Supplementary Material: Full Derivations for MacroIR Interval Analysis

S1. Boundary-State Statistics

Let $\mu_0^{\text{prior}} \in \mathbb{R}^K$ and $\Sigma_0^{\text{prior}} \in \mathbb{R}^{K\times K}$ denote the start-of-interval state distribution. The Markov transition matrix over the interval is

$$P(t) = e^{Qt}, \qquad P_{i_0\to i_t}(t).$$

Define $N_{i_0\to i_t}$ as the number of channels starting in i_0 and ending in i_t.

Mean.

$$\mu^{\text{prior}}_{0\to t,(i_0\to i_t)} = \mu^{\text{prior}}_{0,i_0}\, P_{i_0\to i_t}(t).$$

Covariance. Using multinomial splitting and the law of total covariance,

$$\begin{aligned}\Sigma^{\text{prior}}_{0\to t,(i_0\to i_t)(j_0\to j_t)} &= P_{i_0\to i_t}(t)\big(\Sigma^{\text{prior}}_{0,i_0 j_0} - \delta_{i_0 j_0}\mu^{\text{prior}}_{0,i_0}\big)P_{j_0\to j_t}(t)\\ &\quad + \delta_{i_0 j_0}\delta_{i_t j_t}\,\mu^{\text{prior}}_{0,i_0} P_{i_0\to i_t}(t).\end{aligned} \tag{1}$$

S2. Predicted Interval-Averaged Current

Define the boundary-conditioned current expectation

$$(\gamma_{0\to t})_{(i_0\to i_t)} = \overline{\Gamma}_{i_0\to i_t}.$$

Marginalizing over end states gives

$$(\overline{\gamma}_0)_{i_0} = \sum_{i_t} P_{i_0\to i_t}(t)\,\overline{\Gamma}_{i_0\to i_t}.$$

Thus,

$$\overline{y}^{\text{pred}}_{0\to t} = N_{\text{ch}} \sum_{i_0,i_t} \mu^{\text{prior}}_{0,i_0} P_{i_0\to i_t}(t)\overline{\Gamma}_{i_0\to i_t}.$$

S3. Derivation of $\widetilde{\gamma^T\Sigma\gamma}$

Start from

$$\gamma^T_{0\to t}\Sigma^{\text{prior}}_{0\to t}\gamma_{0\to t} = \sum_{i_0,i_t}\sum_{j_0,j_t} \overline{\Gamma}_{i_0\to i_t}\,\Sigma^{\text{prior}}_{0\to t,(i_0\to i_t)(j_0\to j_t)}\,\overline{\Gamma}_{j_0\to j_t}.$$

Substitute the covariance and split into:

$$T_1 = \overline{\gamma}_0^T (\Sigma_0^{\text{prior}} - \text{diag}(\mu_0^{\text{prior}}))\overline{\gamma}_0,$$

$$T_2 = \sum_{i_0} \mu_{0,i_0}^{\text{prior}} \sum_{i_t} (\overline{\Gamma}_{i_0 \to i_t} P_{i_0 \to i_t}(t))^2.$$

Thus,

$$\widetilde{\gamma^T \Sigma \gamma} = T_1 + T_2.$$

S4. Derivation of the Vector $\widetilde{\gamma^T \Sigma}$

Define

$$(\widetilde{\gamma^T \Sigma})_{i_t} = \sum_{i_0, j_0, j_t} \overline{\Gamma}_{j_0 \to j_t} \Sigma^{\text{prior}}_{0 \to t, (j_0 \to j_t)(i_0 \to i_t)}.$$

Inserting the boundary covariance yields:

$$(\widetilde{\gamma^T \Sigma})_{i_t} = \overline{\gamma}_0^T (\Sigma_0^{\text{prior}} - \text{diag}(\mu_0^{\text{prior}}))P(:, i_t) + \sum_{i_0} \mu_{0,i_0}^{\text{prior}} P_{i_0 \to i_t}(t) \overline{\Gamma}_{i_0 \to i_t}.$$

S5. Final Covariance Update

The propagated covariance is

$$\Sigma^{\text{prop}}(t) = P(t)^T (\Sigma_0^{\text{prior}} - \text{diag}(\mu_0^{\text{prior}}))P(t) + \text{diag}(\mu^{\text{prior}}(t)), \qquad \mu^{\text{prior}}(t) = \mu_0^{\text{prior}} P(t).$$

The posterior covariance (scalar Gaussian update) is

$$\Sigma^{\text{prior}}(t) = \Sigma^{\text{prop}}(t) - \frac{1}{\sigma^2_{\overline{y}^{\text{pred}}_{0 \to t}}} \widetilde{\gamma^T \Sigma}^T \widetilde{\gamma^T \Sigma}.$$