Extensions to the Logic of *All* x *are* y: Verbs, Relative Clauses, and *Only*

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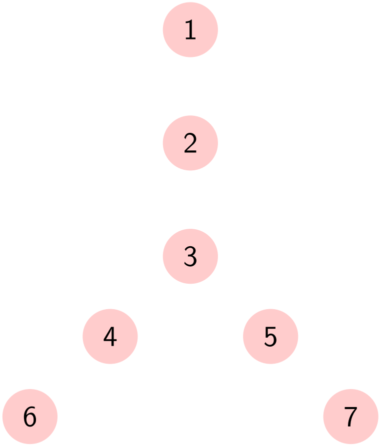
An example that we'll see a few times

Consider the model $\mathcal{M}$ with universe $\{1, \ldots, 7\}$, and with interpretations of three nouns: hawks, birds, and turtles,

$$[\![\text{hawks}]\!] = \{1\}$$

$$[\![\text{birds}]\!] = \{1, 6\}$$

$$[\![\text{turtles}]\!] = \{3, 5, 7\}$$



We want to use see as a (generic) transitive verb,
that is, a verb which takes a direct object.

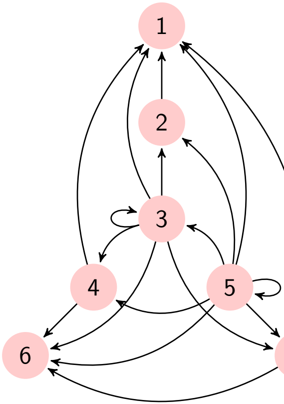
The interpretation $[\![\text{see}]\!]$ should not be a subset of the model.
It should be a set of ordered pairs.

Here's one way to do it

$[\![\text{hawks}]\!] = \{1\}$

$[\![\text{birds}]\!] = \{1, 6\}$

$[\![\text{turtles}]\!] = \{3, 5, 7\}$



$$[\![\text{see}]\!] = \{(2,1),(3,1),(3,2),(3,4),(3,6),(3,7),(4,1),(4,6),$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(5,7),(7,1),(7,6)\}$$

The formal specification of the semantics is the one just above. But the picture is clearer, and so we prefer to use it whenever we can.

Our next language in this lecture, $\mathcal{A}(\mathcal{RC})$

We start with two sets:

- a set N of nouns.
- a set V of verbs.

We make terms as follows:

- If x is a noun, then x is a term.
- If r is a verb and x is a term, then r all x is a term.

We make sentences as follows:

- If x and y are terms, then

  All x y

  is a sentence.
  These are the only sentences in the language.
  We frequently use parentheses to make things more readable.

EXAMPLES OF THE SYNTAX OF $\mathcal{A}(\mathcal{RC})$

Let's say

- N = {dogs, cats, birds, ants, ...}
- V = {see, like, hate, fear, respect, ...}

Here are some terms of $\mathcal{A}(\mathcal{RC})$:

- dogs
- see all cats
- respect all (see all birds)
- love all (respect all (see all dogs))

Note that there are infinitely many terms, and terms may occur in terms.

Let's say

- N = {dogs, cats, birds, ants, ...}
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Here are some terms of $\mathcal{A}(\mathcal{RC})$:

- dogs
- see all cats
- respect all (see all birds)
  read as respect all who see all birds
- love all (respect all (see all dogs))
  read as love all who respect all who see all birds

Note that there are infinitely many terms, and terms may occur in terms.

We read these in English using relative clauses.

Every senator who likes the president sang.
Every senator who the president likes sang.

The relative clauses are in red just above.

The first is a subject relative clause,
the second is an object relative clause

Every senator who likes the president sang.
Every senator who the president likes sang.

The relative clauses are in red just above.

The first is a subject relative clause,
the second is an object relative clause
Like adjectives, relative clauses modify a head noun:

Every female senator sang.

In English, adjectives come before their head noun,
and relative clauses come after it.

Every senator who likes the president sang.
Every senator who the president likes sang.

The relative clauses are in red just above.

The first is a subject relative clause,
the second is an object relative clause

In this lecture, all the relative clauses will be subject relative clauses.

And there will be no head nouns.

(Both restrictions can be lifted with more work.)

Examples of the syntax of $\mathcal{A}(\mathcal{RC})$

Here are some sentences of $\mathcal{A}(\mathcal{RC})$:

- all dogs animals
- all dogs cats
- all dogs (see all cats)
- all (see all cats) (fear all tigers)
  We read this one as "all who see all cats fear all tigers".
- all (love all dogs) (hate all (love all cats))
  We read this one as
  "all who love all dogs (also) hate (all who love all cats)".

Comment

The important point here is that our syntax deviates from the standard syntax in linguistics and of course from the standard syntax in logic.

A model $\mathcal{M}$ for $\mathcal{A}(\mathcal{RC})$ is

$$\mathcal{M} \quad = \quad (M, [\![\ ]\!]).$$

where M is a set, called (as before) the universe,
together with interpretations of the nouns and verbs.

For each noun p, we have an interpretation $[\![p]\!] \subseteq M$.

And for each verb r, we have an interpretation $[\![r]\!] \subseteq M \times M$.

We use an inductive definition to interpret the terms.

The model comes with interpretations of the nouns,
the "base case" of terms.

And the general case is

$$[\![\text{r all x}]\!] \quad = \quad \{m \in M : \text{for all } n \in [\![x]\!], \, (m,n) \in [\![r]\!]\}.$$

And then we say

$$\mathcal{M} \models \text{All } x \; y \quad \text{iff} \quad [\![x]\!] \subseteq [\![y]\!]$$

Here x and y are any terms.

If $[\![x]\!] = \emptyset$ in a given model, then according to our definition

$$\begin{aligned} [\![\text{r all x}]\!] &= \{m \in M : \text{for all } n \in [\![x]\!], (m,n) \in [\![r]\!]\} \\ &= \{m \in M : \text{for all } n \in \emptyset, (m,n) \in [\![r]\!]\} \\ &= M \end{aligned}$$

This means that if $[\![x]\!] = \emptyset$ in a given model $\mathcal{M}$, then every sentence like

all birds (like all x)

comes out true in this model $\mathcal{M}$.

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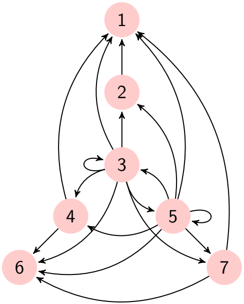
$[\![\text{hawks}]\!] = \{1\}$

$[\![\text{birds}]\!] = \{1, 6\}$

$[\![\text{turtles}]\!] = \{3, 5, 7\}$

$[\![\text{see}]\!]$ as shown



In this model,

$$\begin{array}{lcl} [\![\text{see all hawks}]\!] & = & \{2,3,4,5,7\} \\ [\![\text{see all birds}]\!] & = & \{3,4,5,7\} \\ [\![\text{see all turtles}]\!] & = & \{3,5\} \\ [\![\text{see all (see all hawks)}]\!] & = & \{3,5\} \end{array}$$

Here are some examples of sentences false and true in the model:

$\mathcal{M} \not\models$ All (see all hawks) (see all birds)
$\mathcal{M} \models$ All (see all turtles) (see all (see all hawks))
$\mathcal{M} \models$ All (see all (see all hawks)) (see all turtles)

We say that $\mathcal{M} \models \Gamma$ iff $\mathcal{M} \models \varphi$ for every $\varphi \in \Gamma$.

We say that $\Gamma \models \varphi$ iff for all $\mathcal{M}$: if $\mathcal{M} \models \Gamma$, then also $\mathcal{M} \models \varphi$.

We read this as

	Γ logically implies φ
or	Γ semantically implies φ
or	φ is a semantic consequence of Γ.

This point will apply to all the logics in this course.

Examples of the semantic consequence relation $\models$

All skunks mammals $\models$ All (love all mammals) (love all skunks)

All skunks mammals $\not\models$ All (love all skunks) (love all mammals)

$$\begin{array}{ll} & \text{All skunks mammals} \\ \models & \text{All (hate all (love all skunks)) (hate all (love all mammals))} \end{array}$$

More examples of the semantic consequence relation $\models$

All skunks mammals, All dogs chase all mammals $\models$ All dogs chase all skunks

More generally, for all nouns x, y, and z, and all verbs r,

$$\text{All } x\ y, \text{All } z\ r \text{ all } y \models \text{All } z\ r \text{ all } x$$

The logic A(RC)

$$\frac{}{\mathsf{All}\ x\ x}\ \textsc{axiom} \qquad \frac{\mathsf{All}\ x\ y \quad \mathsf{All}\ y\ z}{\mathsf{All}\ x\ z}\ \textsc{barbara}$$

$$\frac{\mathsf{All}\ x\ (r\ \mathsf{all}\ y) \quad \mathsf{All}\ z\ y}{\mathsf{All}\ x\ (r\ \mathsf{all}\ z)}\ \textsc{down}$$

The logic A(RC)'

$$\frac{}{\mathsf{All}\ x\ x}\ \textsc{axiom} \qquad \frac{\mathsf{All}\ x\ y \quad \mathsf{All}\ y\ z}{\mathsf{All}\ x\ z}\ \textsc{barbara}$$

$$\frac{\mathsf{All}\ y\ x}{\mathsf{All}\ (r\ \mathsf{all}\ x)\ (r\ \mathsf{all}\ y)}\ \textsc{anti}$$

Note that we are using this with x, y, and z as variables ranging over terms,
not as variables ranging over nouns the way we did with $\mathcal{A}$.

We use a different font for the names of the logical systems than for the name of the logical language, $\mathcal{A}(\mathcal{RC})$.

Two examples of derivations in A(RC)':

First, a derivation showing that every instance of (DOWN) is provable in A(RC)':

$$\dfrac{\text{All } x \text{ (see all } y) \qquad \dfrac{\text{All } z\ y}{\text{All (see all } y)\text{(see all } z)}\ \textsc{anti}}{\text{All } x \text{ (see all } z)}\ \textsc{barbara}$$

Second, a proof tree corresponding to an example from earlier in this lecture:

$$\dfrac{\dfrac{\text{All skunks mammals}}{\text{All (love all mammals) (love all skunks)}}\ \textsc{anti}}{\text{All (hate all (love all skunks)) (hate all (love all mammals))}}\ \textsc{anti}$$

We also can prove every instance of (anti) in A(RC):

The logic A(RC)

$$\frac{}{\text{All } x\ x}\ \textsc{axiom} \qquad \frac{\text{All } x\ y \quad \text{All } y\ z}{\text{All } x\ z}\ \textsc{barbara}$$

$$\frac{\text{All } x\ (r \text{ all } y) \quad \text{All } z\ y}{\text{All } x\ (r \text{ all } z)}\ \textsc{down}$$

$$\frac{\dfrac{}{\text{All } (r \text{ all } x)\ (r \text{ all } x)}\ \textsc{axiom} \qquad \text{All } y\ x}{\text{All } (r \text{ all } x)\ (r \text{ all } y)}\ \textsc{down}$$

In (down), we took x to be (r all x),
y to be x,
z to be y.

THE TWO LOGICS PROVE EACH OTHER'S RULES

THE LOGIC A(RC)

$$\frac{}{\text{All } x\ x}\ \text{AXIOM} \qquad \frac{\text{All } x\ y \quad \text{All } y\ z}{\text{All } x\ z}\ \text{BARBARA}$$

$$\frac{\text{All } x\ (r \text{ all } y) \quad \text{All } z\ y}{\text{All } x\ (r \text{ all } z)}\ \text{DOWN}$$

THE LOGIC A(RC)'

$$\frac{}{\text{All } x\ x}\ \text{AXIOM} \qquad \frac{\text{All } x\ y \quad \text{All } y\ z}{\text{All } x\ z}\ \text{BARBARA}$$

$$\frac{\text{All } y\ x}{\text{All } (r \text{ all } x)\ (r \text{ all } y)}\ \text{ANTI}$$

In the first logic, we can prove all instances of the rules of the second logic.

In the second logic, we can prove all instances of the rules of the first logic.

In a few minutes, we'll use this to show the following result:

THEOREM

If $\Gamma \vdash \varphi$ *in* A(RC)*, then* $\Gamma \vdash \varphi$ *in* A(RC)'.
If $\Gamma \vdash \varphi$ *in* A(RC)'*, then* $\Gamma \vdash \varphi$ *in* A(RC).

The first hard derivation in this course

Let us assume the following:

All who hate all ducks are cats
All who love all pigs see all cats
All who see all who hate all ducks are cats

and then prove that

All who love all pigs are cats.

You should try this yourself for a few minutes.

The first hard derivation in this course

Let us assume the following:

All who hate all ducks are cats
All who love all pigs see all cats
All who see all who hate all ducks are cats

and then prove that

All who love all pigs are cats.

Here is a derivation in A(RC):

$$\dfrac{\dfrac{\text{All (hate all d) c} \quad \text{All (love all p) (see all c)}}{\text{All (love all p) (see all (hate all d))}}\ \textsc{down} \qquad \text{All (see all (hate all d)) c}}{\text{All (love all p) c}}\ \textsc{barbara}$$

As you can see from trying it yourself, finding a derivation is difficult.

One problem is that when we build trees from the bottom up
in a system with (BARBARA),
we don't know what the "middle term" should be:
it could be anything!

THEOREM

Let Γ *be any set of sentences in the logic* $\mathcal{A}(\mathcal{RC})$.

Suppose that $\mathcal{T}$ *is a proof tree over* Γ,
and let the root of $\mathcal{T}$ *be* All x are y.

Then in all models $\mathcal{M}$, $[\![\mathrm{x}]\!] \subseteq [\![\mathrm{y}]\!]$.

We have

- a syntax of of terms and sentences
- a semantics
- two proof systems, and the proof that they are basically the same
- soundness of the logics

Still to come

- completeness of our system
- algorithmic work to tell whether or not a given set Γ implies a given sentence φ.

The canonical model of a set Γ of sentences in $\mathcal{A}(\mathcal{RC})$

Suppose that we are given a set Γ in this language $\mathcal{A}(\mathcal{RC})$.

We aim to build a canonical model $\mathcal{M} = \mathcal{M}_\Gamma$
with two properties:

- $\mathcal{M} \models \Gamma$
- For all φ, if $\mathcal{M} \models \varphi$, then $\Gamma \vdash \varphi$.

Why do we do this?

This would prove the completeness theorem for our logic A(RC).
Please be sure that you can do this step yourself.

The canonical model of a set Γ of sentences in $\mathcal{A}(\mathcal{RC})$

Here is the definition.
In it, p is a noun and r a verb.

And then we define

$$
\begin{array}{lcl}
M & = & \text{all terms built from nouns and verbs} \\
[\![p]\!] & = & \{x : \Gamma \vdash \text{All } x\ p\} \\
[\![r]\!] & = & \{(x, y) : \Gamma \vdash \text{All } x\ (r \text{ all } y)\}
\end{array}
$$

The interpretation of nouns is just like
we saw it in the canonical model construction for $\mathcal{A}$.

But the interpretation of verbs is new, and it is more subtle.

Before we do any of the theory, let's do a simple example

Let N have just two nouns: p and q.
Let V have just one verb: see.

Let's draw the picture of the canonical model of

$$\Gamma \quad = \quad \{\text{All p q}\}$$

There are infinitely many terms:

p	q
see all p	see all q
see all (see all p)	see all (see all q)
see all (see all (see all p))	see all (see all (see all q))
$\vdots$	$\vdots$

Before we do any of the theory, let's do a simple example

Let N have just two nouns: p and q.
Let V have just one verb: see.

Let's draw the picture of the canonical model of

$$\Gamma \;=\; \{\text{All p q}\}$$

There are infinitely many terms, and we abbreviate them:

p_0	p	q_0	q
p_1	see all p	q_1	see all q
p_2	see all (see all p)	q_2	see all (see all q)
p_3	see all (see all (see all p))	q_3	see all (see all (see all q))
$\vdots$		$\vdots$	

The picture of the "all" relation $\{(x, y) : \Gamma \vdash \text{All } x\ y\}$

Remember that Γ here is $\{\text{All } p\ q\}$.

$$\begin{tikzcd}
p_0 \arrow[loop above] \arrow[d] & p_1 \arrow[loop above] & p_2 \arrow[loop above] \arrow[d] & p_3 \arrow[loop above] & p_4 \arrow[loop above] \arrow[d] & \cdots \\
q_0 \arrow[loop below] & q_1 \arrow[loop below] \arrow[u] & q_2 \arrow[loop below] & q_3 \arrow[loop below] \arrow[u] & q_4 \arrow[loop below] & \cdots
\end{tikzcd}$$

An arrow $x \to y$ above means that $\Gamma \vdash$ all $x\ y$.

These arrow tell us that in the canonical model

$$\begin{array}{rcl} [\![p]\!] & = & \{p_0\} \\ [\![q]\!] & = & \{p_0, q_0\} \end{array}$$

But the picture on this slide is not the interpretation of the verb see.

Picture of $[\![\mathsf{see}]\!]$ in the canonical model of Γ

Remember that Γ here is $\{\mathsf{All}\ p\ q\}$.

$$
\begin{tikzcd}
p_0 & p_1 \arrow[l] & p_2 \arrow[l] \arrow[dl] & p_3 \arrow[l] & p_4 \arrow[l] \arrow[dl] & \cdots \\
q_0 & q_1 \arrow[l] \arrow[ul] & q_2 \arrow[l] & q_3 \arrow[l] \arrow[ul] & q_4 \arrow[l] & \cdots
\end{tikzcd}
$$

An arrow $x \rightarrow y$ in the picture above means that $\Gamma \vdash$ all x (see all y).

The arrow $x \rightarrow y$ above does not mean that $\Gamma \vdash$ all x y.

Picture of ⟦see⟧ in the canonical model of Γ

Remember that Γ here is {All p q}.

$$\begin{tikzcd}
p_0 & p_1 \arrow[l] & p_2 \arrow[l] \arrow[dl] & p_3 \arrow[l] & p_4 \arrow[l] \arrow[dl] & \cdots \\
q_0 & q_1 \arrow[l] \arrow[ul] & q_2 \arrow[l] & q_3 \arrow[l] \arrow[ul] & q_4 \arrow[l] & \cdots
\end{tikzcd}$$

Here is the idea:

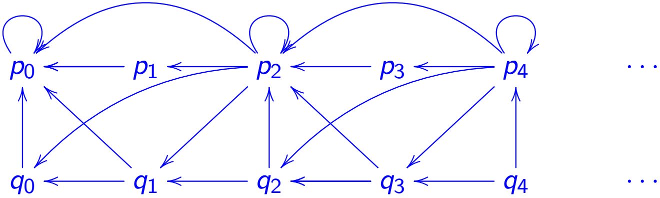
Think of the node x as a “random” x.

If we can prove from Γ that All x see all y,
then we want to put a ⟦see⟧-arrow from x to y.

Here is another set

$$\Gamma \quad = \quad \{\text{All } p \text{ are } q, \text{All } q \text{ (see all } p)\} \ .$$

Its canonical model is



$$\begin{aligned} [\![p]\!] &= \{p_0\} \\ [\![q]\!] &= \{p_0, q_0\} \end{aligned}$$

LEMMA (TRUTH LEMMA)

Let Γ be a set of sentences in $\mathcal{A}(\mathcal{RC})$,
and let $\mathcal{M}$ be the canonical model of Γ.
For all terms t,

$$[\![t]\!] \quad = \quad \{y : \Gamma \vdash \mathsf{All}\ y\ t\}$$

PROOF.

By induction on terms (what else?). $\square$

Lemma

$\mathcal{M} \models \Gamma$.

Proof.

Take a sentence in Γ, say All u v.

To see that this sentence holds in $\mathcal{M}$, let $y \in [\![u]\!]$.

By the Truth Lemma, $\Gamma \vdash$ All y u.

And then using (BARBARA), we have $\Gamma \vdash$ All y v.

Therefore, $y \in [\![v]\!]$.

So we have shown that $[\![u]\!] \subseteq [\![v]\!]$, just as desired. $\square$

Lemma

If $\mathcal{M} \models \varphi$, then $\Gamma \vdash \varphi$.

Proof.

Let φ be All a b.

Let $\mathcal{M}$ be the canonical model.
Suppose that $[\![a]\!] \subseteq [\![b]\!]$ in $\mathcal{M}$.

By (Axiom), $\Gamma \vdash$ All a a.
So by the Truth Lemma, $a \in [\![a]\!]$.

And so $a \in [\![b]\!]$.

By the Truth Lemma again, we have $\Gamma \vdash$ All a b. $\square$

THEOREM

If $\Gamma \models \varphi$*, then* $\Gamma \vdash \varphi$*.*

PROOF.

Let $\mathcal{M}$ be the canonical model of Γ.
By the second lemma, $\mathcal{M} \models \Gamma$.
Thus $\mathcal{M} \models \varphi$.

And so by the third lemma, $\Gamma \vdash \varphi$. $\square$

We have

- a syntax of of terms and sentences
- a semantics
- a proof system
- a hint about soundness
- we just did the completeness of our system

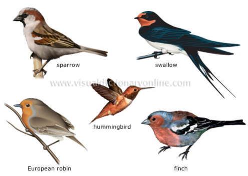
Still to come

- algorithmic work to tell whether or not a given set Γ implies a given sentence φ.

Here is a question: does the conclusion follow?

All hawks birds
All (see all turtles) (see all (see all hawks))
All (see all (see all hawks)) turtles
All turtles (see all birds)

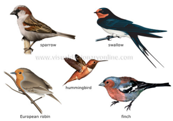
All (see all hawks) (see all birds)



Motivation

Here is a question: does the conclusion follow?

All hawks birds
All (see all turtles) (see all (see all hawks))
All (see all (see all hawks)) turtles
All turtles (see all birds)
———
All (see all hawks) (see all birds)



This is too hard to do by guessing,
and anyways we need a systematic approach.

We can't examine all the models in the world:
there are too many.

The canonical model is infinite.

What to do???????

All hawks birds
All (see all turtles) (see all (see all hawks))
All (see all (see all hawks)) turtles
All turtles (see all birds)

??? All (see all hawks) (see all birds)

All hawks birds
All (see all turtles) (see all (see all hawks))
All (see all (see all hawks)) turtles
All turtles (see all birds)

??? All (see all hawks) (see all birds)

Suppose (questionably) that if there were a proof,
there would be one that only uses the terms above:

hawks	see all hawks	see all (see all hawks)
birds	see all birds	
turtles	see all turtles	

Then our life would be much easier, because we can
simply generate all the consequences of our rules
on this set of seven terms.

$$\frac{\begin{array}{l}\text{All hawks birds}\\ \text{All (see all turtles) (see all (see all hawks))}\\ \text{All (see all (see all hawks)) turtles}\\ \text{All turtles (see all birds)}\end{array}}{\text{??? All (see all hawks) (see all birds)}}$$

The problem is that there might be a proof that used some humongous term $t_{\text{humongous}}$

$$\dfrac{\begin{array}{c}\vdots\\ \text{All (see all hawks) } t_{\text{humongous}}\end{array} \quad \begin{array}{c}\vdots\\ \text{All } t_{\text{humongous}} \text{ (see all birds)}\end{array}}{\text{All (see all hawks) (see all birds)}}\ b$$

Our idea might miss that proof.

We'll see the general algorithm after we do an example of it.

Let's take M to be the terms listed below:

hawks	see all turtles
see all hawks	birds
see all (see all hawks)	turtles
see all birds	

These are just the terms that occur in Γ and φ.

Let's try this together

Start with

$$\Gamma \cup \{\text{All } x\ x : x \in M\}$$

Generate all the sentences which we can prove from the set above by using (BARBARA) and (DOWN).

We call this set Γ^*.

The main point is that
we only generate finitely many sentences in this way
(note that we are using (DOWN) and not (ANTI)),
and we can do all of this work algorithmically.

Here's what we get

We're asking whether $\Gamma \vdash$ All (see all hawks) (see all birds) or not

All hawks birds
All (see all turtles) (see all (see all hawks))
All (see all (see all hawks)) turtles
All turtles (see all birds)

All x x   for $x \in M$

All (see all birds) (see all hawks)
All (see all turtles) (see all (see all (see all hawks)))
All (see all turtles) turtles
All turtles (see all hawks)
All (see all (see all hawks)) (see all birds)
All (see all (see all hawks)) (see all (see all birds))
All (see all (see all hawks)) (see all turtles)
All (see all turtles) (see all (see all turtles))
All (see all (see all hawks)) (see all hawks)
All (see all turtles) (see all birds)
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Here's what we get

We're asking whether $\Gamma \vdash$ All (see all hawks) (see all birds) or not

All hawks birds
All (see all turtles) (see all (see all hawks))
All (see all (see all hawks)) turtles
All turtles (see all birds)

All x x for $x \in M$
All (see all birds) (see all hawks)
All (see all turtles) (see all (see all (see all hawks)))
All (see all turtles) turtles
All turtles (see all hawks)
All (see all (see all hawks)) (see all birds)
All (see all (see all hawks)) (see all (see all birds))
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All (see all turtles) (see all birds)
All (see all turtles) (see all (see all birds))
All (see all turtles) (see all hawks)
All (see all (see all hawks)) (see all (see all (see all hawks)))

The set of sentences above is called Γ^*.
If Γ^* contained the sentence that we asked about at the start, we'd be done.

It doesn't.

And so we'll now show how to make a counter-model from Γ^*.

Here's what we get

We're asking whether $\Gamma \vdash$ All (see all hawks) (see all birds) or not

All hawks birds
All (see all turtles) (see all (see all hawks))
All (see all (see all hawks)) turtles
All turtles (see all birds)

All x x for $x \in M$
All (see all birds) (see all hawks)
All (see all turtles) (see all (see all (see all hawks)))
All (see all turtles) turtles
All turtles (see all hawks)
All (see all (see all hawks)) (see all birds)
All (see all (see all hawks)) (see all (see all birds))
All (see all (see all hawks)) (see all turtles)
All (see all turtles) (see all (see all turtles))
All (see all (see all hawks)) (see all hawks)
All (see all turtles) (see all birds)
All (see all turtles) (see all (see all birds))
All (see all turtles) (see all hawks)
All (see all (see all hawks)) (see all (see all (see all hawks)))

Now we make a finite version of the canonical model from Γ^*.

The universe is M.
$[\![u]\!] = \{v : \Gamma^* \text{ contains All } v\ u\}$.
$[\![r]\!] = \{(u, v) : \Gamma^* \text{ contains All } u\ (r \text{ all } v)\}$.

Here's what we get

We're asking whether $\Gamma \vdash$ All (see all hawks) (see all birds) or not

All hawks birds
All (see all turtles) (see all (see all hawks))
All (see all (see all hawks)) turtles
All turtles (see all birds)

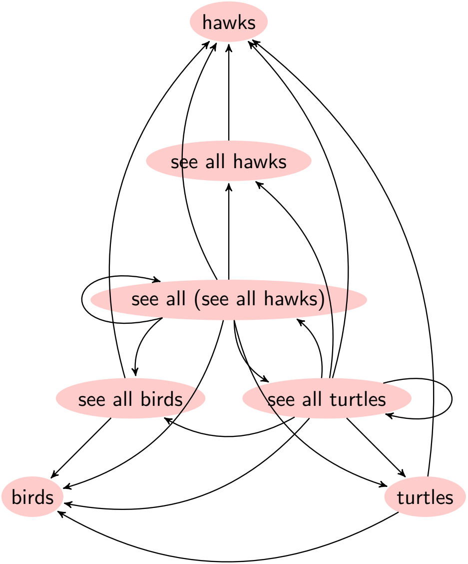
$$\frac{}{\text{All x x}} \quad \text{for } x \in M$$

All (see all birds) (see all hawks)
All (see all turtles) (see all (see all (see all hawks)))
All (see all turtles) turtles
All turtles (see all hawks)
All (see all (see all hawks)) (see all birds)
All (see all (see all hawks)) (see all (see all birds))
All (see all (see all hawks)) (see all turtles)
All (see all turtles) (see all (see all turtles))
All (see all (see all hawks)) (see all hawks)
All (see all turtles) (see all birds)
All (see all turtles) (see all (see all birds))
All (see all turtles) (see all hawks)
All (see all (see all hawks)) (see all (see all (see all hawks)))

To begin,

$$\begin{aligned} [\![\text{hawks}]\!] &= \{\text{hawks}\} \\ [\![\text{birds}]\!] &= \{\text{birds}, \text{hawks}\} \\ [\![\text{turtles}]\!] &= \{\text{turtles}, \text{see all turtles}, \text{see all (see all hawks)}\} \end{aligned}$$

The interpretation of see in this model



Using numbers instead of terms

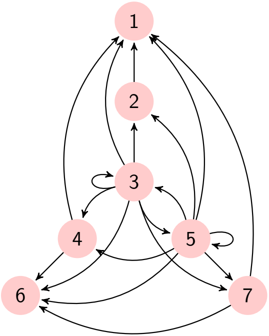
hawks	1
see all hawks	2
see all (see all hawks)	3
see all birds	4
see all turtles	5
birds	6
turtles	7

hawks	1
see all hawks	2
see all (see all hawks)	3
see all birds	4
see all turtles	5
birds	6
turtles	7

$[\![\text{hawks}]\!] = \{1\}$

$[\![\text{birds}]\!] = \{1, 6\}$

$[\![\text{turtles}]\!] = \{3, 5, 7\}$



The models are isomorphic, and most people would prefer to see the numbers at the end.

Here is a question: does the conclusion follow?

All hawks birds
All (see all turtles) (see all (see all hawks))
All (see all (see all hawks)) turtles
All turtles (see all birds)

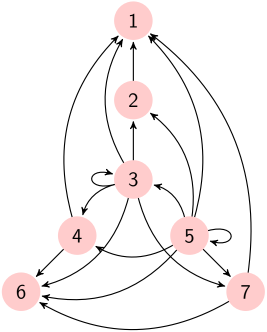
All (see all hawks) (see all birds)

Now we can answer "No" definitively.

$[\![\text{hawks}]\!] = \{1\}$

$[\![\text{birds}]\!] = \{1, 6\}$

$[\![\text{turtles}]\!] = \{3, 5, 7\}$



ALGORITHM TO TELL WHETHER OR NOT $\Gamma \vdash \varphi$

IF $\Gamma \vdash \varphi$, THE ALGORITHM OUTPUTS "YES".
IF $\Gamma \not\vdash \varphi$, IT OUTPUTS "NO" AND A MODEL OF Γ WHERE φ FAILS.

1: $M \leftarrow$ the set of principal terms in $\Gamma \cup \{\varphi\}$, plus the subterms of these terms
2: $\textsc{ax}_M \leftarrow \{\text{All } x\ x : x \in M\}$
3: $\Gamma^* \leftarrow$ the smallest set with $\Gamma \subseteq \Gamma^*$ and $\textsc{ax}_M \subseteq \Gamma^*$,
   and closed under the logic A(RC).
   If Γ^* contains All x y and also All y z, then Γ^* contains All x z.
   If Γ^* contains All x y and All z (r all y), then Γ^* contains All z (r all x).
4: **if** $\varphi \in \Gamma^*$ **then**
5:     output "Yes" $\triangleright\ \Gamma \vdash \varphi$
6: **else** $\triangleright$ we construct a counter-model with universe M
7:     **for all** $u \in \mathsf{N}$, $r \in \mathsf{V}$ **do**
8:         $[\![u]\!] = \{v \in M : \Gamma^* \text{ contains All } v\ u\}$.
9:         $[\![r]\!] = \{(u, v) \in M \times M : \Gamma^* \text{ contains All } u\ (r \text{ all } v)\}$.
10:     **end for**
11:     output "No" and the model $\mathcal{M} = (M, [\![\ ]\!])$ described in lines 7–10.
12: **end if**

I'm going to skip the proof that the algorithm works

Educational point

Small examples could be given as homework or on an exam.

This contrasts with its cousin, filtration in modal logic:
It is unfortunately not possible to ask students
to come up with actual examples of filtration.

I assume for the moment that you know
the relational (Kripke) semantics of modal logic

A model is a set W of "worlds",
a relation that we are going to write as $[\![\text{see}]\!]$
(think of it as the "seeing" relation in a model),
and some way to interpret atomic propositions.

The syntax of modal logic has a "box operator" $\Box$, and we shall see its semantics below.

But there is a second, less-studied operator, $\boxminus$, 'window'.

Let us now compare the main clause in the semantics of both operators.

We associate with the verb see two operators, $\Box_{\mathsf{see}}$ and $\boxminus_{\mathsf{see}}$. Their semantics are

$$w \models \Box_{\mathsf{see}}\varphi \quad \text{iff} \quad w[\![\mathsf{see}]\!]v \text{ implies } v \models \varphi$$

"w sees only φ worlds"

$$w \models \boxminus_{\mathsf{see}}\varphi \quad \text{iff} \quad v \models \varphi \text{ implies } w[\![\mathsf{see}]\!]v$$

"w sees all φ worlds"

We also add the universal modality U, with semantics

$$w \models U\varphi \quad \text{iff} \qquad v \models \varphi \text{ for all } v$$

We translate $\mathcal{A}(\mathcal{RC})$ into propositional logic $+ \boxminus + U$; e.g.,

all skunks mammals $\models$ all (see all mammals) (see all skunks)

translates to

$$U(\text{skunks} \ \rightarrow \text{mammals}) \models U(\boxminus_{see}\text{mammals} \ \rightarrow \boxminus_{see}\text{skunks})$$

and for that matter, we also have

$$U(\text{skunks} \ \rightarrow \text{mammals}) \models U(\boxminus_{hate}\boxminus_{see}\text{skunks} \ \rightarrow \boxminus_{hate}\boxminus_{see}\text{mammals})$$

$\mathcal{O}(\mathcal{RC})$: ANOTHER ADDITION TO $\mathcal{A}$

$\mathcal{O}(\mathcal{RC})$ is the logical system defined the same way as $\mathcal{A}(\mathcal{RC})$ was, except that we include terms of the form

$$r \text{ only } x$$

rather than r all x.

Terms may have terms inside (recursion).

Sentences of $\mathcal{O}(\mathcal{RC})$ are the expressions

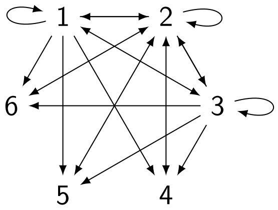
$$\text{all } x\ y,$$

where x and y are terms.

Models are the same.

$$[\![r \text{ only } x]\!] \quad = \quad \{m \in M : \text{for all } n \text{ if } m[\![r]\!]n, \text{ then } n \in [\![x]\!]\}.$$

Consider the model shown below.



$[\![p]\!] = \{2\}$

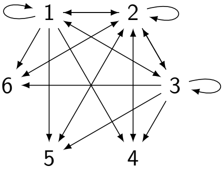
$[\![q]\!] = \{1, 2\}$

We have one verb, see.

1. Find $[\![$see only $p]\!]$.
2. Find $[\![$see only $q]\!]$.
3. Is the sentence All p q true in the model?
4. Is the sentence All q (see only q) true in the model?
5. Is the sentence All (see only p) (see only q) true in the model?
6. Is the sentence All (see only q) (see only p) true in the model?

Example of the semantics

Consider the model shown below.



$[\![p]\!] = \{2\}$

$[\![q]\!] = \{1, 2\}$

We have one verb, see.

1. Find $[\![$see only $p]\!]$. Answer: $\{4, 6\}$.
2. Find $[\![$see only $q]\!]$. Answer: $\{4, 5, 6\}$.
3. Is All p q true in the model?
   Answer: Yes, because $\{2\} \subseteq \{1, 2\}$.
4. Is All q (see only q) true in the model?
   Answer: No, because $\{1, 2\} \not\subseteq \{4, 5, 6\}$.
5. Is All (see only p) (see only q) true in the model?
   Answer: Yes, because $\{4, 6\} \subseteq \{4, 5, 6\}$.
6. Is All (see only q) (see only p) true in the model?
   Answer: No, because $\{4, 5, 6\} \not\subseteq \{4, 6\}$.

$$\frac{}{\textsf{all } x\ x}\ \text{AXIOM} \qquad \frac{\textsf{all } x\ y \quad \textsf{all } y\ z}{\textsf{all } x\ z}\ \text{BARBARA}$$

$$\frac{\textsf{all } x\ (\textsf{r only } y) \quad \textsf{all } y\ z}{\textsf{all } x\ (r\ \textsf{only } z)}\ \text{ONLY} \qquad \frac{\textsf{all } x\ y}{\textsf{all } (r\ \textsf{only } x)\ (r\ \textsf{only } y)}\ \text{MONO}$$

We can use either (ONLY) or (MONO); they are inter-derivable.

Given a set Γ, we define a model $\mathcal{M} = \mathcal{M}_\Gamma$ in several steps.

We have seen the preorder $\leq$ on terms by

$$s \leq t \quad \text{iff} \quad \Gamma \vdash \text{all } s \ t.$$

We take M to be the set of all sets $\mathcal{S}$ of terms which are up-closed in this preorder.

And then we define, for all nouns p and verbs r,

$$\begin{array}{rcl} [\![p]\!] & = & \{\mathcal{S} \in M : p \in \mathcal{S}\} \\ [\![r]\!] & = & \{(\mathcal{S}, \mathcal{T}) \in M \times M : \text{if } (r \text{ only } x) \in \mathcal{S}, \text{ then } x \in \mathcal{T}\} \end{array}$$

Example: $\Gamma = \{\text{all } p\ q\}$

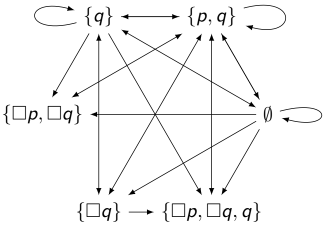
This time as a set with a sentence in $\mathcal{O}(\mathcal{RC})$.

To simplify matters, we assume that the only nouns are p and q, and that the only verb is r.

The points in the canonical model are the sets $\mathcal{S}$ of sentences with the property that for all n,

if r only^n $p = \Box^n p$ belongs to $\mathcal{S}$, then so does r only^n $q = \Box^n q$.

This model is uncountable.
Here are six of its points, together with the arrows (interpreting r) between them.



We have written r only p as $\Box p$, and similarly for q.
What are the interpretations of p and q?

Lemma (Truth Lemma)

Let $\mathcal{M}$ be the canonical model of some set Γ.
Then for all terms x,

$$[\![t]\!] \quad = \quad \{\mathcal{S} \in M : t \in \mathcal{S}\}.$$

Theorem (Completeness)

$\Gamma \vdash \varphi$ *iff* $\Gamma \models \varphi$.

Theorem

This logic $\mathcal{O}(\mathcal{RC})$ has the finite model property:
if $\Gamma \not\models \varphi$, there is a finite counter-model.