\
&= N_{\text{ch}} \sum_{i_0} \mu_{0,i_0} \left(\sum_{i_t} \overline{\Gamma}_{i_0\to i_t} P_{i_0\to i_t}(t) \right) \\
&= N_{\text{ch}}\, \boldsymbol{\mu}_0^\top \overline{\boldsymbol{\gamma}}_0.
\end{aligned} \tag{16}$$

The spec writes this as

$$\boxed{\overline{y}^{\text{pred}}_{0\to t} = N_{\text{ch}}\, \boldsymbol{\mu}_0^\top \overline{\boldsymbol{\gamma}}_0,}$$

where $\overline{y}^{\text{pred}}_{0\to t}$ is the predictive mean total current.

Instrument/binning noise is added later at the variance level, not to the mean.

6 Predictive variance and the bilinear tilde operator

We now derive the predictive variance of the ensemble current and the bilinear tilde operator $\widetilde{\gamma^\top \Sigma \gamma}$.

6.1 Variance decomposition

Let η denote the measurement noise over the interval $[0, t]$, with variance

$$\mathrm{Var}(\eta) = \epsilon^2_{0\to t} = \frac{\epsilon^2}{t} + \nu^2.$$

Write the total current as

$$\overline{y}^{\mathrm{tot}}_{0\to t} = Y_{\mathrm{macro}} + Y_{\mathrm{intr}} + \eta$$

where

- Y_{macro} captures the variability due to uncertainty in the *occupancy distribution* at the start of the interval, and the randomness of which boundary states are realised (through $\mathbf{N}_0$ and $\mathbf{B}$).
- Y_{intr} captures the *intrinsic* interval-current variability $\overline{V}_{i_0\to i_t}$ for each realised boundary state.
- η is the instrument/binning noise.

Under the usual assumptions of independence (instrument noise independent of channel dynamics; intrinsic fluctuations independent across channels conditional on boundary states), we obtain

$$\mathrm{Var}(\overline{y}^{\mathrm{tot}}_{0\to t}) = \mathrm{Var}(Y_{\mathrm{macro}}) + \mathrm{Var}(Y_{\mathrm{intr}}) + \epsilon^2_{0\to t}.$$

We now compute the first two terms separately.

6.2 Macro-variance via boundary covariance

The macro part is the component that would remain if each boundary state (i_0, i_t) contributed deterministically $\overline{\Gamma}_{i_0\to i_t}$ given $N_{i_0\to i_t}$. That is,

$$Y_{\mathrm{macro}} = \sum_{i_0, i_t} \overline{\Gamma}_{i_0\to i_t} N_{i_0\to i_t}.$$

Define the flattened weight vector

$$w_{(i_0\to i_t)} := \overline{\Gamma}_{i_0\to i_t}.$$

Then

$$Y_{\mathrm{macro}} = \mathbf{w}^\top \mathbf{B}.$$

Its variance is

$$\mathrm{Var}(Y_{\mathrm{macro}}) = \mathbf{w}^\top \, \mathrm{Cov}(\mathbf{B}) \mathbf{w} = N_{\mathrm{ch}} \, \mathbf{w}^\top \mathbf{\Sigma}^{\mathrm{prior}}_{0\to t} \mathbf{w},$$

since $\mathrm{Cov}(\mathbf{B})/N_{\mathrm{ch}} = \mathbf{\Sigma}^{\mathrm{prior}}_{0\to t}$.

This suggests defining the *bilinear tilde operator* for the interval current as

$$\widetilde{\gamma^\top \Sigma \gamma} := \mathbf{w}^\top \mathbf{\Sigma}^{\mathrm{prior}}_{0\to t} \mathbf{w},$$

so that

$$\mathrm{Var}(Y_{\mathrm{macro}}) = N_{\mathrm{ch}}\,\widetilde{\gamma^\top \Sigma \gamma}.$$

We now show that this definition leads to the compact formula

$$\widetilde{\gamma^\top \Sigma \gamma} = \overline{\boldsymbol{\gamma}}_0^\top \big(\boldsymbol{\Sigma}_0 - \mathrm{diag}(\boldsymbol{\mu}_0)\big)\overline{\boldsymbol{\gamma}}_0 + \boldsymbol{\mu}_0^\top \big[\mathbf{H}\mathbf{1}\big],$$

with a suitable matrix $\mathbf{H}$.

6.3 Explicit evaluation of $\widetilde{\gamma^\top \Sigma \gamma}$tilde(gammaTSigmagamma)

Recall (**??**):

$$\Sigma^{\mathrm{prior}}_{0\to t,(i_0\to i_t)(j_0\to j_t)} = P_{i_0\to i_t}(t)\big[\Sigma_{0,i_0 j_0} - \delta_{i_0 j_0}\mu_{0,i_0}\big]P_{j_0\to j_t}(t) + \delta_{i_0 j_0}\delta_{i_t j_t}\mu_{0,i_0}P_{i_0\to i_t}(t).$$

Then

$$\begin{aligned}\widetilde{\gamma^\top \Sigma \gamma} &= \sum_{i_0,i_t}\sum_{j_0,j_t}\overline{\Gamma}_{i_0\to i_t}\Sigma^{\mathrm{prior}}_{0\to t,(i_0\to i_t)(j_0\to j_t)}\overline{\Gamma}_{j_0\to j_t} \\ &= T_1 + T_2, \end{aligned} \tag{17}$$

where

$$\begin{aligned} T_1 &= \sum_{i_0,i_t}\sum_{j_0,j_t}\overline{\Gamma}_{i_0\to i_t}P_{i_0\to i_t}(t)\big[\Sigma_{0,i_0 j_0} - \delta_{i_0 j_0}\mu_{0,i_0}\big]P_{j_0\to j_t}(t)\overline{\Gamma}_{j_0\to j_t}, \\ T_2 &= \sum_{i_0,i_t}\sum_{j_0,j_t}\overline{\Gamma}_{i_0\to i_t}\delta_{i_0 j_0}\delta_{i_t j_t}\mu_{0,i_0}P_{i_0\to i_t}(t)\overline{\Gamma}_{j_0\to j_t}. \end{aligned}$$

Term T_1. Introduce the start-conditioned mean interval current

$$(\overline{\gamma}_0)_{i_0} = \sum_{i_t}\overline{\Gamma}_{i_0\to i_t}P_{i_0\to i_t}(t),$$

so that

$$\overline{\gamma}_{0,i_0} = \sum_{i_t}\overline{\Gamma}_{i_0\to i_t}P_{i_0\to i_t}(t).$$

Then

$$\begin{aligned} T_1 &= \sum_{i_0,j_0}\left(\sum_{i_t}\overline{\Gamma}_{i_0\to i_t}P_{i_0\to i_t}(t)\right)\big[\Sigma_{0,i_0 j_0} - \delta_{i_0 j_0}\mu_{0,i_0}\big]\left(\sum_{j_t}\overline{\Gamma}_{j_0\to j_t}P_{j_0\to j_t}(t)\right) \\ &= \sum_{i_0,j_0}\overline{\gamma}_{0,i_0}\big[\Sigma_{0,i_0 j_0} - \delta_{i_0 j_0}\mu_{0,i_0}\big]\overline{\gamma}_{0,j_0} \\ &= \overline{\boldsymbol{\gamma}}_0^\top\big(\boldsymbol{\Sigma}_0 - \mathrm{diag}(\boldsymbol{\mu}_0)\big)\overline{\boldsymbol{\gamma}}_0. \end{aligned}$$

Term T_2. Here the Kronecker deltas collapse the sums:

$$\begin{aligned} T_2 &= \sum_{i_0, i_t} \overline{\Gamma}_{i_0 \to i_t} \mu_{0,i_0} P_{i_0 \to i_t}(t) \overline{\Gamma}_{i_0 \to i_t} \\ &= \sum_{i_0} \mu_{0,i_0} \sum_{i_t} \overline{\Gamma}^2_{i_0 \to i_t} P_{i_0 \to i_t}(t). \end{aligned}$$

Define the matrix

$$H_{i_0 i_t} := \overline{\Gamma}^2_{i_0 \to i_t} P_{i_0 \to i_t}(t), \qquad \mathbf{H} = (H_{i_0 i_t}),$$

and note that

$$(\mathbf{H1})_{i_0} = \sum_{i_t} H_{i_0 i_t} = \sum_{i_t} \overline{\Gamma}^2_{i_0 \to i_t} P_{i_0 \to i_t}(t).$$

Then

$$T_2 = \boldsymbol{\mu}_0^\top (\mathbf{H1}).$$

Combine both terms. From (**??**) we obtain

$$\boxed{\widetilde{\gamma^\top \Sigma \gamma} = \overline{\boldsymbol{\gamma}}_0^\top \big(\boldsymbol{\Sigma}_0 - \operatorname{diag}(\boldsymbol{\mu}_0)\big) \overline{\boldsymbol{\gamma}}_0 + \boldsymbol{\mu}_0^\top [\mathbf{H1}].}$$

6.4 Intrinsic interval variance

The intrinsic contribution Y_{intr} comes from the random deviations of each channel's interval current around its boundary-conditioned mean $\overline{\Gamma}_{i_0 \to i_t}$. For a single channel starting in state i_0, the intrinsic variance is

$$\operatorname{Var}(\overline{y}_{0 \to t} \mid \text{start in } i_0) = \sum_{i_t} \overline{V}_{i_0 \to i_t} P_{i_0 \to i_t}(t) = (\sigma^2_{\overline{\gamma}_0})_{i_0}.$$

For an ensemble of N_{ch} independent channels, the intrinsic variance contribution to the total current is

$$\operatorname{Var}(Y_{\text{intr}}) = N_{\text{ch}}\, \mathbb{E}[\operatorname{Var}(\overline{y}_{0 \to t} \mid \text{start state})] = N_{\text{ch}} \sum_{i_0} \mu_{0,i_0} (\sigma^2_{\overline{\gamma}_0})_{i_0}.$$

In vector form,

$$\operatorname{Var}(Y_{\text{intr}}) = N_{\text{ch}}\, \boldsymbol{\mu}_0^\top \boldsymbol{\sigma}^2_{\overline{\gamma}_0}.$$

6.5 Full predictive variance

Putting macro, intrinsic, and measurement components together,

$$\operatorname{Var}(\overline{y}^{\text{tot}}_{0 \to t}) = \epsilon^2_{0 \to t} + N_{\text{ch}}\, \widetilde{\gamma^\top \Sigma \gamma} + N_{\text{ch}}\, \boldsymbol{\mu}_0^\top \boldsymbol{\sigma}^2_{\overline{\gamma}_0}.$$

This is the predictive variance formula quoted in the spec:

$$\boxed{\sigma^2_{\overline{y}^{\text{pred}}} = \epsilon^2_{0 \to t} + N_{\text{ch}}\, \widetilde{\gamma^\top \Sigma \gamma} + N_{\text{ch}}\, \boldsymbol{\mu}_0^\top \boldsymbol{\sigma}^2_{\overline{\gamma}_0}.}$$

7 Vector tilde and the cross-covariance gg

We finally derive the cross-covariance vector $\mathbf{g} = \widetilde{\gamma^\top \Sigma}$ between the state counts at time t and the ensemble interval current.

7.1 Definition and normalisation

We are interested in the per-channel cross-covariance

$$\mathbf{g} := \frac{1}{N_{\text{ch}}}\,\mathrm{Cov}\big(\mathbf{N}(t),\, \overline{y}^{\text{tot}}_{0\to t}\big) \in \mathbb{R}^K.$$

Writing $\overline{y}^{\text{tot}}_{0\to t} = \mathbf{w}^\top\mathbf{B} + (\text{intrinsic} + \text{noise})$ and noting that intrinsic and instrument noise are independent of the counts, we only need the covariance between $\mathbf{N}(t)$ and $\mathbf{w}^\top\mathbf{B}$. Thus

$$\mathbf{g} = \frac{1}{N_{\text{ch}}}\,\mathrm{Cov}\big(\mathbf{N}(t),\, \mathbf{w}^\top\mathbf{B}\big).$$

7.2 Expressing $\mathbf{N}(t)\mathbf{N(t)}$ in terms of boundary counts

Recall

$$N_t(a) = \sum_{i_0} N_{i_0\to a}.$$

Thus

$$\mathrm{Cov}(N_t(a), \mathbf{w}^\top\mathbf{B}) = \sum_{i_0}\sum_{j_0,j_t} \mathrm{Cov}(N_{i_0\to a}, N_{j_0\to j_t})\overline{\Gamma}_{j_0\to j_t}.$$

Dividing by N_{ch} and using $\mathrm{Cov}(N_{i_0\to a}, N_{j_0\to j_t})/N_{\text{ch}} = \Sigma^{\text{prior}}_{0\to t,(i_0\to a)(j_0\to j_t)}$, we have

$$g_a = \sum_{i_0,j_0,j_t} \Sigma^{\text{prior}}_{0\to t,(i_0\to a)(j_0\to j_t)}\overline{\Gamma}_{j_0\to j_t}.$$

Substituting (**??**),

$$\begin{aligned} g_a = &\sum_{i_0,j_0,j_t} P_{i_0\to a}(t)\big[\Sigma_{0,i_0j_0} - \delta_{i_0j_0}\mu_{0,i_0}\big]P_{j_0\to j_t}(t)\overline{\Gamma}_{j_0\to j_t} \\ &+ \sum_{i_0,j_0,j_t} \delta_{i_0j_0}\delta_{aj_t}\mu_{0,i_0}P_{i_0\to a}(t)\overline{\Gamma}_{j_0\to j_t}. \end{aligned}$$

First term. Introduce $\overline{\boldsymbol{\gamma}}_0$ as before:

$$\overline{\gamma}_{0,j_0} = \sum_{j_t} \overline{\Gamma}_{j_0\to j_t}P_{j_0\to j_t}(t).$$

Then

$$\begin{aligned} (\text{first term}) &= \sum_{i_0,j_0} P_{i_0\to a}(t)\big[\Sigma_{0,i_0j_0} - \delta_{i_0j_0}\mu_{0,i_0}\big]\overline{\gamma}_{0,j_0} \\ &= \big[\mathbf{P}(t)^\top\big(\boldsymbol{\Sigma}_0 - \mathrm{diag}(\boldsymbol{\mu}_0)\big)\overline{\boldsymbol{\gamma}}_0\big]_a. \end{aligned}$$

Second term. Here

$$\sum_{i_0,j_0,j_t} \delta_{i_0j_0}\delta_{aj_t}\mu_{0,i_0}P_{i_0\to a}(t)\overline{\Gamma}_{j_0\to j_t} = \sum_{i_0} \mu_{0,i_0}P_{i_0\to a}(t)\overline{\Gamma}_{i_0\to a}.$$

This is the a-th component of

$$\mathbf{G}^\top\boldsymbol{\mu}_0,$$

since $(\mathbf{G}^\top\boldsymbol{\mu}_0)_a = \sum_{i_0} G_{i_0a}\mu_{0,i_0}$ with $G_{i_0a} = \overline{\Gamma}_{i_0\to a}P_{i_0\to a}(t)$.

Combine both terms. Thus

$$\boxed{\mathbf{g} = \widetilde{\gamma^\top \Sigma} = \mathbf{P}(t)^\top \big(\mathbf{\Sigma}_0 - \mathrm{diag}(\boldsymbol{\mu}_0)\big)\overline{\gamma}_0 + \mathbf{G}^\top \boldsymbol{\mu}_0.}$$

This is precisely the vector tilde formula in the spec.

8 Kalman-style measurement update (sketch)

Once we have

- the propagated state $\boldsymbol{\mu}^{\mathrm{prop}}(t), \mathbf{\Sigma}^{\mathrm{prop}}(t)$,
- the predictive current mean and variance $\overline{y}^{\mathrm{pred}}_{0\to t}, \sigma^2_{\overline{y}^{\mathrm{pred}}}$,
- the cross-covariance $\mathbf{g}$,

the Gaussian update is exactly the standard scalar Kalman filter in a K-dimensional state space.

The joint (approximate) Gaussian of $(\mathbf{N}(t)/N_{\mathrm{ch}}, \overline{y}^{\mathrm{tot}}_{0\to t})$ has mean $\big(\boldsymbol{\mu}^{\mathrm{prop}}(t), \overline{y}^{\mathrm{pred}}_{0\to t}\big)$ and covariance

$$\begin{pmatrix} \mathbf{\Sigma}^{\mathrm{prop}}(t) & \mathbf{g} \\ \mathbf{g}^\top & \sigma^2_{\overline{y}^{\mathrm{pred}}} \end{pmatrix}.$$

Conditioning on the observed value $\overline{y}^{\mathrm{obs}}_{0\to t}$ yields

$$\boldsymbol{\mu}^{\mathrm{post}}(t) = \boldsymbol{\mu}^{\mathrm{prop}}(t) + \frac{\mathbf{g}}{\sigma^2_{\overline{y}^{\mathrm{pred}}}}\,\delta,$$

$$\mathbf{\Sigma}^{\mathrm{post}}(t) = \mathbf{\Sigma}^{\mathrm{prop}}(t) - \frac{\mathbf{g}\mathbf{g}^\top}{\sigma^2_{\overline{y}^{\mathrm{pred}}}},$$

where $\delta = \overline{y}^{\mathrm{obs}}_{0\to t} - \overline{y}^{\mathrm{pred}}_{0\to t}$. These are the update formulas in the spec.

9 How to generalise and extend the theory

For further development, a student can use the same structural pattern:

1. **Specify new boundary-weighted observables.** For any observable that can be expressed as a weighted sum of boundary counts, e.g. higher-order moments of the current or other measurement channels, identify the appropriate boundary weight matrices $\overline{\mathbf{U}}, \overline{\mathbf{W}}$.
2. **Reuse the boundary covariance.** The formula (**??**) for the per-channel boundary covariance is the core building block. Any quadratic form in boundary counts reduces to combinations of:
$$\mathbf{P}(t)^\top \mathbf{\Sigma}_0 \mathbf{P}(t), \qquad \mathrm{diag}(\boldsymbol{\mu}_0), \qquad \mathrm{diag}(\boldsymbol{\mu}^{\mathrm{prop}}(t)).$$
3. **Define general bilinear tilde operators.** For two boundary-weight matrices $\overline{\mathbf{U}}, \overline{\mathbf{W}}$, one can define $\widetilde{u^\top \Sigma w}$ exactly as in Section 6, replacing $\overline{\mathbf{\Gamma}}$ with $\overline{\mathbf{U}}$ and $\overline{\mathbf{W}}$, and obtain general cross-variance formulas.
4. **Derive vector tildes for new observables.** If a new observable couples to the state at time t, compute its cross-covariance with $\mathbf{N}(t)$ exactly as in Section 7.

In all cases the structure is the same:

- lift from macroscopic state space to boundary space,
- modulate with the transition matrix $\mathbf{P}(t)$ and the prior covariance $\mathbf{\Sigma}_0$,
- collapse back to scalar or vector via the chosen weights.

The tilde operator is simply a compact notation for this “lift–modulate–collapse” computation.