Mean and Variance of Integrated Conductance in a Continuous-Time Markov Chain

1 Setup

Let X_t be a continuous-time Markov chain on a finite state space $\{1, \ldots, K\}$ with generator $Q \in \mathbb{R}^{K \times K}$. We write the state-dependent conductance as a vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_K)^\top$ and let $\Gamma = \mathrm{diag}(\boldsymbol{\gamma})$.

Given a time horizon $t > 0$ and a sample path of the chain, we define the *integrated conductance*

$$A(t) \;=\; \int_0^t \gamma\big(X_s\big)\, ds. \tag{1}$$

We are interested in the first two moments of $A(t)$, and in particular its mean and variance, conditional on the chain starting in state i and ending in state j:

$$X_0 = i, \qquad X_t = j.$$

Note that the interval-averaged conductance used in the main text is

$$\bar{\gamma}_{i \to j}(t) \;=\; \frac{1}{t}\, \mathbb{E}\big[A(t) \mid X_0 = i, X_t = j\big], \tag{2}$$

so moments of $\bar{\gamma}_{i \to j}(t)$ follow directly from moments of $A(t)$ by dividing by powers of t.

2 Tilted generator and Feynman–Kac representation

Define the *tilted generator*

$$Q(\alpha) = Q + \alpha\, \Gamma, \qquad \alpha \in \mathbb{R}, \tag{3}$$

and the corresponding transition matrix

$$P^{(\alpha)}(t) = e^{Q(\alpha)t}. \tag{4}$$

A standard Feynman–Kac argument for continuous-time Markov chains yields

$$P^{(\alpha)}_{ij}(t) = \mathbb{E}\Big[\exp\Big(\alpha \int_0^t \gamma(X_s)\,ds\Big)\,\mathbf{1}_{\{X_t=j\}}\,\Big|\,X_0 = i\Big]. \tag{5}$$

We abbreviate $A(t) = \int_0^t \gamma(X_s)\,ds$ and write $P(t) = P^{(0)}(t) = e^{Qt}$.

Differentiating with respect to α at $\alpha = 0$ gives

$$U_{ij}(t) := \frac{\partial}{\partial\alpha} P^{(\alpha)}_{ij}(t)\Big|_{\alpha=0} = \mathbb{E}\big[A(t)\,\mathbf{1}_{\{X_t=j\}} \mid X_0 = i\big], \tag{6}$$

$$W_{ij}(t) := \frac{\partial^2}{\partial\alpha^2} P^{(\alpha)}_{ij}(t)\Big|_{\alpha=0} = \mathbb{E}\big[A^2(t)\,\mathbf{1}_{\{X_t=j\}} \mid X_0 = i\big]. \tag{7}$$

3 Conditional mean and variance of integrated conductance

By conditioning on the final state, we obtain

$$\mathbb{E}\big[A(t) \mid X_0 = i, X_t = j\big] = \frac{U_{ij}(t)}{P_{ij}(t)}, \tag{8}$$

$$\mathbb{E}\big[A^2(t) \mid X_0 = i, X_t = j\big] = \frac{W_{ij}(t)}{P_{ij}(t)}. \tag{9}$$

Therefore the conditional variance of the integrated conductance is

$$\mathrm{Var}\big(A(t) \mid X_0 = i, X_t = j\big) = \frac{W_{ij}(t)}{P_{ij}(t)} - \left(\frac{U_{ij}(t)}{P_{ij}(t)}\right)^2. \tag{10}$$

In terms of the interval-averaged conductance $\bar{\gamma}_{i\to j}(t) = A(t)/t$, we have

$$\mathbb{E}\big[\bar{\gamma}_{i\to j}(t)\big] = \frac{1}{t}\,\frac{U_{ij}(t)}{P_{ij}(t)}, \tag{11}$$

$$\mathrm{Var}\big(\bar{\gamma}_{i\to j}(t)\big) = \frac{1}{t^2}\left[\frac{W_{ij}(t)}{P_{ij}(t)} - \left(\frac{U_{ij}(t)}{P_{ij}(t)}\right)^2\right]. \tag{12}$$

4 Block-matrix representation

The derivatives in (6)–(7) can be computed without explicit diagonalization of Q using matrix exponentials of block matrices.

4.1 First derivative via a $2K \times 2K$ block matrix

Consider the augmented generator

$$\mathcal{Q}_2 = \begin{pmatrix} Q & \Gamma \\ 0 & Q \end{pmatrix} \in \mathbb{R}^{2K\times 2K}. \tag{13}$$

Its exponential has the block form

$$e^{\mathcal{Q}_2 t} = \begin{pmatrix} e^{Qt} & F_1(t) \\ 0 & e^{Qt} \end{pmatrix}, \tag{14}$$

where

$$F_1(t) = \int_0^t e^{Qu}\,\Gamma\, e^{Q(t-u)}\,du = \frac{\partial}{\partial\alpha} e^{(Q+\alpha\Gamma)t}\Big|_{\alpha=0}. \tag{15}$$

Identifying $F_1(t)$ with the first derivative of the tilted semigroup, we have

$$U(t) = F_1(t), \tag{16}$$

so the mean of $A(t)$ conditional on (i, j) can be written as

$$\mathbb{E}[A(t) \mid X_0 = i, X_t = j] = \frac{F_{1,ij}(t)}{[e^{Qt}]_{ij}}. \tag{17}$$

4.2 Second derivative via a $3K \times 3K$ block matrix

To access the second derivative simultaneously, define the $3K \times 3K$ block matrix

$$\mathcal{Q}_3 = \begin{pmatrix} Q & \Gamma & 0 \\ 0 & Q & \Gamma \\ 0 & 0 & Q \end{pmatrix}. \tag{18}$$

Its exponential has the form

$$e^{\mathcal{Q}_3 t} = \begin{pmatrix} e^{Qt} & F_1(t) & F_2(t) \\ 0 & e^{Qt} & F_1(t) \\ 0 & 0 & e^{Qt} \end{pmatrix}, \tag{19}$$

where $F_1(t)$ is as above, and

$$F_2(t) = \int_0^t ds_1 \int_0^{s_1} ds_2 \, e^{Qs_2} \, \Gamma \, e^{Q(s_1-s_2)} \, \Gamma \, e^{Q(t-s_1)}. \tag{20}$$

The second derivative of the tilted semigroup can be written as the symmetric double integral

$$W(t) = \frac{\partial^2}{\partial\alpha^2} e^{(Q+\alpha\Gamma)t}\bigg|_{\alpha=0} = \int_0^t \int_0^t e^{Qs_<} \, \Gamma \, e^{Q(s_>-s_<)} \, \Gamma \, e^{Q(t-s_>)} \, ds_1 \, ds_2, \tag{21}$$

where $s_< = \min(s_1, s_2)$ and $s_> = \max(s_1, s_2)$. Since the square $[0,t] \times [0,t]$ splits into two identical triangular regions $s_2 < s_1$ and $s_1 < s_2$, we obtain

$$W(t) = 2F_2(t). \tag{22}$$

Combining this with (8)–(10), we find

$$\mathbb{E}\big[A(t) \mid X_0 = i, X_t = j\big] = \frac{F_{1,ij}(t)}{[e^{Qt}]_{ij}}, \tag{23}$$

$$\mathbb{E}\big[A^2(t) \mid X_0 = i, X_t = j\big] = \frac{2F_{2,ij}(t)}{[e^{Qt}]_{ij}}, \tag{24}$$

$$\mathrm{Var}\big(A(t) \mid X_0 = i, X_t = j\big) = \frac{2F_{2,ij}(t)}{[e^{Qt}]_{ij}} - \left(\frac{F_{1,ij}(t)}{[e^{Qt}]_{ij}}\right)^2. \tag{25}$$