$\mathcal{S}^\dagger$ and Orthoposets

Larry Moss

Indiana University

North American Summer School on
Logic, Language, and Information
June 23-27, 2025

$\mathcal{S}^\dagger$: SYLLOGISTIC LOGIC WITH COMPLEMENTED NOUNS

We formulate a language $\mathcal{S}^\dagger$ is syllogistic logic
with noun-level negation (i.e., complements on the nouns).

In the syntax, we again begin with a set N of nouns.

Let

$$\textbf{Lit} \quad = \quad \mathsf{N} \cup \{\overline{p} : p \in \mathsf{N}\}.$$

In other words, we have two copies of N, using the "overline" $^-$ to distinguish the copies.

We call the elements of this set literals.

Once again, the elements of **Lit** are either nouns p, q, etc.,
or complemented nouns $\overline{p}$, $\overline{q}$, ...

IMPORTANT CHOICE

We always assume that p and $\overline{\overline{p}}$ are the same.

We consider sentences

All p are q and Some p are q.

Here p and q are any literals.
They could be nouns, they could have the "bar" as in $\overline{p}$.

We call this language $\mathcal{S}^\dagger$.

We again use letters like φ to denote sentences.

We consider sentences

All p are q and Some p are q.

Here p and q are any literals.
They could be nouns, they could have the "bar" as in $\overline{p}$.

We call this language $\mathcal{S}^\dagger$.

We again use letters like φ to denote sentences.

What about No x are y?

It is an abbreviation for All x are $\overline{y}$!

We consider sentences

All p are q and Some p are q.

Here p and q are any literals.
They could be nouns, they could have the "bar" as in $\overline{p}$.

We call this language $\mathcal{S}^\dagger$.

We again use letters like φ to denote sentences.

$\mathcal{S}^\dagger$ goes beyond standard syllogistic logic:

All $\overline{x}$ are $\overline{y}$
Some $\overline{x}$ are y

A model $\mathcal{M}$ is a set M
together with a function

$$[\![\ ]\!] : \mathbf{Lit} \to \mathcal{P}(M)$$

subject to the requirement that

$$[\![\overline{p}]\!] = M \setminus [\![p]\!]$$

for all p.

(In words, we have subsets $[\![p]\!] \subseteq M$ for each literal p,
satisfying the requirement above.)

This gives a model $\mathcal{M} = (M, [\![\quad]\!])$.

Then we define

$$\begin{array}{lll} \mathcal{M} \models \text{All } p \text{ are } q & \text{iff} & [\![p]\!] \subseteq [\![q]\!] \\ \mathcal{M} \models \text{Some } p \text{ are } q & \text{iff} & [\![p]\!] \cap [\![q]\!] \neq \emptyset \end{array}$$

And it follows that

$$\mathcal{M} \models \text{No } p \text{ are } q \qquad \text{iff} \qquad [\![p]\!] \cap [\![q]\!] = \emptyset$$

We also have derived notions such as $\Gamma \models \varphi$.

Let
$\Gamma = \{\text{All } b \text{ are } a, \text{All } \overline{b} \text{ are } a, \text{All } \overline{c} \text{ are } b, \text{All } c \text{ are } \overline{b}, \text{All } c \text{ are } d\}.$

It is not true that

$$\Gamma \models \text{All } b \text{ are } d.$$

Find a model $\mathcal{M} \models \Gamma$ where $[\![b]\!] \not\subseteq [\![d]\!]$.

The details of the completeness proof for our logic
will give us a way of automatically solving problems like this!

THE LOGICAL SYSTEM $\mathsf{S}^{\dagger}$

IN THIS SYSTEM, THE LETTERS p, q, n, ETC. ARE LITERALS

$$\frac{}{\text{All } p \text{ are } p}\ \textsc{axiom} \qquad \frac{\text{Some } p \text{ are } q}{\text{Some } p \text{ are } p}\ \textsc{some}_1 \qquad \frac{\text{Some } p \text{ are } q}{\text{Some } q \text{ are } p}\ \textsc{some}_2$$

$$\frac{\text{All } p \text{ are } n \quad \text{All } n \text{ are } q}{\text{All } p \text{ are } q}\ \textsc{barbara} \qquad \frac{\text{All } q \text{ are } n \quad \text{Some } p \text{ are } q}{\text{Some } p \text{ are } n}\ \textsc{darii}$$

$$\begin{array}{c}[\varphi] \\ \vdots \\ \bot \\ \hline\hline \varphi\end{array}\ \textsc{raa}$$

What is proof by contradiction?

The current logic in has something we have not seen so far, the rule (RAA).

This gives us the ability to do proof by contradiction.

What is proof by contradiction?

Each sentence of the form Some r are $\overline{r}$ is a contradiction.

We'll indicate contradictions with the symbol $\perp$.

If we have a node in a proof tree that's a contradiction,
we "take back" any assumption that was made
"toward a contradiction."

In fact, our taking this back entitles us to withdraw
that assumption and indeed to conclude the opposite.

So we put brackets around the withdrawn assumption,
and we add a new root to the tree:

$$\dfrac{\dfrac{\text{All } p \text{ are } \overline{p} \qquad \dfrac{[\text{Some } p \text{ are } \overline{q}]}{\text{Some } p \text{ are } p}\,\textsc{some}}{\text{Some } p \text{ are } \overline{p}}\,\textsc{darii}}{\text{All } p \text{ are } q}\,\textsc{raa}$$

Semantic negations (opposite sentences)

φ	$\overline{\varphi}$
All x are y	Some x are $\overline{y}$
Some x are y	All x are $\overline{y}$

Two facts

$\mathcal{M} \not\models \varphi$ iff $\mathcal{M} \models \overline{\varphi}$

$\overline{\overline{\varphi}} = \varphi$ for all φ.

$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \color{red}{\bot}\end{array}}{\color{blue}{\overline{\varphi}}}\ \text{RAA}$$

The sentences of the form Some p are $\overline{p}$ are called contradictions in $\mathcal{S}^{\dagger}$.

We use $\bot$ ("bottom") as a symbol for any of these contradictions.

The rule (RAA) tells us that if we can prove a contradiction with some tree $\mathcal{T}$,
then we may take any sentence φ,
withdraw some or all of the occurrences of φ
in the leaves of our derivation by putting brackets around them,
and then using the rule (RAA) to infer $\overline{\varphi}$ at the root.

$$\cfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \bot\end{array}}{\varphi}\ \text{RAA}$$

We obtain a new tree $\mathcal{T}^+$.

We allow the case when φ does not actually occur in the leaves of the tree $\mathcal{T}$.
So in this case, $\mathcal{T}$ and $\mathcal{T}^+$ would have the same set of non-withdrawn leaves.

We write $\Gamma \vdash \varphi$ if there is a proof tree $\mathcal{T}$ whose root is φ and all of whose non-withdrawn leaves belong to Γ.

No p are $p \vdash$ All p are q

A proof using RAA

$$\text{No } p \text{ are } p \vdash \text{All } p \text{ are } q$$

We first temporarily assume the opposite of our goal:

$$\dfrac{\text{All } p \text{ are } \overline{p} \qquad \dfrac{\text{Some } p \text{ are } \overline{q}}{\text{Some } p \text{ are } p}\ \textsc{some}}{\text{Some } p \text{ are } \overline{p}}\ \textsc{darii}$$

$$\text{No } p \text{ are } p \vdash \text{All } p \text{ are } q$$

We first temporarily assume the opposite of our goal:

$$\dfrac{\text{All } p \text{ are } \overline{p} \qquad \dfrac{\text{Some } p \text{ are } \overline{q}}{\text{Some } p \text{ are } p}\ \textsc{some}}{\text{Some } p \text{ are } \overline{p}}\ \textsc{darii}$$

Once we reach a contradiction, we withdraw our assumption:

$$\dfrac{\dfrac{\text{All } p \text{ are } \overline{p} \qquad \dfrac{[\text{Some } p \text{ are } \overline{q}]}{\text{Some } p \text{ are } p}\ \textsc{some}}{\text{Some } p \text{ are } \overline{p}}\ \textsc{darii}}{\text{All } p \text{ are } q}\ \textsc{raa}$$

We have the possibility that a set Γ is inconsistent.

This means that $\Gamma \vdash$ Some a are b and also $\Gamma \vdash$ No a are b

Equivalently: $\Gamma \vdash \varphi$ for all φ.

Note that an inconsistent Γ
is unsatisfiable: it has no models.
(Is this soundness or completeness?)

Hence if Γ is inconsistent, then $\Gamma \models \varphi$ for all φ.

At this point, we take a temporary detour into a topic related to abstract algebra.

Actually, what we are going to do would be called algebraic logic.

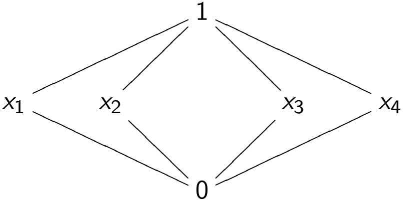
DEFINITION

An orthoposet is a tuple $\mathcal{P} = (P, \leq, 0, ^{-})$ such that

1. $(P, \leq)$ is a partial order: $\leq$ is a reflexive, transitive, and antisymmetric relation on the set P.
2. minimum property : $0 \leq p$ for all $p \in P$.
3. antitone property: if $x \leq y$, then $\overline{y} \leq \overline{x}$.
4. involutive property: $\overline{\overline{x}} = x$.
5. complement-order property: If $x \leq y$ and $x \leq \overline{y}$, then $x = 0$.

The Chinese lantern M_2

I apologize for this name



What we mean here is that the poset is the set of six points above, with the order as shown.

The 0 is at the bottom.

We define the operation $^{-}$ by: $\overline{0} = 1$, $\overline{1} = 0$, $\overline{x}_1 = x_2$, $\overline{x}_2 = x_1$, $\overline{x}_3 = x_4$, and $\overline{x}_4 = x_3$.

You might like to verify the conditions the definition of an orthoposet.

EXAMPLE

For all sets X we have an orthoposet

$$(\mathcal{P}(X), \subseteq, \emptyset, ^{-},)$$

where

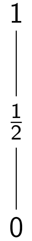
$\mathcal{P}(X)$, the power set of X is the set of all subsets of X,

$\leq$ is the inclusion relation $\subseteq$,

0 is the empty set $\emptyset$,

and $\overline{A} = X \setminus A$ for all subsets A of p.

A poset which cannot be made into an orthoposet



Let Γ be any set of sentences in the fragment.
Let **Lit** be the set of literals

We already know the preorder $\leq$:

$$x \leq y \quad \text{iff} \quad \Gamma \vdash \text{All x are y.}$$

(so Some plays no role)
We have an induced equivalence relation $\equiv$.

That is, we define $y \equiv z$ to mean that $z \leq y \leq z$
and we take $\textbf{Lit}_\Gamma$ to be the quotient $\textbf{Lit}/\!\equiv$.

If there is some x such that $x \leq \overline{x}$, then set 0 to be $[x]$.

We finally define $\overline{[x]} = [\overline{x}]$.

If there is no x such that $x \leq \overline{x}$ we add fresh 0 and 1 to $\textbf{Lit}/\!\equiv$.

We have an orthoposet which we call $\mathbb{P}_\Gamma$.

Orthoposets from logic, concretely

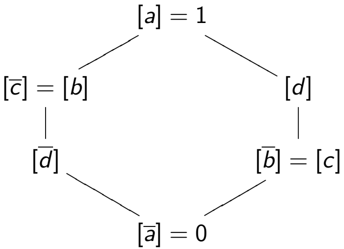
Let

$$\Gamma = \{\text{All } b \text{ are } a, \text{All } \overline{b} \text{ are } a, \text{All } \overline{c} \text{ are } b, \text{All } c \text{ are } \overline{b}, \text{All } c \text{ are } d\}$$

Then (after we do a lot of work proving things from Γ) here are the equivalence classes under $\equiv$, where

$$x \equiv y \quad \text{iff} \quad x \leq y \text{ and } y \leq x$$

$$\begin{array}{ll}
[a] = \{a\} & [\overline{a}] = \{\overline{a}\} \\
[b] = \{b, \overline{c}\} & [\overline{b}] = \{\overline{b}, c\} \\
[c] = \{\overline{b}, c\} & [\overline{c}] = \{b, \overline{c}\} \\
[d] = \{d\} & [\overline{d}] = \{\overline{d}\}
\end{array}$$



Let Γ be

$$\left\{\begin{array}{l} \text{All } \overline{a} \text{ are } c, \\ \text{All } b \text{ are } c, \\ \text{All } c \text{ are } d, \\ \text{All } c \text{ are } \overline{e} \end{array}\right\}$$

Try to draw the picture of this orthoposet.
To start, the equivalence classes are
all singletons (one-element sets):

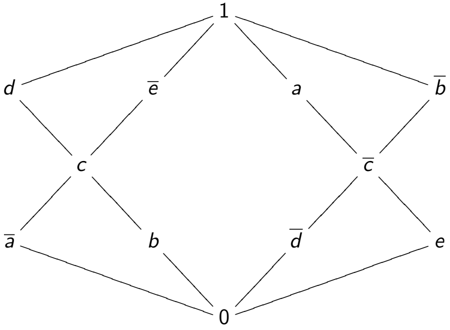
$$\begin{array}{ll} [a] = \{a\} & [\overline{a}] = \{\overline{a}\} \\ [b] = \{b\} & [\overline{b}] = \{\overline{b}\} \\ [c] = \{c\} & [\overline{c}] = \{\overline{c}\} \\ [d] = \{d\} & [\overline{d}] = \{\overline{d}\} \\ [e] = \{e\} & [\overline{e}] = \{\overline{e}\} \end{array}$$

And none of these serve as the 0.

More on this: the orthoposet $\mathbb{P}_\Gamma$

Γ again is

$$\left\{\begin{array}{l} \text{All } \overline{a} \text{ are } c, \\ \text{All } b \text{ are } c, \\ \text{All } c \text{ are } d, \\ \text{All } c \text{ are } \overline{e} \end{array}\right\}$$



Note that we add a fresh 0 and 1,
since none of the literals x has $x \leq y$ for all y.

Let's think about the following "nouns" related to ice cream.

v	favorite flavor of ice cream is vanilla
c	favorite flavor of ice cream is chocolate
s	favorite flavor of ice cream is strawberry
g	favorite flavor of ice cream is garbage-flavored
o	favorite is a flavor other than the four above
w	eats ice cream at least once a week

Let's think about the following "nouns" related to ice cream.

v	favorite flavor of ice cream is vanilla
c	favorite flavor of ice cream is chocolate
s	favorite flavor of ice cream is strawberry
g	favorite flavor of ice cream is garbage-flavored
o	favorite is a flavor other than the four above
w	eats ice cream at least once a week

Let's make up a "reasonable but random" set Γ.

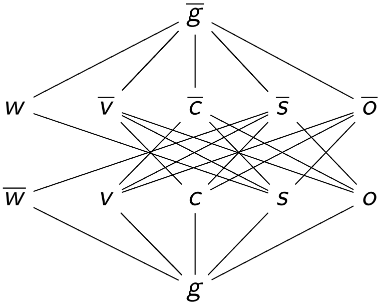
All v are $\overline{c}$	All v are $\overline{o}$	All s are w
All s are $\overline{c}$	All c are $\overline{o}$	
All s are $\overline{v}$	All s are $\overline{o}$	
All g are $\overline{g}$	All g are $\overline{o}$	

This is our theory of ice cream flavors and eating habits.

All v are $\overline{c}$
All s are $\overline{c}$
All s are $\overline{v}$
All g are $\overline{g}$

All v are $\overline{o}$
All c are $\overline{o}$
All s are $\overline{o}$
All g are $\overline{o}$

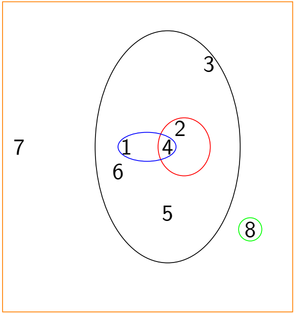
All s are w



A state of a orthoposet $P = (P, \leq, 0, ^{-})$ is a non-empty subset $S \subseteq P$ with the following properties:

- **up-closed** If $x \in S$ and $x \leq y$, then $y \in S$.
- **complete** For all x, either $x \in S$ or $\overline{x} \in S$.
- **consistent** For all x, S does not contain both x and $\overline{x}$.

Let $M = \{1, 2, 3, 4, 5, 6, 7, 8\}$: the orange rectangle.
Let $[\![a]\!] = \{1, 2, 3, 4, 5, 6\}$, in the black oval.
Let $[\![x]\!] = \{1, 4\}$, shown in the blue oval.
Let $[\![y]\!] = \{2, 4\}$, in the red oval.
Let $[\![z]\!] = \{8\}$, in the green oval.



Let's take various points x and calculate $S_x = \{u : x \in [\![u]\!]\}$ as

u ranges over the literals (= nouns and complemented nouns)

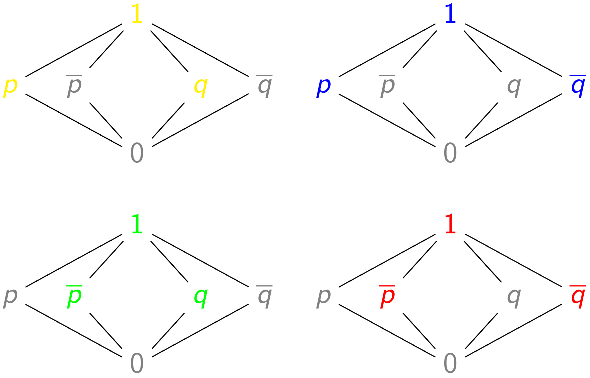
$S_1 = \{x, \overline{y}, \overline{z}, a\}$.
$S_2 = \{\overline{x}, y, \overline{z}, a\}$.
$S_3 = \{\overline{x}, \overline{y}, \overline{z}, a\}$.

The idea of a state is that it should capture the properties of every set S_x.

Most important:
each S_x contains exactly one of u or $\overline{u}$

Look back at the Chinese lantern.

There are four states here: the sets marked •, •, •, and •:



UP-CLOSED If $x \in S$ and $x \leq y$, then $y \in S$.

COMPLETE For all x, either $x \in S$ or $\overline{x} \in S$.

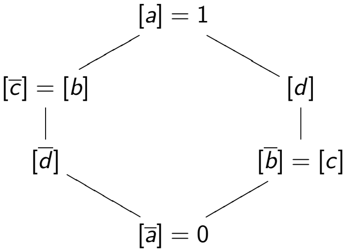
CONSISTENT For all x, S does not contain both x and $\overline{x}$.

$[a] = \{a\}$ $\quad$ $[\overline{a}] = \{\overline{a}\}$

$[b] = \{b, \overline{c}\}$ $\quad$ $[\overline{b}] = \{\overline{b}, c\}$

$[c] = \{\overline{b}, c\}$ $\quad$ $[\overline{c}] = \{b, \overline{c}\}$

$[d] = \{d\}$ $\quad$ $[\overline{d}] = \{\overline{d}\}$



UP-CLOSED If $x \in S$ and $x \leq y$, then $y \in S$.
COMPLETE For all x, either $x \in S$ or $\overline{x} \in S$.
CONSISTENT For all x, S does not contain both x and $\overline{x}$.

It is not obvious, but there are exactly three states.

$$[a] = \{a\} \qquad [\bar{a}] = \{\bar{a}\}$$

$$[b] = \{b, \bar{c}\} \qquad [\bar{b}] = \{\bar{b}, c\}$$

$$[c] = \{\bar{b}, c\} \qquad [\bar{c}] = \{b, \bar{c}\}$$

$$[d] = \{d\} \qquad [\bar{d}] = \{\bar{d}\}$$

$$\begin{tikzcd}
& {[a] = 1} \arrow[dl, dash] \arrow[dr, dash] & \\
{[\bar{c}] = [b]} \arrow[d, dash] & & {[d]} \arrow[d, dash] \\
{[\bar{d}]} \arrow[dr, dash] & & {[\bar{b}] = [c]} \arrow[dl, dash] \\
& {[\bar{a}] = 0} &
\end{tikzcd}$$

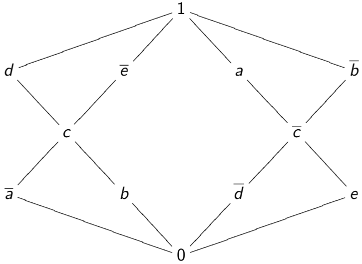
The three states are:

$$S = \{[\bar{d}], [b], [a]\}, \; T = \{[\bar{b}], [d], [a]\}, \; U = \{[b], [d], [a]\}.$$

Return to a previous example

We saw this set Γ:

$$\left\{\begin{array}{l} \text{All } \overline{a} \text{ are } c, \\ \text{All } b \text{ are } c, \\ \text{All } c \text{ are } d, \\ \text{All } c \text{ are } \overline{e} \end{array}\right\}$$



$$\begin{array}{lcl}
S_1 & = & \{a, b, c, d, \overline{e}, 1\} \\
S_2 & = & \{\overline{a}, \overline{b}, c, d, \overline{e}, 1\} \\
S_3 & = & \{a, \overline{b}, \overline{c}, \overline{d}, \overline{e}, 1\} \\
S_4 & = & \{a, \overline{b}, \overline{c}, d, e, 1\} \\
S_5 & = & \{a, \overline{b}, \overline{c}, d, \overline{e}, 1\} \\
S_6 & = & \{a, \overline{b}, \overline{c}, \overline{d}, e, 1\} \\
S_7 & = & \{a, \overline{b}, \overline{c}, \overline{d}, \overline{e}, 1\} \\
S_8 & = & \{\overline{a}, b, c, d, \overline{e}, 1\}
\end{array}$$

DEFINITION

Suppose that $\Gamma \subseteq \mathcal{S}^\dagger$.

The $\mathcal{S}^\dagger$ canonical model $\mathcal{M}^*$ is defined as follows:

Recall $\mathbb{P}_\Gamma$, the orthoposet defined from Γ.

Let M^* be *states*$(\mathbb{P}_\Gamma)$, the set of states of $\mathbb{P}_\Gamma$.

We interpret nouns by

$$[\![u]\!] \quad = \quad \{S \in M^* : [u] \in S\}.$$

Let
$\Gamma = \{$All b are a, All $\overline{b}$ are a, All $\overline{c}$ are b, All c are $\overline{b}$, All c are $d\}$.
We have already seen the $\mathbb{P}_\Gamma$ and the set of states on it.

$S = \{[\overline{d}], [b], [a]\}$

$T = \{[\overline{b}], [d], [a]\}$

$U = \{[b], [d], [a]\}$

$$
\begin{tikzcd}[column sep=small]
 & {[a] = 1} \arrow[dl, no head] \arrow[dr, no head] & \\
{[\overline{c}] = [b]} \arrow[d, no head] & & {[d]} \arrow[d, no head] \\
{[\overline{d}]} \arrow[dr, no head] & & {[\overline{b}] = [c]} \arrow[dl, no head] \\
 & {[\overline{a}] = 0} &
\end{tikzcd}
$$

$M^* = \{S, T, U\}$

$\llbracket a \rrbracket = \{S, T, U\}$
$\llbracket b \rrbracket = \{S, U\}$
$\llbracket c \rrbracket = \{T\}$
$\llbracket d \rrbracket = \{T, U\}$

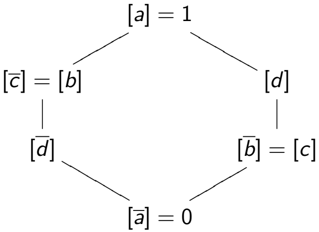
Example of $[\![u]\!] = \{C \in M^* : [u] \in C\}$

Let
$\Gamma = \{$All b are a, All $\overline{b}$ are a, All $\overline{c}$ are b, All c are $\overline{b}$, All c are $d\}$.
We have already seen the $\mathbb{P}_\Gamma$ and the set of states on it.

$S = \{[\overline{d}], [b], [a]\}$

$T = \{[\overline{b}], [d], [a]\}$

$U = \{[b], [d], [a]\}$



$M^* = \{S, T, U\}$

$[\![a]\!] = \{S, T, U\}$
$[\![b]\!] = \{S, U\}$
$[\![c]\!] = \{T\}$
$[\![d]\!] = \{T, U\}$

What are $[\![\overline{a}]\!]$, $[\![\overline{b}]\!]$, $[\![\overline{c}]\!]$, and $[\![\overline{d}]\!]$?

They have to be the complement sets
since this how the overall semantics of $\mathcal{S}^\dagger$ works

EXAMPLE OF $[\![u]\!] = \{C \in M^* : [u] \in C\}$

Let
$\Gamma = \{$All b are a, All $\overline{b}$ are a, All $\overline{c}$ are b, All c are $\overline{b}$, All c are $d\}$.
We have already seen the $\mathbb{P}_\Gamma$ and the set of states on it.

$$S = \{[\overline{d}], [b], [a]\}$$

$$T = \{[\overline{b}], [d], [a]\}$$

$$U = \{[b], [d], [a]\}$$

$$\begin{tikzcd}
& {[a] = 1} \arrow[dl, dash] \arrow[dr, dash] & \\
{[\overline{c}] = [b]} \arrow[d, dash] & & {[d]} \arrow[d, dash] \\
{[\overline{d}]} \arrow[dr, dash] & & {[\overline{b}] = [c]} \arrow[dl, dash] \\
& {[\overline{a}] = 0} &
\end{tikzcd}$$

$$M^* = \{S, T, U\}$$

$$\begin{array}{l|l}
[\![a]\!] = \{S, T, U\} & [\![\overline{a}]\!] = \emptyset \\
[\![b]\!] = \{S, U\} & [\![\overline{b}]\!] = \{T\} \\
[\![c]\!] = \{T\} & [\![\overline{c}]\!] = \{S, U\} \\
[\![d]\!] = \{T, U\} & [\![\overline{d}]\!] = \{S\}
\end{array}$$

Check that $\mathcal{M}^* \models \Gamma$

DEFINITION

Given Γ:

$$\begin{array}{rcl} \Gamma_{all} & = & \text{the "All" sentences in } \Gamma \\ \Gamma_{some} & = & \text{the "Some" sentences in } \Gamma \end{array}$$

LEMMA

$\mathcal{M}^* \models \Gamma_{all}$.

Definition

Given Γ:

$$\begin{aligned}\Gamma_{all} &= \text{the "All" sentences in } \Gamma\\ \Gamma_{some} &= \text{the "Some" sentences in } \Gamma\end{aligned}$$

Lemma

$\mathcal{M}^* \models \Gamma_{all}$.

If Γ is consistent: $\mathcal{M}^* \models \Gamma_{some}$.

Definition

A subset $S \subseteq P$ is extendible if

$$\text{for all } p, q \in S,\ p \nleq \overline{q}.$$

Lemma

Let $S \subseteq P$ be extendible: for all $p, q \in S$, $p \not\leq \overline{q}$.

Then for all $x \in P$, either $S \cup \{x\}$ or $S \cup \{\overline{x}\}$ is again extendible.

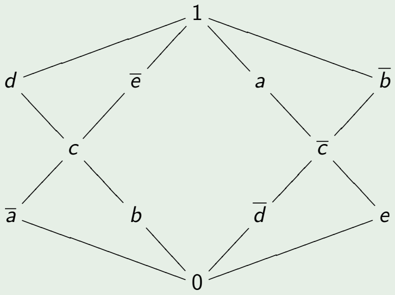
Lemma

For a subset S_0 of an orthoposet $P = (P, \leq, 0, ^-)$, the following are equivalent:

1. *S_0 is a subset of a state $\mathcal{S}$ in P.*
2. *S_0 is extendible.*

EXAMPLE

Extend $\{a, b\}$ to a state.



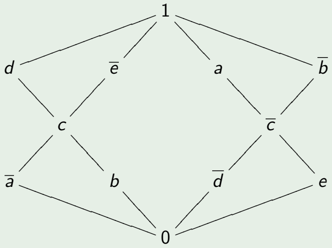
We list our elements in a sequence any way we like. For example

$$a \quad \overline{a} \quad b \quad \overline{b} \quad c \quad \overline{c} \quad d \quad \overline{d} \quad e \quad \overline{e} \quad 0 \quad 1$$

Go through them one at a time, either adding x or $\overline{x}$: whichever preserves extendibility.

EXAMPLE

Extend $\{a, b\}$ to a state.



We list our elements in a sequence any way we like. For example

$$a \quad \overline{a} \quad b \quad \overline{b} \quad c \quad \overline{c} \quad d \quad \overline{d} \quad e \quad \overline{e} \quad 0 \quad 1$$

Go through them one at a time, either adding x or $\overline{x}$: whichever preserves extendibility.

You get $\{a, b, c, d, \overline{e}, 1\}$.

Note that the order of our listing makes a difference.

DEFINITION

Let $\mathbb{P}_\Gamma$ be the orthoposet for Γ.

Let $M^* = \mathit{states}(\mathbb{P}_\Gamma)$.

The interpretation function is given by

$$[\![u]\!] = \{\mathcal{S} \in M^* : [u] \in \mathcal{S}\}$$

LEMMA

$\mathcal{M}^* \models \Gamma_{all}$

Definition

Γ is inconsistent if $\Gamma \vdash \perp$.
(That is, there is a proof tree over Γ whose root is a contradiction like Some x are $\overline{x}$.)

Γ is consistent if it is not inconsistent.

Our next result

If Γ is consistent, then $\mathcal{M}^* \models \Gamma_{some}$

Definition

Γ is satisfiable if it has a model.

Lemma (equivalent to soundness)

If Γ is satisfiable, then Γ is consistent.

Lemma (equivalent to completeness)

Let Γ be consistent in $\mathcal{S}^\dagger$.
Then $\mathcal{M}^* \models \Gamma$.

Another way to say it: every consistent set has a model.

Proof.

First, suppose that Γ contains All p are q.
We check that $[\![p]\!] \subseteq [\![q]\!]$.

$$\begin{array}{rcll} [\![p]\!] & = & \{S \in M^* : [p] \in M^*\} & \\ & \subseteq & \{S \in M^* : [q] \in M^*\} & \text{Why?? This is the key step!} \\ & = & [\![q]\!] & \end{array}$$

$\square$

The $\mathcal{S}^\dagger$ canonical model, finishing the story

Lemma (equivalent to completeness)

Let Γ be consistent in $\mathcal{S}^\dagger$.
Then $\mathcal{M}^* \models \Gamma$.

Another way to say it: every consistent set has a model.

Proof.

Second, suppose that Γ contains Some u are v.

Key point There is a state S such that $\{[u],[v]\} \subseteq S$.

If no state S has $\{[u],[v]\} \subseteq S$, then $\{[u],[v]\}$ is not extendible.
And so $u \leq \overline{v}$.

This means that $\Gamma \vdash$ All u are $\overline{v}$.
Using (DARII) and (RAA), $\Gamma \vdash$ Some v are $\overline{v}$.

So we get a contradiction!

And now $S \in [\![u]\!] \cap [\![v]\!]$, so $\mathcal{M}^* \models$ Some u are v.

□

Lemma

If $\Gamma \cup \{\varphi\} \vdash \bot$, *then* $\Gamma \vdash \overline{\varphi}$.

If $\Gamma \nvdash \varphi$, *then* $\Gamma \cup \{\overline{\varphi}\} \nvdash \bot$.

Proof.

For the first one, this is exactly (RAA)!

We have a proof tree that ends in $\bot$, and the leaves are all in $\Gamma \cup \{\varphi\} \vdash \bot$.

So we can withdraw φ and infer $\overline{\varphi}$ thus:

$$\dfrac{\begin{array}{c}[\varphi]\\ \vdots \\ \bot\end{array}}{\overline{\varphi}}\ \mathrm{RAA}$$

$\square$

One more lemma

Lemma

If $\Gamma \cup \{\varphi\} \vdash \bot$, *then* $\Gamma \vdash \overline{\varphi}$.

If $\Gamma \nvdash \varphi$, *then* $\Gamma \cup \{\overline{\varphi}\} \nvdash \bot$.

Proof.

For the second part, let's fix φ and write the first part with $\overline{\varphi}$ instead of φ:

If $\Gamma \cup \{\overline{\varphi}\} \vdash \bot$, then $\Gamma \vdash \overline{\overline{\varphi}}$.

That is,
If $\Gamma \cup \{\overline{\varphi}\} \vdash \bot$, then $\Gamma \vdash \varphi$.

Now we take the contrapositive to get the first part. $\square$

Completeness: If $\Gamma \not\models \varphi$, there is a model of Γ falsifying φ.

Here is the proof:

Assume that $\Gamma \not\models \varphi$.

By our last lemma, $\Gamma \cup \{\overline{\varphi}\}$ is consistent.

And we also proved that every consistent set has a model.

So, there is a model of Γ where $\overline{\varphi}$ is true
and hence φ is false!

We did it!