Logic for Natural Language, Logic in Natural Language

Welcome to the Course!

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This course presents logical systems tuned to natural language

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- The raison d'être of logic is the study of inference in language.
- However, modern logic was developed in connection with the foundations of mathematics.
- So we have a mismatch, leading to
  — neglect of language in the first place
  — use of first-order logic and no other tools
- First-order logic is both too big and too small:
  — cannot handle many interesting phenomena
  — is undecidable

Natural logic: restore natural language inference as a centerpiece of logic

Program

Show that significant parts of natural language inference
can be carried out in decidable logical systems,
preferably in "light" systems.

To axiomatize as much as possible,
because the resulting logical systems are likely to be interesting.

To ask how much of language could have been done
if the traditional logicians
had today's mathematical tools.

What will you learn in this class?

The class will have a lot of technical material connected to the basic notions of topics such as

- model theory
- algebraic logic
- modal logic
- decidable fragments of first-order logic
- the typed lambda calculus and its connection to grammar and semantics

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The course will also present a lot of educational material. This could be an introduction to logic or a "bridge to mathematical proofs" course.

I comment on educational points in green boxes.

Examples of inferences which we will see in this course

These are the basic data that the course will account for

First, a few examples from the classical syllogistic:

$$\frac{\text{All men are mortal} \quad \text{Socrates is a man}}{\text{Socrates is mortal}} \tag{1}$$

$$\frac{\text{All auctioneers are curmudgeons} \quad \text{No bartenders are curmudgeons}}{\text{No auctioneers are bartenders}} \tag{2}$$

Syllogistic logic is under-appreciated!
My aim in the first two days of the course is to convince you that extended syllogistic logics are very interesting indeed.

Our "syntax" of sentences will give us

All X are Y
Some X are Y
No X are Y
but no boolean connectives (!), at least not yet

We adopt the evident semantics.

We craft a logical system which has formal proofs
using our syntax of sentences and nothing else.

After this, we want to extend the idea of syllogistic logic.

BASIC SYLLOGISTIC LOGIC: ALL, AND SOME

Syntax: *All p are q*, *Some p are q*

Semantics: A model $\mathcal{M}$ is a set M,
and for each noun p we have an interpretation $[\![p]\!] \subseteq M$.

$$\begin{array}{lcl} \mathcal{M} \models \textit{All p are q} & \text{iff} & [\![p]\!] \subseteq [\![q]\!] \\ \mathcal{M} \models \textit{Some p are q} & \text{iff} & [\![p]\!] \cap [\![q]\!] \neq \emptyset \end{array}$$

Proof system:

$$\frac{}{\textit{All p are p}} \qquad \frac{\textit{All p are n} \quad \textit{All n are q}}{\textit{All p are q}}$$

$$\frac{\textit{Some p are q}}{\textit{Some q are p}} \qquad \frac{\textit{Some p are q}}{\textit{Some p are p}} \qquad \frac{\textit{All q are n} \quad \textit{Some p are q}}{\textit{Some p are n}}$$

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We could also add the word *No*.
What rules do you think we would need?

If Γ is a set of formulas, we write $\mathcal{M} \models \Gamma$ if for all $\varphi \in \Gamma$, $\mathcal{M} \models \varphi$.

$\Gamma \models \varphi$ means that every $\mathcal{M} \models \Gamma$ also has $\mathcal{M} \models \varphi$.

A proof tree over Γ is a finite tree $\mathcal{T}$
whose nodes are labeled with sentences
and each node is either an element of Γ,
or comes from its parent(s) by an application of one of the rules.

$\Gamma \vdash S$ means that there is a proof tree $\mathcal{T}$ for over Γ
whose root is labeled S.

English:
If there is an n, and if all ns are ps and also qs, then some p are q.

Semantic assertion:
$\textit{Some } n \textit{ are } n, \textit{ All } n \textit{ are } p, \textit{ All } n \textit{ are } q \models \textit{Some } p \textit{ are } q.$

Proof-theoretic assertion:
$\textit{Some } n \textit{ are } n, \textit{ All } n \textit{ are } p, \textit{ All } n \textit{ are } q \vdash \textit{Some } p \textit{ are } q.$

English:

If there is an n, and if all ns are ps and also qs, then some p are q.
This is something we could check against
human intuition and performance.

Semantic assertion:

Some n are n, *All n are p*, *All n are q* $\models$ *Some p are q*.
The reasoning here would be a mathematical proof.

Proof-theoretic assertion:

Some n are n, *All n are p*, *All n are q* $\vdash$ *Some p are q*.
The proof tree is

$$\dfrac{\textit{All n are q} \qquad \dfrac{\dfrac{\textit{All n are p} \quad \textit{Some n are n}}{\textit{Some n are p}}}{\textit{Some p are n}}}{\textit{Some p are q}}$$

Example of a conclusion which doesn't follow

$$\frac{\textit{All frogs are reptiles.} \quad \textit{All frogs are animals.}}{\textit{All reptiles are animals.}}$$

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We can define a model $\mathcal{M}$ by

$$\begin{array}{lcl} M & = & \{1,2,3,4,5,6\} \\ [\![\mathit{frogs}]\!] & = & \{1,2\} \\ [\![\mathit{reptiles}]\!] & = & \{1,2,3,4\} \\ [\![\mathit{animals}]\!] & = & \{1,2,4,5,6\} \end{array}$$

In this model, the assumptions are true but the conculsion is false. So the argument is invalid.

$\textit{All frogs are reptiles}, \textit{All frogs are animals} \not\models \textit{All reptiles are animals.}$

Soundness/Completeness Theorem

$\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$

References to related work:
Łukasiewicz 1951, Westerståhl 1989.

All the logical systems in this course are complete.
If you follow most of the details, you'll learn a lot of technical material.

A first course in logic could contain the completeness results
which we'll see in the first few days
plus parallel material on propositional logic
and also *something* on first-order logic.

$$\frac{\begin{array}{l}\text{Every giraffe is taller than every gnu}\\ \text{Some gnu is taller than every lion}\\ \underline{\text{Some lion is taller than some zebra}}\end{array}}{\text{Every giraffe is taller than some zebra}} \tag{3}$$

$$\frac{\text{More students than professors run} \quad \text{More professors than deans run}}{\text{More students than deans run}} \tag{4}$$

$$\frac{\begin{array}{l}\text{At most as many xenophobics as yodelers are zookeepers}\\ \text{At most as many zookeepers as alcoholics are yodelers}\\ \text{At most as many yodelers as xenophobics are alcoholics}\end{array}}{\text{At most as many zookeepers as alcoholics are xenophobics}} \tag{5}$$

Example

Assume:

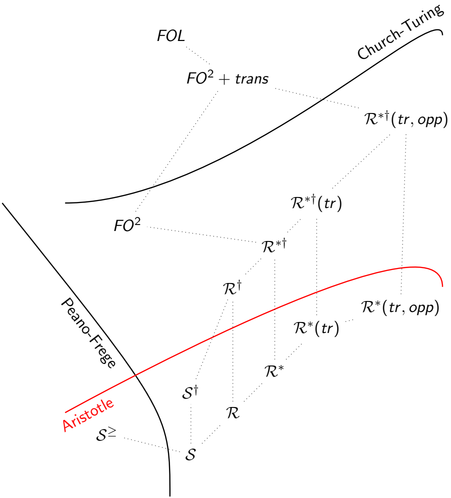
1. *There are at least as many non-y as y*
2. *There are at least as many non-z as z*
3. *All non-y are z*
4. *All x are z*

Then prove from these that *No x are y*.

Here is a formal proof in the logical system which we'll see:

$$\dfrac{\forall(x,z) \qquad \dfrac{\forall(\overline{y},z) \qquad \dfrac{\exists^{\geq}(\overline{y},y) \quad \exists^{\geq}(\overline{z},z)}{\exists^{\geq}(\overline{y},z)}\ (\textsc{Half})}{\forall(z,\overline{y})}\ (\textsc{Card Mix})}{\forall(x,\overline{y})}\ (\textsc{Barbara})$$

The Aristotle boundary is the dividing line between fragments which are formulated syllogistically and those which are not. Reductio proofs are ok. Infinitely many rules are not.



first-order logic

FO^2 + "R is trans"

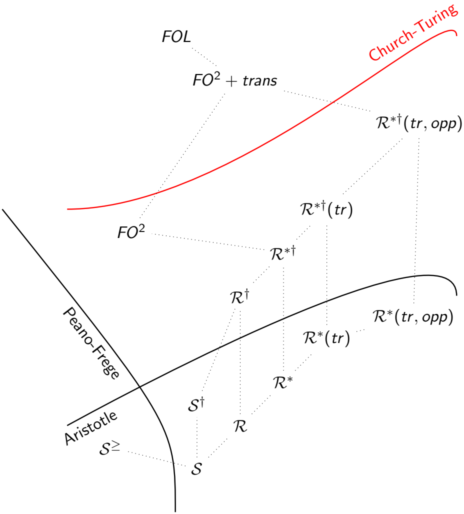
2 variable FO logic

† adds full N-negation

$\mathcal{R}^*(tr)$ + opposites
$\mathcal{R}^*$ + (transitive) comparative adjs
$\mathcal{R}$ + relative clauses
$\mathcal{S}$ + full N-negation
$\mathcal{R}$ = relational syllogistic
$\mathcal{S}^{\geq}$ adds $|p| \geq |q|$
$\mathcal{S}$: all/some/no p are q

Map of Some Natural Logics

The Church-Turing boundary is the dividing line between decidable and undecidable fragments.



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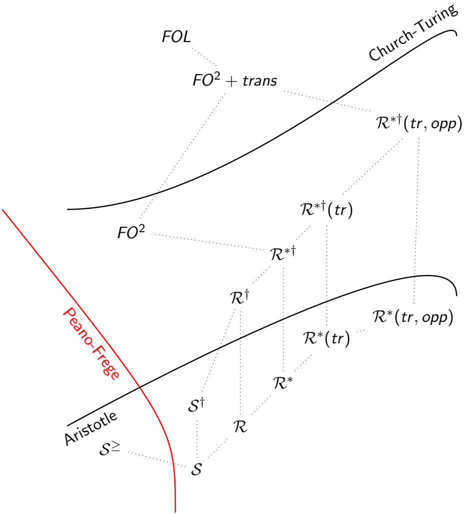
Map of Some Natural Logics

The Peano-Frege boundary divides the fragments according to whether they may be formulated in first-order logic.



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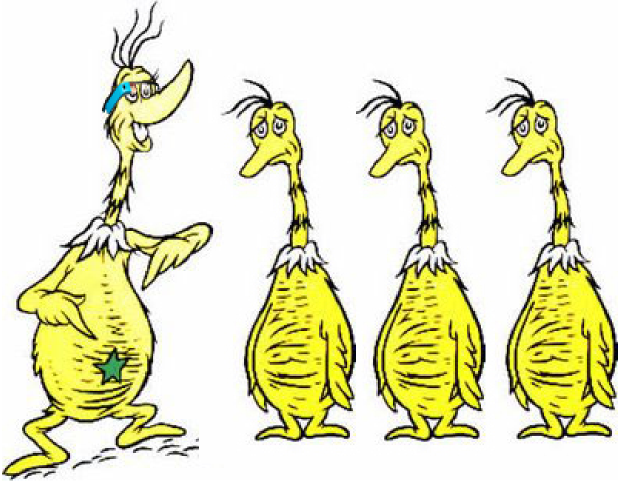
Example of where we would want derivations with variables

All xenophobics see all astronauts
All yodelers see all zookeepers
All non-yodelers see all non-astronauts
All wardens are xenophobics

All wardens see all zookeepers

1		*All xenophobics see all astronauts*	Hyp
2		*All yodelers see all zookeepers*	Hyp
3		*All non-yodelers see all non-astronauts*	Hyp
4		*All wardens are xenophobics*	Hyp
5	*Jane*	*Jane is a warden*	Hyp
6		*All wardens are xenophobics*	R, 4
7		*Jane is a xenophobic*	All Eliim, 6
8		*All xenophobics see all astronauts*	R, 2
9		*Jane sees all astronauts*	All Elim, 8
10		*Jane is a yodeler*	Hyp
11		*Jane sees all zookeepers*	Easy from 2
12		*Jane is not a yodeler*	Hyp
13		*Jane sees all zookeepers*	See below
14		*Jane sees all zookeepers*	Cases 10-11, 12-13
15		*All wardens see all zookeepers*	All Intro

1		*Jane is not a yodeler*	Hyp
2		*Jane sees all astronauts*	R, above
3		*All non-yodelers see all non-astronauts*	R, above
4		*Jane sees all non-astronauts*	All Elim, 1, 3
5	*Bob*	*Bob is a zookeeper*	Hyp
6		*Bob is astronaut*	Hyp
7		*Jane sees Bob*	All Elim, 2
8		*Bob is not astronaut*	Hyp
9		*Jane sees Bob*	All Elim, 4
10		*Jane sees Bob*	Cases
11		*Jane sees all zookeepers*	All Intro



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What "follows from" means

One sentence, A, follows from a second sentence, B,
if every time we use B in a true way,
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If we say

$$B : \text{every animal hops}$$

then it follows that

$$A : \text{every Sneetch moves}$$

animal
Sneetch
Star-Belly Sneetch

move
dance
waltz

animal
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Let's talk about a situation where

all Sneetches dance.

Which one would be true?

- all Star-Belly Sneetches dance
- all animals dance

animal
Sneetch
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- all Star-Belly Sneetches dance   true
- all animals dance   false

We write

all Sneetches$^{\downarrow}$ dance

$\uparrow$ animal
Sneetch
Star-Belly Sneetch

$\uparrow$ move
dance
waltz

---

all Sneetches$^{\downarrow}$ dance

What arrow goes on "dance"?

- all Sneetches waltz
- all Sneetches move

$\uparrow$

animal
Sneetch
Star-Belly Sneetch

$\uparrow$

move
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We write

$$\text{all Sneetches}^{\downarrow}\ \text{dance}^{\uparrow}$$

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Let's put the arrows on the words Sneetches and dance.

1. No Sneetches dance.
2. If you play loud enough music, any Sneetch will dance.
3. Any Sneetch in Zargonia would prefer to live in Yabistan.
4. If any Sneetch dances, McBean will dance, too.

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$$f(x,y) = y - x \tag{6}$$

$$g(x,y) = x + \frac{2}{y} \tag{7}$$

$$h(v,w,x,y,z) = \frac{x-y}{2^{z-(v+w)}} \tag{8}$$

What goes up? What goes down?

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The $\uparrow$ and $\downarrow$ notations have the same meaning in language as in math!

This is not an accident!

Let's look at an (easy) inference in algebra

Which is bigger, $-(7 + 2^{-3})$ or $-(7 + 2^{-4})$?

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$$\cfrac{\cfrac{\cfrac{\cfrac{3<4}{-4<-3}\ -x\text{ is antitone}}{2^{-4}<2^{-3}}\ 2^{x}\text{ is monotone}}{7+2^{-4}<7+2^{-3}}\ 7+x\text{ is monotone}}{-(7+2^{-3})<-(7+2^{-4})}\ -x\text{ is antitone}$$

$f(x)$ monotone means	if $x \leq y$, then $f(x) \leq f(y)$
$f(x)$ antitone means	if $x \leq y$, then $f(y) \leq f(x)$
	i.e., $f(x) \geq f(y)$

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$$
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$$

Occasionally in this week's lectures, I'll use

blue for syntax,
and red for semantics.

$$f(x, y^{\uparrow}) = -(x + 2^{-y})$$

Let's look at a parallel inference in language

Background: skunks $\leq$ mammals.

What do you think about this one?

$$\frac{\textit{All skunks are mammals}}{\textit{All who fear all who respect all skunks fear all who respect all mammals}}$$

Based only on our assumption, which set is bigger?

those who fear all who respect all skunks

or

those who fear all who respect all mammals

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those who fear all who respect all skunks

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$$\dfrac{\dfrac{\text{skunks} \leq \text{mammals}}{\text{respect all mammals} \leq \text{respect all skunks}}\ \lambda x.\text{respect all } x \text{ is antitone}}{\text{fear all who respect all skunks} \leq \text{fear all who respect all mammals}}\ \lambda x.\text{fear all } x \text{ is antitone}$$

respect all $x^{\downarrow}$

fear all who respect all $x^{\uparrow}$

Day-by-day plan for this course

I have arranged the course material in a number of units:

- overview + examples (today, done) + objections
- the simplest logic in the world (today)
- all + verbs + relative clauses (Tuesday)
- negation, definite descriptions, sizes of sets (Tuesday/Wednesday)
- basics on monotonicity $^{\uparrow}$ and $^{\downarrow}$, and connections to categorial grammar and semantics (Wednesday/Thursday)
- implementations and experience with language models (Friday)

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I can't cover everything without rushing, so there will be times when I mainly mention results without proofs.

There are also some worksheets and/or suggested homework exercises.

I like talking to you on the course topics.

Objections to the program of natural logic

Most natural language phenomena are not addressed:
anything "pragmatic"
vagueness, intent of speakers, poetic language

I agree with this objection!

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I don't agree with this; see below.

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To do logic fully, we need resources to handle the worst-possible phenomena

I don't agree with this; see below.

Quine, from *Word and Object*:
If we were to devise a logic of ordinary language
for direct use on sentences as they come,
we would have to complicate our rules of inference
in sundry unilluminating ways.

This is something we'll talk about throughout the week.

You decide

Consider three activites:

- A mathematics: prove the Pythagorean Theorem, $a^2 + b^2 = c^2$.
- B syntax: parse John knows his mother saw him at her house.
- C semantics: tell whether a reader of the UW Fight Song should infer that the speaker wants the Carnegie team to win.

A: mathematics B: syntax

Where would you put C: semantics?

Bow Down to Washington, Bow Down to Washington.
Mighty are the men who wear the Purple and the Gold,
Joyfully we welcome them within the Victor's fold.
We will carve our name in the Hall of Fame,
To preserve the memory of our Devotion.

So, heaven help the foes of Washington,
They're trembling at the feet of mighty Washington.
Our boys are there with bells, their fighting blood excels,
It's harder to push them over the lines than pass the Dardanelles.
So Victory's the cry of Washington
Our leather lungs together with a Rah! Rah! Rah!
And o'er the land, the loyal band
Will sing the glory of Washington forever!.

YOU DECIDE

Consider three activites:

- A mathematics: prove the Pythagorean Theorem, $a^2 + b^2 = c^2$.
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A: mathematics   B: syntax

Where would you put C: semantics?

MY VIEW: B AND C GO TOGETHER; A DOESN'T BELONG

We should not reduce formal linguistic inference to
formal inference in standard logical systems.
(Also, we really can't carry out that reduction,
for several different reasons.)