Channel-Aware Laplace Approximation for the Posterior over p

This note collects the derivations for a channel-aware Laplace approximation of the posterior over the macroscopic occupancy vector p of an ensemble of Markov channels, based on a single interval-averaged current measurement.

The main goals are:

- to obtain an explicit formula for the *Gaussian (Laplace) posterior covariance* Σ_{post} in terms of the prior covariance Σ_p and a rank-1 update defined by the measurement;
- to write the *mean update* in a form that does not involve any explicit inverse of the (singular) prior covariance;
- to emphasise that the posterior mean used in practice is the *MAP estimate* obtained by Newton iteration, and the covariance is the *Laplace covariance* given by the curvature at the MAP.

Throughout, we keep the dependence of all intermediate objects on p explicit: $\delta(p)$, $V(p)$, $v(p)$, etc. Whenever we drop the argument and write δ, V, v, they are implicitly evaluated at the current expansion point (either a fixed p_0, or the current iterate $p^{(k)}$ in Newton's method).

Coordinates, simplex and singular covariance. The macroscopic occupancy vector p lives on the simplex $\{p \in \mathbb{R}^{1\times d} : p_i \geq 0, \ \sum_i p_i = 1\}$, a $(d-1)$-dimensional manifold embedded in $\mathbb{R}^{1\times d}$. The prior covariance Σ_p is therefore singular as a $d \times d$ matrix in ambient coordinates. Formally, one can view Σ_p^{-1} as the inverse on the $(d-1)$-dimensional tangent subspace or as a pseudo-inverse, but *in all final expressions we use in implementation, Σ_p^{-1} cancels algebraically.* We never need to numerically invert a singular covariance.

1. Setup and Notation

We consider:

- A macroscopic occupancy vector
$$p \in \mathbb{R}^{1\times d},$$
a row vector of state probabilities.
- A per-state conductance vector
$$\gamma \in \mathbb{R}^{d\times 1},$$
a column vector.
- A per-state intrinsic-variance vector
$$\sigma^2 \in \mathbb{R}^{d\times 1}.$$

The prior over p (restricted to the simplex) is Gaussian:

$$p \sim \mathcal{N}(\mu, \Sigma_p),$$

where $\mu \in \mathbb{R}^{1\times d}$ is the prior mean and $\Sigma_p \in \mathbb{R}^{d\times d}$ is the prior covariance. As noted above, Σ_p is singular in the ambient coordinates.

We observe a scalar interval-averaged current y:

$$y \approx N_{\text{ch}}(p \cdot \gamma) + \text{noise},$$

with N_{ch} channels and:

- extrinsic (measurement) noise variance ϵ^2;
- intrinsic ensemble variance $N_{\text{ch}}(p \cdot \sigma^2)$ arising from channel stochasticity.

Given a candidate state p, define

$$\delta(p) = y - N_{\text{ch}}(p \cdot \gamma),$$
$$V(p) = \epsilon^2 + N_{\text{ch}}(p \cdot \sigma^2),$$

and the *tilted* vector

$$v(p) = \gamma + \frac{\delta(p)}{V(p)}\,\sigma^2 \in \mathbb{R}^{d\times 1}.$$

We work with an energy function $E(p)$ proportional to the negative log-posterior,

$$p(p \mid y) \propto \exp\bigl(-\tfrac{1}{2}E(p)\bigr).$$

We will denote:

- $\mathbf{g}(p) = \nabla E(p)$, a *row* vector in $\mathbb{R}^{1\times d}$;
- $\mathbf{H}(p) = \nabla^2 E(p)$, a $d \times d$ Hessian.

2. Laplace Approximation Around an Expansion Point

We approximate the posterior around an expansion point p_0 via a second-order Taylor expansion:

$$E(p) \approx E(p_0) + (p - p_0)\,\mathbf{g}_0^\top + \frac{1}{2}(p - p_0)\,\mathbf{H}_0\,(p - p_0)^\top,$$

with

$$\mathbf{g}_0 = \nabla E(p_0), \qquad \mathbf{H}_0 = \nabla^2 E(p)\big|_{p=p_0}.$$

Plugging this into

$$p(p \mid y) \propto \exp\bigl(-\tfrac{1}{2}E(p)\bigr)$$

gives a Gaussian approximation

$$p(p \mid y) \approx \mathcal{N}\bigl(p_{\text{post}}, \Sigma_{\text{post}}\bigr),$$

with

$$\Sigma_{\text{post}} \approx 2\,\mathbf{H}_0^{-1},$$

$$p_{\text{post}} \approx p_0 - \mathbf{g}_0\,\mathbf{H}_0^{-1} = p_0 - \frac{1}{2}\mathbf{g}_0\,\Sigma_{\text{post}}.$$

Conceptually, the Laplace covariance is the inverse curvature at the chosen expansion point, and the mean is the Newton step from p_0. In practice, we will:

- use a special rank-1 structure of $\mathbf{H}_0$ to obtain a closed-form Σ_{post} in terms of Σ_p;
- eliminate any explicit Σ_p^{-1} from the mean update;
- iterate the Newton step until convergence to obtain the MAP.

3. Energy, Gradient and Hessian as Functions of p

We decompose the energy into likelihood and prior terms:

$$E(p) = M(p) + P(p),$$

with

$$M(p) = \frac{\delta(p)^2}{V(p)},$$
$$P(p) = (p-\mu)\,(N_{\mathrm{ch}}\Sigma_p^{-1})(p-\mu)^\top.$$

Here

$$\delta(p) = y - N_{\mathrm{ch}}(p\cdot\gamma), \qquad V(p) = \epsilon^2 + N_{\mathrm{ch}}(p\cdot\sigma^2).$$

The full gradient and Hessian, derived in detail in Section 7, can be compactly written as:

$$\mathbf{g}(p) = 2N_{\mathrm{ch}}(p-\mu)\Sigma_p^{-1} - N_{\mathrm{ch}}\left[\frac{2\delta(p)}{V(p)}\,\gamma^\top + \frac{\delta(p)^2}{V(p)^2}\,(\sigma^2)^\top\right],$$

and

$$\mathbf{H}(p) = 2N_{\mathrm{ch}}\left[\Sigma_p^{-1} + \frac{N_{\mathrm{ch}}}{V(p)}\,v(p)\,v(p)^\top\right],$$

where

$$v(p) = \gamma + \frac{\delta(p)}{V(p)}\,\sigma^2.$$

These expressions are valid for all p on the simplex. They show explicitly that δ, V and v are functions of p.

4. Rank-1 Structure and Closed-Form Covariance

The key structural observation is that the Hessian is a rank-1 update of the prior precision:

$$\mathbf{H}(p) = 2N_{\mathrm{ch}}\left[\Sigma_p^{-1} + \frac{N_{\mathrm{ch}}}{V(p)}\,v(p)v(p)^\top\right].$$

Define

$$u(p) = \sqrt{\frac{N_{\mathrm{ch}}}{V(p)}}\,v(p), \quad\Rightarrow\quad u(p)u(p)^\top = \frac{N_{\mathrm{ch}}}{V(p)}\,v(p)v(p)^\top.$$

Then

$$\mathbf{H}(p) = 2N_{\mathrm{ch}}\big(\Sigma_p^{-1} + u(p)u(p)^\top\big).$$

4.1 Sherman–Morrison Formula

Formally, for an invertible matrix A and a rank-1 update $uu^\top$, the Sherman–Morrison formula says

$$(A + uu^\top)^{-1} = A^{-1} - \frac{A^{-1}uu^\top A^{-1}}{1 + u^\top A^{-1}u}.$$

We apply this with $A = \Sigma_p^{-1}$ and $u = u(p_0)$ at the expansion point p_0. On the tangent subspace of the simplex, Σ_p has an inverse; at the algebraic level we write $A^{-1} = \Sigma_p$ and obtain

$$(\Sigma_p^{-1} + uu^\top)^{-1} = \Sigma_p - \frac{\Sigma_p uu^\top \Sigma_p}{1 + u^\top \Sigma_p u}.$$

Since

$$\mathbf{H}_0 = 2N_{\text{ch}}\big(\Sigma_p^{-1} + uu^\top\big),$$

we get

$$\mathbf{H}_0^{-1} = \frac{1}{2N_{\text{ch}}}\left[\Sigma_p - \frac{\Sigma_p uu^\top \Sigma_p}{1 + u^\top \Sigma_p u}\right].$$

The Laplace covariance at p_0 is

$$\Sigma_{\text{post}} = 2\,\mathbf{H}_0^{-1} = \frac{1}{N_{\text{ch}}}\left[\Sigma_p - \frac{\Sigma_p uu^\top \Sigma_p}{1 + u^\top \Sigma_p u}\right].$$

4.2 Expressing Everything in Terms of $v(p_0)$

At p_0 we have $u = \sqrt{\frac{N_{\text{ch}}}{V}}\, v$ with $V = V(p_0)$ and $v = v(p_0)$, so:

$$uu^\top = \frac{N_{\text{ch}}}{V} vv^\top,$$

and we define

$$s = v^\top \Sigma_p v.$$

Then:

$$\Sigma_p uu^\top \Sigma_p = \Sigma_p \left(\frac{N_{\text{ch}}}{V} vv^\top\right) \Sigma_p = \frac{N_{\text{ch}}}{V}\, \Sigma_p vv^\top \Sigma_p,$$

$$1 + u^\top \Sigma_p u = 1 + \frac{N_{\text{ch}}}{V} v^\top \Sigma_p v = 1 + \frac{N_{\text{ch}}}{V} s = \frac{V + N_{\text{ch}} s}{V}.$$

Substituting:

$$\Sigma_{\text{post}} = \frac{1}{N_{\text{ch}}} \left[\Sigma_p - \frac{\frac{N_{\text{ch}}}{V} \Sigma_p v v^\top \Sigma_p}{\frac{V + N_{\text{ch}} s}{V}} \right] = \frac{1}{N_{\text{ch}}} \left[\Sigma_p - \frac{N_{\text{ch}}}{V + N_{\text{ch}} s} \Sigma_p v v^\top \Sigma_p \right].$$

Thus we obtain the closed-form Laplace covariance:

$$\boxed{\Sigma_{\text{post}}(p_0) = \frac{1}{N_{\text{ch}}} \Sigma_p - \frac{1}{V(p_0) + N_{\text{ch}}\, s(p_0)} \Sigma_p v(p_0) v(p_0)^\top \Sigma_p}$$

with $s(p_0) = v(p_0)^\top \Sigma_p v(p_0)$.

Singular covariance and implementation. Note that:

- The derivation uses Σ_p^{-1} only as a formal device on the intrinsic $(d-1)$-dimensional subspace.
- The final expression for Σ_{post} involves only Σ_p, v and s; no explicit inverse of Σ_p appears.
- This formula remains valid even when Σ_p is singular in ambient coordinates, so it is suitable for direct implementation.

5. Mean Update and Elimination of Σ_p^{-1}

The Laplace mean at p_0 is

$$p_{\text{post}} \approx p_0 - \mathbf{g}_0\, \mathbf{H}_0^{-1} = p_0 - \frac{1}{2} \mathbf{g}_0\, \Sigma_{\text{post}},$$

with

$$\mathbf{g}_0 = 2N_{\text{ch}}(p_0 - \mu)\Sigma_p^{-1} - N_{\text{ch}} \left[\frac{2\delta}{V}\, \gamma^\top + \frac{\delta^2}{V^2}\, (\sigma^2)^\top \right],$$

where $\delta = \delta(p_0)$ and $V = V(p_0)$.

Introduce

$$\Delta p = p_0 - \mu,$$

and rewrite:

$$\mathbf{g}_0 = 2N_{\text{ch}} \Delta p\, \Sigma_p^{-1} - N_{\text{ch}} \left[\frac{2\delta}{V}\, \gamma^\top + \frac{\delta^2}{V^2}\, (\sigma^2)^\top \right].$$

Then

$$p_{\text{post}} = p_0 - \frac{1}{2}\mathbf{g}_0\,\Sigma_{\text{post}} = p_0 - N_{\text{ch}}\Delta p\,\Sigma_p^{-1}\Sigma_{\text{post}} + \frac{N_{\text{ch}}}{2}\left[\frac{2\delta}{V}\,\gamma^\top + \frac{\delta^2}{V^2}\,(\sigma^2)^\top\right]\Sigma_{\text{post}}.$$

Equivalently,

$$p_{\text{post}} = p_0 - N_{\text{ch}}\Delta p\,\Sigma_p^{-1}\Sigma_{\text{post}} + \frac{N_{\text{ch}}\delta}{V}\,\gamma^\top\Sigma_{\text{post}} + \frac{N_{\text{ch}}\delta^2}{2V^2}\,(\sigma^2)^\top\Sigma_{\text{post}}.$$

5.1 Eliminating Σ_p^{-1} Using Σ_{post}

We now use the explicit expression for

$$\Sigma_{\text{post}} = \frac{1}{N_{\text{ch}}}\Sigma_p - \frac{1}{V + N_{\text{ch}}s}\,\Sigma_p v v^\top \Sigma_p$$

to compute $\Sigma_p^{-1}\Sigma_{\text{post}}$ and thereby eliminate Σ_p^{-1} from the mean.

For notational compactness in this subsection, write $N = N_{\text{ch}}$, remembering they are the same quantity.

First:

$$\begin{aligned}
\Sigma_p^{-1}\Sigma_{\text{post}} &= \Sigma_p^{-1}\left(\frac{1}{N}\Sigma_p - \frac{1}{V + Ns}\,\Sigma_p v v^\top \Sigma_p\right) \\
&= \frac{1}{N}\underbrace{\Sigma_p^{-1}\Sigma_p}_{I} - \frac{1}{V + Ns}\,v v^\top \Sigma_p \\
&= \frac{1}{N}I - \frac{1}{V + Ns}\,v v^\top \Sigma_p.
\end{aligned}$$

Therefore:

$$N\,\Delta p\,\Sigma_p^{-1}\Sigma_{\text{post}} = \Delta p - \frac{N}{V + Ns}\,\Delta p\,v v^\top \Sigma_p.$$

Substituting into the expression for p_{post}:

$$\begin{aligned}
p_{\text{post}} &= p_0 - \left[\Delta p - \frac{N}{V + Ns}\,\Delta p\,v v^\top \Sigma_p\right] + \frac{N\delta}{V}\,\gamma^\top\Sigma_{\text{post}} + \frac{N\delta^2}{2V^2}\,(\sigma^2)^\top\Sigma_{\text{post}} \\
&= (p_0 - \Delta p) + \frac{N}{V + Ns}\,\Delta p\,v v^\top \Sigma_p + \frac{N\delta}{V}\,\gamma^\top\Sigma_{\text{post}} + \frac{N\delta^2}{2V^2}\,(\sigma^2)^\top\Sigma_{\text{post}}.
\end{aligned}$$

Since $p_0 - \Delta p = \mu$, we obtain:

$$\boxed{\begin{aligned}
p_{\text{post}} &= \mu + \frac{N_{\text{ch}}}{V + N_{\text{ch}}s}\,(p_0 - \mu)\,v v^\top \Sigma_p \\
&\qquad + \frac{N_{\text{ch}}\delta}{V}\,\gamma^\top\Sigma_{\text{post}} + \frac{N_{\text{ch}}\delta^2}{2V^2}\,(\sigma^2)^\top\Sigma_{\text{post}}.
\end{aligned}} \tag{2}$$

Here Σ_{post} is already expressed purely in terms of Σ_p, v, s and V. No explicit Σ_p^{-1} survives: it has been completely cancelled.

Interpretation.

- The first line in (2) pulls the mean back towards the prior mean μ, modulated by the rank-1 direction $vv^\top\Sigma_p$ and the competition between V and $N_{\text{ch}}s$.
- The second line captures the direct influence of the residual δ through the conductance and intrinsic variance vectors (γ, σ^2), weighted by the covariance Σ_{post}.
- Equation (2) is the general mean update that should be implemented numerically; it only requires multiplications by Σ_p and Σ_{post}, never their inverses.

5.2 Special Case: Expansion at the Prior Mean $p_0 = \mu$

Choosing $p_0 = \mu$ simplifies the expression significantly:

- $\Delta p = p_0 - \mu = 0$,
- the first line in (2) vanishes,
- the gradient at μ reduces to

  $$\mathbf{g}_0 = -N_{\text{ch}}\frac{\delta}{V}(\gamma^\top + v^\top),$$

  so

  $$p_{\text{post}} = \mu - \frac{1}{2}\mathbf{g}_0\Sigma_{\text{post}} = \mu + \frac{N_{\text{ch}}\delta}{2V}\,(\gamma^\top + v^\top)\Sigma_{\text{post}}.$$

In this special case we obtain the concise one-step update:

$$\boxed{p_{\text{post}} \approx \mu + \frac{N_{\text{ch}}\delta}{2V}\,(\gamma^\top + v^\top)\Sigma_{\text{post}},}$$

with

- $\delta = y - N_{\text{ch}}(\mu \cdot \gamma)$,
- $V = \epsilon^2 + N_{\text{ch}}(\mu \cdot \sigma^2)$,
- $v = \gamma + \dfrac{\delta}{V}\sigma^2$,

- Σ_{post} as in Section 4 with $p_0 = \mu$.

This corresponds to a single Newton step from the prior mean, with a Laplace covariance frozen at that expansion point.

6. Newton–MAP Scheme and MAP–Laplace Posterior

Up to now, p_0 has been a generic expansion point. We now make explicit that:

- the posterior mean of interest is the *MAP* $p^\star$, i.e. the minimum of $E(p)$;
- the covariance we associate with this mean is the *Laplace covariance* Σ_{Lap} given by the curvature at $p^\star$.

6.1 Defining the MAP

The posterior mode (MAP) is defined by

$$p^\star = \arg\min_p E(p)$$

subject to the simplex constraints $p_i \geq 0$ and $\sum_i p_i = 1$. We seek this mode via Newton's method applied to $E(p)$.

6.2 Newton Iteration

Let $p^{(k)}$ denote the current iterate (row vector). One Newton step is:

$$p^{(k+1)} = p^{(k)} - \Delta p^{(k)}, \qquad \mathbf{H}(p^{(k)})\,\Delta p^{(k)\top} = \mathbf{g}(p^{(k)})^\top.$$

Algorithmically:

1. Initialise with a feasible point, typically $p^{(0)} = \mu$.
2. For $k = 0, 1, 2, \ldots$ until convergence:
   (a) Compute $\delta^{(k)} = \delta(p^{(k)})$, $V^{(k)} = V(p^{(k)})$, $v^{(k)} = v(p^{(k)})$.
   (b) Form $\mathbf{g}^{(k)} = \mathbf{g}(p^{(k)})$ and $\mathbf{H}^{(k)} = \mathbf{H}(p^{(k)})$ using the formulas in Section 3.

(c) Compute the Laplace covariance $\Sigma_{\text{post}}^{(k)}$ at $p^{(k)}$ using the closed form in Section 4.

(d) Update the mean using the general formula (2) with $p_0 = p^{(k)}$ and $\Sigma_{\text{post}} = \Sigma_{\text{post}}^{(k)}$:

$$p^{(k+1)} = \mu + \frac{N_{\text{ch}}}{V^{(k)} + N_{\text{ch}} s^{(k)}} \left(p^{(k)} - \mu\right) v^{(k)} v^{(k)\top} \Sigma_p + \frac{N_{\text{ch}} \delta^{(k)}}{V^{(k)}} \gamma^\top \Sigma_{\text{post}}^{(k)} + \frac{N_{\text{ch}} \delta^{(k)2}}{2V^{(k)2}} (\sigma^2)^\top \Sigma_{\text{post}}^{(k)}.$$

Here $s^{(k)} = v^{(k)\top} \Sigma_p v^{(k)}$.

(e) Optionally project $p^{(k+1)}$ back onto the simplex (enforce $p_i^{(k+1)} \geq 0$ and renormalise $\sum_i p_i^{(k+1)} = 1$).

If we stop after a single iteration from $p^{(0)} = \mu$, we recover the one-step expression of Section 5.2. In general, Newton generates a sequence $p^{(k)}$ converging to the MAP $p^\star$, and we then set:

$$p_{\text{MAP}} = p^\star.$$

6.3 MAP–Laplace Posterior

Once Newton has converged to $p^\star$, we define the *MAP–Laplace posterior* as:

$$p \mid y \approx \mathcal{N}\big(p^\star, \Sigma_{\text{Lap}}\big),$$

with covariance given by the inverse curvature at $p^\star$:

$$\Sigma_{\text{Lap}} = \mathbf{H}(p^\star)^{-1}.$$

Using the rank-1 structure, we compute Σ_{Lap} via the same closed form as $\Sigma_{\text{post}}(p_0)$, simply evaluated at $p_0 = p^\star$:

$$\Sigma_{\text{Lap}} = \frac{1}{N_{\text{ch}}} \Sigma_p - \frac{1}{V(p^\star) + N_{\text{ch}} s(p^\star)} \Sigma_p v(p^\star) v(p^\star)^\top \Sigma_p,$$

with $s(p^\star) = v(p^\star)^\top \Sigma_p v(p^\star)$.

Thus the computational recipe is:

- iterate Newton using (2) until convergence to $p^\star$;
- evaluate the rank-1 covariance formula at $p^\star$ to obtain Σ_{Lap}.

7. Detailed Derivation of Gradient and Hessian

For completeness, we include the full derivation of the gradient and Hessian of $E(p)$.

Recall:

$$E(p) = M(p) + P(p), \qquad M(p) = \frac{\delta^2}{V}, \qquad P(p) = (p - \mu)\,(N_{\mathrm{ch}}\Sigma_p^{-1})(p - \mu)^\top,$$

with

$$\delta = y - N_{\mathrm{ch}}(p \cdot \gamma), \qquad V = \epsilon^2 + N_{\mathrm{ch}}(p \cdot \sigma^2).$$

Here p is a row vector, γ and σ^2 are column vectors.

7.1 Prior Gradient and Hessian

Define the precision matrix $\Lambda_p = N_{\mathrm{ch}}\Sigma_p^{-1}$. Then:

$$P(p) = (p - \mu)\Lambda_p(p - \mu)^\top.$$

Using the standard formula for the gradient of a quadratic form $f(p) = (p - \mu)A(p - \mu)^\top$, with symmetric A, we have:

$$\nabla f(p) = 2(p - \mu)A,$$

and

$$\nabla^2 f(p) = 2A.$$

Thus:

$$\nabla P(p) = 2(p - \mu)\Lambda_p = 2N_{\mathrm{ch}}(p - \mu)\Sigma_p^{-1},$$
$$\nabla^2 P(p) = 2\Lambda_p = 2N_{\mathrm{ch}}\Sigma_p^{-1}.$$

7.2 Likelihood Gradient

We have:

$$M(p) = \frac{\delta^2}{V}, \qquad \delta = y - N_{\mathrm{ch}}(p \cdot \gamma), \qquad V = \epsilon^2 + N_{\mathrm{ch}}(p \cdot \sigma^2).$$

First compute the gradients of δ and V:

$$\nabla \delta(p) = -N_{\mathrm{ch}}\gamma^\top, \qquad \nabla V(p) = N_{\mathrm{ch}}(\sigma^2)^\top,$$

both row vectors.

Now apply the quotient rule to $M(p) = \delta^2/V$:

$$\nabla M(p) = \frac{2\delta}{V}\,\nabla\delta - \frac{\delta^2}{V^2}\,\nabla V.$$

Substituting:

$$\nabla M(p) = \frac{2\delta}{V}(-N_{\mathrm{ch}}\gamma^\top) - \frac{\delta^2}{V^2}N_{\mathrm{ch}}(\sigma^2)^\top = -N_{\mathrm{ch}}\left[\frac{2\delta}{V}\,\gamma^\top + \frac{\delta^2}{V^2}\,(\sigma^2)^\top\right].$$

7.3 Likelihood Hessian

Write:

$$M(p) = \delta^2 V^{-1}.$$

We already have:

$$\nabla M(p) = a(p)\,\nabla\delta(p) - b(p)\,\nabla V(p),$$

with:

$$a(p) = \frac{2\delta}{V}, \qquad b(p) = \frac{\delta^2}{V^2}.$$

Because δ and V are affine in p, their Hessians vanish: $\nabla^2\delta = 0$, $\nabla^2 V = 0$. Thus:

$$\nabla^2 M(p) = (\nabla a)\,\nabla\delta - (\nabla b)\,\nabla V.$$

It remains to compute ∇a and ∇b.

Gradient of $a(p)$.

$$a(p) = \frac{2\delta}{V}.$$

Using the quotient rule:

$$\nabla a = \frac{2}{V}\,\nabla\delta - \frac{2\delta}{V^2}\,\nabla V.$$

Substitute:

$$\nabla a = \frac{2}{V}(-N_{\mathrm{ch}}\gamma^\top) - \frac{2\delta}{V^2}N_{\mathrm{ch}}(\sigma^2)^\top = -\frac{2N_{\mathrm{ch}}}{V}\gamma^\top - \frac{2N_{\mathrm{ch}}\delta}{V^2}(\sigma^2)^\top.$$

Gradient of $b(p)$.

$$b(p) = \frac{\delta^2}{V^2}.$$

Differentiating:

$$\nabla b = 2\frac{\delta}{V^2}\,\nabla\delta - 2\frac{\delta^2}{V^3}\,\nabla V.$$

Substitute:

$$\nabla b = 2\frac{\delta}{V^2}(-N_{\mathrm{ch}}\gamma^\top) - 2\frac{\delta^2}{V^3}N_{\mathrm{ch}}(\sigma^2)^\top = -\frac{2N_{\mathrm{ch}}\delta}{V^2}\gamma^\top - \frac{2N_{\mathrm{ch}}\delta^2}{V^3}(\sigma^2)^\top.$$

Assembling the Hessian. Recall:

$$\nabla^2 M(p) = (\nabla a)\,\nabla\delta - (\nabla b)\,\nabla V.$$

Using:

$$\nabla\delta = -N_{\mathrm{ch}}\gamma^\top, \qquad \nabla V = N_{\mathrm{ch}}(\sigma^2)^\top,$$

we obtain:

$$\nabla^2 M(p) = (\nabla a)(-N_{\mathrm{ch}}\gamma^\top) - (\nabla b)N_{\mathrm{ch}}(\sigma^2)^\top.$$

Substituting the expressions for ∇a and ∇b and collecting terms, we arrive at:

$$\nabla^2 M(p) = 2N_{\mathrm{ch}}^2\left[\frac{1}{V}\gamma\gamma^\top + \frac{\delta}{V^2}(\gamma(\sigma^2)^\top + \sigma^2\gamma^\top) + \frac{\delta^2}{V^3}\sigma^2(\sigma^2)^\top\right].$$

7.4 Full Gradient and Hessian

Combining prior and likelihood:

$$\begin{aligned}\mathbf{g}(p) &= \nabla E(p) = \nabla M(p) + \nabla P(p)\\ &= 2N_{\mathrm{ch}}(p-\mu)\Sigma_p^{-1} - N_{\mathrm{ch}}\left[\frac{2\delta}{V}\,\gamma^\top + \frac{\delta^2}{V^2}\,(\sigma^2)^\top\right],\end{aligned}$$

and:

$$\begin{aligned}\mathbf{H}(p) &= \nabla^2 E(p) = \nabla^2 M(p) + \nabla^2 P(p)\\ &= 2N_{\mathrm{ch}}\Sigma_p^{-1} + 2N_{\mathrm{ch}}^2\left[\frac{1}{V}\gamma\gamma^\top + \frac{\delta}{V^2}(\gamma(\sigma^2)^\top + \sigma^2\gamma^\top) + \frac{\delta^2}{V^3}\sigma^2(\sigma^2)^\top\right].\end{aligned}$$

7.5 Rank-1 Factorisation of the Hessian

Define:

$$v = \gamma + \frac{\delta}{V}\sigma^2.$$

Compute:

$$vv^\top = \gamma\gamma^\top + \frac{\delta}{V}(\gamma(\sigma^2)^\top + \sigma^2\gamma^\top) + \frac{\delta^2}{V^2}\sigma^2(\sigma^2)^\top.$$

Then:

$$\frac{N_{\text{ch}}}{V}vv^\top = \frac{N_{\text{ch}}}{V}\gamma\gamma^\top + \frac{N_{\text{ch}}\delta}{V^2}(\gamma(\sigma^2)^\top + \sigma^2\gamma^\top) + \frac{N_{\text{ch}}\delta^2}{V^3}\sigma^2(\sigma^2)^\top,$$

which matches exactly the bracketed term in the expression for $\nabla^2 M(p)$, scaled by $2N_{\text{ch}}^2$.

Thus:

$$\nabla^2 M(p) = 2N_{\text{ch}}^2 \frac{1}{V} vv^\top,$$

and so the full Hessian is:

$$\mathbf{H}(p) = 2N_{\text{ch}}\Sigma_p^{-1} + 2N_{\text{ch}}^2\frac{1}{V}vv^\top = 2N_{\text{ch}}\left[\Sigma_p^{-1} + \frac{N_{\text{ch}}}{V}vv^\top\right].$$

Introducing

$$u = \sqrt{\frac{N_{\text{ch}}}{V}}\, v,$$

we recover:

$$\mathbf{H}(p) = 2N_{\text{ch}}(\Sigma_p^{-1} + uu^\top),$$

which is the starting point for the Sherman–Morrison inversion in Section 4.

This completes the detailed derivation with consistent vector orientations, explicit dependence of δ, V and v on p, and with all explicit uses of Σ_p^{-1} eliminated from the implementable formulas for p_{post} and Σ_{post}.

8. Exact MAP Energy with $\log V$ and Rank-2 Hessian

In the previous sections we defined the energy as

$$E_{\text{quasi}}(p) = \frac{\delta(p)^2}{V(p)} + (p-\mu)\big(N_{\text{ch}}\Sigma_p^{-1}\big)(p-\mu)^\top,$$

which corresponds to a *quasi-likelihood* that uses $V(p)$ only as a weight for the squared residual. This choice yields a rank-1 Hessian update and a simple closed form for Σ_{post}.

If we take seriously the generative model

$$y \mid p \sim \mathcal{N}\big(N_{\text{ch}}(p \cdot \gamma),\, V(p)\big),$$

then the *exact* negative log-likelihood (up to an additive constant) is

$$E_{\text{like}}(p) = \log V(p) + \frac{\delta(p)^2}{V(p)}.$$

In this section we incorporate the $\log V(p)$ term and work with the *exact* negative log-posterior

$$E_{\text{exact}}(p) = \underbrace{\log V(p) + \frac{\delta(p)^2}{V(p)}}_{\text{data}} + \underbrace{(p-\mu)\big(N_{\text{ch}}\Sigma_p^{-1}\big)(p-\mu)^\top}_{\text{prior}} + \text{const}.$$

As before:

$$\delta(p) = y - N_{\text{ch}}(p \cdot \gamma), \qquad V(p) = \epsilon^2 + N_{\text{ch}}(p \cdot \sigma^2),$$

and

$$v(p) = \gamma + \frac{\delta(p)}{V(p)}\,\sigma^2.$$

The prior log-determinant term (pseudo-determinant of the singular Σ_p) is a constant in p and therefore *does not* affect the MAP; we ignore it here.

8.1 Exact Gradient

We decompose

$$E_{\text{exact}}(p) = L(p) + M(p) + P(p),$$

with

$$L(p) = \log V(p), \quad M(p) = \frac{\delta(p)^2}{V(p)}, \quad P(p) = (p-\mu)\big(N_{\text{ch}}\Sigma_p^{-1}\big)(p-\mu)^\top.$$

The gradients of M and P were already computed in Section 7:

$$\nabla M(p) = -N_{\text{ch}}\left[\frac{2\delta(p)}{V(p)}\,\gamma^\top + \frac{\delta(p)^2}{V(p)^2}\,(\sigma^2)^\top\right],$$

$$\nabla P(p) = 2N_{\mathrm{ch}}(p-\mu)\Sigma_p^{-1}.$$

For the new term $L(p) = \log V(p)$ we have

$$\nabla L(p) = \frac{1}{V(p)}\,\nabla V(p) = \frac{N_{\mathrm{ch}}}{V(p)}\,(\sigma^2)^\top,$$

since $\nabla V(p) = N_{\mathrm{ch}}(\sigma^2)^\top$.

Thus the *exact* gradient becomes

$$\mathbf{g}_{\mathrm{exact}}(p) = \nabla E_{\mathrm{exact}}(p) = 2N_{\mathrm{ch}}(p-\mu)\Sigma_p^{-1} - N_{\mathrm{ch}}\left[\frac{2\delta(p)}{V(p)}\,\gamma^\top + \frac{\delta(p)^2}{V(p)^2}\,(\sigma^2)^\top - \frac{1}{V(p)}\,(\sigma^2)^\top\right].$$

Equivalently, grouping the $(\sigma^2)^\top$ terms:

$$\boxed{\mathbf{g}_{\mathrm{exact}}(p) = 2N_{\mathrm{ch}}(p-\mu)\Sigma_p^{-1} - N_{\mathrm{ch}}\left[\frac{2\delta(p)}{V(p)}\,\gamma^\top + \left(\frac{\delta(p)^2}{V(p)^2} - \frac{1}{V(p)}\right)(\sigma^2)^\top\right].}$$

This is what must be used in a true Newton iteration targeting the exact MAP.

8.2 Exact Hessian and Rank-2 Structure

We now add the Hessian of $L(p)$ to the Hessian derived in Section 7 for $M(p) + P(p)$.

From Section 7:

$$\nabla^2 P(p) = 2N_{\mathrm{ch}}\Sigma_p^{-1},$$

$$\nabla^2 M(p) = 2N_{\mathrm{ch}}^2\left[\frac{1}{V(p)}\gamma\gamma^\top + \frac{\delta(p)}{V(p)^2}\left(\gamma(\sigma^2)^\top + \sigma^2\gamma^\top\right) + \frac{\delta(p)^2}{V(p)^3}\,\sigma^2\sigma^{2\top}\right].$$

For $L(p) = \log V(p)$, using $\nabla V(p) = N_{\mathrm{ch}}(\sigma^2)^\top$ and $\nabla^2 V(p) = 0$:

$$\nabla^2 L(p) = -\frac{1}{V(p)^2}\left(\nabla V(p)\right)^\top \nabla V(p) = -\frac{N_{\mathrm{ch}}^2}{V(p)^2}\,\sigma^2\sigma^{2\top}.$$

Therefore:

$$\nabla^2\big(M(p)+L(p)\big) = 2N_{\mathrm{ch}}^2\left[\frac{1}{V(p)}\gamma\gamma^\top + \frac{\delta(p)}{V(p)^2}\left(\gamma(\sigma^2)^\top + \sigma^2\gamma^\top\right) + \left(\frac{\delta(p)^2}{V(p)^3} - \frac{1}{2V(p)^2}\right)\sigma^2\sigma^{2\top}\right].$$

Adding the prior Hessian,

$$\nabla^2 P(p) = 2N_{\mathrm{ch}}\Sigma_p^{-1},$$

we obtain the full exact Hessian:

$$\mathbf{H}_{\text{exact}}(p) = 2N_{\text{ch}}\Sigma_p^{-1} + 2N_{\text{ch}}^2 \left[\frac{1}{V(p)}\gamma\gamma^\top + \frac{\delta(p)}{V(p)^2}\left(\gamma(\sigma^2)^\top + \sigma^2\gamma^\top\right) + \left(\frac{\delta(p)^2}{V(p)^3} - \frac{1}{2V(p)^2}\right)\sigma^2\sigma^{2\top}\right].$$

It is convenient to rewrite this as a *rank-2* update of the prior precision. As before, define

$$v(p) = \gamma + \frac{\delta(p)}{V(p)}\sigma^2,$$

and let

$$u_1(p) = v(p), \qquad u_2(p) = \sigma^2.$$

We will write the Hessian in the form

$$\mathbf{H}_{\text{exact}}(p) = 2N_{\text{ch}}\big[\Sigma_p^{-1} + U(p)C(p)U(p)^\top\big],$$

with

$$U(p) = \big[u_1(p), u_2(p)\big] \in \mathbb{R}^{d\times 2},$$

and $C(p)$ a 2×2 diagonal matrix.

From the rank-1 case we know that

$$\frac{N_{\text{ch}}}{V(p)}v(p)v(p)^\top = \frac{N_{\text{ch}}}{V(p)}u_1(p)u_1(p)^\top$$

generates the M-part of the Hessian without $\log V$. The extra term from $L(p)$ is

$$-\frac{N_{\text{ch}}^2}{V(p)^2}\sigma^2\sigma^{2\top} = 2N_{\text{ch}}\left(-\frac{N_{\text{ch}}}{2V(p)^2}\right)u_2(p)u_2(p)^\top.$$

Thus we can take

$$C(p) = \begin{pmatrix}\alpha(p) & 0\\ 0 & \beta(p)\end{pmatrix}, \qquad \alpha(p) = \frac{N_{\text{ch}}}{V(p)}, \qquad \beta(p) = -\frac{N_{\text{ch}}}{2V(p)^2},$$

so that

$$U(p)C(p)U(p)^\top = \alpha(p)u_1(p)u_1(p)^\top + \beta(p)u_2(p)u_2(p)^\top.$$

We have therefore

$$\boxed{\mathbf{H}_{\text{exact}}(p) = 2N_{\text{ch}}\big[\Sigma_p^{-1} + U(p)C(p)U(p)^\top\big],}$$

with a *rank-2* update governed by $v(p)$ and σ^2.

8.3 Woodbury Inversion and Exact Laplace Covariance

The Woodbury identity for a rank-k update says that, for invertible A,

$$(A + UCU^\top)^{-1} = A^{-1} - A^{-1}U\big(C^{-1} + U^\top A^{-1}U\big)^{-1}U^\top A^{-1},$$

where U is $d \times k$ and C is $k \times k$.

We apply this in the intrinsic $(d-1)$-dimensional subspace with

$$A = \Sigma_p^{-1}, \qquad A^{-1} = \Sigma_p, \qquad U = U(p_0), \qquad C = C(p_0),$$

for an expansion point p_0. Then

$$\big(\Sigma_p^{-1} + UCU^\top\big)^{-1} = \Sigma_p - \Sigma_p UKU^\top \Sigma_p,$$

where

$$K = \big(C^{-1} + U^\top \Sigma_p U\big)^{-1} \in \mathbb{R}^{2\times 2}.$$

Since

$$\mathbf{H}_{\text{exact}}(p_0) = 2N_{\text{ch}}\big(\Sigma_p^{-1} + UCU^\top\big),$$

we get

$$\mathbf{H}_{\text{exact}}(p_0)^{-1} = \frac{1}{2N_{\text{ch}}}\Big[\Sigma_p - \Sigma_p UKU^\top \Sigma_p\Big].$$

The *exact* Laplace covariance at p_0 is

$$\Sigma_{\text{post}}^{\text{exact}}(p_0) = 2\,\mathbf{H}_{\text{exact}}(p_0)^{-1} = \frac{1}{N_{\text{ch}}}\Big[\Sigma_p - \Sigma_p UKU^\top \Sigma_p\Big].$$

Explicitly, at p_0:

$$U = \big[v(p_0),\ \sigma^2\big], \qquad C = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

with

$$\alpha = \frac{N_{\text{ch}}}{V(p_0)}, \qquad \beta = -\frac{N_{\text{ch}}}{2V(p_0)^2}.$$

The 2×2 matrix

$$M = C^{-1} + U^\top \Sigma_p U$$

is

$$M = \begin{pmatrix} \alpha^{-1} + v^\top \Sigma_p v & v^\top \Sigma_p \sigma^2 \\ \sigma^{2\top} \Sigma_p v & \beta^{-1} + \sigma^{2\top} \Sigma_p \sigma^2 \end{pmatrix} = \begin{pmatrix} \dfrac{V}{N_{\text{ch}}} + s & b \\ b & -\dfrac{2V(p_0)^2}{N_{\text{ch}}} + c \end{pmatrix},$$

where

$$s = v^\top \Sigma_p v, \qquad b = v^\top \Sigma_p \sigma^2, \qquad c = \sigma^{2\top} \Sigma_p \sigma^2.$$

Then

$$K = M^{-1}$$

is the inverse of a *2-by-2* matrix, which can always be computed analytically or numerically with negligible cost relative to the channel-ensemble computations.

Thus the exact MAP covariance is

$$\boxed{\Sigma_{\text{post}}^{\text{exact}}(p_0) = \frac{1}{N_{\text{ch}}} \left[\Sigma_p - \Sigma_p U(p_0) K(p_0) U(p_0)^\top \Sigma_p \right],}$$

with $K(p_0) = \left(C(p_0)^{-1} + U(p_0)^\top \Sigma_p U(p_0) \right)^{-1}$.

Remark (singular prior). Again, Σ_p^{-1} appears only formally in the Woodbury algebra. The implementable expression for $\Sigma_{\text{post}}^{\text{exact}}(p_0)$ contains only Σ_p, $U(p_0)$ and the small 2×2 inverse $K(p_0)$.

8.4 Exact Mean Update and Elimination of Σ_p^{-1}

The exact Laplace mean at p_0 is

$$p_{\text{post}}^{\text{exact}} = p_0 - \mathbf{g}_{\text{exact}}(p_0)\, \mathbf{H}_{\text{exact}}(p_0)^{-1} = p_0 - \frac{1}{2} \mathbf{g}_{\text{exact}}(p_0)\, \Sigma_{\text{post}}^{\text{exact}}(p_0).$$

Write

$$\Delta p = p_0 - \mu,$$

and decompose the gradient as

$$\mathbf{g}_{\text{exact}}(p_0) = 2 N_{\text{ch}} \Delta p\, \Sigma_p^{-1} - N_{\text{ch}}\, q^\top,$$

where

$$q^\top = \frac{2\delta}{V} \gamma^\top + \left(\frac{\delta^2}{V^2} - \frac{1}{V} \right) (\sigma^2)^\top,$$

with $\delta = \delta(p_0)$ and $V = V(p_0)$.

Then

$$p_{\text{post}}^{\text{exact}} = p_0 - N_{\text{ch}} \Delta p\, \Sigma_p^{-1} \Sigma_{\text{post}}^{\text{exact}}(p_0) + \frac{N_{\text{ch}}}{2} q^\top \Sigma_{\text{post}}^{\text{exact}}(p_0).$$

Using the exact covariance expression

$$\Sigma_{\text{post}}^{\text{exact}}(p_0) = \frac{1}{N_{\text{ch}}}\left[\Sigma_p - \Sigma_p U K U^\top \Sigma_p\right],$$

we compute

$$\begin{aligned}
\Sigma_p^{-1}\Sigma_{\text{post}}^{\text{exact}}(p_0) &= \Sigma_p^{-1}\left[\frac{1}{N_{\text{ch}}}\big(\Sigma_p - \Sigma_p U K U^\top \Sigma_p\big)\right] \\
&= \frac{1}{N_{\text{ch}}}\big(I - U K U^\top \Sigma_p\big).
\end{aligned}$$

Thus

$$N_{\text{ch}}\Delta p\,\Sigma_p^{-1}\Sigma_{\text{post}}^{\text{exact}}(p_0) = \Delta p - \Delta p\, U K U^\top \Sigma_p.$$

Substituting back:

$$\begin{aligned}
p_{\text{post}}^{\text{exact}} &= p_0 - \left[\Delta p - \Delta p\, U K U^\top \Sigma_p\right] + \frac{N_{\text{ch}}}{2} q^\top \Sigma_{\text{post}}^{\text{exact}}(p_0) \\
&= (p_0 - \Delta p) + \Delta p\, U K U^\top \Sigma_p + \frac{N_{\text{ch}}}{2} q^\top \Sigma_{\text{post}}^{\text{exact}}(p_0).
\end{aligned}$$

Since $p_0 - \Delta p = \mu$, we obtain the exact mean update in a form with no explicit Σ_p^{-1}:

$$\boxed{p_{\text{post}}^{\text{exact}} = \mu + \Delta p\, U(p_0) K(p_0) U(p_0)^\top \Sigma_p + \frac{N_{\text{ch}}}{2}\, q^\top \Sigma_{\text{post}}^{\text{exact}}(p_0),}$$

with

$$\Delta p = p_0 - \mu, \quad q^\top = \frac{2\delta}{V}\,\gamma^\top + \left(\frac{\delta^2}{V^2} - \frac{1}{V}\right)(\sigma^2)^\top,$$

and $\Sigma_{\text{post}}^{\text{exact}}(p_0)$ given above.

8.5 Summary: Rank-1 vs Rank-2 Schemes

We now have two related Laplace schemes:

- **Quasi-Laplace (rank-1)**:
  - Energy: $E_{\text{quasi}}(p) = \delta(p)^2/V(p) + P(p)$.
  - Hessian: rank-1 update, closed form $\Sigma_{\text{post}}(p_0) = \dfrac{1}{N_{\text{ch}}}\Sigma_p - \dfrac{1}{V(p_0) + N_{\text{ch}} s(p_0)}\Sigma_p v(p_0) v(p_0)^\top$
  - Mean update: Eq. (2) in Section 5, no explicit Σ_p^{-1}.

- **Exact MAP + Laplace (rank-2)**:
  - Energy: $E_{\text{exact}}(p) = \log V(p) + \delta(p)^2/V(p) + P(p)$.
  - Hessian: rank-2 update, $\mathbf{H}_{\text{exact}}(p) = 2N_{\text{ch}}\big[\Sigma_p^{-1}{+}U(p)C(p)U(p)^\top\big]$.
  - Covariance: $\Sigma_{\text{post}}^{\text{exact}}(p_0) = \dfrac{1}{N_{\text{ch}}}\big[\Sigma_p - \Sigma_p U(p_0)K(p_0)U(p_0)^\top \Sigma_p\big]$, with a 2×2 matrix $K(p_0)$.
  - Mean update: the exact expression above, again without explicit Σ_p^{-1}.

From an implementation perspective, the heavy lifting in MacroIR/MacroDR remains the computation of the channel-ensemble statistics that define γ, σ^2, and thus $V(p)$ and $v(p)$. Going from a rank-1 to a rank-2 Hessian only requires a 2×2 matrix inversion per interval, which is negligible compared to the cost of the boundary-state and tilde-operator computations.

8. Exact MAP from the Gaussian Likelihood (with $\frac{1}{2}$ factors)

Up to now we have worked with an "energy"

$$E(p) = M(p) + P(p)$$

(and in Section 8 with $E_{\text{exact}}(p) = L(p) + M(p) + P(p)$) without explicit $\frac{1}{2}$ factors. In this section we start from the *actual* Gaussian likelihood and prior, write the *true* negative log-posterior, and then explain how it relates to the energies used in the rest of the note.

8.1 Gaussian likelihood and prior in standard form

Given a macroscopic state p, the scalar interval-averaged current is modelled as

$$y \mid p \sim \mathcal{N}\big(\mu_y(p),\, V(p)\big), \qquad \mu_y(p) = N_{\text{ch}}(p \cdot \gamma),$$

with

$$V(p) = \epsilon^2 + N_{\text{ch}}(p \cdot \sigma^2).$$

The Gaussian likelihood is

$$p(y \mid p) = \frac{1}{\sqrt{2\pi V(p)}} \exp\left(-\frac{1}{2} \frac{\big(y - \mu_y(p)\big)^2}{V(p)} \right).$$

Define

$$\delta(p) = y - N_{\mathrm{ch}}(p \cdot \gamma),$$

so that

$$p(y \mid p) = \frac{1}{\sqrt{2\pi V(p)}} \exp\!\left(-\frac{1}{2}\frac{\delta(p)^2}{V(p)}\right).$$

The log-likelihood is therefore

$$\log p(y \mid p) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log V(p) - \frac{1}{2}\frac{\delta(p)^2}{V(p)}.$$

Ignoring the constant $-\frac{1}{2}\log(2\pi)$, the *negative* log-likelihood is

$$F_{\mathrm{like}}(p) = \frac{1}{2}\log V(p) + \frac{1}{2}\frac{\delta(p)^2}{V(p)} \quad (+\text{ const}).$$

For the prior we assume

$$p \sim \mathcal{N}(\mu, \Sigma_p)$$

on the simplex, with precision

$$\Lambda_p = N_{\mathrm{ch}}\Sigma_p^{-1}.$$

The prior density is

$$p(p) \propto \exp\!\left(-\frac{1}{2}(p-\mu)\Lambda_p(p-\mu)^\top\right),$$

so the negative log-prior (up to a constant) is

$$F_{\mathrm{prior}}(p) = \frac{1}{2}(p-\mu)\Lambda_p(p-\mu)^\top = \frac{1}{2}(p-\mu)\big(N_{\mathrm{ch}}\Sigma_p^{-1}\big)(p-\mu)^\top.$$

8.2 Exact negative log-posterior and scaling to "energy"

The (unnormalised) posterior is

$$p(p \mid y) \propto p(y \mid p)\, p(p),$$

so the *exact* negative log-posterior is

$$F_{\mathrm{exact}}(p) = F_{\mathrm{like}}(p) + F_{\mathrm{prior}}(p) \quad (+\text{ const}),$$

that is

$$F_{\text{exact}}(p) = \frac{1}{2}\log V(p) + \frac{1}{2}\frac{\delta(p)^2}{V(p)} + \frac{1}{2}(p-\mu)\big(N_{\text{ch}}\Sigma_p^{-1}\big)(p-\mu)^\top \quad (+\text{ const}).$$

This is what one should minimise to obtain the true MAP:

$$p^\star = \arg\min_p F_{\text{exact}}(p).$$

For algebraic convenience in the rest of the note we work with an *energy* that is simply twice the negative log-posterior:

$$E_{\text{exact}}(p) := 2F_{\text{exact}}(p).$$

Up to an additive constant,

$$E_{\text{exact}}(p) = \underbrace{\log V(p) + \frac{\delta(p)^2}{V(p)}}_{L(p)+M(p)} + \underbrace{(p-\mu)\big(N_{\text{ch}}\Sigma_p^{-1}\big)(p-\mu)^\top}_{P(p)}.$$

This is exactly the decomposition used in Section 8:

$$E_{\text{exact}}(p) = L(p) + M(p) + P(p).$$

Important: multiplying F_{exact} by a positive constant does not change the MAP. Moreover, since

$$E_{\text{exact}}(p) = 2F_{\text{exact}}(p),$$

their Hessians satisfy

$$\nabla^2 E_{\text{exact}}(p) = 2\,\nabla^2 F_{\text{exact}}(p),$$

and the Laplace covariance of the posterior,

$$\Sigma_{\text{Lap}} \approx \big[\nabla^2 F_{\text{exact}}(p^\star)\big]^{-1},$$

can be computed as

$$\Sigma_{\text{Lap}} = 2\big[\nabla^2 E_{\text{exact}}(p^\star)\big]^{-1}.$$

This is exactly the convention used earlier in the note, where we wrote

$$\Sigma_{\text{post}} = 2\,\mathbf{H}^{-1}$$

with $\mathbf{H} = \nabla^2 E$.

Thus:

- all gradient and Hessian formulas derived for E_{exact} (including the rank-2 Woodbury structure in Section 8) are correct and correspond to $2\times$ the exact negative log-posterior;
- the MAP is obtained by minimising F_{exact} or equivalently E_{exact};
- the Laplace covariance is always $\Sigma_{\text{Lap}} = 2\,\mathbf{H}_{\text{exact}}^{-1}$ when $\mathbf{H}_{\text{exact}} = \nabla^2 E_{\text{exact}}$.

8.3 Exact vs quasi energy

For comparison:

- The *exact* data term from the Gaussian likelihood is

$$F_{\text{like}}(p) = \frac{1}{2}\log V(p) + \frac{1}{2}\frac{\delta(p)^2}{V(p)} \quad (+\text{ const}),$$

  and the corresponding energy contribution in this note is

$$L(p) + M(p) = \log V(p) + \frac{\delta(p)^2}{V(p)} = 2F_{\text{like}}(p) \quad (+\text{ const}).$$

- The earlier "quasi" scheme drops the $\log V(p)$ term and keeps only $\delta(p)^2/V(p)$ (again up to a factor 2), which yields a rank-1 Hessian and the simpler covariance formula.

In other words:

- if you want the *exact* Gaussian MAP, work with F_{exact} defined above; all rank-2 Hessian and Woodbury formulas in Section 8 apply directly via $E_{\text{exact}} = 2F_{\text{exact}}$ and $\Sigma_{\text{Lap}} = 2\mathbf{H}_{\text{exact}}^{-1}$;
- if you are happy with the quasi-likelihood (no $\log V$ term), you can use the rank-1 scheme from the earlier sections.