MacroTaylor: Rank-2 Laplace Approximation for the Posterior over p

This note collects the derivations for a *channel-aware* Laplace approximation of the posterior over the macroscopic occupancy vector p of an ensemble of Markov channels, based on a single interval-averaged current measurement.

The **main scheme** is the *rank-2* Laplace approximation based on the *exact* Gaussian likelihood (including the $\frac{1}{2}\log V$ term). As a *pedagogical limit* and for connection to EKF-style approximations, we also present a *rank-1* "quasi-likelihood" scheme in which the $\log V$ term is dropped.

The goals are:

- to define the exact Gaussian likelihood and prior in the macroscopic coordinates p;
- to derive the exact gradient and Hessian of the negative log-posterior, and show that the Hessian is a rank-2 update of the prior precision;
- to obtain a closed-form expression for the *Laplace covariance* via a rank-2 Woodbury inversion;
- to derive a mean-update (MAP) formula in which all explicit Σ_p^{-1} terms are eliminated;
- to show how the rank-1 (no $\log V$) scheme appears as a limiting case and how it connects to quasi-likelihood/EKF practice.

Coordinates, simplex and singular covariance. The macroscopic occupancy vector p lives on the simplex

$$\mathcal{S} = \Big\{ p \in \mathbb{R}^{1\times d} : p_i \geq 0,\ \sum_i p_i = 1 \Big\},$$

a $(d-1)$-dimensional manifold embedded in $\mathbb{R}^{1\times d}$. The prior covariance Σ_p is therefore singular as a $d \times d$ matrix in ambient coordinates. Formally, one may view Σ_p^{-1} as the inverse on the intrinsic $(d-1)$-dimensional subspace, or as a pseudo-inverse. In what follows:

- we freely use Σ_p^{-1} *algebraically* when applying Sherman–Morrison–Woodbury;
- all *implementable* expressions for the mean and covariance are written purely in terms of Σ_p (no explicit inverse);
- the log-determinant of Σ_p (or its pseudo-determinant) is a constant in p and does not affect the MAP.

0. Setup and Notation

We consider:

- Macroscopic occupancy vector

$$p \in \mathbb{R}^{1\times d}$$

  (row vector of state probabilities).

- Per-state conductance (or current) vector

$$\gamma \in \mathbb{R}^{d\times 1}$$

  (column vector).

- Per-state intrinsic-variance vector

$$\sigma^2 \in \mathbb{R}^{d\times 1}.$$

The prior over p (restricted to the simplex) is Gaussian:

$$p \sim \mathcal{N}(\mu, \Sigma_p),$$

where $\mu \in \mathbb{R}^{1\times d}$ is the prior mean and $\Sigma_p \in \mathbb{R}^{d\times d}$ is the prior covariance. We often use the prior precision

$$\Lambda_p = N_{\mathrm{ch}}\Sigma_p^{-1},$$

with N_{ch} the number of channels.

Over an interval we observe a scalar averaged current y which, for a given p, we model as Gaussian:

$$y \mid p \sim \mathcal{N}\big(\mu_y(p), V(p)\big),$$

with

$$\mu_y(p) = N_{\mathrm{ch}}(p \cdot \gamma), \qquad V(p) = \epsilon^2 + N_{\mathrm{ch}}(p \cdot \sigma^2),$$

where ϵ^2 is the extrinsic (instrument) variance and $N_{\mathrm{ch}}(p \cdot \sigma^2)$ is the intrinsic interval variance from channel stochasticity.

Define the residual

$$\delta(p) = y - \mu_y(p) = y - N_{\mathrm{ch}}(p \cdot \gamma).$$

We will also use the *tilted* vector

$$v(p) = \gamma + \frac{\delta(p)}{V(p)}\,\sigma^2 \in \mathbb{R}^{d\times 1},$$

which will factor the Hessian.

1. Exact Gaussian Likelihood and Prior

We start from the standard Gaussian forms and derive the exact negative log-posterior. We then introduce an "energy" that is twice the negative log-posterior, which simplifies later algebra.

1.1 Exact likelihood

Given p, the likelihood is

$$p(y \mid p) = \frac{1}{\sqrt{2\pi V(p)}} \exp\left(-\frac{1}{2}\frac{\delta(p)^2}{V(p)}\right).$$

The log-likelihood is

$$\log p(y \mid p) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log V(p) - \frac{1}{2}\frac{\delta(p)^2}{V(p)}.$$

Up to an additive constant $-\frac{1}{2}\log(2\pi)$, the *negative* log-likelihood is

$$F_{\mathrm{like}}(p) = \frac{1}{2}\log V(p) + \frac{1}{2}\frac{\delta(p)^2}{V(p)} \quad (+\ \mathrm{const}).$$

1.2 Exact prior

The prior density is

$$p(p) \propto \exp\left(-\frac{1}{2}(p-\mu)\,\Lambda_p\,(p-\mu)^\top \right),$$

with $\Lambda_p = N_{\text{ch}}\Sigma_p^{-1}$. The corresponding negative log-prior (up to a constant) is

$$F_{\text{prior}}(p) = \frac{1}{2}(p-\mu)\,\Lambda_p\,(p-\mu)^\top = \frac{1}{2}(p-\mu)\big(N_{\text{ch}}\Sigma_p^{-1}\big)(p-\mu)^\top \quad (+\text{ const}).$$

The log-determinant term $\frac{1}{2}\log\det\Sigma_p$ is a constant in p and therefore does not affect the MAP; we drop it throughout. However, we should note that the posterior covariance does depend on p, so, this is actually a simplification to make the calculations maneagable.

1.3 Exact negative log-posterior and energy

The (unnormalised) posterior is

$$p(p \mid y) \propto p(y \mid p)\,p(p),$$

so the exact negative log-posterior (up to a constant) is

$$F_{\text{exact}}(p) = F_{\text{like}}(p) + F_{\text{prior}}(p).$$

Explicitly:

$$F_{\text{exact}}(p) = \frac{1}{2}\log V(p) + \frac{1}{2}\frac{\delta(p)^2}{V(p)} + \frac{1}{2}(p-\mu)\big(N_{\text{ch}}\Sigma_p^{-1}\big)(p-\mu)^\top \quad (+\text{ const}).$$

We will minimise $F_{\text{exact}}(p)$ to obtain the exact MAP. For algebraic convenience, define the associated "energy"

$$E_{\text{exact}}(p) := 2F_{\text{exact}}(p).$$

Up to an additive constant,

$$E_{\text{exact}}(p) = \underbrace{\log V(p) + \frac{\delta(p)^2}{V(p)}}_{L(p)+M(p)} + \underbrace{(p-\mu)\big(N_{\text{ch}}\Sigma_p^{-1}\big)(p-\mu)^\top}_{P(p)}.$$

Thus:

$$E_{\text{exact}}(p) = L(p) + M(p) + P(p).$$

Multiplying by a positive constant does not change the MAP; moreover, if we write $\mathbf{H}(p) = \nabla^2 E_{\text{exact}}(p)$, then the Laplace covariance of the posterior is

$$\Sigma_{\text{Lap}} \approx \left[\nabla^2 F_{\text{exact}}(p^\star)\right]^{-1} = 2\,\mathbf{H}(p^\star)^{-1},$$

where $p^\star$ is the MAP.

2. Gradient and Hessian of the Exact Energy

We now derive the gradient and Hessian of $E_{\text{exact}}(p)$. We keep track of three contributions:

$$L(p) = \log V(p), \quad M(p) = \frac{\delta(p)^2}{V(p)}, \quad P(p) = (p-\mu)\big(N_{\text{ch}}\Sigma_p^{-1}\big)(p-\mu)^\top.$$

Here p is a row vector, γ and σ^2 are column vectors.
Recall:

$$\delta(p) = y - N_{\text{ch}}(p\cdot\gamma), \qquad V(p) = \epsilon^2 + N_{\text{ch}}(p\cdot\sigma^2).$$

Then

$$\nabla\delta(p) = -N_{\text{ch}}\gamma^\top, \qquad \nabla V(p) = N_{\text{ch}}(\sigma^2)^\top,$$

both row vectors, and $\nabla^2\delta = 0$, $\nabla^2 V = 0$.

2.1 Prior gradient and Hessian

Write

$$P(p) = (p-\mu)\Lambda_p(p-\mu)^\top, \qquad \Lambda_p = N_{\text{ch}}\Sigma_p^{-1}.$$

Since Λ_p is symmetric,

$$\nabla P(p) = 2(p-\mu)\Lambda_p = 2N_{\text{ch}}(p-\mu)\Sigma_p^{-1},$$

and

$$\nabla^2 P(p) = 2\Lambda_p = 2N_{\text{ch}}\Sigma_p^{-1}.$$

2.2 Likelihood gradient: $M(p)$ and $L(p)$

For

$$M(p) = \frac{\delta(p)^2}{V(p)},$$

the gradient (using the quotient rule) is

$$\nabla M(p) = \frac{2\delta}{V}\,\nabla\delta - \frac{\delta^2}{V^2}\,\nabla V,$$

so

$$\nabla M(p) = \frac{2\delta}{V}(-N_{\rm ch}\gamma^\top) - \frac{\delta^2}{V^2}N_{\rm ch}(\sigma^2)^\top = -N_{\rm ch}\left[\frac{2\delta}{V}\,\gamma^\top + \frac{\delta^2}{V^2}\,(\sigma^2)^\top\right].$$

For

$$L(p) = \log V(p),$$

$$\nabla L(p) = \frac{1}{V}\,\nabla V = \frac{N_{\rm ch}}{V}\,(\sigma^2)^\top.$$

Thus the *exact* gradient of $E_{\rm exact} = L + M + P$ is

$$\begin{aligned}\mathbf{g}_{\rm exact}(p) &= \nabla E_{\rm exact}(p) = \nabla P(p) + \nabla M(p) + \nabla L(p)\\ &= 2N_{\rm ch}(p-\mu)\Sigma_p^{-1} - N_{\rm ch}\left[\frac{2\delta}{V}\,\gamma^\top + \frac{\delta^2}{V^2}\,(\sigma^2)^\top\right] + \frac{N_{\rm ch}}{V}\,(\sigma^2)^\top.\end{aligned}$$

Grouping $(\sigma^2)^\top$ terms:

$$\boxed{\mathbf{g}_{\rm exact}(p) = 2N_{\rm ch}(p-\mu)\Sigma_p^{-1} - N_{\rm ch}\left[\frac{2\delta(p)}{V(p)}\,\gamma^\top + \left(\frac{\delta(p)^2}{V(p)^2} - \frac{1}{V(p)}\right)(\sigma^2)^\top\right].}$$

2.3 Likelihood Hessian: $M(p)$ and $L(p)$

The Hessian of $M(p)$ was derived in detail in the rank-1 setting and does not change:

$$\nabla^2 M(p) = 2N_{\rm ch}^2\left[\frac{1}{V}\gamma\gamma^\top + \frac{\delta}{V^2}\left(\gamma(\sigma^2)^\top + \sigma^2\gamma^\top\right) + \frac{\delta^2}{V^3}\,\sigma^2\sigma^{2\top}\right].$$

For $L(p) = \log V(p)$,

$$\nabla^2 L(p) = -\frac{1}{V^2}\left(\nabla V\right)^\top\nabla V = -\frac{N_{\rm ch}^2}{V^2}\,\sigma^2\sigma^{2\top}.$$

So

$$\nabla^2\big(M(p)+L(p)\big) = 2N_{\rm ch}^2\left[\frac{1}{V}\gamma\gamma^\top + \frac{\delta}{V^2}\left(\gamma(\sigma^2)^\top + \sigma^2\gamma^\top\right) + \left(\frac{\delta^2}{V^3} - \frac{1}{2V^2}\right)\sigma^2\sigma^{2\top}\right].$$

2.4 Full exact Hessian and rank-2 structure

Adding the prior Hessian:

$$\nabla^2 P(p) = 2N_{\mathrm{ch}}\Sigma_p^{-1},$$

we obtain

$$\mathbf{H}_{\mathrm{exact}}(p) := \nabla^2 E_{\mathrm{exact}}(p) = 2N_{\mathrm{ch}}\Sigma_p^{-1} + 2N_{\mathrm{ch}}^2\left[\frac{1}{V}\gamma\gamma^\top + \frac{\delta}{V^2}\left(\gamma(\sigma^2)^\top + \sigma^2\gamma^\top\right) + \left(\frac{\delta^2}{V^3} - \frac{1}{2V^2}\right)\sigma^2\sigma^{2\top}\right]$$

It is convenient to express this as a rank-2 update of the prior precision. Define

$$v(p) = \gamma + \frac{\delta(p)}{V(p)}\sigma^2.$$

Set

$$u_1(p) = v(p), \qquad u_2(p) = \sigma^2.$$

We seek a representation

$$\mathbf{H}_{\mathrm{exact}}(p) = 2N_{\mathrm{ch}}\left[\Sigma_p^{-1} + U(p)C(p)U(p)^\top\right],$$

with

$$U(p) = \left[u_1(p), u_2(p)\right] \in \mathbb{R}^{d\times 2},$$

and $C(p)$ a diagonal 2×2 matrix.

From the rank-1 analysis without $\log V$ we know that

$$\frac{N_{\mathrm{ch}}}{V}vv^\top = \frac{N_{\mathrm{ch}}}{V}u_1u_1^\top$$

reconstructs the $\gamma\gamma^\top$, $\gamma(\sigma^2)^\top$ and $\sigma^2\sigma^{2\top}$ terms in M alone. The extra contribution from L is

$$-\frac{N_{\mathrm{ch}}^2}{V^2}\sigma^2\sigma^{2\top} = 2N_{\mathrm{ch}}\left(-\frac{N_{\mathrm{ch}}}{2V^2}\right)u_2u_2^\top.$$

Thus we can take

$$C(p) = \begin{pmatrix}\alpha(p) & 0\\ 0 & \beta(p)\end{pmatrix}, \qquad \alpha(p) = \frac{N_{\mathrm{ch}}}{V(p)}, \qquad \beta(p) = -\frac{N_{\mathrm{ch}}}{2V(p)^2},$$

so that

$$U(p)C(p)U(p)^\top = \alpha(p)u_1u_1^\top + \beta(p)u_2u_2^\top.$$

We have therefore:

$$\boxed{\mathbf{H}_{\mathrm{exact}}(p) = 2N_{\mathrm{ch}}\left[\Sigma_p^{-1} + U(p)C(p)U(p)^\top\right],}$$

a rank-2 update of the prior precision.

3. Rank-2 Woodbury, Exact Laplace Covariance and Mean

We now fix an expansion point p_0 and compute the Laplace covariance and mean using the rank-2 structure. In practice p_0 will be taken as the current Newton iterate or the final MAP $p^\star$.

3.1 Rank-2 Woodbury inversion

The Woodbury identity for a rank-k update says:

$$(A + UCU^\top)^{-1} = A^{-1} - A^{-1}U\big(C^{-1} + U^\top A^{-1}U\big)^{-1}U^\top A^{-1},$$

with A invertible, U of shape $d \times k$, and C of shape $k \times k$.

Formally, on the intrinsic $(d-1)$-dimensional subspace we put

$$A = \Sigma_p^{-1}, \qquad A^{-1} = \Sigma_p, \qquad U = U(p_0), \qquad C = C(p_0).$$

Then

$$\big(\Sigma_p^{-1} + UCU^\top\big)^{-1} = \Sigma_p - \Sigma_p UKU^\top \Sigma_p,$$

where

$$K = \big(C^{-1} + U^\top \Sigma_p U\big)^{-1} \in \mathbb{R}^{2\times 2}.$$

At p_0 we have

$$\mathbf{H}_{\text{exact}}(p_0) = 2N_{\text{ch}}\big[\Sigma_p^{-1} + U(p_0)C(p_0)U(p_0)^\top\big],$$

so

$$\mathbf{H}_{\text{exact}}(p_0)^{-1} = \frac{1}{2N_{\text{ch}}}\left[\Sigma_p - \Sigma_p U(p_0)K(p_0)U(p_0)^\top \Sigma_p\right].$$

The exact Laplace covariance at p_0 is

$$\Sigma_{\text{post}}^{\text{exact}}(p_0) = 2\,\mathbf{H}_{\text{exact}}(p_0)^{-1} = \frac{1}{N_{\text{ch}}}\left[\Sigma_p - \Sigma_p U(p_0)K(p_0)U(p_0)^\top \Sigma_p\right].$$

Explicitly,

$$U(p_0) = \big[v(p_0),\ \sigma^2\big], \qquad C(p_0) = \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix},$$

with

$$\alpha = \frac{N_{\text{ch}}}{V(p_0)}, \qquad \beta = -\frac{N_{\text{ch}}}{2V(p_0)^2},$$

and

$$K(p_0) = \Big(C(p_0)^{-1} + U(p_0)^\top \Sigma_p U(p_0)\Big)^{-1} \in \mathbb{R}^{2\times 2}.$$

The 2×2 matrix

$$M = C^{-1} + U^\top \Sigma_p U$$

can be written in terms of scalar contractions

$$s = v^\top \Sigma_p v, \qquad b = v^\top \Sigma_p \sigma^2, \qquad c = \sigma^{2\top} \Sigma_p \sigma^2,$$

as

$$M = \begin{pmatrix} \alpha^{-1} + s & b \\ b & \beta^{-1} + c \end{pmatrix} = \begin{pmatrix} \dfrac{V(p_0)}{N_{\mathrm{ch}}} + s & b \\ b & -\dfrac{2V(p_0)^2}{N_{\mathrm{ch}}} + c \end{pmatrix}.$$

Then $K(p_0) = M^{-1}$ is obtained with the usual 2×2 inverse, either analytically or numerically.

Remark (singular prior). Again, Σ_p^{-1} appears only formally within the Woodbury identity. The implementable covariance $\Sigma_{\mathrm{post}}^{\mathrm{exact}}(p_0)$ contains only Σ_p, $U(p_0)$ and the small 2×2 inverse $K(p_0)$.

3.2 Exact Laplace mean and elimination of Σ_p^{-1}

The Laplace mean at p_0 is

$$p_{\mathrm{post}}^{\mathrm{exact}} = p_0 - \mathbf{g}_{\mathrm{exact}}(p_0)\, \mathbf{H}_{\mathrm{exact}}(p_0)^{-1} = p_0 - \frac{1}{2} \mathbf{g}_{\mathrm{exact}}(p_0)\, \Sigma_{\mathrm{post}}^{\mathrm{exact}}(p_0),$$

with $\mathbf{g}_{\mathrm{exact}}$ from Section 2.2.

Introduce

$$\Delta p = p_0 - \mu$$

and write

$$\mathbf{g}_{\mathrm{exact}}(p_0) = 2N_{\mathrm{ch}} \Delta p\, \Sigma_p^{-1} - N_{\mathrm{ch}}\, q^\top,$$

where

$$q^\top = \frac{2\delta}{V}\, \gamma^\top + \left(\frac{\delta^2}{V^2} - \frac{1}{V} \right) (\sigma^2)^\top,$$

with $\delta = \delta(p_0)$ and $V = V(p_0)$.

Then

$$p_{\mathrm{post}}^{\mathrm{exact}} = p_0 - N_{\mathrm{ch}} \Delta p\, \Sigma_p^{-1} \Sigma_{\mathrm{post}}^{\mathrm{exact}}(p_0) + \frac{N_{\mathrm{ch}}}{2}\, q^\top \Sigma_{\mathrm{post}}^{\mathrm{exact}}(p_0).$$

Using

$$\Sigma_{\text{post}}^{\text{exact}}(p_0) = \frac{1}{N_{\text{ch}}}\Big[\Sigma_p - \Sigma_p U K U^\top \Sigma_p\Big],$$

we obtain

$$\begin{aligned}\Sigma_p^{-1}\Sigma_{\text{post}}^{\text{exact}}(p_0) &= \Sigma_p^{-1}\left[\frac{1}{N_{\text{ch}}}\big(\Sigma_p - \Sigma_p U K U^\top \Sigma_p\big)\right]\\ &= \frac{1}{N_{\text{ch}}}\big(I - U K U^\top \Sigma_p\big),\end{aligned}$$

so

$$N_{\text{ch}}\Delta p\,\Sigma_p^{-1}\Sigma_{\text{post}}^{\text{exact}}(p_0) = \Delta p - \Delta p\, U K U^\top \Sigma_p.$$

Therefore

$$\begin{aligned}p_{\text{post}}^{\text{exact}} &= p_0 - \Big[\Delta p - \Delta p\, U K U^\top \Sigma_p\Big] + \frac{N_{\text{ch}}}{2} q^\top \Sigma_{\text{post}}^{\text{exact}}(p_0)\\ &= (p_0 - \Delta p) + \Delta p\, U K U^\top \Sigma_p + \frac{N_{\text{ch}}}{2} q^\top \Sigma_{\text{post}}^{\text{exact}}(p_0).\end{aligned}$$

Since $p_0 - \Delta p = \mu$, we arrive at the exact mean update with no explicit Σ_p^{-1}:

$$\boxed{p_{\text{post}}^{\text{exact}} = \mu + \Delta p\, U(p_0) K(p_0) U(p_0)^\top \Sigma_p + \frac{N_{\text{ch}}}{2}\, q^\top \Sigma_{\text{post}}^{\text{exact}}(p_0),}$$

with

$$\Delta p = p_0 - \mu, \quad q^\top = \frac{2\delta}{V}\,\gamma^\top + \left(\frac{\delta^2}{V^2} - \frac{1}{V}\right)(\sigma^2)^\top,$$

and $\Sigma_{\text{post}}^{\text{exact}}(p_0)$ given above.

3.3 Newton–MAP scheme and MAP–Laplace posterior

The exact MAP is defined as

$$p^\star = \arg\min_{p\in\mathcal{S}} F_{\text{exact}}(p) = \arg\min_{p\in\mathcal{S}} E_{\text{exact}}(p),$$

subject to the simplex constraints. We locate $p^\star$ via Newton's method applied to E_{exact}.

Let $p^{(k)}$ be the current iterate. One Newton step is

$$p^{(k+1)} = p^{(k)} - \Delta p^{(k)}, \qquad \mathbf{H}_{\text{exact}}(p^{(k)})\,\Delta p^{(k)\top} = \mathbf{g}_{\text{exact}}(p^{(k)})^\top.$$

Algorithmically:

1. Initialise with a feasible point, typically $p^{(0)} = \mu$.
2. For $k = 0, 1, 2, \dots$ until convergence:
   (a) Evaluate $\delta^{(k)} = \delta(p^{(k)})$, $V^{(k)} = V(p^{(k)})$, $v^{(k)} = v(p^{(k)})$.
   (b) Form $\mathbf{g}_{\text{exact}}(p^{(k)})$ and $\mathbf{H}_{\text{exact}}(p^{(k)})$ using the formulas above.
   (c) Compute the exact Laplace covariance $\Sigma_{\text{post}}^{\text{exact}}(p^{(k)})$ using the rank-2 Woodbury formula.
   (d) Update the mean using the exact mean formula with $p_0 = p^{(k)}$ and $\Sigma_{\text{post}}^{\text{exact}}(p_0) = \Sigma_{\text{post}}^{\text{exact}}(p^{(k)})$:

   $$p^{(k+1)} = \mu + \Delta p^{(k)} U^{(k)} K^{(k)} U^{(k)\top} \Sigma_p + \frac{N_{\text{ch}}}{2} q^{(k)\top} \Sigma_{\text{post}}^{\text{exact}}(p^{(k)}),$$

   where $\Delta p^{(k)} = p^{(k)} - \mu$, $U^{(k)} = U(p^{(k)})$ and $q^{(k)}$ is defined by $\delta^{(k)}$, $V^{(k)}$.
   (e) Optionally project $p^{(k+1)}$ back to the simplex (enforce $p_i^{(k+1)} \geq 0$ and renormalise).

After convergence, $p^{(k)} \to p^\star$ and we define the MAP–Laplace posterior as

$$p \mid y \approx \mathcal{N}\big(p^\star, \Sigma_{\text{Lap}}\big),$$

with

$$\Sigma_{\text{Lap}} = 2\,\mathbf{H}_{\text{exact}}(p^\star)^{-1} = \frac{1}{N_{\text{ch}}}\left[\Sigma_p - \Sigma_p U(p^\star) K(p^\star) U(p^\star)^\top \Sigma_p\right].$$

4. Rank-1 Quasi-Laplace Scheme as a Special Case

We now present the *quasi* (rank-1) scheme obtained by *dropping* the $\log V(p)$ term from the likelihood. This corresponds to a quasi-likelihood / weighted least-squares objective. It is useful pedagogically and connects to EKF-like approximations.

4.1 Quasi energy and Hessian

Define the quasi energy

$$E_{\text{quasi}}(p) = M(p) + P(p) = \frac{\delta(p)^2}{V(p)} + (p - \mu)\big(N_{\text{ch}}\Sigma_p^{-1}\big)(p - \mu)^\top.$$

The gradient is

$$\mathbf{g}_{\text{quasi}}(p) = 2N_{\text{ch}}(p-\mu)\Sigma_p^{-1} - N_{\text{ch}}\left[\frac{2\delta(p)}{V(p)}\,\gamma^\top + \frac{\delta(p)^2}{V(p)^2}\,(\sigma^2)^\top\right].$$

The Hessian is

$$\mathbf{H}_{\text{quasi}}(p) = 2N_{\text{ch}}\Sigma_p^{-1} + 2N_{\text{ch}}^2\left[\frac{1}{V}\gamma\gamma^\top + \frac{\delta}{V^2}\left(\gamma(\sigma^2)^\top + \sigma^2\gamma^\top\right) + \frac{\delta^2}{V^3}\,\sigma^2\sigma^{2\top}\right].$$

Introduce

$$v(p) = \gamma + \frac{\delta(p)}{V(p)}\sigma^2, \qquad u(p) = \sqrt{\frac{N_{\text{ch}}}{V(p)}}\,v(p).$$

Then

$$\mathbf{H}_{\text{quasi}}(p) = 2N_{\text{ch}}\left[\Sigma_p^{-1} + u(p)u(p)^\top\right].$$

This is a rank-1 update of the prior precision.

4.2 Rank-1 covariance and mean

Applying Sherman–Morrison with $A = \Sigma_p^{-1}$ and $u = u(p_0)$ at an expansion point p_0:

$$\left(\Sigma_p^{-1} + uu^\top\right)^{-1} = \Sigma_p - \frac{\Sigma_p uu^\top\Sigma_p}{1 + u^\top\Sigma_p u}.$$

Therefore

$$\mathbf{H}_{\text{quasi}}(p_0)^{-1} = \frac{1}{2N_{\text{ch}}}\left[\Sigma_p - \frac{\Sigma_p uu^\top\Sigma_p}{1 + u^\top\Sigma_p u}\right].$$

The quasi Laplace covariance at p_0 is

$$\Sigma_{\text{post}}^{\text{quasi}}(p_0) = 2\,\mathbf{H}_{\text{quasi}}(p_0)^{-1} = \frac{1}{N_{\text{ch}}}\left[\Sigma_p - \frac{\Sigma_p uu^\top\Sigma_p}{1 + u^\top\Sigma_p u}\right].$$

Substituting $u = \sqrt{N_{\text{ch}}/V}\,v$ and defining $s = v^\top\Sigma_p v$, we obtain

$$\boxed{\Sigma_{\text{post}}^{\text{quasi}}(p_0) = \frac{1}{N_{\text{ch}}}\,\Sigma_p - \frac{1}{V(p_0) + N_{\text{ch}}s(p_0)}\,\Sigma_p v(p_0)v(p_0)^\top\Sigma_p.}$$

The quasi mean at p_0 is

$$p_{\text{post}}^{\text{quasi}} = p_0 - \frac{1}{2}\mathbf{g}_{\text{quasi}}(p_0)\,\Sigma_{\text{post}}^{\text{quasi}}(p_0).$$

Writing $\Delta p = p_0 - \mu$ and using the identity

$$\Sigma_p^{-1}\Sigma_{\text{post}}^{\text{quasi}} = \frac{1}{N_{\text{ch}}} I - \frac{1}{V + N_{\text{ch}} s}\, v v^\top \Sigma_p,$$

we obtain, after the same algebra as in the rank-1 derivation:

$$\boxed{\begin{aligned} p_{\text{post}}^{\text{quasi}} &= \mu + \frac{N_{\text{ch}}}{V(p_0) + N_{\text{ch}} s(p_0)}\,(p_0 - \mu)\, v(p_0) v(p_0)^\top \Sigma_p \\ &\quad + \frac{N_{\text{ch}}\delta(p_0)}{V(p_0)}\,\gamma^\top \Sigma_{\text{post}}^{\text{quasi}}(p_0) + \frac{N_{\text{ch}}\delta(p_0)^2}{2V(p_0)^2}\,(\sigma^2)^\top \Sigma_{\text{post}}^{\text{quasi}}(p_0). \end{aligned}}$$

4.3 Special case $p_0 = \mu$ and connection to EKF

If we choose $p_0 = \mu$, then $\Delta p = 0$ and the first line of the quasi mean vanishes. In that case:

$$\mathbf{g}_{\text{quasi}}(\mu) = -N_{\text{ch}} \left[\frac{2\delta}{V}\,\gamma^\top + \frac{\delta^2}{V^2}\,(\sigma^2)^\top \right],$$

and

$$p_{\text{post}}^{\text{quasi}} = \mu + \frac{N_{\text{ch}}\delta}{2V}\,(\gamma^\top + v^\top)\Sigma_{\text{post}}^{\text{quasi}},$$

where $v = \gamma + \frac{\delta}{V}\sigma^2$ and $\Sigma_{\text{post}}^{\text{quasi}}$ is evaluated at $p_0 = \mu$.

This can be interpreted as a single quasi-Laplace (or EKF-like) update from the prior mean, using only the mean-matching part of the likelihood and ignoring the $\log V$ term.

5. Summary and Interpretation

We now summarise the two schemes:

5.1 Rank-2 (exact) scheme

- Likelihood:

$$p(y \mid p) = \mathcal{N}\big(N_{\text{ch}}(p\cdot\gamma), V(p)\big), \qquad V(p) = \epsilon^2 + N_{\text{ch}}(p\cdot\sigma^2).$$

- Negative log-posterior (up to const):

$$F_{\text{exact}}(p) = \frac{1}{2}\log V(p) + \frac{1}{2}\frac{\delta(p)^2}{V(p)} + \frac{1}{2}(p-\mu)\big(N_{\text{ch}}\Sigma_p^{-1}\big)(p-\mu)^\top.$$

- Energy:
$$E_{\text{exact}}(p) = 2F_{\text{exact}}(p).$$
- Hessian:
$$\mathbf{H}_{\text{exact}}(p) = 2N_{\text{ch}}\left[\Sigma_p^{-1} + U(p)C(p)U(p)^\top\right],$$
with $U(p) = [v(p), \sigma^2]$ and diagonal $C(p)$.
- Covariance:
$$\Sigma_{\text{post}}^{\text{exact}}(p_0) = \frac{1}{N_{\text{ch}}}\left[\Sigma_p - \Sigma_p U(p_0)K(p_0)U(p_0)^\top \Sigma_p\right],$$
with $K(p_0) = \left(C(p_0)^{-1} + U(p_0)^\top \Sigma_p U(p_0)\right)^{-1}$.
- Mean:
$$p_{\text{post}}^{\text{exact}} = \mu + \Delta p\, U(p_0)K(p_0)U(p_0)^\top \Sigma_p + \frac{N_{\text{ch}}}{2} q^\top \Sigma_{\text{post}}^{\text{exact}}(p_0).$$
- Newton iteration uses E_{exact}, converges to the true MAP $p^\star$, and the Laplace covariance at $p^\star$ is $\Sigma_{\text{Lap}} = \Sigma_{\text{post}}^{\text{exact}}(p^\star)$.

This scheme fully incorporates the heteroscedastic variance $V(p)$ via both the δ^2/V term and the $\log V$ term of the Gaussian likelihood. Even when $\delta = 0$, the $\log V$ term can shift the MAP towards states with smaller predicted variance, in competition with the prior.

5.2 Rank-1 (quasi) scheme

- Energy:
$$E_{\text{quasi}}(p) = \frac{\delta(p)^2}{V(p)} + (p-\mu)\left(N_{\text{ch}}\Sigma_p^{-1}\right)(p-\mu)^\top.$$
- Hessian:
$$\mathbf{H}_{\text{quasi}}(p) = 2N_{\text{ch}}\left[\Sigma_p^{-1} + \frac{N_{\text{ch}}}{V(p)} v(p)v(p)^\top\right],$$
a rank-1 update.
- Covariance:
$$\Sigma_{\text{post}}^{\text{quasi}}(p_0) = \frac{1}{N_{\text{ch}}}\Sigma_p - \frac{1}{V(p_0) + N_{\text{ch}}s(p_0)}\Sigma_p v(p_0)v(p_0)^\top \Sigma_p.$$

- Mean: the quasi mean $p_{\text{post}}^{\text{quasi}}$ is given by the rank-1 formula above, and in the special case $p_0 = \mu$ reduces to a single-step update proportional to $(\gamma^\top + v^\top)\Sigma_{\text{post}}^{\text{quasi}}$.

This scheme ignores the $\log V(p)$ contribution to the likelihood and treats $V(p)$ only as a weight in the squared residual. It is analogous to using a weighted least-squares / quasi-likelihood objective in GLMs, and connects to EKF practice where a state-dependent variance is plugged into an effective R but not differentiated.

5.3 Practical recommendation

In MacroTaylor we take the rank-2 scheme as the *main method*:

- it is the correct Gaussian MAP for the interval model;
- the rank-2 Woodbury cost is negligible relative to the cost of computing the MacroIR quantities;
- the heteroscedastic variance $V(p)$ is physically meaningful and carries information about p.

The rank-1 scheme is retained as a *pedagogical device* and as a limiting case:

- it illustrates how a simpler quasi-likelihood/EKF update arises if one drops $\log V(p)$;
- it provides a useful check in regimes where $V(p)$ changes slowly with p or N_{ch} is large.